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On the Set of Points at Infinity of a Polynomial Image of $\mathbb{R}^n$

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Abstract In this work we prove that the *set of points at infinity* $S_\infty := \mathrm{Cl}_{\mathbb{RP}^m}(S) \cap \mathsf{H}_\infty$ *of a semialgebraic set* $S \subset \mathbb{R}^m$ that is the image of a polynomial map $f : \mathbb{R}^n \to \mathbb{R}^m$ is connected. This result is no longer true in general if f is a regular map. However, it still works for a large family of regular maps that we call *quasi-polynomial maps*.

Keywords Polynomial and regular maps and images · Quasi-polynomial maps · Set of points at infinity · Connectedness

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1 Introduction

A map $f := (f_1, \ldots, f_m) : \mathbb{R}^n \to \mathbb{R}^m$ is a *polynomial map* if each component $f_i \in \mathbb{R}[\mathrm{x}] := \mathbb{R}[\mathrm{x}_1, \ldots, \mathrm{x}_n]$. A subset S of $\mathbb{R}^m$ is a *polynomial image* of $\mathbb{R}^n$ if there exists a polynomial map $f : \mathbb{R}^n \to \mathbb{R}^m$ such that $S = f(\mathbb{R}^n)$. More generally, the map f is *regular* if each component f_i is a regular function of $\mathbb{R}(\mathrm{x}) := \mathbb{R}(\mathrm{x}_1, \ldots, \mathrm{x}_n)$,

Dedicated to Ignacio Luengo on the occasion of his 60th birthday.

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that is, $f_i := \frac{g_i}{h_i}$ is a quotient of polynomials such that the zero set of h_i is empty. Analogously, a subset S of $\mathbb{R}^m$ is a *regular image* of $\mathbb{R}^n$ if it is the image $S = f(\mathbb{R}^n)$ of $\mathbb{R}^n$ given by a regular map f.

The present work continues the general study of polynomial and regular images of Euclidean spaces already began in [3,4]. A celebrated Theorem of Tarski–Seidenberg [1, 1.4] says that the image of any polynomial map (and more generally of a regular map) $f : \mathbb{R}^m \to \mathbb{R}^n$ is a semialgebraic subset S of $\mathbb{R}^n$, that is, it can be written as a finite boolean combination of polynomial equations and inequalities, which we will call a *semialgebraic* description. By *elimination of quantifiers* S is semialgebraic if it has a description by a first order formula *possibly with quantifiers*. Such a freedom gives easy semialgebraic descriptions for topological operations: interiors, closures, borders of semialgebraic sets are again semialgebraic.

In an *Oberwolfach* week [10] Gamboa proposed to characterize the semialgebraic sets of $\mathbb{R}^m$ that are polynomial images of $\mathbb{R}^n$ for some $n \geq 1$. The open ones deserve a special attention in connection with the real Jacobian Conjecture [14,15,18]. The interest of polynomial (and also regular) images arises because there exist certain problems in Real Algebraic Geometry that can be reduced for such sets to the case $S = \mathbb{R}^n$ (see [8,9]). Examples of such problems are:

- Optimization of polynomial (and/or regular) functions on S,
- Characterization of the polynomial (or regular functions) that are positive semidefinite on S (Hilbert's 17th problem and Positivstellensatz),
- Computation of trajectories inside S parametrizable by polynomial (or regular) maps.

1.1 Main Result

We denote the projective space of coordinates $(x_0 : x_1 : \cdots : x_m)$ with $\mathbb{RP}^m$. It contains $\mathbb{R}^m$ as the set of points with $x_0 = 1$. The hyperplane at infinity H_∞ has equation $x_0 = 0$. Given a semialgebraic set $S \subset \mathbb{R}^m$, the *set of points at infinity of* S *is* $S_\infty := \mathrm{Cl}_{\mathbb{RP}^m}(S) \cap \mathsf{H}_\infty$. Our main result in this work is the following.

Theorem 1.1 *Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be a non-constant polynomial map and denote $S := f(\mathbb{R}^m)$. Then S_∞ is non-empty and connected.*

It seems a difficult matter to provide a full geometric characterization of all polynomial and/or regular images $S \subset \mathbb{R}^m$. We only know it for the 1-dimensional case [2]. Nevertheless, we have approached the representation as polynomial or regular images of ample families of n-dimensional semialgebraic sets whose boundaries are piecewise linear. We have focused on: convex polyhedra, their interiors, their exteriors and the closure of their exteriors [6,8,9,23]. The proofs are constructive but the arguments are developed *ad hoc*. Two main difficulties arise:

- To develop a strategy to produce an either polynomial or regular map whose image is the desired semialgebraic set.
- To prove the surjectivity of the constructed map.

In [3] appear some straightforward properties that a polynomial (resp. regular) image $S \subset \mathbb{R}^m$ must satisfy: S must be pure dimensional, connected, semialgebraic and its Zariski closure must be irreducible. It follows from [5, 3.1] that S must be *irreducible* in the sense proposed in [5]. All these properties follow readily from the fact [7, 3.6]:

$(*)$ Given two points $p, q \in S$, there exists a polynomial (resp. regular) image L of $\mathbb{R}$ (also known as *parametric semiline*) contained in S and passing through p, q.

There are many examples of semialgebraic sets with property $(*)$ that are polynomial images of no $\mathbb{R}^n$. Take $S := \{0 \leq x \leq 1, 0 \leq y\} \cup \{0 \leq y \leq x\} \subset \mathbb{R}^2$, which satisfies $(*)$. By Theorem 1.1 S is a polynomial image of no $\mathbb{R}^n$ because its set of points at infinity is disconnected. Consequently Theorem 1.1 provides a new obstruction to be a polynomial image of $\mathbb{R}^n$.

We wondered in [4, 7.3] about the number of connected components of the exterior of a polynomial image of dimension ≥ 2. The first author was convinced that the answer was one, but the second author showed in [22] that this number can be arbitrarily large. Nevertheless, Theorem 1.1 is in the vein of the wrong initial position concerning the number of connected components of the exterior of a polynomial image.

1.2 Strategy of the Proof and Structure of the Article

The proof of Theorem 1.1 involves techniques inspired by those employed by Jelonek in his works [14,15] where he studies the geometry of the set of points $\mathcal{S}_f$ at which an either complex or real polynomial map $f : \mathbb{K}^n \to \mathbb{K}^m$ is not proper ($\mathbb{K}$ denotes either $\mathbb{R}$ or $\mathbb{C}$). We highlight the following:

- *Resolution of indeterminacy of rational maps* defined on projective surfaces.
- Sufficient conditions to guarantee that the intersection of two connected complex projective curves of a complex projective surface is either empty or a singleton.
- A *"rational" curve selection lemma*.

For the sake of the reader we include a careful exposition of these techniques in Sect. 2. The reader can proceed directly to Sect. 3 and refer to the Preliminaries only when needed. In Sect. 3 we prove Theorem 1.1 in the more general setting of *quasi-polynomial maps*. In Sect. 4 we show that the set of points at infinity of the image of a general regular map does not need to be connected and we provide some enlightening examples.

2 Preliminaries

We write $\mathbb{K}$ to refer indistinctly to $\mathbb{R}$ or $\mathbb{C}$. We denote the hyperplane at infinity of $\mathbb{KP}^m$ with $\mathsf{H}_\infty(\mathbb{K}) := \{x_0 = 0\}$. Clearly, $\mathbb{KP}^m$ contains $\mathbb{K}^m$ as the set $\mathbb{KP}^m \setminus \mathsf{H}_\infty(\mathbb{K}) = \{x_0 = 1\}$. If $m = 1$, we denote the point at infinity $\mathbb{KP}^1$ with $\{p_\infty\} := \{x_0 = 0\}$ and if $m = 2$, we write $\ell_\infty(\mathbb{K}) := \{x_0 = 0\}$ for the line at infinity of $\mathbb{KP}^2$. We use freely that the real projective space $\mathbb{RP}^m$ can be immersed in $\mathbb{R}^k$ for k large enough as an affine non-singular real algebraic variety [1, 3.4.4]. Thus, the closure in $\mathbb{RP}^m$ of a

semialgebraic subset of $\mathbb{R}^m$ is again a semialgebraic set. It will be useful to understand real algebraic objects as fixed parts under conjugation of complex algebraic objects that are invariant under conjugation.

2.1 Invariant Projective Objects

For each $n \geq 1$ denote the complex conjugation with

$$\sigma := \sigma_n : \mathbb{CP}^n \to \mathbb{CP}^n,\ z = (z_0 : z_1 : \cdots : z_n) \mapsto \overline{z} = (\overline{z_0} : \overline{z_1} : \cdots : \overline{z_n}).$$

Clearly, $\mathbb{RP}^n$ is the set of fixed points of σ. A set $A \subset \mathbb{CP}^n$ is called *invariant* if $\sigma(A) = A$. If $Z \subset \mathbb{CP}^n$ is an invariant non-singular (complex) projective variety, then $Z \cap \mathbb{RP}^n$ is a non-singular (real) projective variety. We say that a rational map $h : \mathbb{CP}^n \dashrightarrow \mathbb{CP}^m$ is *invariant* if $h \circ \sigma_n = \sigma_m \circ h$. Of course, h is invariant if its components are homogeneous polynomials with real coefficients, so it provides by restriction a real rational map $h|_{\mathbb{RP}^n} : \mathbb{RP}^n \dashrightarrow \mathbb{RP}^m$. We use freely usual concepts of Algebraic Geometry such as: rational map, regular map, divisor, blow-up, etc. and refer the reader to [11,17,19,20] for further details. For the sake of the reader we denote complex dimension with $\dim_{\mathbb{C}}(\cdot)$ and real dimension with $\dim_{\mathbb{R}}(\cdot)$. Recall the following fact concerning the regularity of rational maps defined on a non-singular projective curve [17, 7.1].

Lemma 2.1 *Let $Z \subset \mathbb{CP}^n$ be a non-singular projective curve and $F : Z \dashrightarrow \mathbb{CP}^m$ a rational map. Then F extends to a regular map $F' : Z \to \mathbb{CP}^m$. In addition, if Z, F are invariant, so is F'.*

One of the main tools is the resolution of indeterminacy of an invariant rational map. We provide a careful presentation of this well-known tool keeping track of the invariance properties along the process.

2.2 Resolution of Indeterminacy of an Invariant Rational Map

Let $Z_0 \subset \mathbb{CP}^n$ be an invariant non-singular projective variety of dimension d and $F_{\mathbb{C}} := (F_1 : \cdots : F_m) : Z_0 \dashrightarrow \mathbb{CP}^m$ an invariant rational map. To compute the *set of indeterminacy* of $F_{\mathbb{C}}$ one proceeds as follows [19, III.1.4]. Consider for each $i = 0, \ldots, m$ the divisor D_i in Z_0 defined by F_i and let $|D| := \operatorname{hcd}\{D_0, D_1, \ldots, D_m\}$ be the highest common divisor of the divisors $D_0, D_1, \ldots, D_m$. The divisors $D_i' := D_i - |D|$ are relatively prime. By [19, III.1.4.Theorem 2] the map $F_{\mathbb{C}}$ fails to be regular exactly at the points of the invariant set $Y_{\mathbb{C}} := \bigcap_{i=0}^m \operatorname{supp}(D_i')$, which has dimension $\leq d - 2$. As $F_{\mathbb{C}}$ is invariant, it can be restricted to a real rational map $F_{\mathbb{R}} : Z_0 \cap \mathbb{RP}^n \dashrightarrow \mathbb{RP}^m$ whose set of indeterminacy is $Y_{\mathbb{R}} := Y_{\mathbb{C}} \cap \mathbb{RP}^2$.

We assume that Z_0 has dimension 2. As it is well-known, $F_{\mathbb{C}} : Z_0 \dashrightarrow \mathbb{CP}^m$ admits an invariant resolution. Namely,

2.2.1. There exist

(i) *An invariant non-singular projective surface $Z_1 \subset \mathbb{CP}^k$ for some $k \geq 2$.*

(ii) *An invariant (composition of a) sequence of blow-ups* $\pi_{\mathbb{C}} : Z_1 \to Z_0 \subset \mathbb{CP}^n$ *such that* $\pi_{\mathbb{C}}|_{Z_1\setminus\pi_{\mathbb{C}}^{-1}(Y_{\mathbb{C}})} : Z_1\setminus\pi_{\mathbb{C}}^{-1}(Y_{\mathbb{C}}) \to Z_0\setminus Y_{\mathbb{C}}$ *is a biregular isomorphism and* $Y_{\mathbb{C}} = \{p \in Z_0 : \#\pi_{\mathbb{C}}^{-1}(p) > 1\}$.

(iii) *An invariant regular map* $\widehat{F}_{\mathbb{C}} : Z_1 \to \mathbb{CP}^m$ *such that*

$$\widehat{F}_{\mathbb{C}}|_{Z_1\setminus\pi_{\mathbb{C}}^{-1}(Y_{\mathbb{C}})} = F_{\mathbb{C}} \circ \pi_{\mathbb{C}}|_{Z_1\setminus\pi_{\mathbb{C}}^{-1}(Y_{\mathbb{C}})}.$$

In addition, for each $y \in Y_{\mathbb{C}}$ *the irreducible components of* $\pi_{\mathbb{C}}^{-1}(y)$ *are non-singular projective curves* $K_{i,y}$ *that are biregularly equivalent to* $\mathbb{CP}^1$ *(via regular maps* $\Phi_{i,y} : \mathbb{CP}^1 \to K_{i,y}$ *that are invariant for invariant* $K_{i,y}$*) and satisfy*

(iv) *If* $y \in Y_{\mathbb{C}}\setminus Y_{\mathbb{R}}$, *then* $\sigma(K_{i,y}) = K_{i,\sigma(y)}$ *and* $K_{i,y} \cap \mathbb{RP}^k = \varnothing$,

(v) *If* $y \in Y_{\mathbb{R}}$, *then either* $K_{i,y} \cap \mathbb{RP}^k = \varnothing$ *and there exists* $j \neq i$ *such that* $\sigma(K_{i,y}) = K_{j,y}$ *or* $\sigma(K_{i,y}) = K_{i,y}$ *and* $C_{i,y} = K_{i,y} \cap \mathbb{RP}^k$ *is a non-singular projective curve biregularly equivalent to* $\mathbb{RP}^1$ *(via the restriction map* $\phi_{i,y} := \Phi_{i,y}|_{\mathbb{RP}^1} : \mathbb{RP}^1 \to C_{i,y}$*).*

A triple $(Z_1, \pi_{\mathbb{C}}, \widehat{F}_{\mathbb{C}})$ satisfying the previous properties is an *invariant resolution for* $F_{\mathbb{C}}$.

