Discrete Comput Geom (2014) 51:859–884
DOI 10.1007/s00454-014-9592-7

Duality Properties of Strong Isoperimetric Inequalities on a Planar Graph and Combinatorial Curvatures

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Received: 2 July 2013 / Revised: 20 March 2014 / Accepted: 31 March 2014 /
Published online: 13 May 2014

Abstract This paper is about hyperbolic properties on planar graphs. First, we study the relations among various kinds of strong isoperimetric inequalities on planar graphs and their duals. In particular, we show that a planar graph satisfies a strong isoperimetric inequality if and only if its dual has the same property, if the graph satisfies some minor regularity conditions and we choose an appropriate notion of strong isoperimetric inequalities. Second, we consider planar graphs where negative combinatorial curvatures dominate, and use the outcomes of the first part to strengthen the results of Higuchi, Żuk, and, especially, Woess. Finally, we study the relations between Gromov hyperbolicity and strong isoperimetric inequalities on planar graphs, and give a proof that a planar graph satisfying a proper kind of a strong isoperimetric inequality must be Gromov hyperbolic if face degrees of the graph are bounded. We also provide some examples to support our results.

Keywords Isoperimetric inequality · Planar graph · Combinatorial curvature · Gromov hyperbolicity

Mathematics Subject Classification Primary: 05C10 · 05B45 · 52C20 · Secondary 20F67

1 Introduction

The main topic of this paper is strong isoperimetric inequalities on planar graphs, as one can guess from the title. In fact, we have studied the relations of three kinds of

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strong isoperimetric inequalities on planar graphs and their dual graphs, and as an application we have strengthened the results of [22,37,39]. We believe that some of our works can be considered a 'similar effort' for showing that a planar graph satisfies a strong isoperimetric inequality if and only of its dual has the same property, as suggested in [37].

To describe our results precisely, let G be a connected simple planar graph embedded into $\mathbb{R}^2$ locally finitely such that its dual graph G^* is also simple. See Sect. 2 for details of the terminologies. We denote by $V(G)$, $E(G)$, and $F(G)$ the vertex set, the edge set, and the face set, respectively, of G. For each $v \in V(G)$, $\deg(v)$ is the number of edges with one end at v. Similarly for each $f \in F(G)$, $\deg(f)$ is the number of edges surrounding f. In this paper we assume that $3 \le \deg(v), \deg(f) < \infty$ for every $v \in V(G)$ and $f \in F(G)$, unless otherwise stated.

Next suppose S is a finite subgraph of G, and we consider three types of boundaries of S. The first one is ∂S, the set of edges in $E(G)$ such that each element of ∂S has one end on $V(S)$ and the other end on $V(G) \setminus V(S)$. The second one is $\partial_v S \subset V(S)$, each of whose element is an end vertex of some edges in ∂S. The last boundary $\partial_e S$ is the set of edges surrounding S; i.e., $e \in \partial_e S$ if and only if $e \in E(S)$ and e belongs to $E(f)$ for some $f \in F(G) \setminus F(S)$. Now we define three different strong isoperimetric constants by

$$\imath(G) := \inf_S \frac{|\partial S|}{\mathrm{Vol}(S)}, \quad \jmath(G) := \inf_S \frac{|\partial_v S|}{|V(S)|}, \quad \kappa(G) := \inf_S \frac{|\partial_e S|}{|F(S)|},$$

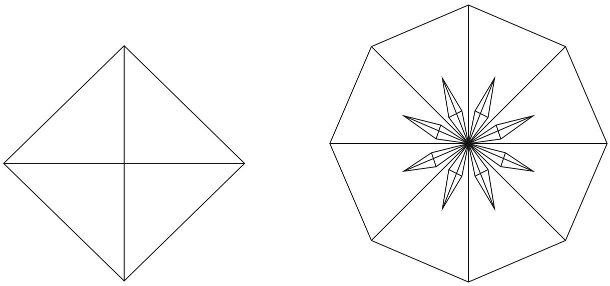
where $|\cdot|$ denotes the cardinality, $\mathrm{Vol}(S) = \sum_{v \in V(S)} \deg(v)$, and S runs over all the nonempty finite subgraphs of G.

The above constants $\imath(G)$, $\jmath(G)$, $\kappa(G)$ characterize some properties of the edge set, the vertex set, and the face set, respectively, of G, and are discrete analogues of Cheeger's constant [9]. The condition $\imath(G) > 0$ is of particular interest in spectral theory on graphs, since this condition is equivalent to the positivity of the smallest eigenvalue of the negative Laplacian [14,15], implying the simple random walk on G is transient. For more about this subject, see for instance [6,16,17,25,29,33,38] and the references therein. The constant $\kappa(G)$ is essentially dealt with in the geometric(combinatorial) group theory [18,20], and it was also investigated in [23,28]. The constant $\jmath(G)$ appears in the geometric group theory as well [12,18,20], and early versions of spectral theory on graphs [14,34]. Note that $\jmath(G)$ is quantitatively equivalent to $\imath(G)$ if vertex degrees of G are bounded, and to $\kappa(G)$ if face degrees of G are bounded (cf. Theorem 1 below).

We will call a simple planar graph *proper* if every face of the graph is a topological closed disk. Now we are ready to describe our main result.

Theorem 1 *Suppose G is a proper planar graph as described above, and G^* is its dual graph. Then*

(a) $\imath(G) > 0$ *if and only if* $\kappa(G^*) > 0$*, and* $\kappa(G) > 0$ *if and only if* $\imath(G^*) > 0$*;*
(b) $\jmath(G) > 0$ *if and only if* $\jmath(G^*) > 0$*;*
(c) *if* $\jmath(G) > 0$*, then* $\imath(G) > 0$*,* $\imath(G^*) > 0$*,* $\kappa(G) > 0$*, and* $\kappa(G^*) > 0$*.*



(a) the graph Λ

(b) attaching eight copies of Λ to a vertex of degree 8

Fig. 1 Constructing the graph G from Γ

Of course the main part of the above theorem is (b). We believe that the part (a) is well known to experts, but we have decided to contain a proof of it for the sake of completeness. Moreover, it is not long. For (c), one can easily deduce it from (a) and (b) as explained in Sect. 4. The reason why we state our results as above, instead of emphasizing (b) alone, is because this way could help one seeing the whole picture easily.

In Theorem 1 we did not require any upper bound for vertex degrees or face degrees of G, which makes the theorem useful. All the statements in Theorem 1 become trivial if both vertex and face degrees of G are bounded, since in this case G is roughly isometric(quasi-isometric) to its dual G^*, hence (b) follows from Theorem (7.34) of [34]. (For rough isometries, see Sect. 6.)

In the course of proving Theorem 1(b) we obtained the following result as a byproduct, which might be interesting by itself (compare it with [37, Reduction Lemma 2]).

Theorem 2 *Suppose G is a proper planar graph such that $|V(S)| \leq C|\partial_v S|$ for every polygon $S \subset G$, where C is a constant not depending on S. If either G is normal or face degrees of G are bounded, then $\jmath(G) > 0$.*

We call a planar graph *normal* if it is proper and the intersection of every two different faces is exactly one of the following: the empty set, a vertex, or an edge. We chose the terminology 'normal' since we adopted the first two properties of *normal tilings* defined in [21]. For polygons, they are basically finite unions of faces in $F(G)$ with simply connected interiors; for the precise definition, see Sect. 2.

One cannot omit the properness condition in Theorem 1, because without it the statement (b) is no longer true. For example, let Γ be a triangulation of the plane such that $\deg v \geq 7$ for every $v \in V(\Gamma)$. Then it is well known that $\jmath(\Gamma) > 0$ and $\jmath(\Gamma^*) > 0$. (Also see Corollary 4 below for this fact.) Furthermore, let us assume that there exists a sequence of vertices $v_k \in V(\Gamma)$ such that $n_k := \deg v_k \to \infty$ as $k \to \infty$. The essential property of Γ is that face degrees of Γ are bounded since it is a triangulation of the plane, while vertex degrees are not bounded. Now for each k, we attach n_k copies of the graph Λ in Fig. 1a to v_k so that a degree 3 vertex of each copy is identified with v_k and each face $f \in F(\Gamma)$ with $V(f) \ni v_k$ contains exactly one copy of Λ in it (Fig. 1b). Let G denote this new graph, which is definitely not proper but satisfies all the other properties we require; that is, $3 \le \deg v, \deg f < \infty$ for every $v \in V(G)$ and $f \in F(G)$, and both G and G^* are simple.

It is not difficult to see $\jmath(G) = 0$ since if we denote by S_k the union of n_k copies of Λ sharing the vertex v_k, then we have $\partial_v(S_k) = \{v_k\}$ and $|V(S_k)| = 4n_k + 1$. To see that $\jmath(G^*) > 0$, first note that G^* and Γ^* are roughly isometric. Moreover, face degrees of G and Γ, or vertex degrees of G^* and Γ^*, are both bounded and we chose Γ so that $\jmath(\Gamma^*) > 0$. Thus the inequality $\jmath(G^*) > 0$ follows from Theorem (7.34) of [34].

To obtain an application of Theorem 1, let us introduce so-called *combinatorial curvatures* defined on planar graphs. Suppose G is a proper planar graph as before. For each $e \in E(G)$, $v \in V(G)$, and $f \in F(G)$ we define *edge curvature* ϕ, *vertex curvature* ψ, and *face curvature* χ by

$$\phi(e) = \sum_{w \in V(e)} \frac{1}{\deg(w)} + \sum_{g: e \in E(g)} \frac{1}{\deg(g)} - 1,$$

$$\psi(v) = 1 - \frac{\deg(v)}{2} + \sum_{g: v \in V(g)} \frac{1}{\deg(g)}, \quad \text{and}$$

$$\chi(f) = 1 - \frac{\deg(f)}{2} + \sum_{w \in V(f)} \frac{1}{\deg(w)}.$$

In the above w and g stand for a vertex and a face, respectively. Remark that the notations ϕ, ψ, and χ are those used in [37], but we have changed the signs. Other than the above combinatorial curvatures for planar graphs, there is another one called *corner curvature* [26,27].

Recently combinatorial curvatures have been extensively studied by various researchers [2,3,6,11,13,22,25–27,35,37,39], but this concept was introduced more than seven decades ago. In [30, Chap.XII], Nevanlinna introduced a characteristic number called *excess*, which is essentially equal to the vertex curvature. Moreover, there might be older literature in this line than [30], since in [30] Nevanlinna mentioned a work of Teichmüller [36] related to excess. In fact, excess was defined for a special type of bipartite regular planar graphs, called Speiser graphs, which capture the combinatorial properties of meromorphic functions defined on some simply connected Riemann surfaces and ramified only over finitely many points in the extended complex plane $\overline{\mathbb{C}}$. For more about Speiser graphs and excess, see for example [5,30,32].

