Discrete Comput Geom (2013) 49:703–705
DOI 10.1007/s00454-013-9487-z

Regular Polytopes of Nearly Full Rank: Addendum

Peter McMullen

Received: 5 November 2012 / Accepted: 1 February 2013 / Published online: 23 February 2013

Abstract A small family of regular polytopes of nearly full rank was omitted from the earlier paper with this title. This omission is rectified here.

Keywords Abstract regular polytope, Realization, Nearly full rank

Mathematics Subject Classification (2010) Primary, 51M20

1 Introduction

At the end of [2], we expressed the hope that our enumeration of the regular polytopes of nearly full rank was then complete. In retrospect, it is fortunate that we did not make an absolute claim for completeness; it turns out that we overlooked a family which is closely related to others that we did describe. In this note, we shall repair the omission.

Perhaps a brief word is in order about how we found the new family. In writing [5], it has seemed useful to present a wider range of illustrations of realization theory than we felt appropriate to put in papers. In particular, the new techniques of [3,4] have enabled us to describe realization domains of polytopes which were out of reach of the theory in [6, Chap. 5]. One example (which actually has a rather complicated realization domain) is the dual J_5^δ of the polytope that in [2, Sect. 13] we called J_5; this six-dimensional regular polytope of rank 5 has 270 vertices, which it shares with the difference body $D(2_{21})$ of the Gosset polytope 2_{21}. At an early stage of looking at this polytope, we discovered that the corresponding abstract regular polytope had

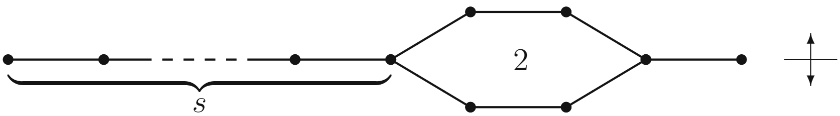
P. McMullen (✉)
University College London, Gower Street, London, WC1E 6BT, UK
e-mail: p.mcmullen@ucl.ac.uk

another six-dimensional realization K_5 (in the notation of Sect. 2), with the 27 vertices of 2_{21} itself. This realization is our starting point.

It is worth pointing out as well that a further opportunity to come across this family was missed. In [1, Sect. 10], we eliminated two possible candidates as regular polytopes of full rank. What we failed to appreciate subsequently is that the facet of the six-dimensional case, namely K_5, was actually a polytope.

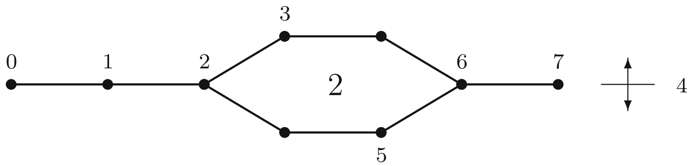
2 The New Family

The new family of regular polytopes (or apeirotopes) is derived by twisting the diagram below by an improper inner automorphism. If we omit the rightmost node (and corresponding branch), then we obtain the diagram for the polytope called $G^{\pi}_{s+1,3}$ in [2, Sect. 11]; it is the Petrial of the polytope $G_{s+1,3}$ of the first Gosset class, whose details are given in [2, Proposition 11.1]. (We have changed the index from the original for future convenience.)



It is clear that we cannot extend the diagram by any more nodes to the right, because we would then obtain the diagram of the infinite group $[3^{2,2,2}]$ as a subdiagram. However, as we shall shortly see, we can allow any $s \leq 2$, in spite of the fact that there is again an 'infinite' subdiagram; we encountered such a situation several times in [2, Sect. 12].

So, let us draw the diagram in the case $s = 2$, now with labels attached which indicate the corresponding generators of the symmetry group $\mathbf{G}$.



