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Metric Properties of Semialgebraic Mappings

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Abstract We give various quantitative versions of Łojasiewicz inequalities for semialgebraic sets and mappings, both in the local and global case.

Keywords Łojasiewicz exponent · Semialgebraic set · Semialgebraic mapping · Polynomial mapping

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1 Introduction

Łojasiewicz inequalities emerged in the late 1950s as the main tool in the division of distributions by a real polynomial (Hörmander [17]) and by a real analytic function (Łojasiewicz [25,26]). Since then they have turned out to be of use in numerous branches of mathematics, including differential equations, dynamical systems and

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singularity theory (see for instance [24,28,38]). Quantitative versions of these inequalities, involving e.g. computing or estimating the relevant exponents, are of importance in real and complex algebraic geometry (see [43] and also [31–33]). Recently a strong demand for explicit estimates of the Łojasiewicz exponent comes from optimization theory (see for instance [23,37]) and also from estimates for global error bounds [27].

Our goal is to give various quantitative versions of these inequalities in the real case both in the local and global context. We denote by $\mathbb{K}$ the field $\mathbb{R}$ of real numbers or the field $\mathbb{C}$ of complex numbers.

Let $X \subset \mathbb{K}^N$ be a closed semialgebraic set. (If $\mathbb{K} = \mathbb{C}$ we consider X as a subset of $\mathbb{R}^{2N}$.) Assume that $0 \in X$ is an accumulation point of X and $f, g : X \to \mathbb{R}$ are two continuous semialgebraic functions such that $f^{-1}(0) \subset g^{-1}(0)$. Then there are positive constants C, η, ε such that the following *Łojasiewicz inequality* holds (see e.g. [4]):

$$|f(x)| \geq C|g(x)|^{\eta} \quad \text{if} \quad x \in X,\ |x| < \varepsilon. \tag{1.1}$$

The infimum of the exponents η in (1.1) is called the *Łojasiewicz exponent* of the pair (f, g) *on the set* X *at* 0 and is denoted by $\mathcal{L}_0(f, g|X)$. It is known (see [3]) that $\mathcal{L}_0(f, g|X)$ is a rational number; moreover, inequality (1.1) holds actually with $\eta = \mathcal{L}_0(f, g|X)$ for some $\varepsilon, C > 0$ (see [41]). An asymptotic estimate for $\mathcal{L}_0(f, g|X)$ was obtained by Solernó [39]; we shall discuss it in Remark 2.4. Inequality (1.1) is valid in a more general setting of functions definable in an o-minimal polynomially bounded structure (in particular for subanalytic functions) (see [12,16]).

From the point of view of applications the most interesting case of inequality (1.1) is when f is a semialgebraic function and $g(x) = \operatorname{dist}(x, X \cap f^{-1}(0))$. We shall consider the distance induced by the Euclidean norm. By convention $\operatorname{dist}(x, \emptyset) = 1$. More precisely, we shall consider the following case. Let $F = (f_1, \ldots, f_m) : \mathbb{K}^N \to \mathbb{K}^m$ be a semialgebraic mapping and $X \subset \mathbb{K}^N$ a closed semialgebraic set such that $0 \in X$ is an accumulation point of X. So we have the following Łojasiewicz inequality:

$$|F(x)| \geq C \operatorname{dist}(x, F^{-1}(0) \cap X)^{\eta} \quad \text{if } x \in X,\ |x| < \varepsilon. \tag{1.2}$$

The smallest exponent η in (1.2) is called the *Łojasiewicz exponent* of F *on the set* X *at* 0 and is denoted by $\mathcal{L}_0^{\mathbb{K}}(F|X)$. In Sect. 2 we shall give explicit bounds for $\mathcal{L}_0^{\mathbb{K}}(F|X)$ in terms of the degrees of the data involved. The main result of this section is an explicit estimate for the local Łojasiewicz exponent for separation of semialgebraic sets (see Theorem 1.1).

The second aim of this article is to obtain similar results but for the Łojasiewicz exponent at infinity. Assume now that a closed semialgebraic set $X \subset \mathbb{K}^N$ is unbounded. By the *Łojasiewicz exponent at infinity of a mapping* $F : X \to \mathbb{K}^m$ we mean the supremum of the exponents ν in the following Łojasiewicz inequality:

$$|F(x)| \geq C|x|^{\nu} \quad \text{for} \quad x \in X,\ |x| \geq R, \tag{1.3}$$

for some positive constants C, R; we denote it by $\mathcal{L}_\infty^{\mathbb{K}}(F|X)$. If $X = \mathbb{K}^N$ we call the exponent $\mathcal{L}_\infty^{\mathbb{K}}(F|X)$ the *Łojasiewicz exponent at infinity* of F and denote it by

$\mathcal{L}_{\infty}^{\mathbb{K}}(F)$. Clearly $\mathcal{L}_{\infty}^{\mathbb{K}}(F|X)$ may be negative. Note that inequality (1.3) holds only when $X \cap F^{-1}(0)$ is compact.

The next inequality, called the Hörmander–Łojasiewicz inequality [17], is always valid for a continuous semialgebraic mapping:

$$|F(x)| \geq C\Big(\frac{\operatorname{dist}(x, F^{-1}(0) \cap X)}{1+|x|^2}\Big)^{\theta} \quad \text{for } x \in X, \tag{1.4}$$

where C, θ are some positive constants. In Sect. 3 we state Theorem 3.2 which is a global quantitative version of regular separation at infinity of semialgebraic sets. It implies, in particular, an estimate for the exponent (see Corollaries 3.3, 3.4).

The paper is organized as follows: in Sects. 2 and 3 we discuss Łojasiewicz inequalities respectively in the local and global case. The proofs of the main results are given in the last section.