For finite subsets $E \subset E(G)$, let $\overline{\phi}(E) = (1/|E|) \sum_{e \in E} \phi(e)$ and define the *upper average* of ϕ on G by

$$\overline{\phi}(G) = \limsup_{|E(S)| \to \infty} \overline{\phi}(E(S)),$$

where limit superior is taken over all simply connected finite subgraphs S of G. Similarly, for finite subsets $V \subset V(G)$ and $F \subset F(G)$, let $\overline{\psi}(V) = (1/|V|) \sum_{v \in V} \psi(v)$ and $\overline{\chi}(F) = (1/|F|) \sum_{f \in F} \chi(f)$, and define the upper averages of ψ and χ on G, respectively, by

$$\overline{\psi}(G) = \limsup_{|V(S)|\to\infty} \overline{\psi}(V(S)), \quad \overline{\chi}(G) = \limsup_{|F(S)|\to\infty} \overline{\chi}(F(S)),$$

where the limit superiors are also taken over all simply connected finite subgraphs S of G. Our second main result is the following.

Theorem 3 *Suppose G is a proper planar graph.*

(a) *If $\overline{\phi}(G) < 0$ or $\overline{\chi}(G) < 0$, then $\jmath(G) > 0$.*
(b) *If $\overline{\psi}(G) < 0$, and if either G is normal or the vertex degrees of G are bounded, then $\jmath(G) > 0$.*
(c) *It is possible to have $\overline{\psi}(G) < 0$ and $\jmath(G) = \kappa(G) = 0$.*

The most surprising part in Theorem 3 might be (c), because it is very tempting to believe that $\overline{\psi}(G) < 0$ if and only if $\overline{\chi}(G^*) < 0$. However, (a) and (c) of Theorem 3, when combined with Theorem 1(b), show that it cannot be true. This discrepancy comes from the fact that the definition of $\overline{\chi}(G^*)$ requires some subgraphs of G^* whose corresponding subgraphs of G are *disconnected*. This will be explained in the subsequent sections in detail. For Theorem 3(a) and the second part (the case when vertex degrees are bounded) of Theorem 3(b), their credits should be addressed to Woess [37]. In fact, Woess showed that $\imath(G) > 0$ if one of the following conditions holds: $\overline{\phi}(G) < 0$, or $\overline{\psi}(G) < 0$, or $\overline{\chi}(G) < 0$. Note that this result is already enough for the second part of Theorem 3(b), because $\imath(G)$ is quantitatively equivalent to $\jmath(G)$ when vertex degrees of G are bounded. Also one can check that Woess's arguments are enough to show (a) only with some minor modifications. This will be explained in Sect. 5.

It was observed independently in [22,39] that the condition $\psi(v) < 0$ actually implies $\psi(v) \le -\varepsilon_0$ for some positive constant ε_0. Higuchi also showed in [22] that one can choose $\varepsilon_0 = 1/1806$. Thus we obtain the following immediate corollary of Theorems 1 and 3.

Corollary 4 *Suppose G is a proper planar graph. If $\psi(v) < 0$ for all $v \in V(G)$, or $\chi(f) < 0$ for all $f \in F(G)$, then $\jmath(G) > 0$. Consequently, in either case we have $\imath(G) > 0$, $\kappa(G) > 0$, $\imath(G^*) > 0$, $\jmath(G^*) > 0$, and $\kappa(G^*) > 0$.*

Compare this corollary with [22, Theorem B and Corollary 2.3] and [39, Proposition 4]. Another corollary of Theorem 3 is the following.

Corollary 5 *Suppose G is a graph satisfying the assumptions in* (a) *or* (b) *of Theorem 3. Then G contains a tree T such that $\jmath(T) > 0$.*

Proof This corollary is an immediate consequence of Theorem 3 above, and Theorem 1.1 of [4]. For a given locally finite graph Γ and its subgraph $S \subset \Gamma$, let $\tilde{\partial}_v S$ be the set of all vertices in $V(\Gamma) \setminus V(S)$ that have a neighbor in S. Define

$$\tilde{\jmath}(\Gamma) := \inf_S \frac{|\tilde{\partial}_v S|}{|V(S)|},$$

where S runs over all the nonempty finite subgraphs of Γ as before. Then Benjamini and Schramm showed in [4] that every graph G with $\tilde{\jmath}(G) > 0$ contains a tree T with

$\tilde{\jmath}(T) > 0$. But one can check that $\jmath(\Gamma) = \tilde{\jmath}(\Gamma)/(1 + \tilde{\jmath}(\Gamma))$ for every planar graph Γ, so we have the corollary. The details are left to the readers. □

Our last topic is about the relation between strong isoperimetric inequalities and Gromov hyperbolicity on planar graphs. For the definition of Gromov hyperbolic spaces, see Sect. 6.

Theorem 6 *Suppose G is a planar graph whose face degrees are bounded.*

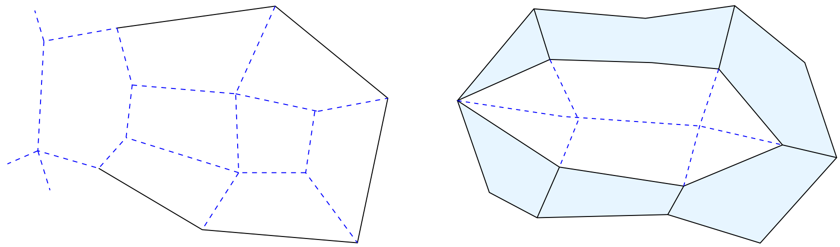
(a) *If $\kappa(G) > 0$, then G is hyperbolic in the sense of Gromov.*
(b) *The converse of* (a) *is not true. That is, G could be Gromov hyperbolic* (*and normal*)*, but $\kappa(G) = 0$.*
(c) *The right isoperimetric constant for Gromov hyperbolicity is $\kappa(G)$. That is, it is possible that G is* (*normal and*) *not Gromov hyperbolic, but $\imath(G) > 0$.*

Theorem 6(a) is widely believed and even considered trivial to some experts, but surprisingly we could not find its proof in the literature. Of course, however, it deserves to be written somewhere since it can save some works like [3, Corollary 5] or [39, Corollary 1]. Also note that the condition $\kappa(G) > 0$ is equivalent to $\jmath(G) > 0$ in the above theorem, since face degrees of G are bounded.

2 Planar Graphs

Let $G = \big(V(G), E(G)\big)$ be a graph, where $V(G)$ is the vertex set and $E(G)$ is the (undirected) edge set of G. Every edge $e \in E(G)$ is associated with two vertices $v, w \in V(G)$, saying that e is incident to v and w, or e connects v and w. In this case we write $e = [v, w]$, and the vertices v and w are called the endpoints of e. Also we say that v and w are neighbors of each other. A graph G is called simple if there is no self-loop nor multiple edges; that is, for every edge $[v, w] \in E(G)$ we have $v \neq w$, and for every two vertices $v, w \in V(G)$ there is at most one edge connecting these two vertices. A graph G is called connected if it is connected as a one-dimensional simplicial complex, and planar if there is a continuous injective map $h : G \to \mathbb{R}^2$. The image $h(G)$ is called an embedded graph, but we will not distinguish G from $h(G)$ when the embedding is fixed, and the embedded graph will be denoted by G instead of $h(G)$. We say that G is embedded into $\mathbb{R}^2$ locally finitely if every compact set in $\mathbb{R}^2$ intersects only finite number of vertices and edges of G. From now on, G will always be a connected simple planar graph embedded into $\mathbb{R}^2$ locally finitely.

The closure of each component of $\mathbb{R}^2 \setminus G$ is called a face of G, and we denote by $F(G)$ the face set of G. The dual graph G^* of G is the planar graph such that the vertex set of G^* is just $V(G^*) = F(G)$, and for the edge set we have $[f_1, f_2] \in E(G^*)$ if and only if f_1 and f_2 share an edge in G. The degree of a vertex $v \in V(G)$, denoted by $\deg v$, is the number of neighbors of v, and the degree of a face $f \in F(G)$, denoted by $\deg f$, is the number of edges in $E(G)$ surrounding f. In this paper, one essential assumption about G is that G^* is simple, and that $3 \leq \deg v, \deg f < \infty$ for every $v \in V(G)$ and $f \in F(G)$. Under this assumption, $\deg f$ is just the degree of f as a vertex in G^*.



(a) a subgraph (solid polygonal line) that is not simply connected

(b) a face graph with simply connected interior but not being a polygon

Fig. 2 A face graph S (*right*) and its dual part S^* (*left*)

A graph S is called a subgraph of G, denoted by $S \subset G$, if $V(S) \subset V(G)$ and $E(S) \subset E(G)$. A subgraph will be always assumed to be an induced subgraph; that is, if $v, w \in V(S)$ and $[v, w] \in E(G)$, then $[v, w] \in E(S)$. A subgraph is called finite if $|V(S)|$ is finite, and a finite subgraph $S \subset G$ is called simply connected if it is connected as a one-dimensional simplicial complex and $G \setminus S$ has only one component. Here we remark that our definition of simply connectedness might be different from the one in some other literature, since a subgraph $S \subset G$ could be simply connected as a subset of $\mathbb{R}^2$, but not as a subgraph of G (Fig. 2a). However, this usually makes no difference when studying strong isoperimetric inequalities, because in the proof we can always add to S all the finite components of $G \setminus S$ (cf. proofs of Lemma 8, Theorem 2, etc.).

For $S \subset G$, we define $F(S)$ as the *subset* of $F(G)$ such that $f \in F(S)$ if and only if $f \in F(G)$ and it is the closure of a component of $\mathbb{R}^2 \setminus S$. This notation is a little bit confusing, since $F(S)$ in fact means the intersection of $F(G)$ and the face set of S. By abuse of the notation, we will treat a face $f \in F(G)$ as a subgraph of G consisting of all the edges and vertices on the topological boundary of f. Thus we have $|E(f)| = \deg f$. Edges are also treated as subgraphs in a similar way. A subgraph $S \subset G$ is called a *face graph* if it is a union of faces; i.e., $V(S) = \bigcup_{f \in F(S)} V(f)$. A finite subgraph is called a *polygon* if it is a simply connected face graph and the *interior* of S,

$$D(S) := \Bigl(\bigcup_{f \in F(S)} f\Bigr)^{\circ}, \tag{2.1}$$

is simply connected as a subset of $\mathbb{R}^2$, where $(\cdot)^{\circ}$ denotes the interior of the considered set with respect to the Euclidean topology. Note that even though a face graph has simply connected interior, the graph itself may not be simply connected (Fig. 2b). Thus in the definition we require a polygon to be simply connected.

