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Asymptotics for Some Combinatorial Characteristics of the Convex Hull of a Poisson Point Process in the Clifford Torus

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Abstract Let $\mathcal{P}_\lambda$ be a homogeneous Poisson point process of rate λ in the Clifford torus $T^2 \subset \mathbb{E}^4$. Let (f_0, f_1, f_2, f_3) be the f-vector of $\operatorname{conv} \mathcal{P}_\lambda$ and let $\bar{v}$ be the mean valence of a vertex of the convex hull. Asymptotic expressions for $\mathsf{E}\, f_1$, $\mathsf{E}\, f_2$, $\mathsf{E}\, f_3$ and $\mathsf{E}\, \bar{v}$ as $\lambda \to \infty$ are proved in this paper.

Keywords Clifford torus · Poisson point process · Random polytope · Poisson–voronoi tessellation

1 Introduction

Recently Poisson–Voronoi tessellations became an object for extensive investigations. The first non-trivial result concerning Poisson–Voronoi tessellations belongs to J. L. Meijering. His paper [6] shows that a typical cell of a Poisson–Voronoi tessellation of the three-dimensional Euclidean space $\mathbb{E}^3$ has an expectation of number of facets equal to

$$\frac{48\pi^2}{35} + 2 = 15.5354\ldots.$$

A survey [9] contains a number of further results concerning Poisson–Voronoi tessellations.

It is possible to consider Voronoi tessellations of a sphere or a hyperbolic space of constant curvature as well as Voronoi tessellations of a Euclidean space. Given a locally finite set A in a sphere, Euclidean space or a hyperbolic space of constant

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curvature, it is possible to consider an associated Delaunay triangulation. It is known (see, for example, [7]) that the following statements are equivalent.

1. A subset $B \subset A$ spans a face of Delaunay triangulation.
2. A set of points equidistant to all points of B contains a face of Voronoi tessellation associated with A.

Therefore, the notions of Voronoi tessellation and Delaunay triangulation are dual to each other.

Consider a finite set A of points in general position in the sphere

$$S_r^d = \{(\xi_1, \xi_2, \ldots, \xi_{d+1}) \subset \mathbb{E}^{d+1} : \xi_1^2 + \xi_2^2 + \cdots + \xi_{d+1}^2 = r^2\}.$$

A subset $B \subset A$ determines a face of the Delaunay triangulation associated with A if and only if conv B is a face of conv A.

N. Dolbilin and M. Tanemura (see [3]) studied convex hulls of finite subsets of the Clifford torus T^2 embedded in $\mathbb{E}^4$. Since $T^2 \subset S^3_{\sqrt{2}}$, this case can be considered as an additional restriction for a finite subset of $S^3_{\sqrt{2}}$ generating the Delaunay triangulation (or the Voronoi tessellation). For a special class of point sets in T^2 called *regular sets* [3] completely describes the combinatorial structure of the convex hull.

In addition, the convex hull of the Poisson point process within T^2 has been explored by numeric methods. Dolbilin and Tanemura considered the average number $\bar{f}$ of 2-faces of a cell in the corresponding Voronoi tessellation of $\mathbb{S}^3$, which is exactly the average degree of a vertex in the Delaunay triangulation. A strong linear relation between $\bar{f}$ and $\log_{10} N$ ($N = 4\pi^2\lambda$ is the average number of points) has been observed, and the obtained regression formula was

$$\bar{f} \approx -2.419308 + 9.971915 \log_{10} N.$$

In other words, the simulation has shown that the mean valence of a vertex of the convex hull (or the mean number of hyperfaces of a Poisson–Voronoi cell) is likely to have an expectation $O^*(\ln \lambda)$, as the rate of the process λ tends to infinity.

Here and further $F_1 = O^*(F_2)$ means that

$$\limsup_{\lambda \to \infty} \max\left(\left|\tfrac{F_1}{F_2}\right|, \left|\tfrac{F_2}{F_1}\right|\right) < \infty.$$

N. Dolbilin suggested the author to prove the conjecture on the logarithmic growth of the mean valence of a vertex.

In this paper this conjecture and several related results are proved.