2 The Łojasiewicz Exponent at a Point

We will give an estimate from above of the Łojasiewicz exponent for the regular separation of closed semialgebraic sets and for a continuous semialgebraic mapping on a closed semialgebraic set. Let us start from some notation. Let $X \subset \mathbb{R}^N$ be a closed semialgebraic set. It is known that X has a decomposition

$$X = X_1 \cup \cdots \cup X_k \tag{2.1}$$

into the union of closed basic semialgebraic sets

$$X_i = \{x \in \mathbb{R}^N : g_{i,1}(x) \geq 0, \dots, g_{i,r_i}(x) \geq 0,\ h_{i,1}(x) = \cdots = h_{i,l_i}(x) = 0\}, \tag{2.2}$$

$i = 1, \dots k$ (see [4]), where $g_{i,1}, \dots, g_{i,r_i}, h_{i,1}, \dots, h_{i,l_i} \in \mathbb{R}[x_1, \dots, x_N]$. Assume that r_i is the smallest possible number of the inequalities $g_{i,j}(x) \geq 0$ in the definition of X_i, for $i = 1, \dots, k$. Denote by $r(X)$ the minimum of $\max\{r_1, \dots, r_k\}$ over all decompositions (2.1) into unions of sets of the form (2.2). As shown by Bröcker [6] (cf. [5,35]),

$$r(X) \leq \frac{N(N+1)}{2}. \tag{2.3}$$

Denote by $\kappa(X)$ the mimimum of the numbers

$$\max\{\deg g_{1,1}, \dots, \deg g_{k,r_k}, \deg h_{1,1}, \dots, \deg h_{k,l_k}\}$$

over all decompositions (2.1) of X into the union of sets of the form (2.2), provided $r_i \leq r(X)$. Obviously $r(X) = 0$ if and only if X is an algebraic set. The numbers $r(X)$ and $\kappa(X)$ characterize the complexity of the semialgebraic set X. For more information about the complexity see for example [2,4,34].

Theorem 2.1 *Let $X, Y \subset \mathbb{R}^N$ be closed semialgebraic sets, and suppose $0 \in X \cap Y$. Set $r = r(X) + r(Y)$ and $d = \max\{\kappa(X), \kappa(Y)\}$. Then there exist a neighbourhood $U \subset \mathbb{R}^N$ of 0 and a positive constant C such that*

$$\operatorname{dist}(x, X) + \operatorname{dist}(x, Y) \geq C \operatorname{dist}(x, X \cap Y)^{d(6d-3)^{N+r-1}} \quad \text{for } x \in U. \tag{2.4}$$

If, additionally, 0 is an isolated point of $X \cap Y$, then for some neighbourhood $U \subset \mathbb{R}^N$ of 0 and some positive constant C,

$$\operatorname{dist}(x, X) + \operatorname{dist}(x, Y) \geq C|x|^{\frac{(2d-1)^{N+r}+1}{2}} \quad \text{for } x \in U. \tag{2.5}$$

The proof of the above theorem will be carried out in Sect. 4. The key point in the proof will be the following inequality [22, Cor. 8]. Let $X = (g_1, \ldots, g_k)^{-1}(0)$ and $Y = (h_1, \ldots, h_l)^{-1}(0) \subsetneq \mathbb{R}^N$, where $g_i, h_j \in \mathbb{R}[x_1, \ldots, x_N]$ are polynomials of degree not greater than d. Let $a \in \mathbb{R}^N$. Then there exists a positive constant C such that

$$\operatorname{dist}(x, X) + \operatorname{dist}(x, Y) \geq C \operatorname{dist}(x, X \cap Y)^{d(6d-3)^{N-1}} \tag{KS$_1$}$$

in a neighbourhood of a. If, additionally, a is an isolated point of $X \cap Y$, then

$$\operatorname{dist}(x, X) + \operatorname{dist}(x, Y) \geq C|x - a|^{\frac{(2d-1)^{N}+1}{2}} \tag{G}$$

in a neighbourhood of a for some positive $C > 0$, which is a consequence of [14].

Theorem 2.1 implies

Corollary 2.2 *Let $F : X \to \mathbb{R}^m$ be a continuous semialgebraic mapping, where $X \subset \mathbb{R}^N$ is a closed semialgebraic set, and suppose $0 \in X$ and $F(0) = 0$. Set $r = r(X) + r(\text{graph } F)$ and $d = \max\{\kappa(X), \kappa(\text{graph } F)\}$. Then*

$$\mathcal{L}_0^{\mathbb{R}}(F|X) \leq d(6d-3)^{N+r-1}. \tag{2.6}$$

If, additionally, 0 is an isolated zero of F, then

$$\mathcal{L}_0^{\mathbb{R}}(F|X) \leq \frac{(2d-1)^{N+r}+1}{2}. \tag{2.7}$$

Remark 2.3 The inequality (2.6) is crucial for estimating the rate of convergence of algorithms (based on semi-definite programming) of minimization of a polynomial on a basic semialgebraic set. Indeed, (2.6) enabled us [23] to reduce effectively the problem of minimizing polynomials on a compact semialgebraic set to the case of minimizing polynomials on a ball, which is much simpler [36].

Remark 2.4 We shall now comment on the result of Solernó [39] concerning the Łojasiewicz exponent $\mathcal{L}_0(f, g|X)$ in the inequality (1.1) for a pair (f, g) of continuous semialgebraic functions on a closed semialgebraic set $X \subset \mathbb{R}^N$. In general his estimate is of the form

$$\mathcal{L}_0(f, g|X) \le D^{M^{ca}}, \qquad (\mathrm{S}_a)$$

where D is a bound for the degrees of the polynomials involved in a description of f, g and X; M is the number of variables in these formulas (so in general $M \ge N$); a is the maximum number of alternating blocs of quantifiers in these formulas; and c is an (unspecified) universal constant. The estimate (S_a) was obtained from the effective Tarski–Seidenberg theorem [15].