Now suppose $S \subset G$ is a face subgraph, and define S^* as the subgraph of G^* such that $V(S^*) = F(S)$ (Fig. 2). Then it is easy to check that for a given face graph $S \subset G$, the interior $D(S)$ is connected if and only if S^* is connected, and such S is a polygon if and only if S^* is simply connected. In particular this suggests that even though S is connected, S^* may not be connected. For example, suppose f_1, f_2 are faces of G such that $f_1 \cap f_2$ is a vertex. Then the face graph $S \subset G$ with $F(S) = \{f_1, f_2\}$ is simply connected, but S^* is definitely disconnected. Thus if we define

$$\overline{\chi}_1(G) = \limsup_{|F(S)|\to\infty} \overline{\chi}(F(S)), \tag{2.2}$$

where at this time the limit superior is taken over all the *polygons* $S \subset G$, then we have $\overline{\psi}(G^*) = \overline{\chi}_1(G) \leq \overline{\chi}(G)$. Therefore the condition $\overline{\chi}(G) < 0$ is stronger than the condition $\overline{\psi}(G^*) < 0$, and this observation has delivered us to Theorem 3.

For a face subgraph of $S \subset G$, there is a bijection between ∂S^* and $\partial_e S$. However, there is no set defined in S which corresponds to $\partial_v S^*$. Thus if S is a subgraph of G, which may or may not be a face graph, let us define the *face boundary* of S by

$$\partial_f S := \{f \in F(S) : E(f) \cap \partial_e S \neq \emptyset\}.$$

That is, $\partial_f S$ is the set of faces in $F(S)$, each of which shares an edge with a face in $F(G) \setminus F(S)$.

We finish this section with the following lemma. Recall that a planar graph G is called proper if every face of G is a topological closed disk, and normal if it is proper and any two different faces of G share at most a vertex or an edge.

Lemma 7 *Suppose G is a proper planar graph. Then for every finite and connected subgraph $S \subset G$ with $|V(S)| \geq 2$, we have $|\partial_v S| \geq 2$. Moreover, if G is normal and $S \subset G$ has the property $|V(S)| \geq 3$, then $|\partial_v S| \geq 3$.*

Proof Suppose $S \subset G$ is a finite and connected subgraph of G. Then we cannot have $\partial_v S = \emptyset$ since G is connected. If $|\partial_v S| = 1$, there must be a unique face $f \in F(G)$ surrounding S; i.e., there is only one face $f \in F(G)\setminus F(S)$ such that $E(f) \cap E(S) \neq \emptyset$. Then f cannot be a topological closed disk because S contains an edge, contradicting the properness condition of G.

Now suppose G is normal, $S \subset G$, $\partial_v S = \{v_1, v_2\}$, and $|V(S)| \geq 3$. In this case there are exactly two faces, say f_1 and f_2, whose union surrounds S. Then we must have $v_1, v_2 \in V(f_1) \cap V(f_2)$, so the normality condition of G implies that $f_1 \cap f_2 = [v_1, v_2]$. This means that S is just an edge because both f_1 and f_2 are topological closed disks, which is definitely a contradiction since $|V(S)| \geq 3$. □

3 Proof of Theorem 1(a)

First, let us derive a useful formula which will be used frequently. Let S be a finite simply connected subgraph of G. Then every vertex $v \in V(S) \setminus \partial_v S$ corresponds to at least three edges in $E(S)$ and each vertex $v \in \partial_v S$ corresponds to at least one edge in $E(S)$, while every edge in $E(S)$ corresponds to exactly two vertices in $V(S)$. Therefore we have

$$2|E(S)| \geq 3(|V(S)| - |\partial_v S|) + |\partial_v S| = 3|V(S)| - 2|\partial_v S|.$$

Thus by Euler's formula we obtain

$$|E(S)| + 1 = |V(S)| + |F(S)| \leq \frac{2}{3} \cdot |E(S)| + \frac{2}{3} \cdot |\partial_v S| + |F(S)|,$$

or

$$|E(S)| \leq 2|\partial_v S| + 3|F(S)| - 3. \tag{3.1}$$

Lemma 8 *For a given planar graph G, the following two conditions are equivalent:*

(a) *for every finite $S \subset G$, there is a constant C_1 such that $|F(S)| \leq C_1|\partial_e S|$;*
(b) *for every finite $S \subset G$, there is a constant C_2 such that $\sum_{f \in F(S)} \deg(f) \leq C_2|\partial_e S|$.*

Furthermore, the constants C_i are quantitative to each other.

Proof We only need to show the implication (a) → (b), since the other direction is trivial. In this case, we can assume without loss of generality that S is a polygon, since otherwise we can add to S all the finite components of $G \setminus S$ and cut from S all the edges not surrounding a face in $F(S)$. Note that these operations make either the set $\partial_e S$ smaller, or the set $F(S)$ bigger, or both. If $D(S)$ consists of several components, we can consider them separately.

Then since S is a polygon we have $|\partial_v S| \leq |\partial_e S|$ and

$$2|E(S)| = \sum_{f \in F(S)} \deg(f) + |\partial_e S|. \tag{3.2}$$

Now Eqs. (3.1) and (3.2) imply that

$$\sum_{f \in F(S)} \deg(f) = 2|E(S)| - |\partial_e S| \leq 4|\partial_v S| + 6|F(S)| - |\partial_e S| \leq 3|\partial_e S| + 6|F(S)|, \tag{3.3}$$

and we obtain the implication (a) → (b) with $C_2 = 6C_1 + 3$. In fact one can check, by modifying (3.1), that a better estimate $C_2 \leq 6C_1 + 1$ holds since S is a polygon hence every vertex in $\partial_v S$ corresponds at least two edges in $E(S)$. This completes the proof. □

Note that the condition (a) of the previous lemma is equivalent to the condition $\kappa(G) > 0$ and, considering the duality property, one can check that (b) is equivalent to $\iota(G^*) > 0$. Thus Lemma 8 implies the equivalence between $\iota(G^*) > 0$ and $\kappa(G) > 0$. The equivalence between $\iota(G) > 0$ and $\kappa(G^*) > 0$ comes from the duality. This completes the proof of Theorem 1(a).

4 Partitioning a Finite Subgraph

In this section we will prove Theorem 2 and the rest of Theorem 1. Our main strategy is partitioning a given finite subgraph into 'nice' subgraphs, and passing strong isoperimetric inequalities of 'nice' subgraphs to the given one.

In the lemma below, the notation $S_1 \cup S_2$, where S_1, S_2 are subgraphs of G, means the subgraph $S \subset G$ with $V(S) = V(S_1) \cup V(S_2)$ and $E(S) = E(S_1) \cup E(S_2)$. A priori, therefore, $S_1 \cup S_2$ does not have to be an *induced* subgraph, but we will only consider the case when it is induced. The graph $S_1 \cap S_2$ is defined similarly.

Lemma 9 *Suppose S is a finite subgraph of G. If $S = S_1 \cup S_2 \cup \cdots \cup S_n$, where $S_1, S_2, \ldots, S_n$ are subgraphs of G such that*

(a) $|V(S_i)| \le C|\partial_v S_i|$ *for each* $i = 1, 2, \ldots, n$,
(b) $S_1 \cup S_2 \cup \cdots \cup S_i$ *is an induced subgraph for each* $i = 1, 2, \ldots, n$,
(c) $|V\big((S_1 \cup S_2 \cup \cdots \cup S_{i-1}) \cap S_i\big)| = 1$ *for each* $i = 2, 3, \ldots, n$, *and*
(d) $|\partial_v S| \ge n/\tau$ *for some* $\tau > 0$,

then we have $|V(S)| \le (1 + 2\tau)C|\partial_v S|$.

Proof From (b) and (c), it is easy to see that

$$|\partial_v(S_1 \cup \cdots \cup S_i)| \ge |\partial_v(S_1 \cup \cdots \cup S_{i-1})| + |\partial_v S_i| - 2. \tag{4.1}$$

Therefore,

$$\begin{aligned} |V(S)| = |V(S_1 \cup \cdots \cup S_n)| &\le |V(S_1)| + |V(S_2)| + \cdots + |V(S_n)| \\ &\le C(|\partial_v S_1| + |\partial_v S_2| + \cdots + |\partial_v S_n|) \\ &\le C(|\partial_v(S_1 \cup S_2)| + |\partial_v S_3| + \cdots + |\partial_v S_n| + 2) \\ &\le \cdots \le C(|\partial_v(S_1 \cup \cdots \cup S_n)| + 2n - 2) \le (1 + 2\tau)C|\partial_v S|, \end{aligned}$$

as desired. □

Suppose S is a finite simply connected subgraph of G. If $T \subset S$ is a polygon such that no T' with $T \subsetneq T' \subset S$ is a polygon, we call T a *leaf* of S. If $T \subset S$ is finite tree such that $E(T) \cap E(f) = \emptyset$ for every $f \in F(S)$, and no T' with $T \subsetneq T' \subset S$ is a tree satisfying the same property, we call such T a *branch* of S. If $T \subset S$ is either a leaf or a branch, we call T a *part* of S (see Fig. 3).

Proof of Theorem 2 (*the normal case*) Let G be a *normal* planar graph satisfying the assumptions in Theorem 2, and suppose a finite subgraph $S \subset G$ is given. To prove Theorem 2, we may assume that S is simply connected. Otherwise we can add to S all the finite components of $G \setminus S$ and consider each component of S separately.

Choose an edge $e_1 \in E(S)$, and note that there exists a unique leaf or branch S_1 of S such that $e \in E(S_1)$, depending on whether $e \in E(f)$ for some $f \in F(S)$ or not, respectively. If $S_1 \ne S$, choose $e_2 \in E(S) \setminus E(S_1)$ with only one end in $V(S_1)$, and let S_2 be the part of S such that $e_2 \in E(S_2)$. Then because S_1 and S_2 are *maximal* polygons or trees of the *simply connected* subgraph S, one can see that $S_1 \cap S_2$ is a single vertex that is an end of e_2, and $S_1 \cup S_2$ is an induced subgraph of S. If $S_1 \cup S_2 \ne S$, choose $e_3 \in E(S) \setminus E(S_1 \cup S_2)$ with only one end in $V(S_1 \cup S_2)$, and repeat the same process. This process cannot run forever, since S is a finite graph. Thus we just have enumerated the parts $S_1, S_2, \ldots, S_n$ of S so that $S = S_1 \cup S_2 \cup \cdots \cup S_n$ and they satisfy the conditions (b) and (c) of Lemma 9.