2 Notation and Main Results

In the four-dimensional Euclidean space $\mathbb{E}^4$ consider the two-dimensional Clifford torus

$$T^2 = \{(\cos\phi, \sin\phi, \cos\psi, \sin\psi) : -\pi < \phi, \psi \le \pi\}.$$

Clearly, T^2 is a submanifold of the three-dimensional sphere

$$S^3_{\sqrt{2}} = \big\{(\xi_1, \xi_2, \xi_3, \xi_4) : \xi_1^2 + \xi_2^2 + \xi_3^2 + \xi_4^2 = 2\big\}.$$

T^2 has a locally Euclidean planar metric and, consequently, the natural Borel measure mes_2, where $\mathrm{mes}_2(T^2) = 4\pi^2$.

Consider a random point set $\mathcal{P} \subset T^2$. For every Borel-measurable set $A \subset T^2$ define a random variable

$$n(A) = n_{\mathcal{P}}(A) = |\mathcal{P} \cap A|.$$

Denote by $\mathrm{Pois}(\nu)$ the Poisson distribution with rate parameter ν, i.e. the distribution of a random variable ζ_ν such that

$$\mathsf{P}(\zeta_\nu = j) = e^{-\nu} \frac{\nu^j}{j!} \quad \text{for} \quad j = 0, 1, 2, \dots.$$

Say that $\mathcal{P} = \mathcal{P}_\lambda$ is the (homogeneous) Poisson point process of rate $\lambda > 0$ if the random variable $n(A)$ is distributed according to $\mathrm{Pois}\big(\lambda\, \mathrm{mes}_2(A)\big)$ law for every Borel-measurable set $A \subset T^2$.

Call a polytope in $\mathbb{E}^4$ *generic* if it is a simplicial 4-polytope, or a simplex of dimension at most 3, or an empty polytope. Remind the notion of f-vector of a 4-polytope and extend it to the cases of other generic polytopes.

The f-vector of a 4-polytope P is a 4-vector (f_0, f_1, f_2, f_3), where f_i is the number of i-faces of P for $i = 0, 1, 2, 3$. By definition, set the f-vectors for the three-dimensional simplex, the two-dimensional simplex, the segment, the one-point set and the empty polytope as $(4, 6, 4, 2)$, $(3, 3, 1, 0)$, $(2, 1, 0, 0)$, $(1, 0, 0, 0)$ and $(0, 0, 0, 0)$ respectively.

If $P = \mathrm{conv}\, \mathcal{P}_\lambda$, then P is almost surely a generic polytope, and therefore (f_0, f_1, f_2, f_3) is a well-defined random vector.

Call the event $n(T^2) \le 4$ a *degenerate case* and the complementary event $n(T^2) > 4$, respectively, a *non-degenerate case*.

Remark The reason to choose $f_3 = 2$ for a three-dimensional simplex is the convenience to treat it as a polytope with 2 hyperfaces equal to this simplex. The other components were chosen to satisfy Dehn–Sommerville equations (see, for example, [1]). The f-vectors for other polytopes occurring in degenerate cases were chosen rather arbitrarily according to the common idea of simplices in dimensions lower than 3.

The main results of this paper are below.

Theorem 2.1 *The number of hyperfaces of* $\mathrm{conv}\, \mathcal{P}_\lambda$ *has a magnitude of expectation* $O^*(\lambda \ln \lambda)$ *as* λ *tends to infinity.*

Theorem 2.2 *The numbers of* 1*-faces and* 2*-faces of* $\mathrm{conv}\, \mathcal{P}_\lambda$ *both have magnitudes of expectation* $O^*(\lambda \ln \lambda)$ *as* λ *tends to infinity.*

In addition, one can easily observe that the value of f_0 (i.e. the number of vertices) for the polytope $\operatorname{conv}\mathcal{P}_\lambda$ is exactly $n(T^2)$. Therefore

$$\mathsf{E}\, f_0 = \mathsf{E}\, n(T^2) = 4\lambda\pi^2,$$

as $n(T^2)$ is $\mathrm{Pois}(4\lambda\pi^2)$-distributed (expectations of Poisson random variables are computed, for example, in [8]).

Remark For a random polytope $\operatorname{conv}\mathcal{P}_\lambda$ the asymptotics of the expectation of f-vector as $\lambda \to \infty$ is now completely described.

The other combinatorial characteristic of a polytope is mean valence of its vertices. More precisely, given a polytope P in $\mathbb{E}^4$ (possibly, empty) with f-vector (f_0, f_1, f_2, f_3), consider the value

$$\bar{v} = \bar{v}(P) = \begin{cases} \frac{2f_1}{f_0}, & \text{if } f_0 \neq 0, \\ 0, & \text{if } f_0 = 0. \end{cases}$$

Then $\bar{v}$ is called *the mean valence of vertex* of P. If $P = \operatorname{conv}\mathcal{P}_\lambda$, then $\bar{v} = \bar{v}(\operatorname{conv}\mathcal{P}_\lambda)$ is a random variable.