Let us recall some terminology and results concerning blow-ups of non-singular projective varieties at non-singular centers, from which 2.2.1. follows readily.

2.3 Blow-Up with a Non-singular Variety as Center

Let $Z_0 \subset \mathbb{CP}^n$ be a non-singular irreducible projective variety and $Y \subset Z_0$ a non-singular subvariety. Let $H_1, \ldots, H_m$ be a system of homogeneous polynomials of the same degree that generates an ideal I whose saturation

$$\overline{I} := \{H \in \mathbb{C}[\mathrm{z}] := \mathbb{C}[\mathrm{z}_0, \mathrm{z}_1, \ldots, \mathrm{z}_n] : (\mathrm{z})^k H \subset I \text{ for some } k \geq 0\}$$

equals the ideal $\mathcal{J}(Y)$ of (homogeneous) polynomials of $\mathbb{C}[\mathrm{z}]$ vanishing identically on Y.

2.3.1. The blow-up $\mathrm{Bl}_Y(Z_0)$ *of* Z_0 *with center* Y is the closure in $Z_0 \times \mathbb{CP}^{m-1}$ of the set

$$\{(z; (H_1(z) : \cdots : H_m(z))) \in (Z_0\setminus Y) \times \mathbb{CP}^{m-1}\}$$

together with the projection $\pi : \mathrm{Bl}_Y(Z_0) \subset Z_0 \times \mathbb{CP}^{m-1} \to Z_0$, $(z; y) \mapsto z$ (see [11, 7.18] and [20, VI.2.2]). Recall the following facts:

- $\mathrm{Bl}_Y(Z_0)$ is a non-singular irreducible projective variety of the same dimension as Z_0, independent of the choices made in the process.
- $\pi^{-1}(Y)$ is a non-singular hypersurface of $\mathrm{Bl}_Y(Z_0)$.
- $\pi|_{\mathrm{Bl}_Y(Z_0)\setminus\pi^{-1}(Y)} : \mathrm{Bl}_Y(Z_0)\setminus\pi^{-1}(Y) \to Z_0\setminus Y$ is a biregular isomorphism.

- Y admits a finite cover by affine open subsets $\{U_\alpha\}_\alpha$ satisfying $\pi^{-1}(U_\alpha) \cong U_\alpha \times \mathbb{CP}^{r-1}$ where $r := \dim_{\mathbb{C}}(Z_0) - \dim_{\mathbb{C}}(Y)$. In particular, the fiber of each $y \in Y$ is a projective space $\mathbb{CP}^{r-1}$.
- If $Y_1, \ldots, Y_r$ are the irreducible components of Y, then they are pairwise disjoint and non-singular and it holds

$$\mathrm{Bl}_Y(Z_0) \cong \mathrm{Bl}_{Y_r}(\cdots \mathrm{Bl}_{Y_2}(\mathrm{Bl}_{Y_1}(Z_0)) \cdots).$$

If Z_0, Y are invariant, $\mathrm{Bl}_Y(Z_0)$ can be assumed invariant, too, by choosing an ideal I with saturation $\mathcal{J}(Y)$ and whose generators are invariant (given any family of generators, consider the real and the imaginary parts of all of them). If we consider the immersion of $\mathrm{Bl}_Y(Z_0)$ in some $\mathbb{CP}^N$ using Segre's map, also the regular map $\pi : \mathrm{Bl}_Y(Z_0) \to Z_0$ is invariant.

2.3.2. Assume that Z_0 is an invariant non-singular projective surface and Y is a finite invariant subset. Consider the invariant blow-up $(\mathrm{Bl}_Y(Z_0), \pi)$ of Z_0 with center Y:

(i) For each $y \in Y$ the fiber $\pi^{-1}(y)$ is a $\mathbb{CP}^1$. If $y \in Y \cap \mathbb{RP}^N$, there exists an invariant biregular equivalence between $\pi^{-1}(y)$ and $\mathbb{CP}^1$. If $y \in Y \backslash \mathbb{RP}^N$, then $\pi^{-1}(y) \cap \mathbb{RP}^N = \varnothing$.

(ii) If $C \subset Z_0$ is a non-singular curve not contained in the center Y, its strict transform

$$\widetilde{C} := \mathrm{Cl}_{\mathrm{Bl}_Y(Z_0)}(\pi^{-1}(C \backslash Y)) \subset \pi^{-1}(C)$$

is as well a non-singular curve. In addition, if there exists an invariant biregular equivalence $\Phi : \mathbb{CP}^1 \to C$, the strict transform $\widetilde{C} \subset \pi^{-1}(C)$ of C under π is invariant and there exists an invariant biregular equivalence $\Psi : \mathbb{CP}^1 \to \widetilde{C}$.

Proof (*Sketch of proof of statement 2.2.1.*) To solve the indeterminacy of the rational map $F_{\mathbb{C}} : Z_0 \dashrightarrow \mathbb{CP}^m$, one blows the set of points of indeterminacy $Y_{\mathbb{C}}$ of $F_{\mathbb{C}}$ up and considers the composition G of $F_{\mathbb{C}}$ with the previous sequence of blowing-ups [19, IV.3.3.Theorem 3]. If G is regular, the process is concluded. Otherwise one applies the previous procedure to G. In finitely many steps one achieves a regular map and the process finishes. By 2.3 we may assume that each rational map involved in the process is invariant, so in the last step of the process we obtain

- An invariant non-singular projective surface $Z_1 \subset \mathbb{CP}^k$ for some $k \geq 2$,
- An invariant sequence of blow-ups $\pi_{\mathbb{C}} : Z_1 \to Z_0 \subset \mathbb{CP}^n$ and
- An invariant regular map $\widehat{F}_{\mathbb{C}} : Z_1 \to \mathbb{CP}^m$ such that

$$\widehat{F}_{\mathbb{C}}|_{Z_1 \backslash \pi_{\mathbb{C}}^{-1}(Y_{\mathbb{C}})} = F_{\mathbb{C}} \circ \pi_{\mathbb{C}}|_{Z_1 \backslash \pi_{\mathbb{C}}^{-1}(Y_{\mathbb{C}})}.$$

The triple $(Z_1, \pi_{\mathbb{C}}, \widehat{F}_{\mathbb{C}})$ satisfies conditions (i) to (iii). The fiber of each point of $Y_{\mathbb{C}}$ under π is a complex projective curve by 2.3 whose irreducible components are non-singular rational curves while the fiber of each point of $Z_0 \backslash Y_{\mathbb{C}}$ under π is a singleton. Thus, $Y_{\mathbb{C}}$ is the fundamental set of $\pi_{\mathbb{C}}$, so (ii) holds. Assertions (iv) and (v) are straightforward consequences of 2.3. □

2.4 Projective Curves Intersecting Each Other in a Singleton

It will be useful to know sufficient conditions that guarantee that the intersection of two connected complex projective curves of a complex projective surface is either empty or a singleton. The following result is a reformulation of [14, 4.6] and [15, 3.1].

Lemma 2.2 *Let X be a complex projective surface. Assume that*

- *$U \subset X$ is a connected orientable manifold such that*

$$H_1(U;\mathbb{Z}) = H_2(U;\mathbb{Z}) = 0,$$

- *U is dense in X and the complement $A := X\backslash U$ is a complex projective curve.*

Let $C_1, C_2 \subset A$ be two connected, complex projective curves without common irreducible components. Then the intersection $C_1 \cap C_2$ is either the empty set or a singleton.

Proof The proof is conducted in several steps:

2.4.1. We prove first that $H^1(A;\mathbb{Z}) = 0$.

Assume that X is a compact polyhedron of dimension 4 and A a closed subpolyhedron of X. As U is an orientable real manifold of dimension 4, by Lefschetz duality [21, 6.1.11 & 6.2.19] we have $H^i(X, A;\mathbb{Z}) \cong H_{4-i}(U;\mathbb{Z})$ for $i = 0, \ldots, 4$. By the long exact sequence of cohomology [21, 5.4.13]

$$H^1(X, A;\mathbb{Z}) \to H^1(X;\mathbb{Z}) \to H^1(A;\mathbb{Z}) \to H^2(X, A;\mathbb{Z}) \cong H_2(U;\mathbb{Z}) = 0,$$

so $H^1(X;\mathbb{Z}) \to H^1(A;\mathbb{Z})$ is an epimorphism. As U is a connected open dense subset of X, there exists by [14, 4.7] an epimorphism

$$0 = H_1(U;\mathbb{Z}) \to H_1(X,\mathbb{Z}),$$

so $H_1(X,\mathbb{Z}) = 0$. By the universal-coefficient theorem for cohomology [21, 5.5.3]

$$H^1(X,\mathbb{Z}) \cong \mathrm{Hom}(H_1(X,\mathbb{Z}),\mathbb{Z}) \oplus \mathrm{Ext}(\mathbb{Z},\mathbb{Z}),$$

so by [21, 5.5.1] $H^1(X;\mathbb{Z}) = 0$. Consequently, $H^1(A;\mathbb{Z}) = 0$.

2.4.2. Let $C \subset A$ be a projective algebraic curve. Then $H^1(C;\mathbb{Z}) = 0$. In particular, it holds $H^1(C_1 \cup C_2;\mathbb{Z}) = 0$.

Let C' be the union of the irreducible components of A not contained in C. Clearly, $\mathcal{F} := C \cap C'$ is a finite set. As C and C' are analytic sets, they are locally contractible, so for each $x \in \mathcal{F}$ there exists a neighborhood V^x in A such that

- $V^x \cap C$ and $V^x \cap C'$ have the singleton $\{x\}$ as a deformation retract,
- $V^x \cap C \cap C' = \{x\}$ and
- $V^{x_1} \cap V^{x_2} = \varnothing$ if $x_1, x_2 \in \mathcal{F}$ and $x_1 \neq x_2$.

It holds that $V := C \cup \bigcup_{x\in\mathcal{F}}(V^x \cap C')$ and $W := C' \cup \bigcup_{x\in\mathcal{F}}(V^x \cap C)$ are open subsets of A such that $V \cup W = A$, $V \cap W = \bigcup_{x\in\mathcal{F}}(V^x \cap (C \cup C'))$ and C, C' are respective deformation retracts of V and W. In addition, $\mathcal{F}$ is a deformation retract of $V \cap W$. By Mayer–Vietoris' exact sequence for cohomology [21, 5.4.9]

$$\begin{aligned} 0 = H^1(A;\mathbb{Z}) = H^1(V \cup W;\mathbb{Z}) &\to H^1(V;\mathbb{Z}) \oplus H^1(W;\mathbb{Z}) \\ &\to H^1(V \cap W;\mathbb{Z}) \cong H^1(\mathcal{F};\mathbb{Z}). \end{aligned}$$

As $\mathcal{F}$ is a finite set, $H^1(\mathcal{F};\mathbb{Z}) = 0$, so $H^1(C;\mathbb{Z}) \cong H^1(V;\mathbb{Z}) = 0$.

2.4.3. $C_1 \cap C_2$ is either empty or a singleton.

Assume $C_1 \cap C_2 \neq \varnothing$. Let V_1, V_2 be two open subsets of $C_1 \cup C_2$ such that

- $C_i \subset V_i$ is a deformation retract of V_i for $i = 1, 2$,
- $V_1 \cup V_2 = C_1 \cup C_2$ and $C_1 \cap C_2$ is a deformation retract of $V_1 \cap V_2$

(for the construction of V_1, V_2 proceed similarly to 2.4.2). By Mayer–Vietoris' exact sequence for reduced cohomology [21, 5.4.8 & p. 240] applied to the open subsets V_1 and V_2 of $C_1 \cup C_2$ (whose intersection $V_1 \cap V_2 \supset C_1 \cap C_2 \neq \varnothing$), we deduce

$$\begin{aligned} &\tilde{H}^0(C_1;\mathbb{Z}) \oplus \tilde{H}^0(C_2;\mathbb{Z}) \cong \tilde{H}^0(V_1;\mathbb{Z}) \oplus \tilde{H}^0(V_2;\mathbb{Z}) \\ &\quad\to \tilde{H}^0(V_1 \cap V_2;\mathbb{Z}) \cong \tilde{H}^0(C_1 \cap C_2;\mathbb{Z}) \\ &\quad\to \tilde{H}^1(V_1 \cup V_2;\mathbb{Z}) \cong \tilde{H}^1(C_1 \cup C_2;\mathbb{Z}) \cong H^1(C_1 \cup C_2;\mathbb{Z}) = 0. \end{aligned}$$

As C_1, C_2 are connected, $\tilde{H}^0(C_i;\mathbb{Z}) = 0$, so $\tilde{H}^0(C_1 \cap C_2;\mathbb{Z}) = 0$. Thus, the finite set $C_1 \cap C_2$ is connected, so it is a singleton. □

Example 2.3 Let $\mathcal{F} \subset \mathbb{C}^2$ be a finite set and $U := \mathbb{C}^2\backslash\mathcal{F}$ its complement. Then $H_1(U,\mathbb{Z}) = H_2(U,\mathbb{Z}) = 0$.

By Hurewicz's theorem $H_1(U,\mathbb{Z})$ is the abelianization of $\pi_1(U) = 0$, so $H_1(U,\mathbb{Z}) = 0$. We identify $\mathbb{C}^2 \equiv \mathbb{R}^4$. To compute $H_2(U,\mathbb{Z})$, we may assume $\mathcal{F} := \{p_1,\dots,p_r\}$ where $p_k := (2k-1,0,0,0)$. Notice that $D_r := \bigcup_{i=1}^r \mathbb{S}^3_{p_i}$ where $\mathbb{S}^3_{p_i} := \{x \in \mathbb{R}^4 : \|x - p_i\| = 1\}$ is a deformation retract of $U \equiv \mathbb{R}^4\backslash\mathcal{F}$. Observe

$$\mathbb{S}^3_{p_i} \cap \mathbb{S}^3_{p_j} = \begin{cases} \{p_{ij} := \frac{p_i+p_j}{2}\} & \text{if } |i-j| = 1, \\ \varnothing & \text{if } |i-j| > 1. \end{cases}$$

Denote $D_k := \bigcup_{i=1}^k \mathbb{S}^3_{p_i}$ and observe $H_2(D_1;\mathbb{Z}) = H_2(\mathbb{S}^3_{p_1};\mathbb{Z}) = 0$. By induction hypothesis, we assume $H_2(D_{r-1};\mathbb{Z}) = 0$. By Mayer–Vietoris' exact sequence for homology [21, Sect. 4.6]

$$\begin{aligned} 0 = H_2(D_{r-1},\mathbb{Z}) \oplus H_2(\mathbb{S}^3_{p_r};\mathbb{Z}) &\to H_2(D_r;\mathbb{Z}) = H_2(U;\mathbb{Z}) \\ &\to H_1(D_{r-1} \cap \mathbb{S}^3_{p_r},\mathbb{Z}) = H_1(\{p_{r-1,r}\};\mathbb{Z}) = 0, \end{aligned}$$

so $H_2(U;\mathbb{Z}) = 0$, as required. □

Remark 2.4 Lemma 2.2 applies if U is homeomorphic to the complement in $\mathbb{C}^2$ of a finite subset.