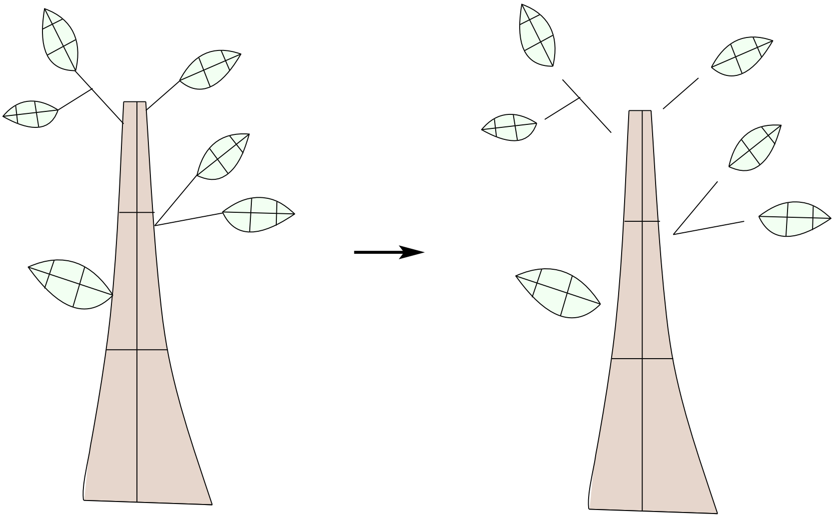


Fig. 3 A subgraph composed of ten parts (seven leaves and three branches) is shown on the *left*, and it is divided into leaves and branches on the *right*. The largest part in the *center* of each figure should be called a trunk, but we accidently call it a leaf

Each S_i is either a branch or a leaf, i.e., a finite tree or a polygon. If S_i is a leaf, the inequality $|V(S_i)| \leq C|\partial_v S_i|$ holds by our assumption. If S_i is a branch, then since $F(S_i) = \emptyset$ and $|V(S_i)| = |E(S_i)| + 1$, we have $|V(S_i)| \leq 2|\partial_v S_i|$ by (3.1). Thus the condition (a) of Lemma 9 is also satisfied for the sequence $S_1, \ldots, S_n$.

It remains to show that $|\partial_v S| \geq n/3$. To see this, let k be the number of S_i such that $|\partial_v S_i| = 2$ and $\partial_v S \cap V(S_i) \neq \emptyset$. If $k \geq n/3$, there is nothing more to prove since such S_i must be an edge by Lemma 7, hence such S_i's are disjoint by our definition of a branch. Now suppose that $k < n/3$. Then a part S_i satisfying $|\partial_v S_i| = 2$ and $\partial_v S \cap V(S_i) = \emptyset$ must be an edge whose both ends are connected to leaves. Thus the number of such S_i's cannot exceed $(n - k)/2$. This means that there are at least $(n-k)/2$ parts with $|\partial_v S_i| \geq 3$. Since such S_i makes the left hand side of (4.1) increase at least by one, we conclude that

$$|\partial_v S| = |\partial_v(S_1 \cup \cdots \cup S_n)| \geq \frac{n-k}{2} \geq \frac{n}{3}$$

since $k < n/3$. This completes the proof. □

The approach used in the previous proof, that is, partitioning S into parts and using Lemma 9, has some problems when G is not normal. Let Λ be the graph of Fig. 1a given in the introduction, and for each $i = 1, 2, \ldots, n$, let S_i be a copy of Λ. We next connect them back to back as in Fig. 4 and obtain a new graph $S := S_1 \cup S_2 \cup \cdots \cup S_n$. Also suppose S is enclosed by the union of two faces with huge face degrees. Then even though each S_i satisfies the inequality $|V(S_i)| \leq 3|\partial_v S_i|$, we cannot pass this inequality to S; i.e., we have $|\partial_v S|/|V(S)| \to 0$ as $n \to \infty$. As we saw in the previous

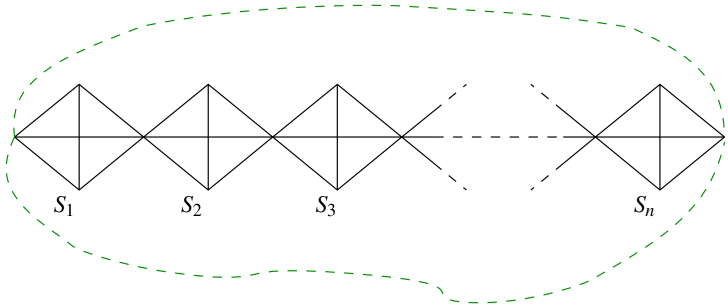


Fig. 4 A problem when partitioning into polygons

proof, however, this problem cannot occur when G is normal. The next lemma says that it cannot happen either if $\jmath(G^*) > 0$.

Lemma 10 *Suppose $\jmath(G^*) > 0$, and let S be a finite simply connected subgraph of G satisfying the following two properties:*

(a) *every branch of S is a path; i.e., a finite union of consecutive edges without self-intersections;*
(b) *$P \cap P' \cap P'' = \emptyset$ for every three distinct parts P, P', P'' of S.*

Then there is an absolute constant C such that $|V(S)| \le C|\partial_v S|$.

Note that the subgraph S in Lemma 10 has the following property: for any vertex $v \in V(S)$ and every sufficiently small neighborhood $U \ni v$ in $\mathbb{R}^2$, $U \setminus (S \cup D(S))$ has at most two components.

Proof Let $\{v_1, v_2, \dots, v_m\}$ be an enumeration of $\partial_v S$ along the boundary of S. In other words, when walking along the topological boundary of $S \cup D(S)$ either clockwise or counterclockwise, we enumerate the vertices in $\partial_v S$ in the order we encounter them, allowing some multiple counts. However, any vertex in $\partial_v S$ will not be counted more than twice by the assumptions (a) and (b). Consequently we have $2|\partial_v S| \ge m$.

Note that S is a simply connected graph with no outbound edge in $E(G) \setminus E(S)$ between v_i and v_{i+1}, where i is in mod m. Thus for each $i \in \{1, 2, \dots, m\}$, there is the face f_i attached to S between v_i and v_{i+1}. To be precise, f_i is the face in $F(G) \setminus F(S)$ such that $E(f_i)$ contains all the edges in $\partial_e S$ between v_i and v_{i+1} (Fig. 5).

Let T be the subgraph of G such that $V(T) = V(S) \cup \bigcup_{i=1}^m V(f_i)$. Definitely T is a face graph with $\partial_f T \subset \{f_1, f_2, \dots, f_m\}$, hence our assumption $\jmath(G^*) > 0$ implies that $|V(T^*)| \le C_1 |\partial_v T^*|$ for $C_1 = \jmath(G^*)^{-1}$, or

$$|F(T)| \le C_1 |\partial_f T| \le C_1 \cdot m \le 2C_1 |\partial_v S|.$$

Thus by (3.1) we have

$$\begin{aligned}|V(S)| = |E(S)| - |F(S)| + 1 &\le 2|\partial_v S| + 2|F(S)| \\ &\le 2|\partial_v S| + 2|F(T)| \le (2 + 4C_1)|\partial_v S|,\end{aligned}$$

as desired. □

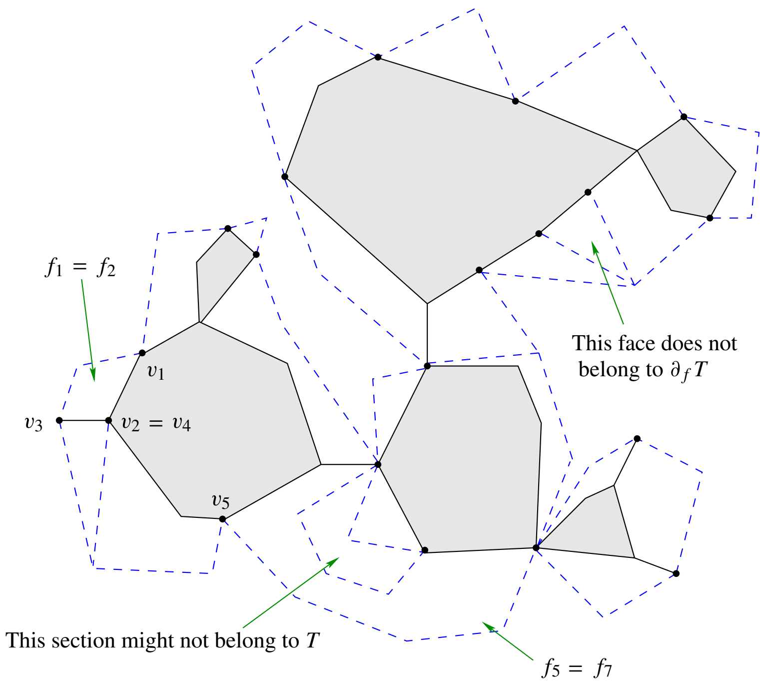


Fig. 5 A sketch of a subgraph satisfying the assumptions in Lemma 10; this graph consists of 6 polygons and 5 edges, and *dotted lines* indicate the faces attached to S; the section that does not belong to T would be some union of faces in $F(G) \setminus F(T)$

Corollary 11 *Suppose G is a proper planar graph with bounded face degrees such that $|V(T)| \le C|\partial_v T|$ for every polygon $T \subset G$. If S is a simply connected subgraph of G satisfying the conditions* (a) *and* (b) *in Lemma 10, then $|V(S)| \le C_1|\partial_v S|$ for some constant C_1 not depending on S.*

Proof We follow the proof of Theorem 2, the normal case, and enumerate the parts $S_1, S_2, \ldots, S_n$ of S so that they satisfy the conditions (a), (b), and (c) of Lemma 9. Then as in Lemma 10, we enumerate $\partial_v S$ by $v_1, v_2, \ldots, v_m$, where some vertices are counted twice, and let $f_1, f_2, \ldots, f_m$ be the faces attached to S between v_i and v_{i+1}, where i is in mod m.