Theorem 2.3 *The expectation of the mean valence of a vertex of* $\operatorname{conv}\mathcal{P}_\lambda$ *has asymptotics* $\mathsf{E}\,\bar{v} = O^*(\ln\lambda)$ *as* λ *tends to infinity.*

Remark Theorem 2.3 provides an answer to the problem proposed by Dolbilin and Tanemura.

Here and further the designations of all combinatorial characteristics apply to the random polytope $\operatorname{conv}\mathcal{P}_\lambda$.

3 Integral Expressions for $\mathsf{E}\, f_3$ and $\mathsf{E}\,\bar{v}$

Let $(T^2)^4$ be the fourth Cartesian power of T^2 with natural measure mes_8. Let $X \subset (T^2)^4$ be the set of all points $x = (x_1, x_2, x_3, x_4)$, where $x_i \in T^2$ such that points x_1, x_2, x_3, x_4 are affinely independent in $\mathbb{E}^4$.

For every $x \in X$ denote by $p(x)$ a hyperplane spanned by points x_1, x_2, x_3, x_4. It is obvious that X is open in $(T^2)^4$. Moreover, it is easily seen that $(T^2)^4 \setminus X$ has a zero measure.

Denote by $\Pi^+(x)$ and $\Pi^-(x)$ the two half-spaces determined by $p(x)$ for every $x \in X$.

The sets

$$C^+(x) = T^2 \cap \Pi^+(x) \quad \text{and} \quad C^-(x) = T^2 \cap \Pi^-(x)$$

are called *caps*.

Without loss of generality, assume that for every $x \in X$

$$\mathrm{mes}_2(C^+(x)) \leq \mathrm{mes}_2(C^-(x)).$$

Let $G : X \to \mathbb{R}$ be a function determined by

$$G(x) = \mathrm{mes}_2(C^+(x)).$$

Clearly, $G(x)$ is continuous on X.

The integral expressions for $E f_3$ and $E \bar{v}$ will be obtained using the famous Slivnyak–Mecke formula. This formula was proved for the first time in [5], and in [2] it is stated as follows.

Proposition 3.1 (Slivnyak–Mecke formula) *Let $\mathcal{X}$ be a space with measure μ. Suppose $\mathcal{N}_{\mathcal{X}}$ is a space of all locally finite point configurations in $\mathcal{X}$. Consider a Poisson point process $\mathcal{P}_\mu$ within $\mathcal{X}$ corresponding to the measure μ. Then for every measurable function $F : \mathcal{X}^s \times \mathcal{N}_{\mathcal{X}} \to [0, \infty)$ holds*

$$\begin{aligned} &\mathsf{E} \sum_{\{x_1, x_2, \ldots, x_s\} \subset \mathcal{P}_\mu}^{\neq} F\left(x_1, x_2, \ldots, x_s, \mathcal{P}_\mu \setminus \{x_1, x_2, \ldots, x_s\}\right) \\ &\quad = \int\limits_{\mathcal{X}^s} \mathsf{E}\left(F\left(x_1, x_2, \ldots, x_s, \mathcal{P}_\mu\right)\right) d\mu(x_1) d\mu(x_2), \ldots, d\mu(x_s). \end{aligned} \tag{1}$$

The sign $\neq$ here stands for summation over all s-tuples of *distinct* points.

Throughout the proofs of Lemmas 3.2 and 3.3 λ is assumed to be a fixed positive real number.

Lemma 3.2

$$\mathsf{E}\, f_3 = \frac{1}{24} \int\limits_{(T^2)^4} \lambda^4 \left(e^{-\lambda G(x)} + e^{-\lambda(4\pi^2 - G(x))}\right) dx. \tag{2}$$

Proof Apply the Slivnyak–Mecke formula (1) for

$$\mathcal{X} = T^2, \quad s = 4, \quad \mu = \lambda \cdot \mathrm{mes}_2$$

and

$$F(x_1, x_2, x_3, x_4, X) = \mathbf{1}_{C^+(x) \cap X \subset \partial C^+(x)} + \mathbf{1}_{C^-(x) \cap X \subset \partial C^-(x)}.$$