In our Corollary 2.2 only the function $g(x) = \mathrm{dist}(x, X \cap F^{-1}(0))$ is defined by a formula which is not quantifier-free, and it has two alternating blocs of quantifiers, hence $a = 2$. So Solernó's estimate (S_a) reads $\mathcal{L}_0^{\mathbb{R}}(F|X) \le d^{(N+2)^{2c}}$, which is comparable with our estimate $\mathcal{L}_0^{\mathbb{R}}(F|X) \le d(6d-3)^{N+r-1}$ since $r(X) \le \frac{1}{2}N(N+1)$ by (2.3). Indeed, we believe that the universal constant c is at least 1, probably $c \gg 1$. Needless to say, our estimate is explicit.

Recall that for a real polynomial mapping $F : \mathbb{R}^N \to \mathbb{R}^m$ such that $d = \deg F$ (where $\deg F$ is the maximum of the degrees of the components of F) we have

$$\mathcal{L}_0^{\mathbb{R}}(F) \le d(6d-3)^{N-1} \qquad (\mathrm{KS}_2)$$

(see [22, Cor. 6] or [29]). Actually both papers are based on an estimate for the Łojasiewicz exponent in the gradient inequality obtained in [11,13].

We now consider a polynomial mapping restricted to an algebraic set. From Corollary 2.2 we obtain an estimation of its local Łojasiewicz exponent, also for a non-isolated zero-set (cf. [30,40] for mappings with isolated zeros).

Corollary 2.5 *Let* $F : (\mathbb{K}^N, 0) \to (\mathbb{K}^m, 0)$ *be a polynomial mapping, let* $X \subset \mathbb{K}^N$ *be an algebraic set defined by a system of equations* $g_1(x) = \cdots = g_r(x) = 0$*, where* $g_1, \ldots, g_r \in \mathbb{K}[x_1, \ldots, x_N]$*, and let* $d = \max\{\deg F, \deg g_1, \ldots, \deg g_r\}$*. Assume that* $d > 0$ *and* $0 \in X$*.*

(a) *If* $\mathbb{K} = \mathbb{R}$*, then* $\mathcal{L}_0^{\mathbb{R}}(F|X) \le d(6d-3)^{N-1}$.
(b) *If* $\mathbb{K} = \mathbb{C}$*, then* $\mathcal{L}_0^{\mathbb{C}}(F|X) \le d^N$.

Indeed, assertion (a) immediately follows from Corollary 2.2. We will prove (b). Let $G = (F, g_1, \ldots, g_r) : \mathbb{C}^N \to \mathbb{C}^{m+r}$. We can assume that $m \ge N$. Similarly to [42, Thm. 1], we prove that there exists a linear mapping $L = (L_1, \ldots, L_m) : \mathbb{C}^{m+r} \to \mathbb{C}^m$ of the form $L_i(y_1, \ldots, y_m) = y_i + \sum_{j=r+1}^{m} \alpha_{i,j} y_j$, $i = 1, \ldots, m$, where $\alpha_{i,j} \in \mathbb{C}$, such that $\mathcal{L}_0^{\mathbb{C}}(G|X) = \mathcal{L}_0^{\mathbb{C}}(L \circ G|X)$. Moreover, $\deg L_j \circ G \le d$ for $j = 1, \ldots, m$. Cygan [8] proved that for analytic sets $Z, Y \subset \mathbb{C}^{N+m}$ the intersection index at 0 of Z and Y is a separation exponent of Z and Y at $0 \in Z \cap Y$. It is known that for $Z = \mathbb{C}^N \times \{0\}$ and $Y =$ graph $L \circ G$, the index does not exceed d^N (see [10,44]), so $\mathcal{L}_0^{\mathbb{C}}(L \circ G) \le d^N$. Since $G^{-1}(0) = F^{-1}(0) \cap X$ and by definition of L we have $G(x) = (F(x), 0)$ for $x \in X$, it follows that $\mathcal{L}_0^{\mathbb{C}}(F|X) \le d^N$, proving (b).

3 The Łojasiewicz Exponent at Infinity

Let us first recall some known results on the Łojasiewicz exponent at infinity of a polynomial mapping $F = (f_1, \ldots, f_m) : \mathbb{C}^N \to \mathbb{C}^m$ on an algebraic set $V \subset \mathbb{C}^N$ (see definition of $\mathcal{L}_\infty^{\mathbb{C}}(F|V)$ and $\mathcal{L}_\infty^{\mathbb{C}}(F)$ in Introduction). Let $\deg f_j = d_j$, $j = 1, \ldots, m$, $d_1 \geq \cdots \geq d_m > 0$ and set

$$B(d_1, \ldots, d_m; k) = \begin{cases} d_1 \cdots d_m & \text{for} \quad m \leq k, \\ d_1 \cdots d_{k-1} d_m & \text{for} \quad m > k. \end{cases}$$

Chądzyński [7] proved that

$$\mathcal{L}_\infty^{\mathbb{C}}(F) \geq d_2 - d_1 d_2 + \sum_{b \in F^{-1}(0)} \mu_b(F), \tag{Ch}$$

where $\mu_b(F)$ is the multiplicity of F at b, provided $N = m = 2$ and $\#F^{-1}(0) < \infty$. For arbitrary $m \geq N$, under the assumption $\#F^{-1}(0) < \infty$, Kollár [21] proved that

$$\mathcal{L}_\infty^{\mathbb{C}}(F) \geq d_m - B(d_1, \ldots, d_m; N); \tag{K}$$

then Cygan et al. [10] improved this to

$$\mathcal{L}_\infty^{\mathbb{C}}(F) \geq d_m - B(d_1, \ldots, d_m; N) + \sum_{b \in F^{-1}(0)} \mu_b(F), \tag{CKT}$$

where $\mu_b(F)$ is the intersection multiplicity (in the sense of Achilles et al. [1]) of the graph of F and $\mathbb{C}^n \times \{0\}$ at the point $(b, 0)$. For a complex k-dimensional algebraic variety $V \subset \mathbb{C}^N$ of degree D the following estimate was obtained by Jelonek [18,19]:

$$\mathcal{L}_\infty^{\mathbb{C}}(F|V) \geq d_m - D \cdot B(d_1, \ldots, d_m; k) + \sum_{b \in F^{-1}(0) \cap V} \mu_b(F), \tag{J}$$

where $\#(F^{-1}(0) \cap V) < \infty$. Cygan [9] gave the following global inequality:

$$|F(x)| \geq C \Big(\frac{\operatorname{dist}(x, F^{-1}(0))}{1 + |x|^2} \Big)^{B(d_1, \ldots, d_m; N)} \quad \text{for} \quad x \in \mathbb{C}^N \tag{C_1}$$

for some positive constant C. Moreover she proved in [8] that for complex algebraic sets $X, Y \subset \mathbb{C}^N$ there exists a positive constant C such that

$$\operatorname{dist}(x, X) + \operatorname{dist}(x, Y) \geq C \Big(\frac{\operatorname{dist}(x, X \cap Y)}{1 + |x|^2} \Big)^{\deg X \cdot \deg Y} \quad \text{for } x \in \mathbb{C}^N. \tag{C_2}$$

A result similar to (C_2) was obtained by Ji et al. [20].

For real algebraic sets we have the following global Łojasiewicz inequality (see [22]). If $X, Y \subset \mathbb{R}^N$ are algebraic sets defined by systems of polynomial equations of degrees at most d, then for some positive constant C,

$$\operatorname{dist}(x, X) + \operatorname{dist}(x, Y) \geq C\Big(\frac{\operatorname{dist}(x, X \cap Y)}{1+|x|^2}\Big)^{d(6d-3)^{N-1}} \quad \text{for } x \in \mathbb{R}^N. \qquad (\mathrm{KS}_3)$$

In particular, we have the following global Łojasiewicz inequality (see [22]). Let $F = (f_1, \ldots, f_m) : \mathbb{R}^N \to \mathbb{R}^m$ be a polynomial mapping of degree d. Then for some positive constant C,

$$|F(x)| \geq C\Big(\frac{\operatorname{dist}(x, F^{-1}(0))}{1+|x|^2}\Big)^{d(6d-3)^{N-1}} \quad \text{for } x \in \mathbb{R}^N. \qquad (\mathrm{KS}_4)$$

Moreover, if the set $F^{-1}(0)$ is compact, then

$$\mathcal{L}_\infty^{\mathbb{R}}(F) \geq -d(6d-3)^{N-1}. \qquad (\mathrm{KS}_5)$$

By using (KS_4) we obtain a global Łojasiewicz inequality for polynomial mappings.

Proposition 3.1 *Let $X \subset \mathbb{R}^N$ be an algebraic set defined by a system of polynomial equations $g_1(x) = \cdots = g_r(x) = 0$, where $g_1, \ldots, g_r \in \mathbb{R}[x_1, \ldots, x_N]$. Let $F : \mathbb{R}^N \to \mathbb{R}^m$ be a polynomial mapping and let $d = \max\{\deg F, \deg g_1, \ldots, \deg g_r\}$. Then for some positive constant C,*

$$|F(x)| \geq C\Big(\frac{\operatorname{dist}(x, F^{-1}(0) \cap X)}{1+|x|^2}\Big)^{d(6d-3)^{N-1}} \quad \textit{for } x \in X.$$

If, additionally, the set X is unbounded and $F^{-1}(0) \cap X$ is compact, then

$$\mathcal{L}_\infty^{\mathbb{R}}(F|X) \geq -d(6d-3)^{N-1}.$$

Indeed, let $G = (g_1, \ldots, g_r) : \mathbb{R}^N \to \mathbb{R}^r$, and let $H : \mathbb{R}^N \to \mathbb{R}^{m+r}$ be a polynomial mapping defined by $H(x) = (F(x), G(x))$ for $x \in \mathbb{R}^N$. Then $H^{-1}(0) = F^{-1}(0) \cap X$, so from ($\mathrm{KS}_4$) we deduce the first assertion. If $F^{-1}(0) \cap X$ is compact, then so is $H^{-1}(0)$, and the second assertion follows immediately from the first (cf. (KS_5)).

In the above proof we cannot apply (Ch), (K), (CKT), (J) or (C_1), because the complexification of a real polynomial mapping with compact real zero-set may have an unbounded zero-set.

The following global Łojasiewicz inequality for semialgebraic sets is the main result of this section. The proof is given in Sect. 4.

Theorem 3.2 *Let $X, Y \subset \mathbb{R}^N$ be closed semialgebraic sets. Set $r = r(X) + r(Y)$ and $d = \max\{\kappa(X), \kappa(Y)\}$. Then there exists a positive constant C such that*

$$\operatorname{dist}(x, X) + \operatorname{dist}(x, Y) \geq C\Big(\frac{\operatorname{dist}(x, X \cap Y)}{1+|x|^d}\Big)^{d(6d-3)^{N+r-1}} \quad \textit{for } x \in \mathbb{R}^N. \qquad (3.1)$$

Theorem 3.2 immediately implies the following.

Corollary 3.3 *Let $F : X \to \mathbb{R}^m$ be a continuous semialgebraic mapping, where $X \subset \mathbb{R}^N$ is a closed semialgebraic set. If $d = \max\{\kappa(X), \kappa(Y)\}$ and $r = r(X)+r(Y)$, where $Y = \operatorname{graph} F$, then there exists a positive constant C such that*

$$|F(x)| \geq C\Big(\frac{\operatorname{dist}(x, F^{-1}(0) \cap X)}{1+|x|^d}\Big)^{d(6d-3)^{N+r-1}} \quad \text{for } x \in X.$$

In particular, if the set X is unbounded and $F^{-1}(0) \cap X$ is compact, then

$$\mathcal{L}_\infty^{\mathbb{R}}(F|X) \geq (1-d)d(6d-3)^{N+r-1}.$$

For a polynomial mapping $F : X \to \mathbb{R}^m$ we have $r(\operatorname{graph} F) = r(X)$ and $\kappa(\operatorname{graph} F) = \max\{\deg F, \kappa(X)\}$, so we obtain