2.5 Rational Curve Selection Lemma

To finish this section we present the following variation of the curve selection lemma adapted to the situations we will approach later. We refer the reader to [16, 4.7] for a result of similar nature.

Lemma 2.5 *Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be a regular map and $S := f(\mathbb{R}^n)$. Let $S' \subset S$ be a semialgebraic dense subset of S and $p \in \mathrm{Cl}_{\mathbb{RP}^m}(S)\backslash S$. Then there exist (after reordering the variables of $\mathbb{R}^n$) a rational path*

$$\alpha := (\pm \mathtt{t}^{k_1}, \mathtt{t}^{k_2}\mathtt{p}_2, \ldots, \mathtt{t}^{k_n}\mathtt{p}_n) \in \mathbb{R}(\mathtt{t})^n,$$

where

- $k_i \in \mathbb{Z}$, $k_1 = \min\{k_1, \ldots, k_n\} < 0$,
- $\mathtt{p}_i \in \mathbb{R}[\mathtt{t}]$ *and* $\mathtt{p}_i(0) \neq 0$ *for* $i = 2, \ldots, n$

and an integer $r \geq 1$ such that for each $\beta \in (\mathtt{t})^r\mathbb{R}[\mathtt{t}]^n$

(i) $p = \lim_{t\to 0^+}(f \circ (\alpha + \beta))(t)$ *and*
(ii) $(f \circ (\alpha + \beta))(t) \subset S'$ *for $t > 0$ small enough.*

Before proving the previous result, we need a technical lemma.

Lemma 2.6 *Let $F \in \mathbb{R}[\mathtt{x}]$ be a polynomial that is not identically zero and let $g \in \mathbb{R}((\mathtt{t}))^n$. Then for each $s \geq 1$ there exists $r \geq 1$ such that if $h \in (\mathtt{t})^r\mathbb{R}[[\mathtt{t}]]^n$, we have $F(g) - F(g + h) \in (\mathtt{t})^s\mathbb{R}[[\mathtt{t}]]$.*

Proof Write $g := \frac{g'}{\mathtt{t}^k}$ where $k \geq 0$ and $g' \in \mathbb{R}[[\mathtt{t}]]^n$. Let $\mathtt{z}$ and $\mathtt{y} := (\mathtt{y}_1, \ldots, \mathtt{y}_n)$ be variables. Write $F(\mathtt{x} + \mathtt{z}\mathtt{y}) = F(\mathtt{x}) + \mathtt{z}H(\mathtt{x}, \mathtt{y}, \mathtt{z})$ where $H \in \mathbb{R}[\mathtt{x}, \mathtt{y}, \mathtt{z}]$ is a polynomial of degree d. Let $r := s + kd$ and observe that if $h \in (\mathtt{t})^r\mathbb{R}[[\mathtt{t}]]^n$, we may write $h := \mathtt{t}^r h'$ where $h' \in \mathbb{R}[[\mathtt{t}]]$ and

$$F(g + h) - F(g) = \mathtt{t}^r H\Big(\frac{g'}{\mathtt{t}^k}, h', \mathtt{t}^r\Big).$$

Observe that the order of the series $F(g + h) - F(g)$ is $\geq r - kd = s$, as required.□

Proof of Lemma 2.5 The proof is conducted in several steps:

2.5.1. We may assume: *S' is open in S.*

As S' is dense in S and S is pure dimensional because it is the image of $\mathbb{R}^n$ under a regular map, it holds that $\mathrm{Cl}_{\mathbb{R}^m}(S\backslash S')$ has dimension $\leq \dim_{\mathbb{R}}(S) - 1$. Thus, $S'' := S\backslash \mathrm{Cl}_{\mathbb{R}^m}(S\backslash S') \subset S'$ is dense and open in S. Changing S' with S'', we may assume that S' is open in S.

2.5.2. There exists a Nash path $\lambda : (-1, 1) \to \mathbb{RP}^n$ such that $(f \circ \lambda)(0, 1) \subset S'$, $\lim_{t\to 0}\lambda(t) = q \in \mathbb{RP}^n\backslash\mathbb{R}^n$ and $\lim_{t\to 0}(f \circ \lambda)(t) = p$.

As $p \in \mathrm{Cl}_{\mathbb{RP}^m}(S')\backslash S'$, there exists by the Nash curve selection Lemma [1, 8.1.13] a Nash path $\gamma : (-1, 1) \to \mathbb{RP}^m$ such that $\gamma((0, 1)) \subset S'$ and $\gamma(0) = p$. Let $\{x_k\}_k \subset \mathbb{R}^n$ and $\{t_k\}_{k\geq 1} \subset (0, 1)$ be sequences such that $\lim_{k\to\infty} t_k = 0$ and $f(x_k) = \gamma(t_k)$ for all $k \geq 1$. We may assume that $\{x_k\}_{k\geq 1}$ converges to $q \in \mathbb{RP}^n$. As $S = f(\mathbb{R}^n)$ and $p = \gamma(0) \in \mathrm{Cl}_{\mathbb{RP}^m}(S)\backslash S$, we have $q \in \mathbb{RP}^n\backslash\mathbb{R}^n = \mathsf{H}_\infty(\mathbb{R})$.

As $\dim_{\mathbb{R}}(f^{-1}(\gamma((0, 1)))) \geq \dim_{\mathbb{R}}(\gamma((0, 1))) = 1$ and

$$q \in \mathrm{Cl}_{\mathbb{RP}^n}(f^{-1}(\gamma((0, 1))))\backslash(f^{-1}(\gamma((0, 1)))),$$

there exists a Nash path $\lambda : (-1, 1) \to \mathbb{RP}^n$ such that $\lambda((0, 1)) \subset f^{-1}(\gamma((0, 1)))$ and $\lambda(0) = q$.

2.5.3. Construction of the integer k_1. After reparametrizing γ, we may assume $f \circ \lambda = \gamma$, so there exist $\lambda_0, \lambda_1, \ldots, \lambda_n, \gamma_0, \gamma_1, \ldots, \gamma_m \in \mathbb{R}[[\mathtt{t}]]$ such that

$$f \circ \lambda = f\Big(\frac{\lambda_1}{\lambda_0}, \ldots, \frac{\lambda_n}{\lambda_0}\Big) = \Big(\frac{\gamma_1}{\gamma_0}, \ldots, \frac{\gamma_m}{\gamma_0}\Big) = \gamma,$$

$\lim_{t\to 0^+}(1 : \lambda(t)) = q$ and $\lim_{t\to 0^+}(1 : \gamma(t)) = p$. As $q \in \mathsf{H}_\infty(\mathbb{R})$, we may assume after reordering the variables $x_1, \ldots, x_n$ that the order

$$k_1 := \omega\Big(\frac{\lambda_1}{\lambda_0}\Big) = \min\Big\{\omega\Big(\frac{\lambda_i}{\lambda_0}\Big) : i = 1, \ldots, n\Big\} < 0.$$

After a change of the type $\mathtt{t} \mapsto \mathtt{t}u(\mathtt{t})$ where $u \in \mathbb{R}[[\mathtt{t}]]$ is a unit, we assume $\frac{\lambda_1}{\lambda_0} = \pm\mathtt{t}^{k_1}$ with $k_1 < 0$. Write

$$\frac{\lambda_i}{\lambda_0} = t^{k_i'}\rho_i \quad \text{where} \quad \begin{cases} k_i' \in \mathbb{Z},\ \rho_i \in \mathbb{R}[[\mathtt{t}]] \text{ and } \rho_i(0) \neq 0 & \text{if } \frac{\lambda_i}{\lambda_0} \neq 0, \\ k_i' = 0 \text{ and } \rho_i = 0 & \text{if } \frac{\lambda_i}{\lambda_0} = 0. \end{cases}$$

2.5.4. Construction of the integer r. Write $f := \big(\frac{f_1}{f_0}, \ldots, \frac{f_m}{f_0}\big)$ where $f_i \in \mathbb{R}[\mathrm{x}]$ and f_0 does not vanish on $\mathbb{R}^n$. By Lemma 2.6 there exists $r_0 \geq 1$ such that if $\mu := (\mu_1, \ldots, \mu_n) \in (\mathtt{t})^{r_0}\mathbb{R}[[\mathtt{t}]]^n$, then

$$\lim_{t\to 0^+}(1 : \lambda(t) + \mu(t)) = q$$
$$\text{and} \quad \lim_{t\to 0^+}\Big(1 : \frac{f_1}{f_0}(\lambda(t) + \mu(t)) : \cdots : \frac{f_m}{f_0}(\lambda(t) + \mu(t))\Big) = p.$$

Since S' is open in S, also $f^{-1}(S')$ is open in $\mathbb{R}^n$. As

$$\lambda((0, 1)) \subset f^{-1}(\gamma((0, 1))) \subset f^{-1}(S'),$$

there exist finitely many polynomials $g_1, \ldots, g_q \in \mathbb{R}[\mathrm{x}]$ such that

$$\lambda((0, \varepsilon_1)) \subset \{g_1 > 0, \ldots, g_q > 0\} \subset f^{-1}(S')$$

for some $\varepsilon_1 > 0$ small enough. Thus, $g_i \circ \lambda = a_i t^{\ell_i} + \cdots$ where $a_i > 0$ and $\ell_i \in \mathbb{Z}$. By Lemma 2.6 there exists $r \geq \max\{r_0, k_2', \ldots, k_n'\} + 1$ such that if $\eta \in (\mathtt{t})^r\mathbb{R}[[\mathtt{t}]]$, then

$$(g_i \circ (\lambda + \eta)) - (g_i \circ \lambda) \in (\mathtt{t})^s\mathbb{R}[[\mathtt{t}]],$$

where $s := \max\{0, \ell_1, \ldots, \ell_q\} + 1$. Consequently, if $\eta \in (\mathtt{t})^r\mathbb{R}[[\mathtt{t}]]$, each series $g_i \circ (\lambda + \eta) > 0$ for $t > 0$ small enough, so $(\lambda + \eta)(t) \in f^{-1}(S)$ for $t > 0$ small enough.

2.5.5. Construction of the rational path α. Choose $\mu := (0, \mu_2, \ldots, \mu_n) \in (\mathtt{t})^r\mathbb{R}[[\mathtt{t}]]^n$ such that $\mathtt{t}^{-k_i'}(\frac{\lambda_i}{\lambda_0} + \mu_i) = \rho_i + \mathtt{t}^{-k_i'}\mu_i \in \mathbb{R}[\mathtt{t}]\backslash\{0\}$. Write

$$\mathtt{t}^{k_i'}(\rho_i + \mathtt{t}^{-k_i'}\mu_i) = \mathtt{t}^{k_i}\mathtt{p}_i,$$

where $k_i \in \mathbb{Z}$, $\mathtt{p}_i \in \mathbb{R}[\mathtt{t}]$ and $\mathtt{p}_i(0) \neq 0$. The rational path

$$\alpha := (\pm\mathtt{t}^{k_1}, \mathtt{t}^{k_2}\mathtt{p}_2, \ldots, \mathtt{t}^{k_n}\mathtt{p}_n) \in \mathbb{R}(\mathtt{t})^n$$

and the integer r satisfy the conditions in the statement. □

3 Connectedness of the Set of Points at Infinity of a Polynomial Image

3.1 Proof of Theorem 1.1

The purpose of this section is to prove Theorem 1.1. We approach this result in the more general framework of *quasi-polynomial maps*. Given a regular map

$$f := \left(\frac{f_1}{f_0}, \cdots, \frac{f_m}{f_0}\right) : \mathbb{R}^n \to \mathbb{R}^m,$$

where each $f_i \in \mathbb{R}[\mathtt{x}]$, consider the invariant rational map

$$F_{\mathbb{C}} : \mathbb{CP}^n \dashrightarrow \mathbb{CP}^m,\ x := (x_0 : x_1 : \cdots : x_n) \mapsto (F_0(x) : F_1(x) : \cdots : F_m(x)),$$

where $F_i(\mathtt{x}_0 : \mathtt{x}_1 : \cdots : \mathtt{x}_n) := \mathtt{x}_0^d f_i(\frac{\mathtt{x}_1}{\mathtt{x}_0}, \ldots, \frac{\mathtt{x}_n}{\mathtt{x}_0})$ and $d := \max_{i=0,\ldots,m}\{\deg(f_i)\}$. Let $Y_{\mathbb{C}}$ be the set of indeterminacy of $F_{\mathbb{C}}$ and write $F_0 = \mathtt{x}_0^e F_0'$ where $e \geq 0$ and $F_0' \in \mathbb{R}[\mathtt{x}_0, \mathtt{x}_1, \ldots, \mathtt{x}_n]$ is a homogeneous polynomial that is not divisible by $\mathtt{x}_0$. Observe that $F_{\mathbb{C}}$ can be restricted to a rational map $F_{\mathbb{R}} : \mathbb{RP}^n \dashrightarrow \mathbb{RP}^m$ whose set of indeterminacy is $Y_{\mathbb{R}} := Y_{\mathbb{C}} \cap \mathbb{RP}^m$.

Definition 3.1 We say that f is a *quasi-polynomial map* if $F_{\mathbb{C}}(\mathsf{H}_\infty^n(\mathbb{C})) \subset \mathsf{H}_\infty^m(\mathbb{C})$ and no real indeterminacy point of $F_{\mathbb{C}}$ belongs to $\{F_0' = 0\}$, that is, $Y_{\mathbb{C}} \cap \mathbb{RP}^n \cap \{F_0' = 0\} = \varnothing$.