By the definition of a part, we have $\partial_e S \cap E(S_i) \ne \emptyset$ for each $i = 1, 2, \ldots, n$, so we must have $|\partial_e S| \ge n$. On the other hand, face degrees of G are bounded above, say by M, so the number of edges in $E(f_i) \cap \partial_e S$ cannot exceed M. Thus the inequality $mM \ge |\partial_e S|$ holds. Since $m \le 2|\partial_v S|$, we have $|\partial_v S| \ge n/(2M)$. The corollary follows from Lemma 9. □

The problem in Fig. 4 is caused by the fact that face degrees of G are not bounded, and we have seen so far that this phenomenon does not occur under the assumptions of either Theorem 1 or Theorem 2. However, there is still a problem in partitioning a subgraph into leaves and branches, since we still have to worry about the unboundedness of *vertex* degrees. For example the proof in Lemma 10 does not work for a

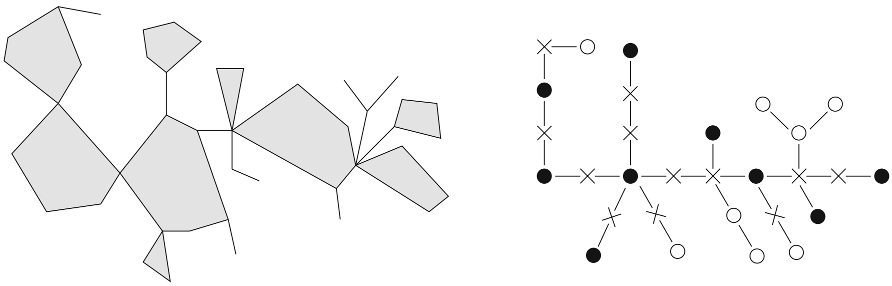


Fig. 6 A sketch of a simply connected graph S (*left*) and the tree T obtained from S (*right*); on the tree T, the symbols *filled circles*, *open circles* and *multiplication signs* indicate the vertices in V_1, V_2, and V_3, respectively

general subgraph $S \subset G$ since when one walks along the topological boundary of $S \cup D(S)$, there may be a vertex in $\partial_v S$ which appears too many times. So to overcome this obstacle, one may need a different kind of partition.

Lemma 12 *Suppose G is a proper planar graph and S is a finite simply connected subgraph of G. Then there exists a partition $S_1 \cup S_2 \cup \cdots \cup S_n = S$ which satisfies the conditions* (b)*,* (c)*, and* (d) *of Lemma* 9*. Furthermore, we can make the partition so that each S_i satisfies the conditions* (a) *and* (b) *of Lemma* 10*.*

Proof Suppose $S \subset G$ is given, and let $V_1 := \{L_1, L_2, \ldots, L_k\}$ be the set of all the leaves of S. Define V^i, $i = 1, 2, \ldots, k$, such that

$$V^i := \{v \in \partial_v L_i : \text{there exists an edge } e \in E(S) \setminus E(L_i) \text{ with one end at } v\}.$$

Then by reducing each L_i to a vertex, connecting it to the vertices in V^i, and keeping the rest of S unchanged, we get a new finite tree T (Fig. 6). The reason we add to T the vertices in V^i is because we do not want to change the combinatorial pattern of S. Formally, T is the graph with $V(T) := V_1 \cup V_2 \cup V_3$, where V_1 is as above, $V_2 := V(S) \setminus V(L_1 \cup L_2 \cup \cdots \cup L_k)$, and $V_3 := \bigcup_{i=1}^k V^i$, and we define the edge set $E(T)$ so that $[v, w] \in E(T)$ if one of the following holds: (1) $v, w \in V_2 \cup V_3$ and $[v, w] \in E(S)$, or (2) $v \in V_1$, $w \in V_3$, and $w \in V(v)$, or (3) $v \in V_3$, $w \in V_1$, and $v \in V(w)$. Such T must be a tree since S is simply connected.

In T, let $A = \{v \in V(T) : \deg_T v \geq 3\}$. Here $\deg_T v$ denotes the number of edges in T with one end at v. Then we consider T a simplicial complex, and let $\{T_1, T_2, \ldots, T_m\}$ be the set of the closures of each components of $T \setminus A$ (Fig. 7). Note that each T_j is isomorphic to a finite path. Moreover for $i \neq j$, $T_i \cap T_j$ is either empty or a vertex in A, hence we can enumerate $\{T_j\}$ so that $T_1 \cup T_2 \cup \cdots \cup T_j$ is connected for $j = 1, 2, \ldots, m$ and $(T_1 \cup T_2 \cup \cdots \cup T_{j-1}) \cap T_j$ is a single vertex for $j = 2, 3, \ldots, m$.

For each j we assign a subgraph $S_j \subset S$ in an obvious way, but some modification is needed. For S_1, we define it so that

$$V(S_1) = \big(V(T_1) \cap (V_2 \cup V_3)\big) \cup \bigcup_{v \in V(T_1) \cap V_1} V(v),$$

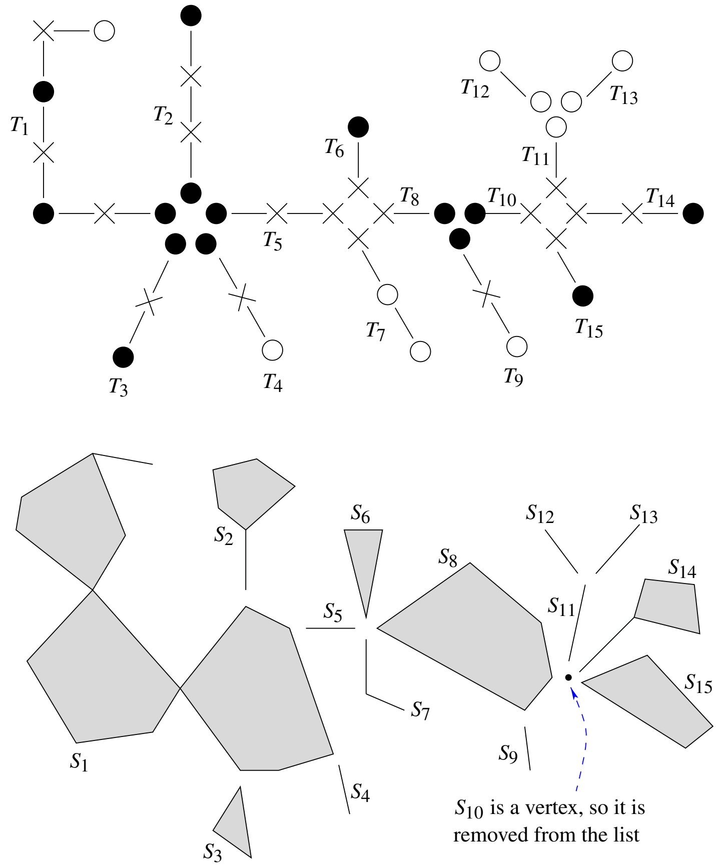


Fig. 7 T is partitioned so that each T_j is a path (*top*), and S is partitioned associated with the partition of T (*bottom*)

and for $j = 2, 3, \dots, m$ we define S_j so that

$$V(S_j) = \big(V(T_j) \cap (V_2 \cup V_3)\big) \cup \bigcup_{v \in V(T_j \setminus (T_1 \cup \cdots \cup T_{j-1})) \cap V_1} V(v).$$

In other words, S_j is the graph consisting of the vertices in $T_j \cap (V_2 \cup V_3)$ and the leaves in $T_j \cap V_1$, but from S_j we remove the leaf in $T_1 \cup \cdots \cup T_{j-1}$, if any. But if a leaf is removed from S_j, it must correspond to a vertex $v = L_i \in V(T)$ such that $\{L_i\} = (T_1 \cup T_2 \cup \cdots \cup T_{j-1}) \cap T_j$. Then since T_j contains a vertex in V^i, $(S_1 \cup \cdots \cup S_{j-1}) \cap S_j$ must be a single vertex. Clearly $(S_1 \cup \cdots \cup S_{j-1}) \cap S_j$ is also a single vertex when no leaf is removed from S_j.

Some of S_j could be just a vertex in V_3, so we eliminate all such S_j's from the list. Now we have just obtained a sequence of subgraphs $\{S_{n_1}, S_{n_2}, \dots, S_{n_s}\}$ which satisfies the condition (c) of Lemma 9. The condition (b) of Lemma 9 easily follows from the construction. Moreover, since each T_{n_i} is a path, each S_{n_i} satisfies the conditions (a) and (b) of Lemma 10.

Now it remains to verify the condition (d) of Lemma 9. Let $B := \{v \in V(T) : \deg_T v = 1\}$. If $v \in B$, then $v \notin V_3$ by the definition of V^i. Thus $v \in V_1 \cup V_2$. If $v \in V_2$, then v is in fact a vertex in $\partial_v S$. If $v \in V_1$, it corresponds to a leaf L_i with $|V^i| = 1$. Then since $|\partial_v L_i| \geq 2$ by Lemma 7, we can assign v to a vertex $w \in \partial_v S \setminus V_3$. Thus there exists an injection map from B into $\partial_v S$, so the inequality $|B| \leq |\partial_v S|$ holds.

Now in T, we replace each T_j by an edge and get another tree T'. Then the number of edges in T' is exactly m, and every vertex in T' has degree either 1 or ≥ 3. Also the number of vertices $v \in V(T')$ with $\deg_{T'} v = 1$ is exactly $|B|$. Therefore by a computation similar to (3.1) we obtain

$$m = |E(T')| \leq 2|B| \leq 2|\partial_v S|.$$

Since $s \leq m$, we conclude that the sequence $\{S_{n_1}, S_{n_2}, \ldots, S_{n_s}\}$ also satisfies the condition (d) of Lemma 9. Since $S = S_{n_1} \cup S_{n_2} \cup \cdots \cup S_{n_s}$, this finishes the proof. □

Proof of Theorems 1 and 2 Theorem 1(a) was already proved. For (b) of Theorem 1, the implication $\jmath(G^*) > 0 \rightarrow \jmath(G) > 0$ is a consequence of Lemmas 9, 10, and 12, and the converse comes from the duality. Theorem 1(c) is an easy consequence of Theorem 1(a) and (b). In fact, for a simply connected subgraph $S \subset G$ we have

$$\mathrm{Vol}(S) = \sum_{v \in V(S)} \deg v \leq 3|\partial S| + 6|V(S)|,$$

which comes from the dual property of (3.3). Thus the inequality $\jmath(G) > 0$ implies $\imath(G) > 0$, because we always have $|\partial_v S| \leq |\partial S|$. Now the other inequalities in Theorem 1(c) follow from Theorem 1(a) and (b), and a computation similar to the above. We leave the details to the readers.

The normal case of Theorem 2 was already proved, and the case when face degrees of G are bounded comes from Lemmas 9, 12, and Corollary 11. This completes the proof of Theorems 1 and 2. □

5 Combinatorial Curvatures and Strong Isoperimetric Inequalities

In this section we deal with combinatorial curvatures and prove Theorem 3. But as explained in the introduction, Theorem 3(a) was essentially proved in [37]. To see it, and to prove Theorem 3(b), suppose S is a subgraph of a proper graph G. As before, we can assume that S is simply connected without loss of generality.