Corollary 3.4 *Let $F : X \to \mathbb{R}^m$ be a polynomial mapping, where $X \subset \mathbb{R}^N$ is a closed semialgebraic set. If $D = \max\{2, \kappa(X)\}$ and $d = \max\{\deg F, D\}$, and $r = 2r(X)$, then*

$$|F(x)| \geq C\Big(\frac{\operatorname{dist}(x, F^{-1}(0) \cap X)}{1+|x|^D}\Big)^{d(6d-3)^{N+r-1}} \quad \text{for } x \in X. \tag{3.2}$$

In particular, if the set X is unbounded and $F^{-1}(0) \cap X$ is compact, then

$$\mathcal{L}_\infty^{\mathbb{R}}(F|X) \geq -\frac{D}{2}d(6d-3)^{N+r-1}.$$

The above corollary is not a direct consequence of Corollary 3.3, so we will prove it separately in Sect. 4.

4 Proofs of Theorems 2.1 and 3.2 and of Corollary 3.4

It suffices to consider the case when X and Y are basic closed semialgebraic sets. So, let

$$X = \{x \in \mathbb{R}^N : g_{1,1}(x) \geq 0, \dots, g_{1,r(X)}(x) \geq 0,\ h_{1,1}(x) = \cdots = h_{1,l}(x) = 0\},$$
$$Y = \{x \in \mathbb{R}^N : g_{2,1}(x) \geq 0, \dots, g_{2,r(Y)}(x) \geq 0,\ h_{2,1}(x) = \cdots = h_{2,l}(x) = 0\},$$

where $g_{i,j}, h_{i,s} \in \mathbb{R}[x_1, \dots, x_N]$. We may indeed assume that X and Y are defined by the same number of equations, because we can repeat the same equations if necessary. Let $r_1 = r(X)$, $r_2 = r(Y)$, $r = r_1 + r_2$, and let $G_i : \mathbb{R}^N \times \mathbb{R}^r \to \mathbb{R}^{r_i}$, $i = 1, 2$, be the polynomial mappings defined by

$$G_1(x, y_1, \dots, y_r) = (g_{1,1}(x) - y_1^2, \dots, g_{1,r_1}(x) - y_{r_1}^2),$$
$$G_2(x, y_1, \dots, y_r) = (g_{2,1}(x) - y_{r_1+1}^2, \dots, g_{2,r_2}(x) - y_{r_1+r_2}^2).$$

Let

$$A = \{(x, y_1, \dots, y_r) \in \mathbb{R}^N \times \mathbb{R}^r : G_1(x, y) = 0,\ h_{1,1}(x) = \dots = h_{1,l}(x) = 0\},$$
$$B = \{(x, y_1, \dots, y_r) \in \mathbb{R}^N \times \mathbb{R}^r : G_2(x, y) = 0,\ h_{2,1}(x) = \dots = h_{2,l}(x) = 0\}.$$

Then the sets A an B are algebraic and $\pi(A) = X, \pi(B) = Y$, where $\pi : \mathbb{R}^N \times \mathbb{R}^r \ni (x, y) \mapsto x \in \mathbb{R}^N$. Moreover, $\deg G_1, \deg G_2 \le d$, provided $d > 1$.

From the definitions of A and B, we immediately obtain

$$\forall_{x_1 \in X}\ \forall_{x_2 \in Y}\ \exists_{y \in \mathbb{R}^r}\ (x_1, y) \in A\ \wedge\ (x_2, y) \in B; \tag{4.1}$$

moreover,

$$\forall_{x \in \mathbb{R}^N \setminus X}\ \exists_{x_1 \in X}\ \forall_{y \in \mathbb{R}^r}\ [\operatorname{dist}(x, X) = |x - x_1|\ \wedge\ (x_1, y) \in A\ \Rightarrow \operatorname{dist}(x, X) \ge \operatorname{dist}((x, y), A)] \tag{4.2}$$

and

$$\forall_{x \in \mathbb{R}^N \setminus Y}\ \exists_{x_2 \in Y}\ \forall_{y \in \mathbb{R}^r}\ [\operatorname{dist}(x, Y) = |x - x_2|\ \wedge\ (x_2, y) \in B\ \Rightarrow \operatorname{dist}(x, Y) \ge \operatorname{dist}((x, y), B)]. \tag{4.3}$$

Indeed, we will prove (4.2); the proof of (4.3) is similar. Take $x \in \mathbb{R}^N \setminus X$ and let $x_1 \in X$ satisfy $\operatorname{dist}(x, X) = |x - x_1|$. So, for any $y \in \mathbb{R}^r$ such that $(x_1, y) \in A$, we have

$$\operatorname{dist}(x, X) = |x - x_1| = |(x, y) - (x_1, y)| \ge \operatorname{dist}((x, y), A).$$

This gives (4.2).

Proof of Theorem 2.1 We will assume that the origin is a non-isolated point of $X \cap Y$; otherwise, we proceed in the same way using formula (G) instead of (KS_1). Let $p = d(6d - 3)^{N+r-1}$.

Claim 1. The assertion (2.4) is equivalent to

$$\operatorname{dist}(x, Y) \ge C' \operatorname{dist}(x, X \cap Y)^p \quad \text{for } x \in (\partial X) \cap U_1 \tag{4.4}$$

for a neighbourhood $U_1 = \{x \in \mathbb{R}^N : |x| < \rho\}$ of the origin, $\rho < 1$, and some positive constant C', where ∂X denotes the boundary of X (cf. [9, Lem. 4.2] and [22, Proof of Theorem 2]). Indeed, the implication (2.4)⇒(4.4) is obvious. Assume that the converse fails. Then for a neighbourhood $U_2 = \{x \in \mathbb{R}^N : |x| < \frac{\rho}{2}\}$ of the origin, there exists a sequence $a_\nu \in U_2$ such that $a_\nu \to 0$ and

$$\operatorname{dist}(a_\nu, X) + \operatorname{dist}(a_\nu, Y) < \frac{1}{\nu} \operatorname{dist}(a_\nu, X \cap Y)^p \quad \text{for } \nu \in \mathbb{N}. \tag{4.5}$$

Taking a subsequence if necessary, it suffices to consider two cases: $a_\nu \notin X$ for $\nu \in \mathbb{N}$ or $a_\nu \in \operatorname{Int} X$ for $\nu \in \mathbb{N}$.