Remarks 3.2 (i) The condition $F_{\mathbb{C}}(\mathsf{H}^n_\infty(\mathbb{C})) \subset \mathsf{H}^m_\infty(\mathbb{C})$ is equivalent to

$$e := \max\{\deg(f_1), \ldots, \deg(f_m)\} - \deg(f_0) > 0.$$

(ii) If the polynomials $f_0, f_1, \ldots, f_m$ are relatively prime, then

$$Y_{\mathbb{C}} \cap \{F'_0 = 0\} = \{F'_0 = 0, F_1 = 0, \ldots, F_m = 0\}.$$

(ii) Quasi-polynomial maps include polynomial maps, but also more general regular maps. If $e > 0$ and $\{F'_0 = 0\} \cap \mathbb{RP}^n = \varnothing$, then f is a quasi polynomial map. This occurs for instance if $e > 0$ and $f_0 := b_0^2 + (b_1^2 \mathrm{x}_1^{2k} + \cdots + b_n^2 \mathrm{x}_n^{2k})^\ell$ where $k, \ell \geq 0$ and $b_i \in \mathbb{R}\backslash\{0\}$.

(iv) If $n = 1$, the condition $e := \max\{\deg(f_1), \ldots, \deg(f_m)\} - \deg(f_0) > 0$ characterizes quasi-polynomial maps.

Theorem 3.3 *Let $S \subset \mathbb{R}^m$ be a semialgebraic set that is the image of a quasi-polynomial map $f : \mathbb{R}^n \to \mathbb{R}^m$. Then S_∞ is connected.*

Proof of Theorem 3.3 for the case $n = 2$ Write

$$f := \left(\frac{f_1}{f_0}, \cdots, \frac{f_m}{f_0}\right) : \mathbb{R}^2 \to \mathbb{R}^m,$$

where each $f_i \in \mathbb{R}[\mathrm{x}_1, \mathrm{x}_2]$ and f_0 has an empty zero set. Keep notations from 3.1 and assume $\gcd(f_0, f_1, \ldots, f_m) = 1$, so the set of points of indeterminacy of $F_{\mathbb{K}}$ (where $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$) is

$$Y_{\mathbb{K}} = \{x_0 F'_0 = 0, F_1 = 0, \ldots, F_m = 0\} \cap \mathbb{KP}^2.$$

By 2.2 $Y_{\mathbb{K}}$ is a finite subset of $\ell_\infty(\mathbb{K}) \cup (\{F'_0 = 0\} \cap \mathbb{KP}^2)$. As f is quasi-polynomial, $Y_{\mathbb{R}} \subset \ell_\infty(\mathbb{R})\backslash\{F'_0 = 0\}$. The proof is conducted in several steps:

Step 1. *Initial Preparation*. Let $(Z_1, \pi_{\mathbb{C}}, \widehat{F}_{\mathbb{C}})$ be an invariant resolution for $F_{\mathbb{C}}$. Keep notations from 2.2.1. and denote

- $X_1 := Z_1 \cap \mathbb{RP}^k$, which is a non-singular real projective surface.
- $\pi_{\mathbb{R}} := \pi_{\mathbb{C}}|_{X_1} : X_1 \to \mathbb{RP}^2$, which is the composition of a sequence of finitely many blow-ups and its restriction $\pi_{\mathbb{R}}|_{X_1\backslash\pi_{\mathbb{R}}^{-1}(Y_{\mathbb{R}})} : X_1\backslash\pi_{\mathbb{R}}^{-1}(Y_{\mathbb{R}}) \to \mathbb{RP}^2\backslash Y_{\mathbb{R}}$ is a biregular isomorphism.
- $\widehat{F}_{\mathbb{R}} := \widehat{F}_{\mathbb{C}}|_{X_1} : X_1 \to \mathbb{RP}^m$, which is a real regular map and satisfies

$$\widehat{F}_{\mathbb{R}}|_{X_1\backslash\pi_{\mathbb{R}}^{-1}(Y_{\mathbb{R}})} = F_{\mathbb{R}} \circ \pi_{\mathbb{R}}|_{X_1\backslash\pi_{\mathbb{R}}^{-1}(Y_{\mathbb{R}})}.$$

As $Y_{\mathbb{R}} \subset \ell_\infty(\mathbb{R})$, it holds $\pi_{\mathbb{R}}(x) \in \mathbb{RP}^2\backslash\ell_\infty(\mathbb{R}) \equiv \mathbb{R}^2$ for each point $x \in X_1\backslash\pi_{\mathbb{R}}^{-1}(\ell_\infty(\mathbb{R}))$. Thus,

$$\widehat{F}_{\mathbb{R}}(x) = F_{\mathbb{R}} \circ \pi_{\mathbb{R}}(x) = f(\pi_{\mathbb{R}}(x)) \in \mathbb{R}^m \equiv \mathbb{RP}^m\backslash\mathsf{H}_\infty(\mathbb{R}),$$

so $\widehat{F}_{\mathbb{R}}^{-1}(\mathsf{H}_\infty(\mathbb{R})) \subset \pi_{\mathbb{R}}^{-1}(\ell_\infty(\mathbb{R}))$.

3.1.1. The strict transform K_∞ under $\pi_\mathbb{C}$ of $\ell_\infty(\mathbb{C})$ is an invariant non-singular rational curve and $\pi_\mathbb{C}|_{\mathsf{K}_\infty} : \mathsf{K}_\infty \to \ell_\infty(\mathbb{C}) \equiv \mathbb{CP}^1$ is an invariant biregular isomorphism. Consequently, $\mathsf{C}_\infty := \mathsf{K}_\infty \cap \mathbb{RP}^k$ is a real non-singular rational curve and it is the strict transform under $\pi_\mathbb{R}$ of $\ell_\infty(\mathbb{R})$.

3.1.2. Define $\mathsf{E} := \{F_0'(z) = 0\} \subset \mathbb{CP}^2$ and let $\widetilde{\mathsf{E}}$ be its strict transform under $\pi_\mathbb{C}$, which is an invariant projective curve. As f_0 has empty zero set and $\mathtt{x}_0$ does not divide F_0', the intersection $\mathsf{E} \cap \mathbb{RP}^2$ is a finite subset of $\ell_\infty(\mathbb{R})$. In addition $\pi_\mathbb{C}^{-1}(\mathsf{E}) = \widetilde{\mathsf{E}} \cup \bigcup_{y \in \mathsf{E} \cap Y_\mathbb{C}} \pi_\mathbb{C}^{-1}(y)$. We have:

3.1.3. $\widetilde{\mathsf{E}} \cap \bigcup_{y \in Y_\mathbb{R}} \pi_\mathbb{C}^{-1}(y) = \varnothing$ and $\widetilde{\mathsf{E}} \cap \mathbb{RP}^k \subset \mathsf{C}_\infty$.

Let $y \in Y_\mathbb{R}$ and let us show $\widetilde{\mathsf{E}} \cap \pi_\mathbb{C}^{-1}(y) = \varnothing$. Otherwise, there exists $x \in \widetilde{\mathsf{E}} \cap \pi_\mathbb{C}^{-1}(y)$, so $y = \pi_\mathbb{C}(x) \in \pi_\mathbb{C}(\widetilde{\mathsf{E}}) = \mathsf{E}$. Consequently,

$$y \in Y_\mathbb{R} \cap \mathsf{E} = Y_\mathbb{R} \cap \{F_0' = 0\} = \varnothing,$$

which is a contradiction. As $\pi_\mathbb{C}(\widetilde{\mathsf{E}} \cap \mathbb{RP}^k) \subset \mathsf{E} \cap \mathbb{RP}^2 \subset \ell_\infty(\mathbb{R})$, we have

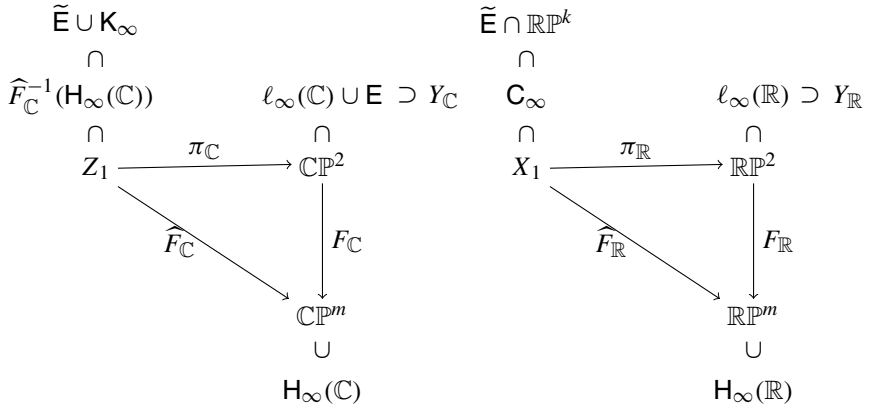
$$\widetilde{\mathsf{E}} \cap \mathbb{RP}^k \subset \pi_\mathbb{R}^{-1}(\ell_\infty(\mathbb{R})) \setminus \bigcup_{y \in Y_\mathbb{R}} \pi_\mathbb{R}^{-1}(y) \subset \mathsf{C}_\infty.$$

3.1.4. As $\widehat{F}_\mathbb{C}|_{Z_1 \setminus \pi_\mathbb{C}^{-1}(Y_\mathbb{C})} = F_\mathbb{C} \circ \pi_\mathbb{C}|_{Z_1 \setminus \pi_\mathbb{C}^{-1}(Y_\mathbb{C})}$, it holds

$$\widetilde{\mathsf{E}} \cup \mathsf{K}_\infty \subset \widehat{F}_\mathbb{C}^{-1}(\mathsf{H}_\infty(\mathbb{C})) \subset \widetilde{\mathsf{E}} \cup \mathsf{K}_\infty \cup \pi_\mathbb{C}^{-1}(Y_\mathbb{C}).$$

Thus, $\widehat{F}_\mathbb{C}^{-1}(\mathsf{H}_\infty(\mathbb{C}))$ is an invariant projective curve whose irreducible components different from $\widetilde{\mathsf{E}}$ are by 2.2.1. either singletons or non-singular rational curves.

3.1.5. The following diagram summarizes the achieved situation:



Step 2. We prove next:

3.1.6. $\widehat{F}_{\mathbb{C}}^{-1}(\mathsf{H}_\infty(\mathbb{C}))$ *is connected*. Consequently, $\widehat{F}_{\mathbb{C}}^{-1}(\mathsf{H}_\infty(\mathbb{C}))$ *does not contain isolated points and its irreducible components different from* $\widetilde{\mathsf{E}}$ *are non-singular rational curves*.

Indeed, by Stein's factorization theorem [13, III.11.5] applied to the projective morphism $\widehat{F}_{\mathbb{C}} : Z_1 \to \mathbb{CP}^m$, there exist a projective variety V and projective morphisms $G_1 : Z_1 \to V$ and $G_2 : V \to \mathbb{CP}^m$ such that

- G_1 is surjective and its fibers are connected,
- G_2 is a finite morphism and
- $\widehat{F}_{\mathbb{C}} = G_2 \circ G_1$.

To prove 3.1.6. it is enough to show that $\mathsf{H} := G_2^{-1}(\mathsf{H}_\infty(\mathbb{C}))$ and $G_1^{-1}(\mathsf{H}) = \widehat{F}_{\mathbb{C}}^{-1}(\mathsf{H}_\infty(\mathbb{C}))$ are connected.

3.1.7. H *is connected.*

As G_2 is finite, it is by [13, II.5.17] an affine morphism. Thus,

$$G_2^{-1}(\mathbb{C}^m) = G_2^{-1}(\mathbb{CP}^m)\backslash G_2^{-1}(\mathsf{H}_\infty(\mathbb{C})) = V\backslash\mathsf{H}$$

is an affine algebraic variety. As V is a complete manifold, we deduce by [12, 6.2, p. 79] that $\mathsf{H} = V\backslash G_2^{-1}(\mathbb{C}^m)$ is connected because it is the complement in V of an affine open subvariety.

3.1.8. $G_1^{-1}(\mathsf{H})$ *is connected.*

Suppose that $G_1^{-1}(\mathsf{H})$ is the disjoint union of two closed subsets A_1, A_2. As G_1 is proper and surjective, $G_1(A_1), G_1(A_2)$ are closed subsets of H and $\mathsf{H} = G_1(A_1) \cup G_1(A_2)$. In case $G_1(A_i) \neq \varnothing$ for $i = 1, 2$, the intersection $G_1(A_1) \cap G_1(A_2) \neq \varnothing$ because H is connected. If $x \in G_1(A_1) \cap G_1(A_2)$, the fiber

$$G_1^{-1}(\{x\}) = (G_1^{-1}(\{x\}) \cap A_1) \sqcup (G_1^{-1}(\{x\}) \cap A_2)$$

is the disjoint union of two non-empty closed sets, so $G_1^{-1}(\{x\})$ is disconnected, which is a contradiction because the fibers of G_1 are connected.

Step 3. In the following we use Lemma 2.2 several times. To ease the procedure we point out the key facts. By 2.3.2. $A := \pi_{\mathbb{C}}^{-1}(\ell_\infty(\mathbb{C}) \cup Y_{\mathbb{C}})$ is an algebraic curve whose irreducible components are non-singular rational curves. Observe that $U := Z_1\backslash A = \pi_{\mathbb{C}}^{-1}(\mathbb{C}^2\backslash Y_{\mathbb{C}})$ is a dense subset of Z_1 biregularly equivalent to $\mathbb{C}^2\backslash Y_{\mathbb{C}}$ (that is, the complement in $\mathbb{C}^2$ of a finite subset).

3.1.9. Let $B_1, B_2 \subset A$ *be two connected compact algebraic curves without common irreducible components. Then the intersection* $B_1 \cap B_2$ *is either empty or a singleton by* Lemma 2.2, Example 2.3 and Remark 2.4.

Step 4. By Zariski's Main Theorem [13, III.11.4] applied to the birational projective morphism $\pi_{\mathbb{C}} : Z_1 \to \mathbb{CP}^2$, *the fiber* $\pi_{\mathbb{C}}^{-1}(y)$ *is connected for each* $y \in Y_{\mathbb{C}}$. We prove next:

3.1.10. If $y \in Y_{\mathbb{R}}$, *the invariant projective variety* $T_y := \pi_{\mathbb{C}}^{-1}(y) \cap \widehat{F}_{\mathbb{C}}^{-1}(\mathsf{H}_\infty(\mathbb{C}))$ *is connected and* $\mathsf{K}_\infty \cap T_y$ *is a singleton. In addition, if* T_y *has dimension* 1 *and* $\mathsf{K}_{1,y}, \ldots, \mathsf{K}_{r_y,y}$ *are the invariant irreducible components of* T_y, *the projective curve* $\bigcup_{i=1}^{r_y} \mathsf{K}_{i,y}$ *is connected and* $\mathsf{K}_\infty \cap T_y = \mathsf{K}_\infty \cap \bigcup_{i=1}^{r_y} \mathsf{K}_{i,y}$ *is a singleton contained in* X_1.