When $\overline{\phi}(G) < 0$, Woess showed the inequality $|E(S)| \leq C|\partial_v S|$, where C is an absolute constant, in the proof of Theorem 1 of [37]. But since $|V(S)| \leq |E(S)|+1 \leq 2|E(S)|$ by Euler's formula, the conclusion $\jmath(G) > 0$ follows immediately. When $\overline{\chi}(G) < 0$, the inequality $|F(S)| \leq C|\partial_v S|$ was obtained in the proof of Theorem 2(b)

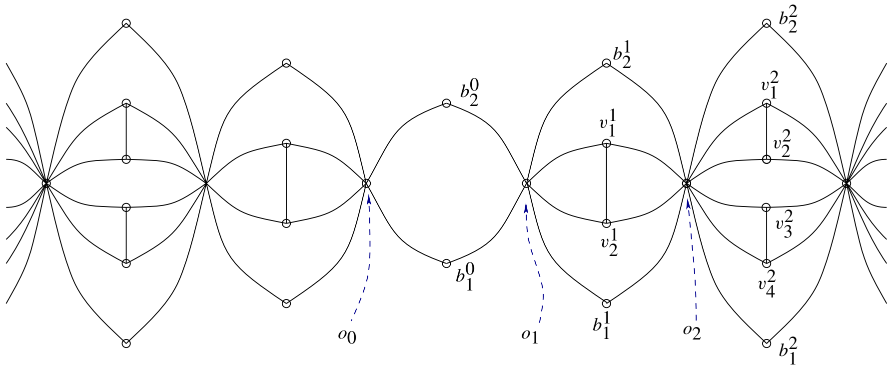


Fig. 8 The graph S

of [37], where C is an absolute constant. Then by (3.1) and Euler's formula we have

$$|V(S)| = |E(S)| - |F(S)| + 1 \le 2|\partial_v S| + 2|F(S)| - 2 \le (2 + 2C)|\partial_v S|, \quad (5.1)$$

showing that $\jmath(G) > 0$.

We next consider Theorem 3(b). When vertex degrees of G are bounded, Theorem 3(b) follows from Theorem 2(a) of [37] as explained in the introduction, but let us prove it in a different way. Since $\overline{\psi}(G) = \overline{\chi}_1(G^*)$, where $\overline{\chi}_1(G)$ is defined in (2.2), the assumption $\overline{\psi}(G) < 0$ implies the inequality $\overline{\chi}_1(G^*) < 0$. Then we follow the proof of Theorem 2(b) of [37] and use (5.1), and confirm that there exists an absolute constant C such that $|V(S)| \le C|\partial_v S|$ for every *polygon* $S \subset G^*$. Thus we have $\jmath(G^*) > 0$ by Theorem 2 if either G^* is normal or face degrees of G^* are bounded. Since G is normal if and only if G^* is normal, and face degrees of G^* are bounded if and only if vertex degrees of G are bounded, Theorem 3(b) follows from Theorem 1(b).

For Theorem 3(c), we construct a planar graph G such that $\overline{\psi}(G) < 0$ but $\jmath(G) = 0$. Note that such graph cannot be normal.

For $n \in \mathbb{Z}$, let S_n be the finite graph such that

$$V(S_n) = \left\{o_n, o_{n+1}, b_1^n, b_2^n, v_1^n, v_2^n, \ldots, v_{2|n|-1}^n, v_{2|n|}^n\right\}$$

with edges $[o_{n+i}, b_j^n]$, $[o_{n+i}, v_{2k-1}^n]$, $[o_{n+i}, v_{2k}^n]$, and $[v_{2k-1}^n, v_{2k}^n]$, where $0 \le i \le 1$, $1 \le j \le 2$, and $1 \le k \le |n|$. Then for each $n \in \mathbb{Z}$, S_n and S_{n+1} share the vertex o_{n+1}, so $S := \bigcup_{n=-\infty}^{\infty} S_n$ is a connected planar graph. Furthermore, we can embed it into $\mathbb{R}^2$ in such a way that each v_k^n is enclosed by the cycle $[o_n, b_1^n] \cup [b_1^n, o_{n+1}] \cup [o_{n+1}, b_2^n] \cup [b_2^n, o_n]$ (see Fig. 8).

On the unbounded faces of S, we add vertices and edges so that the resulting graph G satisfies the following properties: G is a simple proper planar graph with $3 \le \deg v, \deg f < \infty$ for every $v \in V(G)$ and $f \in F(G)$, $\deg v \ge 7$ for every added vertex $v \in V(G) \setminus V(S)$, $\deg b_j^n \ge 7$ for $j = 1, 2$ and $n \in \mathbb{Z}$, and $\deg o_n \ge 14|n| + 14$ for all $n \in \mathbb{Z}$. In other words, we for example triangulate both of the unbounded faces

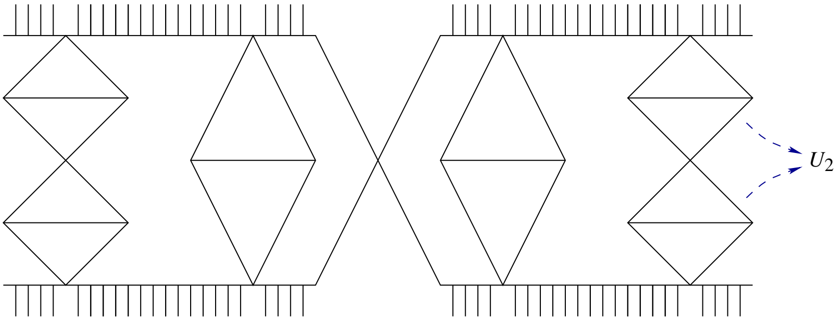


Fig. 9 The dual graph of G; the subgraph U_2 corresponds to the disconnected graph $U_2^* = [v_1^2, v_2^2] \cup [v_3^2, v_4^2]$

of S so that every added vertices and b_j^n's are of degree at least 7 and the degrees of o_n are huge enough. Note that this can be done by mathematical induction.

It is easy to see that $\kappa(G) = \jmath(G) = 0$ since $|\partial_v S_n| = |\partial_e S_n| = 4$ but $|F(S_n)| = 3|n| + 1$ and $|V(S_n)| = 2|n| + 4$. On the other hand, we have $\psi(v) \leq -1/6$ if $v \in V(G) \setminus V(S)$ or $v = b_j^n$ for some $j = 1, 2$ and $n \in \mathbb{Z}$. Also direct computation shows that

$$\psi(v_k^n) = 1 - \frac{3}{2} + \frac{2}{3} + \frac{1}{4} = \frac{5}{12} \quad \text{for all } n \in \mathbb{Z} \quad \text{and} \quad 1 \leq k \leq 2|n|;$$
$$\psi(o_n) \leq 1 - \frac{\deg o_n}{2} + \frac{\deg o_n}{3} \leq -\frac{7|n| + 4}{3} \quad \text{for all } n \in \mathbb{Z}.$$

Thus the vertex curvature function ψ assumes a positive value only at v_k^n, but they are dominated by the negative vertex curvature at o_n. In fact, for each o_n there are at most $4|n| + 2$ neighboring vertices v_k^n and v_k^{n-1}, so

$$\psi(o_n) + \sum_{k=1}^{2|n|} \psi(v_k^n) + \sum_{k=1}^{2|n-1|} \psi(v_k^{n-1}) \leq -\frac{7|n| + 4}{3} + \frac{5(4|n| + 2)}{12} = -\frac{4|n| + 3}{6}.$$

This means that if T is a finite connected subgraph of G with $|V(T)| \geq 3$, then $\sum_{v \in V(T)} \psi(v) \leq -|V(T)|/6$. Consequently we have $\overline{\psi}(G) \leq -1/6 < 0$, and this completes the proof of Theorem 3. Note that the face graph $U_n \subset G^*$ with $F(U_n) = V(U_n^*) = \{v_k^n : 1 \leq k \leq 2|n|\}$ is a connected graph which looks similar to the one in Fig. 4, while U_n^* is a disconnected graph in G (Fig. 9).

6 Gromov Hyperbolicity and Strong Isoperimetric Inequalities

Let X be a geodesic metric space; that is, X is a metric space such that for every $a, b \in X$ there is a geodesic line segment γ from a to b such that $\text{dist}(a, b) = \text{length}(\gamma)$. Then X is called δ-*hyperbolic* if every geodesic triangle $\triangle \subset X$ is δ-*thin*; i.e., any side of $\triangle$ is contained in the δ-neighborhood of the union of the other two sides. If

X is δ-hyperbolic for some $\delta \geq 0$, we just say that X is hyperbolic in the sense of Gromov, or *Gromov hyperbolic*. For other equivalent definitions and general theory about Gromov hyperbolic spaces, we refer [10,18,20]. Note that every connected graph can be realized as a geodesic metric space, where the metric is the simplicial metric such that every edge is of length 1.

Now suppose $\varphi : [\alpha, \beta] \subset \mathbb{R} \to X$ is a path; i.e., a continuous function. We say that φ is *t-detour* if there exists a geodesic segment γ from $\varphi(\alpha)$ to $\varphi(\beta)$ and a point $z \in \gamma$ such that $\mathrm{Im}(\varphi) \cap B(z,t) = \emptyset$. Here $B(z,t)$ denotes the open ball with center z and radius t. The detour growth function $g_X : (0,\infty) \to [0,\infty]$ is defined by

$$g_X(t) := \inf\{\mathrm{length}(\mathrm{Im}\varphi) : \varphi \text{ is a } t\text{-detour map}\}$$

with the convention $g_X(t) = \infty$ when there is no rectifiable t-detour map. Then it is known [7] that a geodesic metric space X is Gromov hyperbolic if and only if $\lim_{t\to\infty} g_X(t)/t = \infty$, which is what we will use for a proof of Theorem 6.

Suppose X_1 and X_2 are metric spaces. A function $f : X_1 \to X_2$ is called a *rough isometry*, or *quasi-isometry*, if there exist constants $A \geq 1$, $B \geq 0$, and $C \geq 0$ such that for all $x, y \in X_1$ we have

$$\frac{1}{A}\,\mathrm{dist}(x,y) - B \leq \mathrm{dist}\big(f(x), f(y)\big) \leq A\,\mathrm{dist}(x,y) + B,$$

and for every $w \in X_2$ there exists $x \in X_1$ such that $\mathrm{dist}(f(x), w) \leq C$. The notion of rough isometries was introduced by Kanai [24] and Gromov [19]. We say that X_1 is roughly isometric to X_2 if there exists a rough isometry from X_1 to X_2, and it is not difficult to see that rough isometries define an equivalence relation on the space of metric spaces. Moreover, it is well known that if X_1 is roughly isometric to X_2, then X_1 is Gromov hyperbolic if and only if X_2 is Gromov hyperbolic (cf. [10, p. 35] or [8, p. 6]), and if G_1 and G_2 are roughly isometric graphs of bounded vertex degree, then $\jmath(G_1) > 0$ if and only if $\jmath(G_2) > 0$ [34, Theorem (7.34)]. Note that we already used the latter fact in the introduction.