Assume that $a_\nu \notin X$ for $\nu \in \mathbb{N}$. Let $x_\nu \in (\partial X) \cap U_1$ be such that $\operatorname{dist}(a_\nu, X) = |a_\nu - x_\nu|$. Since $\rho < 1$, we have $\operatorname{dist}(a_\nu, X)^{\frac{1}{p}} \geq \operatorname{dist}(a_\nu, X)$. So, for some $C'' > 0$,

$$[\operatorname{dist}(a_\nu, X) + \operatorname{dist}(a_\nu, Y)]^{\frac{1}{p}} \geq \operatorname{dist}(a_\nu, X)^{\frac{1}{p}} \geq C'' \operatorname{dist}(a_\nu, X),$$

and, by (4.4),

$$[\operatorname{dist}(a_\nu, X) + \operatorname{dist}(a_\nu, Y)]^{\frac{1}{p}} \geq \operatorname{dist}(x_\nu, Y)^{\frac{1}{p}} \geq C'' \operatorname{dist}(x_\nu, X \cap Y).$$

Since $\operatorname{dist}(a_\nu, X) + \operatorname{dist}(x_\nu, X \cap Y) \geq \operatorname{dist}(a_\nu, X \cap Y)$, by adding the above inequalities, we obtain

$$[\operatorname{dist}(a_\nu, X) + \operatorname{dist}(a_\nu, Y)]^{\frac{1}{p}} \geq \frac{C''}{2} \operatorname{dist}(a_\nu, X \cap Y).$$

This contradicts (4.5) and proves the Claim in this case.

Now assume that all a_ν are in $\operatorname{Int} X$. Let $y_\nu \in Y \cap U_1$ be such that $\operatorname{dist}(a_\nu, Y) = |a_\nu - y_\nu|$. By (4.5) we see that $y_\nu \notin X$, so there exists $x_\nu \in (\partial X) \cap [a_\nu, y_\nu]$, where $[a_\nu, y_\nu]$ is the segment with endpoints a_ν, y_ν.

By (4.5) and the choice of ρ,

$$|a_\nu - x_\nu| \leq \operatorname{dist}(a_\nu, Y) < \frac{1}{\nu} \operatorname{dist}(a_\nu, X \cap Y)^p < \frac{1}{2} \operatorname{dist}(a_\nu, X \cap Y) \quad \text{for } \nu \geq 2.$$

Hence,

$$\operatorname{dist}(x_\nu, X \cap Y) \geq \operatorname{dist}(a_\nu, X \cap Y) - |a_\nu - x_\nu| \geq \frac{1}{2} \operatorname{dist}(a_\nu, X \cap Y) \quad \text{for } \nu \geq 2.$$

This together with (4.5) gives

$$\operatorname{dist}(x_\nu, Y) \leq \operatorname{dist}(a_\nu, Y) < \frac{1}{\nu} \operatorname{dist}(a_\nu, X \cap Y)^p \leq \frac{2^p}{\nu} \operatorname{dist}(x_\nu, X \cap Y)^p \quad \text{for } \nu \geq 2.$$

This contradicts (4.4) and proves the claim in this case. Summing up, we have proved Claim 1.

If $d = 1$, then the assertion is trivial. Assume that $d > 1$. By (KS_1), there exists a positive constant C such that

$$\operatorname{dist}((x, y), A) + \operatorname{dist}((x, y), B) \geq C \operatorname{dist}((x, y), A \cap B)^{d(6d-3)^{N+r-1}} \tag{4.6}$$

in a neighbourhood W of $0 \in \mathbb{R}^{N+r}$. Obviously, for any $(x, y) \in \mathbb{R}^{N+r}$,

$$\operatorname{dist}((x, y), A \cap B) \geq \operatorname{dist}(x, X \cap Y). \tag{4.7}$$

One can assume that $g_{i,j}(0) = 0$ for any i, j. Indeed, if $g_{i,j}(0) < 0$ for some i, j, then $0 \notin X$ or $0 \notin Y$, which contradicts the assumption. If $g_{i,j}(0) > 0$ for some i, j, then we can omit this inequality in the definition of X, respectively Y, and the germ at 0 of X, respectively Y will not change. If $g_{i,j}(0) > 0$ for any i, j, then the assertion reduces to (KS_2). So, there exists a neighbourhood $W_1 = U_3 \times U' \times U'' \subset W$ of $0 \in \mathbb{R}^{N+r}$, where $U_3 \subset \mathbb{R}^N$, $U' \subset \mathbb{R}^{r(X)}$ and $U'' \subset \mathbb{R}^{r(Y)}$ such that:

$$\begin{aligned} &\text{for any } (x_1, y', y'') \in \mathrm{A}, \;\; \text{where } x_1 \in \mathbb{R}^N, y' \in \mathbb{R}^{r(X)}, y'' \in \mathbb{R}^{r(Y)} \\ &\qquad\qquad\qquad\qquad\qquad \text{if } x_1 \in X \cap U_3, \text{ then } y' \in U' \end{aligned} \tag{4.8}$$

and

$$\begin{aligned} &\text{for any } (x_2, y', y'') \in B, \;\; \text{where } x_2 \in \mathbb{R}^N, y' \in \mathbb{R}^{r(X)}, y'' \in \mathbb{R}^{r(Y)} \\ &\qquad\qquad\qquad\qquad\qquad \text{if } x_2 \in Y \cap U_3, \text{ then } y'' \in U''. \end{aligned} \tag{4.9}$$