The clue to prove 3.1.10. is the following:

3.1.11. Let $y \in Y_{\mathbb{R}}$ *and* T *be an invariant connected union of irreducible components of* $\pi_{\mathbb{C}}^{-1}(y)$. *Let* $K_1, \ldots, K_r$ *be the invariant irreducible components of* T *and denote* $C_i := K_i \cap \mathbb{RP}^k$, *which is* by 2.2.1(v) *a real non-singular rational curve for* $i = 1, \ldots, r$. *We have*:

(i) *The projective curve* $\bigcup_{i=1}^{r} K_i$ *is connected.*
(ii) *Suppose moreover* $\mathsf{K}_\infty \cap T \neq \varnothing$. *Then the intersection*

$$\mathsf{K}_\infty \cap T = \mathsf{K}_\infty \cap \bigcup_{i=1}^{r} K_i$$

is a singleton contained in X_1. *In particular,* $\mathsf{K}_\infty \cup \bigcup_{i=1}^{r} K_i$ *is connected.*
(iii) *We may order the indices* $i = 1, \ldots, r$ *in such a way that* $\mathsf{C}_\infty \cap C_1 = \mathsf{K}_\infty \cap T$ *is a singleton and* $C_i \cap \bigcup_{j=1}^{i-1} C_j$ *is a singleton for* $i = 2, \ldots, r$. *In particular, the real projective curve* $C := \bigcup_{i=1}^{r} C_i$ *is connected and* $\mathsf{C}_\infty \cap C \neq \varnothing$.

We prove first 3.1.11.(i). If $T = \bigcup_{i=1}^{r} K_i$, there is nothing to prove.

Otherwise, denote the non-invariant irreducible components of T with K_{r+1}, $\ldots$, K_s. Denote

$$t := \max\left\{\#\mathcal{F} :\ \mathcal{F} \subset \{1, \ldots, r\},\ \bigcup_{i \in \mathcal{F}} K_i \quad \text{is connected}\right\}$$

and let us check $t = r$. Suppose by contradiction $t < r$. We may assume that $K := \bigcup_{i=1}^{t} K_i$ is connected. As each K_ℓ is connected, each intersection $K \cap K_\ell = \varnothing$ for $t < \ell \leq r$. As T is connected and invariant and $K \cap \bigcup_{i=t+1}^{r} K_i = \varnothing$, we may assume $K \cap K_{r+1} \neq \varnothing$ and $\sigma(K_{r+1}) = K_{r+2}$. As K is invariant, $K \cap K_{r+2} \neq \varnothing$, so $K \cup K_{r+1} \cup K_{r+2}$ is connected and invariant. Repeating the previous argument recursively, we find indices $r < r + 2j_0 \leq s$ and $t < \ell \leq r$ such that (after reordering the indices $i = r+1, \ldots, s$)

$$K_\ell \cap \left(K \cup \bigcup_{i=r+1}^{r+2j_0} K_j\right) \neq \varnothing$$

and for each $1 \leq j \leq j_0$ it holds

- $K \cup \bigcup_{i=r+1}^{r+2j} K_i$ is connected and invariant and
- $\sigma(K_{r+2j-1}) = K_{r+2j}$.

Such indices $r + 2j_0, \ell$ exist because $T = \bigcup_{i=1}^{s} K_i$ is connected.

The invariant connected projective curves K_ℓ and $K \cup \bigcup_{i=r+1}^{r+2j} K_j$ are contained in $A := \pi_{\mathbb{C}}^{-1}(\ell_\infty(\mathbb{C}) \cup Y_{\mathbb{C}})$ because $y \in Y_{\mathbb{R}}$. The non-empty invariant intersection $K_\ell \cap (K \cup \bigcup_{i=r+1}^{r+2j} K_j)$ is by 3.1.9. a singleton $\{p_\ell\} \subset X_1 = Z_1 \cap \mathbb{RP}^k$. As $K_i \subset \mathbb{CP}^k \backslash \mathbb{RP}^k$ for $i = r+1, \ldots, s$ (see 2.2.1(iv)), we have $p_\ell \in K_\ell \cap K \neq \varnothing$, which is a contradiction. Then $r = t$, so $\bigcup_{i=1}^{r} K_i$ is connected.

Next we prove 3.1.11.(ii). Since $\mathsf{K}_\infty, T \subset A$ are invariant connected projective curves, the non-empty invariant intersection $\mathsf{K}_\infty \cap T$ is by 3.1.9 a singleton $\{p\} \subset X_1$. Thus, $p \in T \cap \mathbb{RP}^k \subset \bigcup_{i=1}^{r} K_i$ because the non-invariant irreducible components of T are by 2.2.1(v) contained in $\mathbb{CP}^k \backslash \mathbb{RP}^k$. Consequently, $\mathsf{K}_\infty \cap T = \mathsf{K}_\infty \cap \bigcup_{i=1}^{r} K_i = \{p\}$.

Finally, we show 3.1.11(iii). As $p \in X_1 = Z_1 \cap \mathbb{RP}^k$,

$$\mathsf{C}_\infty \cap C = \mathsf{C}_\infty \cap \bigcup_{i=1}^{r} C_i = \Big(\mathsf{K}_\infty \cap \bigcup_{i=1}^{r} K_i\Big) \cap \mathbb{RP}^k = \{p\} \neq \varnothing.$$

We may assume $\mathsf{C}_\infty \cap C_1 \neq \varnothing$, that is, $\mathsf{C}_\infty \cap C_1 = \{p\} = \mathsf{K}_\infty \cap T$. As $\bigcup_{i=1}^{r} K_r$ is connected, we claim: *We may order the indices $i = 2, \ldots, r$ in such a way that K_i intersects the union $\bigcup_{j=1}^{i-1} K_j$ for $i = 2, \ldots, r$.*

Indeed, as $K = \bigcup_{i=1}^{r} K_i$ is connected, the intersection $K_1 \cap \bigcup_{i=2}^{r} K_i \neq \varnothing$ and we assume $K_1 \cap K_2 \neq \varnothing$. As K is connected, the intersection of $K_1 \cup K_2$ and $\bigcup_{i=3}^{r} K_i \neq \varnothing$ is non-empty. We may assume that K_3 intersects $K_1 \cup K_2$ and proceeding this way we prove the claim.

Next, since K_i and $\bigcup_{j=1}^{i-1} K_j$ are invariant connected projective curves contained in A, the non-empty invariant intersection $K_i \cap \bigcup_{j=1}^{i-1} K_j$ is by 3.1.9. a singleton $\{q_i\} \subset X_1$. Thus,

$$q_i \in \mathbb{RP}^k \cap K_i \cap \bigcup_{j=1}^{i-1} K_j = C_i \cap \bigcup_{j=1}^{i-1} C_j,$$

so $C_i \cap \bigcup_{j=1}^{i-1} C_j \neq \varnothing$ for $i = 2, \ldots, r$. As each C_i is a non-singular curve biregularly equivalent to $\mathbb{RP}^1$, we deduce that the projective curve $C := \bigcup_{i=1}^{r} C_i$ is connected.

3.1.12. Now we are ready to prove 3.1.10. As $y \in Y_{\mathbb{R}} \subset \ell_\infty(\mathbb{R}) \subset \ell_\infty(\mathbb{C})$ and $\pi_{\mathbb{C}}(\mathsf{K}_\infty) = \ell_\infty(\mathbb{C})$, we deduce $\pi_{\mathbb{C}}^{-1}(y) \cap \mathsf{K}_\infty \neq \varnothing$. If $y_1 \neq y_2$, the intersection $\pi_{\mathbb{C}}^{-1}(y_1) \cap \pi_{\mathbb{C}}^{-1}(y_2) = \varnothing$ and by 3.1.3. $\widetilde{\mathsf{E}} \cap \pi_{\mathbb{C}}^{-1}(y) = \varnothing$. Consequently, by 3.1.4.

$$\begin{aligned} T_y \cap \mathsf{K}_\infty &= \pi_{\mathbb{C}}^{-1}(y) \cap \widehat{F}_{\mathbb{C}}^{-1}(\mathsf{H}_\infty(\mathbb{C})) \cap \mathsf{K}_\infty \\ &= ((\pi_{\mathbb{C}}^{-1}(y) \cap \mathsf{K}_\infty) \cup \pi_{\mathbb{C}}^{-1}(y)) \cap \mathsf{K}_\infty = \pi_{\mathbb{C}}^{-1}(y) \cap \mathsf{K}_\infty \neq \varnothing, \end{aligned}$$

so by 3.1.11.(ii) $T_y \cap \mathsf{K}_\infty$ is a singleton. Denote

$$R_y := \widetilde{\mathsf{E}} \cup \mathsf{K}_\infty \cup \bigsqcup_{z \in Y_{\mathbb{C}}, z \neq y} (\pi_{\mathbb{C}}^{-1}(z) \cap \widehat{F}_{\mathbb{C}}^{-1}(\mathsf{H}_\infty(\mathbb{C})))$$

and observe $\widehat{F}_{\mathbb{C}}^{-1}(\mathsf{H}_\infty(\mathbb{C})) = T_y \cup R_y$ by 3.1.4. Using again $\widetilde{\mathsf{E}} \cap \pi_{\mathbb{C}}^{-1}(y) = \varnothing$ and $\pi_{\mathbb{C}}^{-1}(y_1) \cap \pi_{\mathbb{C}}^{-1}(y_2) = \varnothing$ if $y_1 \neq y_2$, we deduce $T_y \cap R_y = T_y \cap \mathsf{K}_\infty$, which is a singleton. Consequently, if T_y was disconnected, then $\widehat{F}_{\mathbb{C}}^{-1}(\mathsf{H}_\infty(\mathbb{C}))$ would have been disconnected, which contradicts 3.1.6. Thus, the first part of claim 3.1.10. holds. The second part follows readily from 3.1.11.

Step 5. Next we show:

3.1.13. $\widehat{F}_{\mathbb{R}}^{-1}(\mathsf{H}_\infty(\mathbb{R}))$ is connected. Consequently, $\widehat{F}_{\mathbb{R}}(\widehat{F}_{\mathbb{R}}^{-1}(\mathsf{H}_\infty(\mathbb{R})))$ is also connected.

As $\widehat{F}_{\mathbb{C}}$ is invariant, $\widehat{F}_{\mathbb{R}}^{-1}(\mathsf{H}_\infty(\mathbb{R})) = \widehat{F}_{\mathbb{C}}^{-1}(\mathsf{H}_\infty(\mathbb{C})) \cap \mathbb{RP}^k$. By 3.1.4.

$$\widehat{F}_{\mathbb{C}}^{-1}(\mathsf{H}_\infty(\mathbb{C})) = \widetilde{\mathsf{E}} \cup \mathsf{K}_\infty \cup \bigcup_{y \in Y_{\mathbb{C}}} (\pi_{\mathbb{C}}^{-1}(y) \cap \widehat{F}_{\mathbb{C}}^{-1}(\mathsf{H}_\infty(\mathbb{C}))). \tag{3.1}$$

Fix $y \in Y_{\mathbb{R}}$ and consider $T_y := \pi_{\mathbb{C}}^{-1}(y) \cap \widehat{F}_{\mathbb{C}}^{-1}(\mathsf{H}_\infty(\mathbb{C}))$. If T_y is a singleton, we deduce by 3.1.10 that $T_y \subset \mathsf{K}_\infty$. Otherwise we denote the invariant irreducible components of T_y with $\mathsf{K}_{1,y}, \ldots, \mathsf{K}_{r_y,y}$. As the non-invariant irreducible components of $\pi_{\mathbb{C}}^{-1}(y)$ do not intersect $\mathbb{RP}^k$ by 2.2.1(v), we deduce $T_y \cap \mathbb{RP}^k = \bigcup_{i=1}^{r_y} \mathsf{K}_{i,y} \cap \mathbb{RP}^k$.

In addition by 2.2.1(iv) $\pi_{\mathbb{C}}^{-1}(y) \cap \mathbb{RP}^k = \varnothing$ for all $y \in Y_{\mathbb{C}} \backslash Y_{\mathbb{R}}$. Denote the subset of points of $Y_{\mathbb{R}}$ such that T_y is not a singleton with $Y'_{\mathbb{R}}$. If we intersect expression (3.1) with $\mathbb{RP}^k$, we deduce by 3.1.3

$$\widehat{F}_{\mathbb{R}}^{-1}(\mathsf{H}_\infty(\mathbb{R})) = \widehat{F}_{\mathbb{C}}^{-1}(\mathsf{H}_\infty(\mathbb{C})) \cap \mathbb{RP}^k = \mathsf{C}_\infty \cup \bigcup_{y \in Y'_{\mathbb{R}}} \bigcup_{i=1}^{r_y} \mathsf{K}_{i,y} \cap \mathbb{RP}^k.$$

By 3.1.10. the projective curve $\bigcup_{i=1}^{r_y} \mathsf{K}_{i,y}$ is connected and the intersection $\mathsf{K}_\infty \cap \bigcup_{i=1}^{r_y} \mathsf{K}_{i,y}$ is a singleton $\{p_y\}$ for each $y \in Y'_{\mathbb{R}}$. By 3.1.11(iii) each projective curve $\mathsf{C}_y := \bigcup_{i=1}^{r_y} \mathsf{C}_{i,y}$ is connected and each intersection $\mathsf{C}_\infty \cap \mathsf{C}_y \neq \varnothing$. As $\mathsf{C}_\infty = \mathsf{K}_\infty \cap \mathbb{RP}^k$ is by 3.1.1. connected, we conclude that

$$\widehat{F}_{\mathbb{R}}^{-1}(\mathsf{H}_\infty(\mathbb{R})) = \mathsf{C}_\infty \cup \bigcup_{y \in Y'_{\mathbb{R}}} \bigcup_{i=1}^{r_y} \mathsf{C}_{i,y}$$

is connected too.