It is well known that if G is a planar graph with $\jmath(G) > 0$, then the growth rate of the volume of combinatorial balls in G is exponential. To be precise, let us fix $v \in V(G)$, and suppose B_n is the combinatorial ball of radius n and centered at v; i.e., B_n is the subgraph of G such that $w \in V(B_n)$ if and only if the distance between v and w is less than or equal to n. We also let $B_0 = \{v\}$. Then because $\partial_v B_n \subset V(B_n) \setminus V(B_{n-1})$, we deduce from the definition of $\jmath(G)$ that

$$\begin{aligned}|V(B_n)| &= |V(B_{n-1})| + |V(B_n) \setminus V(B_{n-1})| \geq |V(B_{n-1})| + |\partial_v B_n| \\ &\geq |V(B_{n-1})| + \jmath(G)|V(B_n)| \geq \big(1 + \jmath(G)\big)|V(B_{n-1})|.\end{aligned}$$

Since $|V(B_0)| = 1$, this inequality implies that

$$|V(B_n)| \geq \big(1 + \jmath(G)\big)^n \tag{6.1}$$

for all $n \in \mathbb{N}$.

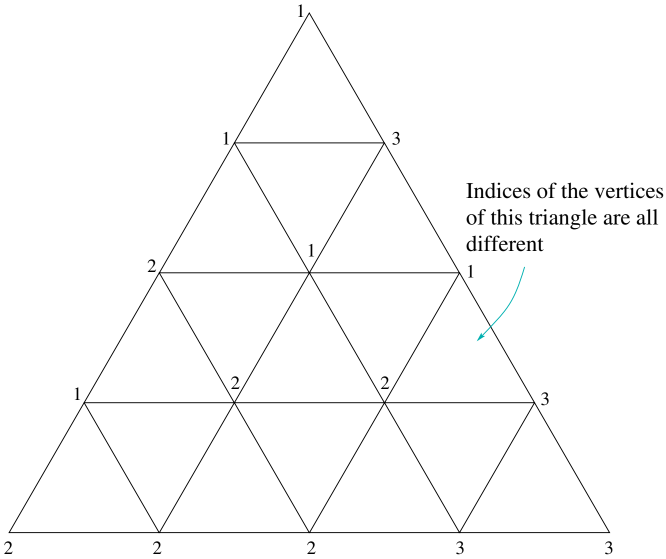


Fig. 10 A Sperner labeled triangulation of a triangle

One more ingredient we need for the proof of Theorem 6 is *Sperner's Lemma* applied to the two-dimensional case. Let T be a triangulation of a topological triangle with vertices A_1, A_2, and A_3. We also assume that a function $h : V(T) \to \{1, 2, 3\}$ satisfies the following properties: $h(A_i) = i$ for $i = 1, 2, 3$, and if $v \in V(T)$ lies on the side $A_i A_j$, $i \neq j$, then $h(v) \in \{i, j\}$. Such h is called a *Sperner Labeling* of T, and Sperner's Lemma [8, p. 124] says that there exists a face (triangle) $f \in F(T)$ such that $h(V(f)) = \{1, 2, 3\}$; i.e., if $v, w \in V(f)$ and $v \neq w$, then $h(v) \neq h(w)$ (see Fig. 10).

Now we are ready to prove Theorem 6. For (a), the following proof is suggested by Mario Bonk. Suppose G is a planar graph of bounded face degree such that $\kappa(G) > 0$, or equivalently, $\jmath(G) > 0$. We assume without loss of generality that G is a triangulation graph of the plane, since otherwise we can add bounded number of vertices and edges on every face of G to obtain a triangulation graph G'. This is possible because face degrees of G are bounded. Then obviously G' is roughly isometric to G, so we have G is Gromov hyperbolic if and only if G' is Gromov hyperbolic. Moreover, in this case $(G')^*$ is also roughly isometric to G^*, so Theorem 1(b) and Theorem (7.34) of [34] imply that $\jmath(G') > 0$ since vertex degrees of $(G')^*$ and G^* are bounded. (Alternatively, it is not difficult to show that $\kappa(G') > 0$ directly.)

Let φ be a t-detour map from $a \in G$ to $b \in G$. By the definition there exist a geodesic segment γ from a to b, and a point $z \in \gamma$ such that $\varphi \cap B(z, t) = \emptyset$, where we denoted $\mathrm{Im}(\varphi)$ by φ for simplicity. Furthermore by shrinking γ and φ if necessary, we can assume that γ meets φ only at a and b. Thus we can treat $\gamma \cup \varphi$ as a topological triangle with vertices a, b, and z (see Fig. 11).

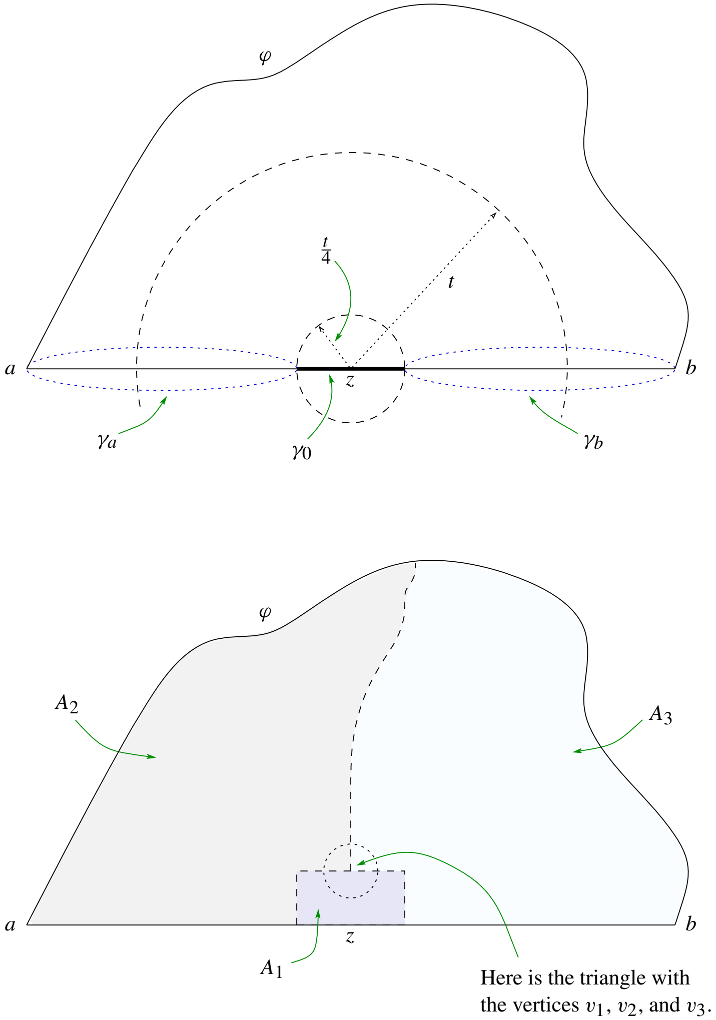


Fig. 11 A detour map φ

Let Γ be the subgraph of G whose vertices are those on γ, φ, and the bounded component of $G \setminus (\gamma \cup \varphi)$. Then since G is a triangulation graph, Γ becomes a triangulation of the 2-dimensional simplex $\triangle$ with $\partial\triangle = \gamma \cup \varphi$. Let $\gamma_0 = \gamma \cap B(z, t/4)$, γ_a be the component of $\gamma \setminus \gamma_0$ containing a, and γ_b be the component of $\gamma \setminus \gamma_0$ containing b.
Define

$$\begin{aligned}
A_1 &:= \{v \in V(\Gamma) : \mathrm{dist}(v, \gamma_0) \leq t/4 \text{ and } \mathrm{dist}(v, \gamma_0) \leq \mathrm{dist}(v, \gamma_a \cup \gamma_b)\},\\
A_2 &:= \{v \in V(\Gamma) \setminus A_1 : \mathrm{dist}(v, \gamma_a) < \mathrm{dist}(v, \gamma_b)\},\\
A_3 &:= \{v \in V(\Gamma) \setminus A_1 : \mathrm{dist}(v, \gamma_b) \leq \mathrm{dist}(v, \gamma_a)\}.
\end{aligned}$$

It is clear that $\varphi \cap A_1 = \emptyset$. Otherwise there exists $v \in \varphi$ and $w \in \gamma_0$ such that $\mathrm{dist}(v, w) \leq t/4$, so we must have $\mathrm{dist}(v, z) \leq \mathrm{dist}(v, w) + \mathrm{dist}(w, z) \leq t/2$, contradicting the assumption $\varphi \cap B(z, t) = \emptyset$. Moreover if $v \in \gamma$ lies between a and z, that is, if v lies on the side opposite the vertex b of $\gamma \cup \varphi$, then v is definitely in either A_1 or A_2. Similarly if $v \in \gamma$ lies between b and z, then v belongs to $A_1 \cup A_3$. Thus if we label every vertex $v \in A_i$ by $i \in \{1, 2, 3\}$, it becomes a Sperner labeling. Hence there exists a triangle in Γ with vertices $v_i \in A_i$, $i = 1, 2, 3$, by Sperner's lemma.

Let $B := B(v_1, t/8)$ and we claim that $B \cap (\varphi \cup \gamma) = \emptyset$ for $t \geq 12$. First, note that $B \cap \varphi = \emptyset$ since otherwise there exists $v \in \varphi$ such that $\mathrm{dist}(v, z) \leq 5t/8$. If $\mathrm{dist}(v_1, \gamma) \leq t/8$, then because $v_1 \in A_1$ we have $\mathrm{dist}(v_1, \gamma_0) \leq t/8$. But because $v_2 \in A_2$ and

$$\mathrm{dist}(v_2, \gamma_0) \leq 1 + \mathrm{dist}(v_1, \gamma_0) \leq 1 + t/8 \leq t/4$$

for $t \geq 8$, there exists $x \in \gamma_a$ such that

$$\mathrm{dist}(v_2, x) = \mathrm{dist}(v_2, \gamma_a) < \mathrm{dist}(v_2, \gamma_0) \leq 1 + t/8.$$

Similarly there exists $y \in \gamma_b$ such that $\mathrm{dist}(v_3, y) \leq 1 + t/8$, so we have

$$\mathrm{dist}(x, y) \leq \mathrm{dist}(x, v_2) + \mathrm{dist}(v_2, v_3) + \mathrm{dist}(v_3, y) < 3 + t/4. \tag{6.2}$$

On the other hand, γ is a geodesic segment and $B(z, t/4)$ separates γ_a and γ_b. Then because $x \in \gamma_a$ and $y \in \gamma_b$, we must have $\mathrm{dist}(x, y) \geq t/2$. This contradicts (6.2) for $t \geq 12$, so the claim follows.