For the present purposes, the best way to list the generators R_j of $\mathbf{G}$ is by giving the equations of their mirrors, in terms of coordinate vectors $(\xi_1, \ldots, \xi_8) \in \mathbb{E}^8$. They are

$$R_j : \begin{cases} \xi_1 + \xi_8 = 2, & \text{if } j = 0, \\ \xi_1 + \cdots + \xi_8 = 0, & \text{if } j = 1, \\ \xi_2 + \xi_7 = 0, & \text{if } j = 2, \\ \xi_2 = \xi_3, & \text{if } j = 3, \\ \xi_1 = \xi_8,\ \xi_2 = \xi_7,\ \xi_3 = \xi_6,\ \xi_4 = \xi_5, & \text{if } j = 4, \\ \xi_5 = \xi_6, & \text{if } j = 5, \\ \xi_4 = \xi_5, & \text{if } j = 6, \\ \xi_1 + \cdots + \xi_4 = \xi_5 + \cdots + \xi_8, & \text{if } j = 7. \end{cases}$$

Thus R_4 is the diagram twist, which just reverses the order of the coordinates $\xi_1, \dots, \xi_8$.

Of course, as we know from [6, Chap. 2], to show that we do obtain polytopes (rather than pre-polytopes), we must verify the intersection property

$$\langle R_i \mid i \in \mathsf{J} \rangle \cap \langle R_i \mid i \in \mathsf{K} \rangle = \langle R_i \mid i \in \mathsf{J} \cap \mathsf{K} \rangle$$

for all subsets $\mathsf{J}, \mathsf{K} \subseteq \{0, \dots, 7\}$. However, the geometric picture given by the generators makes this straightforward, if a little tedious.

For $r = 5, \dots, 8$, the symmetry group of the general member K_r of the family is

$$\mathbf{K}_r := \langle R_{8-r}, \dots, R_7 \rangle.$$

For $r = 5, 6, 7$, K_r is a (finite) $(r + 1)$-dimensional regular polytope of rank r, and thus of nearly full rank; similarly, K_8 is an eight-dimensional apeirotope of nearly full rank.

The initial vertex of K_8 is the origin o; which R_0 takes into the initial vertex $(2, 0^6, 2)$ of the vertex-figure K_7; as usual in this context, α^k denotes a string $\alpha, \dots, \alpha$ of length k. Under the group $\mathbf{G}_0 = \mathbf{K}_7$ of the vertex-figure, we obtain all permutations with an even number of changes of sign of $(2, 2, 0^6)$ and (1^8), namely, the vertex-set of the Gosset polytope 4_{21}. Thus K_8 has the vertices of the semi-regular tiling 5_{21} of $\mathbb{E}^8$.

More generally, the r-coface (that is, coface of rank r) K_r has the same vertices as $(r-3)_{21}$. Moreover, as we said above, the facet of K_r is the Petrial $G^{\pi}_{r-5,3}$ of the regular polytope $G_{r-5,3}$ of the first Gosset class. We pointed out in [2, Sect. 11] that G^{π}_{33} is an apeirotope, whose facets are themselves apeirotopes; these are actually of type J_6 of Sect. 13, rather than of type A_6 of Sect. 12 as mistakenly asserted. Hence K_8 even has ridges which are apeirotopes.

So far as K_5 is concerned, its group is obtained from that of J^{δ}_5 by changing the sign of the diagram twist T (that is, replace the mirror T by its orthogonal complement $T^{\perp}$); this changes a proper outer automorphism to an improper inner one. Since such replacements were used quite often in [2, Sect. 12], this makes the fact that K_5 was overlooked even less excusable.

Let us add one comment about K_7. In spite of the apparent symmetry of the diagram, K_7 is not self-dual; indeed, like each of the polytopes K_r, it has no geometric dual. However, just as with other cases, if we reverse the order of the generators $R_1, \dots, R_7$ and change the sign of the twist R_4, then we obtain the symmetry group of another copy of K_7 (or, rather, the same copy, but with different initial vertex and so on).

References

1. McMullen, P.: Regular polytopes of full rank. Discrete Comput. Geom. **32**, 1–35 (2004)
2. McMullen, P.: Regular polytopes of nearly full rank. Discrete Comput. Geom. **46**, 660–703 (2011)
3. McMullen, P.: Realizations of regular polytopes, III. Aequationes Math. **82**, 35–63 (2011)
4. McMullen, P.: Realizations of regular polytopes, IV. Aequationes Math. (in press)
5. McMullen, P.: Geometric Regular Polytopes (in preparation)
6. McMullen P., Schulte, E.: Abstract Regular Polytopes. Encyclopedia of Mathematics and Its Applications, No. 92. Cambridge University Press, Cambridge (2002)