Final Step. *Conclusion.* As $\widehat{F}_{\mathbb{R}}(\widehat{F}_{\mathbb{R}}^{-1}(\mathsf{H}_\infty(\mathbb{R})))$ is connected, to prove that S_∞ is connected, too, it is enough to show that both sets are equal, that is,

$$\widehat{F}_{\mathbb{R}}(\widehat{F}_{\mathbb{R}}^{-1}(\mathsf{H}_\infty(\mathbb{R}))) = S_\infty.$$

As $X_1 = Z_1 \cap \mathbb{RP}^k$ is compact, $\widehat{F}_{\mathbb{R}}$ is proper, so the restriction

$$\widehat{F}_{\mathbb{R}}|_{\widehat{F}_{\mathbb{R}}^{-1}(\mathbb{R}^m)} : \widehat{F}_{\mathbb{R}}^{-1}(\mathbb{R}^m) \to \mathbb{R}^m$$

is also proper. As $\pi_{\mathbb{R}}^{-1}(\mathbb{R}^2)$ is dense in X_1 and

$$S = f(\mathbb{R}^2) = F_{\mathbb{R}}(\mathbb{R}^2) = \widehat{F}_{\mathbb{R}}(\pi_{\mathbb{R}}^{-1}(\mathbb{R}^2)),$$

we have

$$\begin{aligned}\widehat{F}_{\mathbb{R}}(\widehat{F}_{\mathbb{R}}^{-1}(\mathbb{R}^m)) &= \widehat{F}_{\mathbb{R}}(\mathrm{Cl}_{\widehat{F}_{\mathbb{R}}^{-1}(\mathbb{R}^m)}(\pi_{\mathbb{R}}^{-1}(\mathbb{R}^2)))\\ &= \mathrm{Cl}_{\mathbb{R}^m}(\widehat{F}_{\mathbb{R}}(\pi_{\mathbb{R}}^{-1}(\mathbb{R}^2))) = \mathrm{Cl}_{\mathbb{R}^m}(f(\mathbb{R}^2)) = \mathrm{Cl}_{\mathbb{R}^m}(S),\end{aligned}$$

so $\widehat{F}_{\mathbb{R}}(\widehat{F}_{\mathbb{R}}^{-1}(\mathbb{R}^m)) = \mathrm{Cl}_{\mathbb{R}^m}(S)$. As $\widehat{F}_{\mathbb{R}}$ is proper,

$$\begin{aligned}\mathrm{Cl}_{\mathbb{R}^m}(S) \sqcup S_\infty &= \mathrm{Cl}_{\mathbb{RP}^m}(S) = \mathrm{Cl}_{\mathbb{RP}^m}(\widehat{F}_{\mathbb{R}}(\widehat{F}_{\mathbb{R}}^{-1}(\mathbb{R}^m)))\\ &= \widehat{F}_{\mathbb{R}}(\mathrm{Cl}_{X_1}(\widehat{F}_{\mathbb{R}}^{-1}(\mathbb{R}^m))) = \widehat{F}_{\mathbb{R}}(X_1)\\ &= \widehat{F}_{\mathbb{R}}(\widehat{F}_{\mathbb{R}}^{-1}(\mathbb{R}^m)) \sqcup \widehat{F}_{\mathbb{R}}(\widehat{F}_{\mathbb{R}}^{-1}(\mathsf{H}_\infty(\mathbb{R}))).\end{aligned}$$

Consequenlty, $S_\infty = \widehat{F}_{\mathbb{R}}(\widehat{F}_{\mathbb{R}}^{-1}(\mathsf{H}_\infty(\mathbb{R})))$, which is by 3.1.13. connected, as required. □

We are ready to prove Theorem 1.1. We present an independent proof from the one of Theorem 3.3. This is enlightening for the proof of Theorem 3.3 for $n \geq 3$.

Proof (*Proof of Theorem* 1.1) For $n = 1$ the result follows from [2, 1.1]. To prove that S_∞ is connected if $n \geq 2$, it is enough to show that for any given pair of points $p, q \in S_\infty$ there exists a connected subset of S_∞ containing p and q. By Lemma 2.5 there exist polynomials $\mathtt{p}_i, \mathtt{q}_i \in \mathbb{R}[\mathtt{t}]$ such that $\mathtt{p}_i(0), \mathtt{q}_i(0) \neq 0$ and integers k_i, ℓ_i such that the rational paths $\alpha := (\mathtt{t}^{k_1}\mathtt{p}_1, \ldots, \mathtt{t}^{k_n}\mathtt{p}_n)$ and $\beta := (\mathtt{t}^{\ell_1}\mathtt{q}_1, \ldots, \mathtt{t}^{\ell_n}\mathtt{q}_n)$ satisfy

$$\lim_{t\to 0^+}(f \circ \alpha)(t) = p \quad \text{and} \quad \lim_{t\to 0^+}(f \circ \beta)(t) = q.$$

At least one couple (k_i, ℓ_j) is of negative integers. Consider the polynomials

$$P_i(\mathtt{x}, \mathtt{y}) := \begin{cases} \mathtt{y}^{|k_i|}\mathtt{p}_i(\mathtt{x}) & \text{if } k_i < 0,\\ \mathtt{x}^{k_i}\mathtt{p}_i(\mathtt{x}) & \text{if } k_i \geq 0\end{cases} \quad \text{and} \quad Q_i(\mathtt{x}, \mathtt{y}) := \begin{cases} (-\mathtt{y})^{|\ell_i|}\mathtt{q}_i(\mathtt{x}) & \text{if } \ell_i < 0,\\ \mathtt{x}^{\ell_i}\mathtt{q}_i(\mathtt{x}) & \text{if } \ell_i \geq 0\end{cases}$$

and let

$$h := \frac{xy+1}{2}(P_1, \ldots, P_n) + \frac{1-xy}{2}(Q_1, \ldots, Q_n).$$

Consider the polynomial map $g := f \circ h : \mathbb{R}^2 \to \mathbb{R}^m$ and observe

- $T_0 := \operatorname{im}(g) \subset \operatorname{im}(f) = S$, so $T_{0,\infty} \subset S_\infty$.
- $p = \lim_{t\to 0^+} g(t, \frac{1}{t}) \in T_{0,\infty}$ and $q = \lim_{t\to 0^+} g(t, -\frac{1}{t}) \in T_{0,\infty}$.

As $T_{0,\infty}$ is connected for $n = 2$ by Theorem 3.3, we are done. □

3.2 Proof of Theorem 3.3

Now we prove Theorem 3.3 for an arbitrary n.

Proof (*Proof of Theorem 3.3 for an arbitrary n.*) The case $n = 1$ follows from [2, 1.4]. Assume $n \geq 2$ and write $f := (\frac{f_1}{f_0}, \ldots, \frac{f_m}{f_0})$ where each $f_j \in \mathbb{R}[x]$, $\gcd(f_0, f_1, \ldots, f_m) = 1$ and f_0 does not vanish on $\mathbb{R}^n$. The proof is conducted in several steps:

3.2.1. Initial assumptions to simplify the proof. Assume $\deg(f_1) \geq \deg(f_j)$ for $j = 1, \ldots, m$. After a change of the type $(y_1, \ldots, y_m) \mapsto (y_1, y_2 + b_2 y_1, \ldots, y_m + b_m y_1)$ where $b_j \in \mathbb{R}$ we can suppose

$$\deg(f_1) = \cdots = \deg(f_m) = d > \deg(f_0) = d - e$$

for some $e \geq 1$. Denote

$$F_j := x_0^d f_j\Big(\frac{x_1}{x_0}, \ldots, \frac{x_n}{x_0}\Big)$$

and $F_0 = x_0^e F_0'$ where $e \geq 1$ and $F_0' \in \mathbb{R}[x_0, x_1, \ldots, x_n]$ is not divisible by x_0. Notice that x_0 does not divide F_j for $j = 1, \ldots, m$ because $\deg(f_j) = \deg(F_j) = d$. After a change of the type $(x_1, \ldots, x_m) \mapsto (x_1, x_2 + a_2 x_1, \ldots, x_n + a_n x_1)$ where $a_i \in \mathbb{R}$ we can suppose $\deg(f_j) = \deg_{x_1}(f_j)$ for each j.

3.2.2. Equivalent formulation for the statement. To prove that S_∞ is connected, it is equivalent to show: *There exists a point $p_0 \in S_\infty$ such that for any other point $q \in S_\infty$ there exists a connected subset of S_∞ containing p_0 and q.*

We will prove this last fact. Fix

$$p_0 := \lim_{t\to 0^+} (F_0 : F_1 : \cdots : F_m)\Big(1 : \frac{1}{t} : 0 : \cdots : 0\Big) \in S_\infty(\mathbb{R})$$

and let $q \in S_\infty$. By Lemma 2.5 there exist a rational path

$$\alpha := (t^{k_1} p_1, \ldots, t^{k_n} p_n) \in \mathbb{R}(t)^n,$$

where

- $k_i \in \mathbb{Z}$, $k_{i_0} := \min\{k_1, \ldots, k_n\} < 0$,
- $\mathtt{p}_i \in \mathbb{R}[\mathtt{t}]$ with $\mathtt{p}_i(0) \neq 0$ for $i = 1, \ldots, n$ and $\mathtt{p}_{i_0} = \pm 1$

and an integer $r \geq 1$ such that $q = \lim_{t\to 0^+}(f \circ (\alpha + \beta))(t)$ for each $\beta \in (\mathtt{t})^r\mathbb{R}[\mathtt{t}]^n$. After the change $\mathtt{t} \to \mathtt{t}^6$ we may assume $k_{i_0} \leq -6$ and even. We keep all initial notations.

3.2.3. Construction of an auxiliary regular map. Write $\mathtt{p}_i := \sum_{j=0}^{d_i} a_{ij}\mathtt{t}^j$ where $d_i := \deg(\mathtt{p}_i)$ and consider the formula

$$P_i(\mathtt{x}, \mathtt{y}) := \sum_{j+k_i<0} a_{ij}\mathtt{y}^{|k_i+j|} + \sum_{j+k_i\geq 0} a_{ij}\frac{\mathtt{x}^{e_j}\mathtt{y}}{((\mathtt{xy}-1)^2+\mathtt{y}^4)^{q_j}},$$

where $j + k_i + 1 = 4q_j + e_j$, $q_j \geq 0$ and $0 \leq e_j \leq 3$ for each $0 \leq j \leq d_i$ such that $j + k_i \geq 0$. Observe $P_i(\mathtt{t}, \frac{1}{\mathtt{t}}) = \mathtt{t}^{k_i}\mathtt{p}_i(\mathtt{t})$ for $i = 1, \ldots, n$ and $P_{i_0} = \mathtt{p}_{i_0}\mathtt{y}^{|k_{i_0}|} = \pm\mathtt{y}^{|k_{i_0}|}$.

Denote $\ell_0 := \max_i\{q_{d_i} : d_i + k_i \geq 0\}$ and notice $((\mathtt{xy}-1)^2+\mathtt{y}^4)^\ell P_i \in \mathbb{R}[\mathtt{x}, \mathtt{y}]$ for each $\ell \geq \ell_0$. In addition

$$\begin{aligned}\deg(((\mathtt{xy}-1)^2+\mathtt{y}^4)^\ell P_i) &\leq \max\{-k_i + 4\ell, 4\ell - 4q_j + e_j + 1 : k_i + j \geq 0\}\\ &\leq 4\ell + \max\{-k_i, 4\} \leq 4\ell + |k_{i_0}|\end{aligned}$$

and the equality $\deg(((\mathtt{xy}-1)^2+\mathtt{y}^4)^\ell P_i) = 4\ell + |k_{i_0}|$ holds if and only if $k_i = k_{i_0}$. As $|k_{i_0}| \geq 6$,

$$\begin{aligned}&\deg_{\mathtt{x}}(((\mathtt{xy}-1)^2+\mathtt{y}^4)^\ell P_i)\\ &\leq \left\{\begin{matrix}2(\ell - q_{d_i}) + 3 & \text{if } k_i + d_i \geq 0\\ 2\ell & \text{if } k_i + d_i < 0\end{matrix}\right\} < 2\ell + 4 < 4\ell + |k_{i_0}|\end{aligned}$$

for all $i = 1, \ldots, n$. Let $\ell := \max\{\ell_0, r\}$ and define

$$h_i(\mathtt{x}, \mathtt{y}) := \begin{cases}((\mathtt{xy}-1)^2+\mathtt{y}^4)^\ell & \text{if } i = 0,\\ ((\mathtt{xy}-1)^2+\mathtt{y}^4)^\ell P_1(\mathtt{x}, \mathtt{y}) + \mathtt{p}_1(0)\mathtt{x}^{4\ell+|k_{i_0}|} & \text{if } i = 1,\\ ((\mathtt{xy}-1)^2+\mathtt{y}^4)^\ell P_i(\mathtt{x}, \mathtt{y}) & \text{if } i = 2, \ldots, n.\end{cases}$$

Consider the regular map $h := \left(\frac{h_1}{h_0}, \ldots, \frac{h_n}{h_0}\right) : \mathbb{R}^2 \to \mathbb{R}^n$ and observe

$$\begin{aligned}h(\mathtt{t}, 0) &= (\mathtt{p}_1(0)\mathtt{t}^{4\ell+|k_{i_0}|}, 0, \ldots, 0)\\ \text{and}\quad h\left(\mathtt{t}, \frac{1}{\mathtt{t}}\right) &= \alpha(\mathtt{t}) + (\mathtt{p}_1(0)\mathtt{t}^{8\ell+|k_{i_0}|}, 0, \ldots, 0),\end{aligned}$$

so $\lim_{t\to 0^+}(f \circ h)(\frac{1}{t}, 0) = p_0$ and $\lim_{t\to 0^+}(f \circ h)(t, \frac{1}{t}) = q$.

3.2.4. Construction of an auxiliary quasi-polynomial map such that p_0, q belong to the set of points at infinity of its image. Let $g := f \circ h : \mathbb{R}^2 \to \mathbb{R}^m$ and denote

$g_i := F_i(h_0, h_1, \ldots, h_n) \in \mathbb{R}[\mathtt{u}_1, \mathtt{u}_2]$ for $i = 0, 1, \ldots, m$. Observe $g = (\frac{g_1}{g_0}, \ldots, \frac{g_m}{g_0})$ and g_0 does not vanish on $\mathbb{R}^2$. We claim: *g is a quasi-polynomial map and $p_0, q \in g(\mathbb{R}^2)_\infty$.*

First we have

- $T_0 := \mathrm{im}(g) \subset \mathrm{im}(f) = S$, so $T_{0,\infty} \subset S_\infty$.
- $p_0 = \lim_{t\to 0^+} g(\frac{1}{t}, 0) \in T_{0,\infty}$ and $q = \lim_{t\to 0^+} g(t, \frac{1}{t}) \in T_{0,\infty}$.
- $g_0 = ((\mathtt{u}_1\mathtt{u}_2 - 1)^2 + \mathtt{u}_2^4)^{e\ell} F_0'(h_0, h_1, \ldots, h_n)$.