Let $U \subset U_3$ be a neighbourhood of $0 \in \mathbb{R}^N$. If $(\partial X) \cap U = \emptyset$, then $U \subset X$ and the assertion is obvious. Assume that $(\partial X) \cap U \neq \emptyset$. Take $x \in (\partial X) \cap U$, and let $x' \in Y$ be a point for which $\operatorname{dist}(x, Y) = |x - x'|$. By (4.1) there exists $y \in \mathbb{R}^r$ such that $(x, y) \in A$ and $(x', y) \in B$. Diminishing the neighbourhood U if necessary, we may assume that $x' \in U_3$. By (4.8) and (4.9) we see that $(x, y) \in W$, so, by (4.2) and (4.3),

$$\operatorname{dist}(x, Y) \geq \operatorname{dist}((x, y), A) + \operatorname{dist}((x, y), B).$$

Summing up, (4.6), (4.7) and Claim 1 give the assertion. □

Proof of Theorem 3.2 Let $p = d(6d-3)^{N+r-1}$. If $d = 1$ then the assertion is trivial. If $X \backslash Y = \emptyset$ or $Y \backslash X = \emptyset$, then the assertion is obvious. So, we will assume that $X \backslash Y \neq \emptyset$, $Y \backslash X \neq \emptyset$ and $d > 1$. In particular $\partial X \neq \emptyset$.

By (KS_3) we have

$$\operatorname{dist}((x, y), A) + \operatorname{dist}((x, y), B) \geq C\left(\frac{\operatorname{dist}((x, y), A \cap B)}{1 + |(x, y)|^2}\right)^p \tag{4.10}$$

for $(x, y) \in \mathbb{R}^{N+r}$. Since $\operatorname{dist}((x, y), A \cap B) \geq \operatorname{dist}(x, X \cap Y)$ for any $(x, y) \in \mathbb{R}^{N+r}$ (see (4.7)), the inequality (4.10) gives

$$\operatorname{dist}((x, y), A) + \operatorname{dist}((x, y), B) \geq C\left(\frac{\operatorname{dist}(x, X \cap Y)}{1 + |(x, y)|^2}\right)^p \tag{4.11}$$

for $(x, y) \in \mathbb{R}^{N+r}$.

Claim 2. The assertion (3.1) is equivalent to

$$\operatorname{dist}(x, Y) \geq C'\left(\frac{\operatorname{dist}(x, X \cap Y)}{1 + |x|^d}\right)^p \quad \text{for } x \in \partial X \tag{4.12}$$

for some positive constant C' (cf. [9, Lem. 4.2] and [22, Proof of Theorem 2]). Indeed, the implication (3.1)⇒(4.12) is obvious. Assume that the converse fails. Then there exists a sequence $a_\nu \in \mathbb{R}^N$ such that

$$\operatorname{dist}(a_\nu, X) + \operatorname{dist}(a_\nu, Y) < \frac{1}{\nu}\Big(\frac{\operatorname{dist}(a_\nu, X \cap Y)}{1 + |a_\nu|^d}\Big)^p \quad \text{for } \nu \in \mathbb{N}. \tag{4.13}$$

By using Theorem 2.1 we see that $|a_\nu| \to \infty$. Taking subsequences of a_ν if necessary, it suffices to consider two cases: $a_\nu \notin X$ for $\nu \in \mathbb{N}$ or $a_\nu \in \operatorname{Int} X$ for $\nu \in \mathbb{N}$.

Suppose $a_\nu \notin X$ for $\nu \in \mathbb{N}$. Let $b_\nu \in \partial X$ be such that $\operatorname{dist}(a_\nu, X) = |a_\nu - b_\nu|$. Since $\big(\frac{\operatorname{dist}(a_\nu, X\cap Y)}{1+|a_\nu|^d}\big)^p$ is a bounded sequence, we have $|b_\nu - a_\nu| = \operatorname{dist}(a_\nu, X) \to 0$. So, for some $C'' > 0$ and sufficiently large ν,

$$[\operatorname{dist}(a_\nu, X) + \operatorname{dist}(a_\nu, Y)]^{\frac{1}{p}} \geq \operatorname{dist}(a_\nu, X)^{\frac{1}{p}} \geq C''\Big(\frac{\operatorname{dist}(a_\nu, X)}{1 + |a_\nu|^d}\Big),$$

and, by (4.12) and the fact that $|a_\nu| \to \infty$ and $|b_\nu - a_\nu| \to 0$,

$$[\operatorname{dist}(a_\nu, X) + \operatorname{dist}(a_\nu, Y)]^{\frac{1}{p}} \geq \operatorname{dist}(b_\nu, Y)^{\frac{1}{p}} \geq C''\Big(\frac{\operatorname{dist}(b_\nu, X \cap Y)}{1 + |a_\nu|^d}\Big).$$

Since $\operatorname{dist}(a_\nu, X) + \operatorname{dist}(b_\nu, X\cap Y) \geq \operatorname{dist}(a_\nu, X\cap Y)$, by adding the above inequalities we obtain

$$[\operatorname{dist}(a_\nu, X) + \operatorname{dist}(a_\nu, Y)]^{\frac{1}{p}} \geq \frac{C''}{2}\Big(\frac{\operatorname{dist}(a_\nu, X \cap Y)}{1 + |a_\nu|^d}\Big).$$

This contradicts (4.13) and proves the claim in this case.