Note that

$$|\partial_e \Gamma| = \mathrm{length}(\varphi) + \mathrm{length}(\gamma) \leq 2 \cdot \mathrm{length}(\varphi).$$

Moreover because $v_1 \in V(\Gamma)$ and $B \cap (\varphi \cup \gamma) = \emptyset$ for $t \geq 12$, we must have $B \subset \Gamma$ in this case. Thus if $t \geq 12$ and $8n \leq t < 8(n + 1)$ for some $n \in \mathbb{N}$, the inequality (6.1) implies

$$|V(\Gamma)| \geq |V(B)| \geq \big(1 + \jmath(G)\big)^n \geq \big(1 + \jmath(G)\big)^{t/8-1}.$$

Now because $|\partial_e \Gamma| \geq |\partial_v \Gamma|$, we have

$$\mathrm{length}(\varphi) \geq \frac{1}{2}|\partial_e \Gamma| \geq \frac{1}{2}|\partial_v \Gamma| \geq \frac{1}{2}\jmath(G)|V(\Gamma)| \geq \frac{1}{2}\jmath(G)\big(1 + \jmath(G)\big)^{t/8-1}.$$

This proves that $g_G(t)/t \to \infty$ as $t \to \infty$, where g_G is the detour growth function introduced at the beginning of this section. We conclude that G is Gromov hyperbolic by Ref. [7].

We remark here that the above argument can be considered an alternative proof of (a part of) Theorem 2.1 in [10, Chap. 6], where it is proved that every reasonable complete simply connected Riemannian manifold satisfying a linear isoperimetric inequality

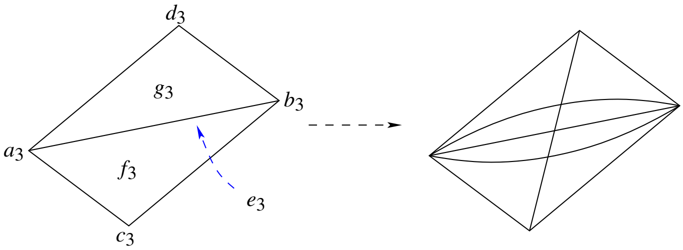


Fig. 12 Replacing e_3 with triple edges and drawing a diagonal line

must be Gromov hyperbolic. In fact, one only needs to modify some terminologies in the above proof so that they are adequate to the continuous case, and use the continuous version of Sperner's lemma [1, p. 378] instead of the combinatorial version. Since it is irrelevant to our subject, we omit the details here.

To prove Theorem 6(b), let us construct a normal planar graph G_1 which is Gromov hyperbolic but $\jmath(G_1) = \kappa(G_1) = 0$. Let Γ_1 be a triangulation of the plane such that $\deg v \geq 7$ for all $v \in V(\Gamma_1)$. Then obviously Γ_1 is Gromov hyperbolic. Choose a sequence of edges $e_n = [a_n, b_n] \in E(\Gamma_1)$ which are far away from each other, for instance $\mathrm{dist}(a_n, a_m) \geq 10$ for $n \neq m$, and let f_n and g_n be the triangular faces of Γ_1 sharing e_n. Also let c_n and d_n be the vertices of f_n and g_n, respectively, which are not lying on e_n.

We obtain a new graph from Γ_1 by replacing e_n by n-multiple edges and drawing a line from c_n to d_n (Fig. 12). We do this operation for all $n \in \mathbb{N}$, and let G_1 denote the resulting graph. G_1 is obviously normal. Moreover, G_1 is Gromov hyperbolic because it is roughly isometric to Γ_1. It is also easy to see that $\jmath(G_1) = \kappa(G_1) = 0$, since if we let S_n be the subgraph of G_1 such that $V(S_n)$ consists of a_n, b_n, c_n, d_n, and all the added vertices in $f_n \cup g_n$, then we have $|F(S_n)| = 2n + 2$ and $|\partial_e S_n| = 4$. This completes the proof of Theorem 6(b).

The graph G constructed in Sect. 5 also satisfies the properties in Theorem 6(b) except being normal. To explain this, first note that we could construct G so that every face is of degree at most 4. Now if we let S be the *infinite* subgraph of G such that

$$V(S) = V(G) \setminus \{v_k^n : n \in \mathbb{Z} \text{ and } 1 \leq k \leq 2|n|\},$$

then one can check that S is a planar graph of bounded face degree such that every vertex has degree at least 7. Thus S is Gromov hyperbolic by Corollary 4 and Theorem 6, so G is also Gromov hyperbolic since it is roughly isometric to S.

For the last, we prove Theorem 6(c); i.e., construct a normal planar graph G_2 of bounded face degree such that $\imath(G_2) > 0$ but not Gromov hyperbolic. The main idea here is to construct a graph with a structure far from being hyperbolic, but the simple random walk on it is transient.

We start with the square lattice graph Γ_2; i.e., $V(\Gamma_2) = \mathbb{Z} \times \mathbb{Z}$, and Γ_2 has an edge between (n_1, m_1) and (n_2, m_2) if and only if $|n_1 - n_2| + |m_1 - m_2| = 1$. Let $O = (0, 0)$ be the origin, and for each $n \in \mathbb{N} \cup \{0\}$ define V_n as the set of vertices of

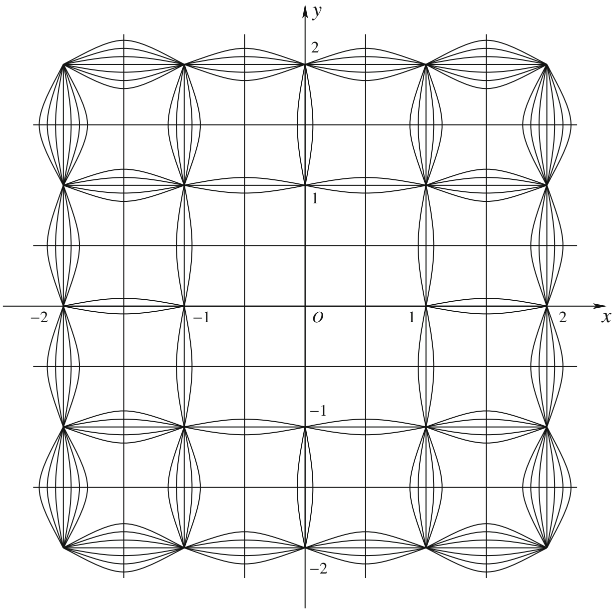


Fig. 13 The graph G_2; it is drawn with $\ell_n = 2n - 1$ for aesthetic reasons, but ℓ_n should increase faster

Γ_2 whose combinatorial distance from O is equal to n. Also for each $n \in \mathbb{N}$, let E_n be the set of edges of Γ_2 connecting a vertex in V_{n-1} to another one in V_n. Then as in Theorem 6(b) we replace each edge in E_n by ℓ_n-multiple edges, where the sequence $\{\ell_n\}$ increases to infinity very fast but will be determined later. Finally we draw the lines $x = m + 1/2$ and $y = m + 1/2$ for all $m \in \mathbb{Z}$. We can do this operation so that each vertical line $x = m + 1/2$ meets no vertical edges, and meets every horizontal edge at most once. Of course we can draw the horizontal lines $y = m + 1/2$ in a similar way, and let G_2 denote the obtained graph (Fig. 13).

It is clear that G_2 is not Gromov hyperbolic, since it is roughly isometric to Γ_2. Also one can immediately see that G_2 is normal, face degrees of G_2 are bounded above by 4, and $\jmath(G_2) = \kappa(G_2) = 0$. So we only need to show that $\imath(G_2) > 0$. Define $h : V(\Gamma_2) \to V(G_2)$ as the natural injection which maps each integer lattice point to itself, and suppose that S is a finite subgraph of G_2. Let N be the largest natural number such that $h(V_N) \cap V(S) \neq \emptyset$, and W_1 be the set of vertices of S which lies in the region $\{(x, y) : |x| + |y| < N + 1/10\}$. If no such N exists, we set $W_1 = \emptyset$. We also define W_2 as the set of vertices in $V(S) \setminus W_1$ lying on the intersection of two lines $x = k_1 + 1/2$ and $y = k_2 + 1/2$ for some $k_1, k_2 \in \mathbb{Z}$, and let $W_3 = V(S) \setminus (W_1 \cup W_2)$.

The vertices in W_3 must be on the 'middle' of some multiple edges, so each of them has a neighbor in V_l for some $l \geq N + 1$. Consequently $W_3 \subset \partial_v S$ and $|W_3| \leq |\partial_v S| \leq |\partial S|$. For W_2, the only neighbors of a vertex $v \in W_2$ are those at the 'middle' of some multiple edges. Then since $(k_1 + 1/2) + (k_2 + 1/2) \geq N + 1/10$ implies $k_1 + k_2 + 1/2 \geq N + 1/10$ for $k_1, k_2 \in \mathbb{N}$, every $v \in W_2$ is either in $\partial_v S$ or has a neighbor in W_3. Thus $|W_2| \leq |W_3| + |\partial_v S| \leq 2|\partial S|$.

Suppose $W_1 \neq \emptyset$, and let $v \in h(V_N) \cap V(S)$. Also let us assume that there are k edges in ∂S with one end at v. Then by our construction v must have at least $\ell_{N+1} - k$ neighbors in W_3, all of which are in $\partial_v S$. So we must have $|\partial S| \geq \ell_{N+1}$. On the other hand, if we denote by C_n the number of vertices in G_2 lying in the region $\{(x, y) : |x| + |y| < n + 1/10\}$, then definitely C_n depends only on $\ell_1, \ldots, \ell_n$ and n. Thus we can choose the sequence $\{\ell_n\}$ so that $\ell_{n+1} \geq C_n$ for all $n \in \mathbb{N}$. This in particular implies that $|W_1| \leq C_N \leq \ell_{N+1} \leq |\partial S|$, and note that the inequality $|W_1| \leq |\partial S|$ is obviously true when $W_1 = \emptyset$.

So far we have shown that $|V(S)| = |W_1| + |W_2| + |W_3| \leq 4|\partial S|$, but this is enough to conclude $\imath(G_2) > 0$ by considering the duality property of Lemma 8. This completes the proof of Theorem 6.

7 Further Remarks

One of the main assumptions of Theorem 1(b) is that G is embedded into the plane locally finitely, so it must be a planar graph with only *one end*. Recently we have extended this result to the case when G has finitely many ends [31]. Furthermore, if G is normal, Theorem 1(b) remains valid even when G has infinitely many ends.

Acknowledgments The author appreciates Mario Bonk for helpful advices about Theorem 6(a), and KIAS (Korea Institute for Advanced Study) for its support through the Associate Member Program. He also thanks to the anonymous referee for helpful comments. This work was supported by the research fund of Hanyang University (HY-2009-N) and by Basic Science Research Program through the National Research Foundation of Korea (NRF) funded by the Ministry of Education, Science and Technology (2010-0004113).

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