In addition,

$$\deg(g_0) = d(4\ell + |k_{i_0}|) - e|k_{i_0}| < d(4\ell + |k_{i_0}|) = \max\{\deg(g_1), \ldots, \deg(g_m)\}.$$

Indeed, since

- $\deg(F_0') = \deg(f_0) = \deg_{\mathtt{x}_1}(f_0) = d - e$,
- $\deg(F_j) = \deg(f_j) = \deg_{\mathtt{x}_1}(f_j) = d$,
- $\deg(h_i) \le 4\ell + |k_{i_0}|$, $\deg(h_1) = 4\ell + |k_{i_0}|$ and
- $(h_0, h_1, \ldots, h_n)(\mathtt{t}, 0) = (1, \mathrm{p}_1(0)\mathtt{t}^{4\ell+|k_{i_0}|}, 0, \ldots, 0)$,

we have

$$\begin{aligned}\deg(g_0) &= 4e\ell + (d-e)(4\ell + |k_{i_0}|) = d(4\ell + |k_{i_0}|) - e|k_{i_0}| \\ &< d(4\ell + |k_{i_0}|) = \deg(g_j)\end{aligned}$$

for each $j = 1, \ldots, m$.

Let $\mu := 4\ell + |k_{i_0}|$ and write $H_i := \mathtt{u}_0^\mu h_i(\frac{\mathtt{u}_1}{\mathtt{u}_0}, \frac{\mathtt{u}_2}{\mathtt{u}_0})$ and $G_j := \mathtt{u}_0^{d\mu} g_j(\frac{\mathtt{u}_1}{\mathtt{u}_0}, \frac{\mathtt{u}_2}{\mathtt{u}_0})$, which are homogeneous polynomials. Notice that

$$\begin{aligned}G_j &= \mathtt{u}_0^{d\mu} F_j\Big(h_0\Big(\frac{\mathtt{u}_1}{\mathtt{u}_0}, \frac{\mathtt{u}_2}{\mathtt{u}_0}\Big), h_1\Big(\frac{\mathtt{u}_1}{\mathtt{u}_0}, \frac{\mathtt{u}_2}{\mathtt{u}_0}\Big), \ldots, h_n\Big(\frac{\mathtt{u}_1}{\mathtt{u}_0}, \frac{\mathtt{u}_2}{\mathtt{u}_0}\Big)\Big) \\ &= F_j(H_0, H_1, \ldots, H_n)\end{aligned}$$

and $G_0 = \mathtt{u}_0^{e|k_{i_0}|} G_0'$ where $G_0' := ((\mathtt{u}_1\mathtt{u}_2 - \mathtt{u}_0^2)^2 + \mathtt{u}_2^4)^{e\ell} F_0'(H_0, H_1, \ldots, H_n)$. Counting degrees one realizes: $\mathtt{u}_0$ *does not divide* G_0'.

In the following all zero sets are considered in $\mathbb{RP}^n$. To prove that g is quasi-polynomial it only remains to check $\{G_0' = 0, G_1 = 0, \ldots, G_m = 0\} = \varnothing$.

Indeed, as $\{F_0' = 0, F_1 = 0, \ldots, F_n = 0\} = \varnothing$, the following equality holds

$$\begin{aligned}\{G_0' = 0, G_1 = 0, \ldots, G_m = 0\} &= \{H_0 = 0, \ldots, H_n = 0\} \\ &\cup \{((\mathtt{u}_1\mathtt{u}_2 - \mathtt{u}_0^2)^2 + \mathtt{u}_2^4)^{e\ell} = 0, G_1 = 0, \ldots, G_m = 0\}.\end{aligned}$$

Notice

$$H_0(\mathrm{u}_0,\mathrm{u}_1,\mathrm{u}_2)=u_0^{|k_{i_0}|}((\mathrm{u}_1\mathrm{u}_2-\mathrm{u}_0^2)^2+\mathrm{u}_2^4)^\ell,$$

$$H_1(0,\mathrm{u}_1,\mathrm{u}_2)=\begin{cases}\mathrm{p}_1(0)\mathrm{u}_1^{4\ell+|k_{i_0}|} & \text{if } k_{i0}<k_1,\\ \mathrm{p}_1(0)\big((\mathrm{u}_1^2\mathrm{u}_2^2+\mathrm{u}_2^4)^\ell\mathrm{u}_2^{|k_{i_0}|}+\mathrm{u}_1^{4\ell+|k_{i_0}|}\big) & \text{if } k_{i0}=k_1,\end{cases}$$

$$H_i(0,0,\mathrm{u}_2)=\begin{cases}0 & \text{if } i\geq 2 \text{ and } k_{i0}<k_i,\\ \mathrm{p}_i(0)\mathrm{u}_2^{4\ell+|k_{i_0}|} & \text{if } i\geq 2 \text{ and } k_{i0}=k_i.\end{cases}$$

Consequently, $\{H_0=0,\dots,H_n=0\}=\varnothing$ because

- $H_0=0$ provides $\mathrm{u}_0=0$,
- $H_1(0,\mathrm{u}_1,\mathrm{u}_2)=0$ provides $\mathrm{u}_1=0$ (we have used here that k_{i_0} is even) and
- $H_{i_0}(0,0,\mathrm{u}_2)=0$ provides $\mathrm{u}_2=0$.

On the other hand $((\mathrm{u}_1\mathrm{u}_2-\mathrm{u}_0^2)^2+\mathrm{u}_2^4)^{e\ell}=0$ provides $\mathrm{u}_0=0,\mathrm{u}_2=0$. As $G_1=0$, we have

$$\begin{aligned}0&=G_1(0,\mathrm{u}_1,0)\\&=F_1(H_0(0,\mathrm{u}_1,0),H_1(0,\mathrm{u}_1,0),\dots,H_n(0,\mathrm{u}_1,0))=F_1(0,\mathrm{u}_1^\mu,0,\dots,0).\end{aligned}$$

For the last equality use that $\deg_{\mathrm{x}}(((\mathrm{x}\mathrm{y}-1)^2+\mathrm{y}^4)^\ell P_i)<4\ell+|k_{i_0}|$ for all $i=1,\dots,n$. As $\deg(F_1)=\deg(f_1)=\deg_{\mathrm{x}_1}(f_1)=\deg_{\mathrm{x}_1}(F_1)$, we obtain $a:=F_1(0,1,0,\dots,0)\neq 0$. As

$$0=F_1(0,\mathrm{u}_1^\mu,0,\dots,0)=a\mathrm{u}_1^{d\mu},$$

we get $\mathrm{u}_1=0$, so $\{((\mathrm{u}_1\mathrm{u}_2-\mathrm{u}_0^2)^2+\mathrm{u}_2^4)^{e\ell}=0,G_1=0,\dots,G_m=0\}=\varnothing$. Therefore $\{G_0'=0,G_1=0,\dots,G_m=0\}=\varnothing$, so $g:\mathbb{R}^2\to\mathbb{R}^m$ is a quasi-polynomial map.

3.2.5. Conclusion. By Theorem 3.3 for the case $n=2$ (already proved in 3.1) applied to the quasi-polynomial map g, we deduce that $T_{0,\infty}=(g(\mathbb{R}^2))_\infty\subset S_\infty$ is connected and since $p_0,q\in T_{0,\infty}$, we are done. □

The set of points at infinity of a semialgebraic set $S\subset\mathbb{R}^m$ is a semialgebraic subset of the hyperplane of infinity $\mathsf{H}_\infty(\mathbb{R})$ of $\mathbb{RP}^m$. It seems reasonable to ask the following.

Question 3.4 Let S_0 be a connected closed semialgebraic subset of $\mathsf{H}_\infty(\mathbb{R})$. *Is there a polynomial (or a quasi-polynomial) map* $f:\mathbb{R}^n\to\mathbb{R}^m$ *such that* $f(\mathbb{R}^n)_\infty=\mathsf{S}_0$?

For $m=2$ the answer is positive but for higher dimension we have no further information.

Example 3.5 For each connected closed semialgebraic subset $\mathsf{S}_0\subset\ell_\infty(\mathbb{R})$ there exists a polynomial map $f:\mathbb{R}^2\to\mathbb{R}^2$ such that $\dim_{\mathbb{R}}(f(\mathbb{R}^2))=2$ and $(f(\mathbb{R}^2))_\infty=\mathsf{S}_0$.

If S_0 is not a singleton, the assertion follows from [11,12,23]. On the other hand, the polynomial map $f:\mathbb{R}^2\to\mathbb{R}^2$, $(x,y)\mapsto(x,y^2+x^2)$ satisfies $f(\mathbb{R}^2))_\infty=\{(0{:}1{:}0)\}$, which is a singleton.

Remarks 3.6 We have introduced quasi-polynomial maps to understand the limitations of the image of a regular map to have a connected set of points at infinity. The following examples show that they do not enjoy a very nice behavior.

(i) The composition of the quasi-polynomial maps

$$g : \mathbb{R}^2 \to \mathbb{R}^2,\ (x, y) \mapsto (x, y^2)$$
$$\text{and}\quad f : \mathbb{R}^2 \to \mathbb{R}^2,\ (x, y) \mapsto \left(\frac{x^3}{1 + x^2 + y^2}, \frac{y^3}{1 + x^2 + y^2}\right)$$

is not a quasi-polynomial map.

(ii) The image of the quasi-polynomial map $g : \mathbb{R} \to \mathbb{R}^2,\ t \mapsto (\frac{1}{1+t^2}, 1 + t^2)$ is the semialgebraic set $S = \{xy = 1, y \geq 1\}$, which is not a polynomial image of $\mathbb{R}^n$.

4 Set of Points at Infinity of a Regular Image of $\mathbb{R}^n$

We have proved in Sect. 3 that the set of points at infinity of the image of a quasi-polynomial map $f : \mathbb{R}^n \to \mathbb{R}^m$ is connected. This is no longer true in general for regular maps even if $n = 1$.

4.1 Preliminary Examples

We present some examples to illustrate the previous fact and to show that the conditions in the statement of Theorem 3.3 are sharp.

Examples 4.1 (i) The image of the regular map

$$f : \mathbb{R}^2 \to \mathbb{R}^2,\ (x, y) \mapsto \left((xy - 1)^2 + x^2, \frac{1}{(xy - 1)^2 + x^2}\right)$$

is $S := \{a > 0, ab = 1\}$, so $S_\infty = \{(0:1:0), (0:0:1)\}$ is disconnected.

(ii) The image of the regular map

$$f : \mathbb{R}^2 \to \mathbb{R}^2,\ (x, y) \mapsto \left(\frac{x^2}{1 + y^2}, \frac{y^2}{1 + x^2}\right)$$

is $S := \{a \geq 0, b \geq 0, ab < 1\}$, so $S_\infty = \{(0:1:0), (0:0:1)\}$ is disconnected. If we write $f := (\frac{f_1}{f_0}, \frac{f_2}{f_0})$ where each f_i is a non-zero polynomial, then $\deg(f_0) = \max\{\deg(f_1), \deg(f_2)\}$.

(iii) The image of the regular map

$$f : \mathbb{R}^2 \to \mathbb{R}^2,\ (x, y) \mapsto \left(\frac{(1 + x^4)y^6}{(1 + y^4)^2}, \frac{(1 + y^4)x^4}{(1 + x^4)^3}\right)$$

is a semialgebraic set S such that $S_\infty = \{(0 : 1 : 0), (0 : 0 : 1)\}$ is disconnected. If we write $f := \left(\frac{f_1}{f_0}, \frac{f_2}{f_0}\right)$ where each f_i is a non-zero polynomial, then $\deg(f_0) <$

$\max\{\deg(f_1), \deg(f_2)\}$. The set $Y_{\mathbb{C}} = \{(0{:}1{:}0), (0{:}0{:}1)\}$ of indeterminacy of the rational map

$$F_{\mathbb{C}} := (F_0{:}F_1{:}F_2) : \mathbb{CP}^2 \dashrightarrow \mathbb{CP}^2$$

is contained in $\{F_0' = 0\} \cap \mathbb{RP}^2$ where $F_0' = (x_0^4+x_1^4)^3(x_0^4+x_2^4)^2$ and $F_0 = x_0^2 F_0'$.

Proof Observe that $f(\mathbb{R}^2) \subset \{a^2 b \leq 1, a \geq 0, b \geq 0\}$ because

$$\Big(\frac{(1+x^4)y^6}{(1+y^4)^2}\Big)^2\Big(\frac{(1+y^4)x^4}{(1+x^4)^3}\Big) = \Big(\frac{y^4}{1+y^4}\Big)^3\Big(\frac{x^4}{1+x^4}\Big) \leq 1.$$

Thus, $S_\infty \subset \{(0{:}1{:}0), (0{:}0{:}1)\}$. To prove the converse inclusion it is enough to pick two rational paths $\alpha_i : (0, 1] \to \mathbb{R}^2$ such that $\lim_{t\to 0^+} \|\alpha_i(t)\|^2 = +\infty$ and

$$\lim_{t\to 0^+} \frac{g_1}{g_0}(\alpha_1(t)) = +\infty, \quad \lim_{t\to 0^+} \frac{g_2}{g_0}(\alpha_1(t)) = 0,$$
$$\lim_{t\to 0^+} \frac{g_1}{g_0}(\alpha_2(t)) = 0, \quad \lim_{t\to 0^+} \frac{g_2}{g_0}(\alpha_2(t)) = +\infty.$$

For instance, $\alpha_1(t) := \big(\frac{1}{t}, 1\big)$ and $\alpha_2(t) := \big(1, \frac{1}{t}\big)$ do the job. □

The following question arises naturally.

Question 4.2 Given a closed semialgebraic subset $\mathsf{S}_0 \subset \ell_\infty(\mathbb{R}) \subset \mathbb{RP}^2$: *Is there a regular map* $f : \mathbb{R}^2 \to \mathbb{R}^2$ *such that* $(f(\mathbb{R}^2))_\infty = \mathsf{S}_0$?

In case S_0 is either connected or a finite set, the answer to Question 4.2 is by Example 3.5 above and Proposition 4.3 below affirmative.

4.2 More Sophisticated Examples

If S_0 is a finite set, we proceed as follows.