Consider now the case $a_\nu \in \operatorname{Int} X$ for $\nu \in \mathbb{N}$. Let $y_\nu \in Y$ be such that $\operatorname{dist}(a_\nu, Y) = |a_\nu - y_\nu|$. By (4.13) we see that $y_\nu \notin X$, so there exist $x_\nu \in (\partial X) \cap [a_\nu, y_\nu]$ for $\nu \in \mathbb{N}$. By (4.13), for sufficiently large ν,

$$|a_\nu - x_\nu| \leq \operatorname{dist}(a_\nu, Y) < \frac{1}{\nu}\Big(\frac{\operatorname{dist}(a_\nu, X \cap Y)}{1 + |a_\nu|^d}\Big)^p < \frac{1}{2}\operatorname{dist}(a_\nu, X \cap Y). \tag{4.14}$$

Hence,

$$\operatorname{dist}(x_\nu, X \cap Y) \geq \operatorname{dist}(a_\nu, X \cap Y) - |a_\nu - x_\nu| \geq \frac{1}{2}\operatorname{dist}(a_\nu, X \cap Y).$$

This together with (4.13) gives

$$\operatorname{dist}(x_\nu, Y) \leq \operatorname{dist}(a_\nu, Y) < \frac{1}{\nu}\Big(\frac{\operatorname{dist}(a_\nu, X \cap Y)}{1 + |a_\nu|^d}\Big)^p \leq \frac{2^p}{\nu}\Big(\frac{\operatorname{dist}(x_\nu, X \cap Y)}{1 + |a_\nu|^d}\Big)^p.$$

By (4.14), for sufficiently large ν we have $|x_\nu| \le 2|a_\nu|$, so, for a positive constant C''',

$$\operatorname{dist}(x_\nu, Y) \le \frac{C'''}{\nu}\Big(\frac{\operatorname{dist}(x_\nu, X\cap Y)}{1+|x_\nu|^d}\Big)^p.$$

This contradicts (4.12) and proves the claim in this case. Summing up, we have proved Claim 2.

Take any $x_0 \in \partial X$. By (4.1), (4.2) and (4.3) there exist $x_2 \in Y$ and $y_0 \in \mathbb{R}^r$ such that $(x_0, y_0) \in A$, $(x_2, y_0) \in B$, and $\operatorname{dist}(x_0, Y) = |x_0 - x_2| \ge \operatorname{dist}((x_0, y_0), B)$. Hence from (4.11),

$$\operatorname{dist}(x_0, Y) \ge C\Big(\frac{\operatorname{dist}(x_0, X\cap Y)}{1+|(x_0, y_0)|^2}\Big)^p. \tag{4.15}$$

It is easy to observe that there exist constants $C_1, R_1 > 0$ such that for $(x, y) \in A$, $|(x, y)| \ge R_1$ we have $C_1|y|^2 \le |x|^d$. Since $d \ge 2$, for a constant $C_2 > 0$ we obtain $|(x, y)| \le C_2|x|^{d/2}$ for $(x, y) \in A$, $|(x, y)| \ge R_1$. Hence from (4.15) we easily deduce

$$\operatorname{dist}(x_0, Y) \ge C\Big(\frac{\operatorname{dist}(x_0, X\cap Y)}{1+C_2^2|x_0|^d}\Big)^p, \tag{4.16}$$

provided $|x_0| \ge R_1$. So, diminishing C if necessary, we obtain (4.16) for all $x_0 \in \partial X$. This together with Claim 2 gives the assertion of Theorem 3.3.

Proof of Corollary 3.4 Let $H : \mathbb{R}^{N+r} \to \mathbb{R}^{m+r+l}$ be a polynomial mapping defined by

$$H(x, y) = (F(x), G_1(x, y), h_{1,1}(x), \dots, h_{1,l}(x)), \quad x \in \mathbb{R}^N,\ y \in \mathbb{R}^r.$$

Then $\deg H \le d$. Let $V = F^{-1}(0) \cap X$ and let $Z = H^{-1}(0)$. By (KS_4), for some positive constant C we have

$$|H(x, y)| \ge C\Big(\frac{\operatorname{dist}((x, y), Z)}{1+|(x, y)|^2}\Big)^{d(6d-3)^{N+r-1}} \quad \text{for} \quad (x, y) \in \mathbb{R}^N \times \mathbb{R}^r.$$

Because $\operatorname{dist}((x, y), Z) \ge \operatorname{dist}(x, V)$, we obtain

$$|H(x, y)| \ge C\Big(\frac{\operatorname{dist}(x, V)}{1+|(x, y)|^2}\Big)^{d(6d-3)^{N+r-1}} \quad \text{for} \quad (x, y) \in \mathbb{R}^N \times \mathbb{R}^r. \tag{4.17}$$

It is easy to observe that there exist constants $C_1, R_1 > 0$ such that for $(x, y) \in A$ with $|(x, y)| \ge R_1$ we have $C_1|y|^2 \le |x|^D$. Since $D \ge 2$, for a constant $C_2 > 0$ we obtain $|(x, y)| \le C_2|x|^{D/2}$ for $(x, y) \in A$, $|(x, y)| \ge R_1$. Hence from (4.17) we easily deduce (3.2) for $x \in X$, $|x| \ge R_1$. So, diminishing C if necessary, we obtain (3.2) for all $x \in X$.

We now show the second assertion of the corollary. Since X is unbounded, we may assume that so is A. Since V is compact, so is $H^{-1}(0)$. By (KS_5) we have $\mathcal{L}_\infty(H) \geq -d(6d-3)^{N+r-1}$, in particular for some constants $C, R > 0$,

$$|H(x, y)| \geq C|(x, y)|^{-d(6d-3)^{N+r-1}} \quad \text{for } (x, y) \in A, \quad |(x, y)| \geq R.$$

Since $|(x, y)| \leq C_2|x|^{D/2}$ for $(x, y) \in A$, $|(x, y)| \geq R_1$, it follows that, for some constant $C_3 > 0$.

$$|F(x)| = |H(x, y)| \geq C_3|x|^{-\frac{D}{2}d(6d-3)^{N+r-1}} \quad \text{for } (x, y) \in A,\ |(x, y)| \geq R,$$

and $\mathcal{L}^{\mathbb{R}}_\infty(F|X) \geq -\frac{D}{2}d(6d-3)^{N+r-1}$. This ends the proof of Corollary 3.4. □

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