Proposition 4.3 *Let* $H_i := c_i x - d_i y$ *be linear equations such that* $c_i d_i \neq 0$ *and the lines* $\ell_i := \{H_i = 0\}$ *are pairwise different. Denote* $p_i := (0 : d_i^2 : c_i^2)$. *Then the image of the regular map*

$$h := (h_1, h_2) : \mathbb{R}^2 \to \mathbb{R}^2, \ (x, y) \mapsto \Big(\frac{x^2}{1+\prod_{i=1}^r H_i(x,y)^2}, \frac{y^2}{1+\prod_{i=1}^r H_i(x,y)^2}\Big)$$

is a semialgebraic set S *such that* $S_\infty = \{p_1, \ldots, p_r\}$.

Proof Observe first

$$h\Big(\frac{d_i}{t}, \frac{c_i}{t}\Big) = \Big(\frac{d_i^2}{t^2}, \frac{c_i^2}{t^2}\Big) \equiv (t^2 : d_i^2 : c_i^2) \overset{t\to 0^+}{\longrightarrow} (0 : d_i^2 : c_i^2) = p_i,$$

so $\{p_1, \ldots, p_r\} \subset S_\infty$. Next, we prove the converse inclusion $S_\infty \subset \{p_1, \ldots, p_r\}$.

The Jacobian of h is not identically zero, so $S := h(\mathbb{R}^2)$ has dimension 2. In addition, S is pure dimensional because it is a regular image of $\mathbb{R}^2$. Thus,

$$S' := S \backslash (h(\{x = 0\}) \cup h(\{y = 0\}) \cup h(\{\mathrm{Jac}(h) = 0\}))$$

is dense in S. Fix a point $p \in S_\infty$. After interchanging the variables and changing $\mathtt{x}$ with $-\mathtt{x}$ if necessary, there exists by Lemma 2.5 a rational path $\alpha := (\mathtt{t}^{-k}, \mathtt{t}^\ell \mathrm{p})$ where $k, \ell \in \mathbb{Z}, -k = \min\{-k, \ell\} < 0, \mathrm{p} \in \mathbb{R}[\mathtt{t}]$ and $\mathrm{p}(0) \neq 0$ such that $\lim_{t\to 0^+}(h \circ \alpha) = p$ and $(h \circ \alpha)((0, \varepsilon)) \subset S'$ for $\varepsilon > 0$ small enough. Our change of variables does not modify the structure of h, so we keep the same notations. As $p \in \ell_\infty(\mathbb{R})$, one of the following limits is infinity:

$$\lim_{t\to 0^+}(h_1 \circ \alpha)(t) = \lim_{t\to 0^+} \frac{t^{2k(r-1)}}{t^{2kr} + \prod_{i=1}^r H_i(1, t^{k+\ell}\mathrm{p}(t))^2},$$
$$\lim_{t\to 0^+}(h_2 \circ \alpha)(t) = \lim_{t\to 0^+} \frac{t^{2kr+2\ell}\mathrm{p}(t)^2}{t^{2kr} + \prod_{i=1}^r H_i(1, t^{k+\ell}\mathrm{p}(t))^2}.$$

Notice the following:

(1) The first limit is infinity if and only if

$$t^{2kr} + \prod_{i=1}^r H_i(1, \mathtt{t}^{k+\ell}\mathrm{p}(\mathtt{t}))^2 = \mathtt{t}^{\nu_1}\mathrm{q}_1(\mathtt{t})$$

for some $\mathrm{q}_1 \in \mathbb{R}[\mathtt{t}]$ and an integer $\nu_1 \geq 2k(r-1) + 1$.

(2) The second limit is infinity if and only if $\ell < 0$ and

$$t^{2kr} + \prod_{i=1}^r H_i(1, \mathtt{t}^{k+\ell}\mathrm{p}(\mathtt{t}))^2 = \mathtt{t}^{\nu_2}\mathrm{q}_2(\mathtt{t})$$

for some $\mathrm{q}_2 \in \mathbb{R}[\mathtt{t}]$ and an integer $\nu_2 \geq 2kr + 2\ell + 1$.

As $k + \ell \geq 0$, we deduce in both cases that there exists an index $i = 1, \ldots, r$ such that $\lim_{t\to 0^+} H_i(1, t^{k+\ell}\mathrm{p}(t)) = 0$. Since by hypothesis $c_i, d_i \neq 0$, we conclude $\ell = -k < 0$ and $\mathrm{p}(0) = \frac{c_i}{d_i}$. Denote $\mu := \nu_j - 2k(r-1) - 1 \geq 0$ and $\mathrm{q} := \mathrm{q}_j$ in both cases, so

$$\mathtt{t}^{2kr} + \prod_{i=1}^r H_i(1, \mathtt{t}^{k+\ell}\mathrm{p}(\mathtt{t}))^2 = \mathtt{t}^{2k(r-1)+1}\mathtt{t}^{\mu}\mathrm{q}.$$

We deduce

$$\begin{aligned} p &= \lim_{t\to 0^+} (t^{2kr} + \prod_{i=1}^{r} H_i(1, \mathrm{p}(t))^2 : t^{2k(r-1)} : t^{2kr+2\ell}\mathrm{p}(t)^2) \\ &= \lim_{t\to 0^+} (t^{2k(r-1)+1} t^{\mu}\mathrm{q}(t) : t^{2k(r-1)} : t^{2k(r-1)}\mathrm{p}(t)^2) \\ &= \lim_{t\to 0^+} (t^{\mu+1}\mathrm{q}(t) : 1 : \mathrm{p}(t)^2) = \Big(0 : 1 : \frac{c_i^2}{d_i^2}\Big), \end{aligned}$$

so $S_\infty \subset \{p_1, \ldots, p_r\}$. □

We present next an example of a regular image S such that S_∞ has exactly two 1-dimensional connected components.

Lemma 4.4 *There exists a regular map $f : \mathbb{R}^2 \to \mathbb{R}^2$ whose image S satisfies*

$$S_\infty = \Big\{(0 : u : 1) :\ 0 \le u \le \frac{1}{2}\Big\} \cup \Big\{(0 : 1 : v) :\ 0 \le v \le \frac{1}{2}\Big\}.$$

Proof We build f as the composition of two regular maps that we construct next: *4.B.1. Let $T := \{0 < a \le 1, b > 0\} \cup \{0 < b \le 1, a > 0\}$. The image of the regular map*

$$g : \mathbb{R}^2 \to \mathbb{R}^2,\ (x, y) \mapsto \Big(\frac{x^2+1}{1+x^2y^2}, \frac{y^2+1}{1+x^2y^2}\Big)$$

is a semialgebraic set S_1 such that $T \subset S_1 \subset T \cup [0, 2]^2 \subset \{a > 0, b > 0\}$. In particular, $S_{1,\infty} = \{(0 : 1 : 0), (0 : 0 : 1)\}$.

We check first $S_1 \subset T \cup [0, 2]^2$. Let $(x, y) \in \mathbb{R}^2$ and write $f_3(x, y) =: (a, b)$. We claim: *If $b > 2$, then $0 < a \le 1$.* If $a > 2$, we will have by symmetry $0 < b \le 1$, so $S_1 \subset T \cup [0, 2]^2$.

It is clear that $a > 0$. Suppose by contradiction $a > 1$. Then $x^2 > x^2y^2$ and $y^2 > 1 + 2x^2y^2$, so

$$x^2 < x^2 + 2x^4y^2 < x^2y^2 < x^2,$$

which is a contradiction.

Next, we check $T \subset S_1$. It is enough to prove by symmetry that

$$T_1 := \{0 < a \le 1, b > 0\} \subset S_1.$$

Let $(a, b) \in T_1$ and consider the system of equations

$$\begin{cases} x^2 + 1 = a(1 + x^2y^2) \\ y^2 + 1 = b(1 + x^2y^2) \end{cases} \rightsquigarrow \begin{cases} b(x^2 + 1) - a(y^2 + 1) = 0, \\ ay^4 + (a - 1 - b)y^2 + b - 1 = 0. \end{cases}$$

A simple discussion shows that both systems are equivalent. The discriminant Δ of the biquadratic equation $ay^4 + (a-1-b)y^2 + b - 1 = 0$ is

$$(a-1-b)^2 - 4a(b-1) = (b-3a+1)^2 + 8a(1-a),$$

which is ≥ 0 because $0 < a \leq 1$. As $a - 1 - b < 0$, the real number

$$z_0 := \frac{b + (1-a) + \sqrt{(b-3a+1)^2 + 8a(1-a)}}{2a}$$

is positive and has a square root y_0, which is a solution of the biquadratic equation

$$ay^4 + (a-1-b)y^2 + b - 1 = 0.$$

The equation $b(x^2+1) - a(y_0^2+1) = 0$ has a real solution x_0 if

$$\begin{aligned} 0 < 2(a(y_0^2+1) - b) \ &= 2az_0 + 2a - 2b = -b + 1 + a \\ &\quad + \sqrt{(b-3a+1)^2 + 8a(1-a)} \end{aligned}$$

or equivalently if

$$0 < (b-3a+1)^2 + 8a(1-a) - (b-1-a)^2 = 4b(1-a).$$

As $b > 0$ and $a < 1$, it holds $4b(1-a) > 0$, so we deduce $(a, b) \in S_1$, as required.

4.2.2. Let $B_1 := \{0 < 2a \leq b\}$ and $B_2 := \{0 < 2b \leq a\}$. Write also $A_1 := \{0 < x \leq 1,\ 4 \leq y\}$ and $A_2 := \{0 < y \leq 1,\ 4 \leq x\}$. *Then the image of* $A := A_1 \cup A_2$ *under the regular map*

$$\begin{aligned} &h : \mathbb{R}^2 \to \mathbb{R}^2, \\ &(x, y) \mapsto \Big(\frac{x((y-1)^2y^2 + 2(x-1)^2x)}{1 + x(x-1)^2y(y-1)^2}, \frac{y((x-1)^2x^2 + 2(y-1)^2y)}{1 + x(x-1)^2y(y-1)^2} \Big), \end{aligned}$$

is a semialgebraic set S_2 *contained in* $B := B_1 \cup B_2$, *which satisfies*

$$S_{2,\infty} = B_{1,\infty} \cup B_{2,\infty} = \Big\{ (0 : u : 1) :\ 0 \leq u \leq \frac{1}{2} \Big\} \cup \Big\{ (0 : 1 : v) :\ 0 \leq v \leq \frac{1}{2} \Big\}.$$

Write $h := \left(\frac{h_1}{h_0}, \frac{h_2}{h_0}\right)$ where

$$\begin{aligned} h_0(\mathrm{x}, \mathrm{y}) &:= 1 + \mathrm{x}(\mathrm{x}-1)^2\mathrm{y}(\mathrm{y}-1)^2, \\ h_1(\mathrm{x}, \mathrm{y}) &:= \mathrm{x}((\mathrm{y}-1)^2\mathrm{y}^2 + 2(\mathrm{x}-1)^2\mathrm{x}), \\ h_2(\mathrm{x}, \mathrm{y}) &:= \mathrm{y}((\mathrm{x}-1)^2\mathrm{x}^2 + 2(\mathrm{y}-1)^2\mathrm{y}). \end{aligned}$$

As $h_1(\mathtt{y},\mathtt{x}) = h_2(\mathtt{x},\mathtt{y})$ and $h_0(\mathtt{y},\mathtt{x}) = h_0(\mathtt{x},\mathtt{y})$, we have $\frac{h_1(\mathtt{y},\mathtt{x})}{h_0(\mathtt{y},\mathtt{x})} = \frac{h_2(\mathtt{x},\mathtt{y})}{h_0(\mathtt{x},\mathtt{y})}$, so it is enough to prove: $h(A_1) \subset B_1$ *and* $(h(A_1))_\infty = B_{1,\infty}$.

Let $(x, y) \in A_1$. It holds $h_1(x, y) > 0$ and

$$\begin{aligned} h_2(x,y) - 2h_1(x,y) &= y((x-1)^2x^2 + 2(y-1)^2y) - 2(x((y-1)^2y^2 + 2(x-1)^2x)) \\ &= 2(1-x)(y-1)^2y^2 + (y-4)(x-1)^2x^2 \geq 0 \end{aligned}$$

because $0 < x \leq 1$ and $y \geq 4$, so $h(x, y) \in B_1$. Therefore, $(h(A_1))_\infty \subset B_{1,\infty}$ and it only remains to check $B_{1,\infty} \subset (h(A_1))_\infty$.

Indeed, for each $0 < \lambda \leq 1$ consider the half-line $x = \lambda$, $y = t \geq 4$ and the curve $C_\lambda \subset h(A_1)$ parametrized by

$$\begin{aligned} \alpha_\lambda(t) &:= (\alpha_{\lambda 1}(t), \alpha_{\lambda 2}(t)) := g(\lambda, t) \\ &= \Big(\frac{\lambda(t-1)^2t^2 + 2\lambda\mu_\lambda)}{1 + \mu_\lambda t(t-1)^2}, \frac{2(t-1)^2t^2 + \lambda\mu_\lambda t}{1 + \mu_\lambda t(t-1)^2}\Big), \end{aligned}$$

where $\mu_\lambda := \lambda(\lambda - 1)^2$. As

$$\lim_{t\to+\infty} \alpha_{\lambda 1}(t) = +\infty, \qquad \lim_{t\to+\infty} \alpha_{\lambda 2}(t) = +\infty \qquad \text{and} \qquad \lim_{t\to+\infty} \frac{\alpha_{\lambda 1}(t)}{\alpha_{\lambda 2}(t)} = \frac{\lambda}{2},$$

we deduce $C_{\lambda,\infty} = \{(0 : \frac{\lambda}{2} : 1)\}$, so

$$B_{1,\infty} = \bigcup_{0<\lambda\leq 1} C_{\lambda,\infty} \subset \Big(\bigcup_{0<\lambda\leq 1} C_\lambda\Big)_\infty = (h(A_1))_\infty,$$

as required.

4.2.3. The image of the regular map $f := h \circ g : \mathbb{R}^2 \to \mathbb{R}^2$ is a semialgebraic set S such that $S_\infty = \{(0 : u : 1) : \ 0 \leq u \leq \frac{1}{2}\} \cup \{(0 : 1 : v) : \ 0 \leq v \leq \frac{1}{2}\}$, as required. □

Question 4.5 Let S_0 be a closed semialgebraic subset of the hyperplane of infinity $\mathsf{H}_\infty(\mathbb{R})$ of $\mathbb{RP}^m$. *Is there a regular map* $f : \mathbb{R}^n \to \mathbb{R}^m$ *such that* $(f(\mathbb{R}^n))_\infty = \mathsf{S}_0$?

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