If $n(T^2) > 4$ and x_1, x_2, x_3, x_4 are distinct points of $\mathcal{P}_\lambda$ then it is not hard to see that almost surely

$$\begin{aligned} &F\left(x_1, x_2, x_3, x_4, \mathcal{P}_\nu \setminus \{x_1, x_2, x_3, x_4\}\right) \\ &\quad = \begin{cases} 1, & \text{if } x_1, x_2, x_3, x_4 \text{ span a hyperface of } \mathcal{P}_\lambda, \\ 0, & \text{otherwise.} \end{cases} \end{aligned}$$

If $n(T^2) = 4$ and x_1, x_2, x_3, x_4 are the four points of $\mathcal{P}_\lambda$ then almost surely

$$F\left(x_1, x_2, x_3, x_4, \mathcal{P}_\nu \setminus \{x_1, x_2, x_3, x_4\}\right) = 2.$$

Finally, if $n(T^2) < 4$ then there are no quadruples in $\mathcal{P}_\lambda$ and the left part of (1) is an empty sum.

Therefore, in every case

$$\sum_{\{x_1,x_2,x_3,x_4\}\subset\mathcal{P}_\lambda}^{\neq} F\left(x_1, x_2, x_3, x_4, \mathcal{P}_\lambda \setminus \{x_1, x_2, x_3, x_4\}\right) = 24 f_3, \tag{3}$$

since the quadruple (x_1, x_2, x_3, x_4) can be ordered in 24 different ways.

Moreover, by definition of a Poisson point process,

$$\begin{aligned} \mathsf{E}\,\mathbf{1}_{C^+(x)\cap X\subset\partial C^+(x)} &= e^{-\lambda\cdot \mathrm{mes}_2(C^+(x))} = e^{-\lambda G(x)}, \\ \mathsf{E}\,\mathbf{1}_{C^-(x)\cap X\subset\partial C^-(x)} &= e^{-\lambda\cdot \mathrm{mes}_2(C^-(x))} = e^{-\lambda(4\pi^2-G(x))}. \end{aligned} \tag{4}$$

Substitution of (3) and (4) into (1) gives the statement of Lemma 3.2. □

For $\nu > 0$ let ζ_ν be distributed as $\mathrm{Pois}(\nu)$. Denote

$$h(\nu) = \mathsf{E}\,\frac{1}{\zeta_\nu+4} = \sum_{j=0}^{\infty} \frac{\nu^j}{j!(j+4)} e^{-\nu}. \tag{5}$$

Direct computation of the sum in (5) gives

$$h(\nu) = \frac{1}{\nu} - \frac{3}{\nu^2} + \frac{6}{\nu^3} - \frac{6-6e^{-\nu}}{\nu^4}. \tag{6}$$

Obviously, $h(\nu)$ is continuous for $\nu > 0$.

Lemma 3.3

$$\begin{aligned} \mathsf{E}\,\bar{v} = \frac{1}{12} \int_{(T^2)^4} & \lambda^4\big(e^{-\lambda G(x)} h\big(4\lambda\pi^2 - \lambda G(x)\big) + e^{-4\lambda\pi^2+\lambda G(x)} h\big(\lambda G(x)\big)\big)\,dx \\ & +2 - \mathsf{P}\big(n(T^2) = 2\big) - 2\mathsf{P}\big(n(T^2) < 2\big). \end{aligned} \tag{7}$$

Proof The Dehn–Sommerville equations [1, Sect. 1.2] hold for conv $\mathcal{P}_\lambda$ almost surely in non-degenerate cases as well as in the case $n(T^2) = 4$. These equations imply $f_1 = f_3 + f_0$. Therefore, in these cases

$$\bar{v} = 2\frac{f_3}{f_0} + 2. \tag{8}$$

Apply the Slivnyak–Mecke formula (1) for

$$\mathcal{X} = T^2, \quad s = 4, \quad \mu = \lambda \cdot \mathrm{mes}_2$$

and

$$F(x_1, x_2, x_3, x_4, X) = \big(\mathbf{1}_{C^+(x)\cap X\subset\partial C^+(x)} + \mathbf{1}_{C^-(x)\cap X\subset\partial C^-(x)}\big) \times \frac{1}{|X \cup \{x_1, x_2, x_3, x_4\}|}.$$

The substitution gives the following identity:

$$24\mathsf{E}\big(\tfrac{f_3}{f_0} \mid n(T^2) \geq 4\big) \cdot P\big(n(T^2) \geq 4\big) = \int\limits_{(T^2)^4} \lambda^4\big(e^{-\lambda G(x)} h\big(4\lambda\pi^2 - \lambda G(x)\big) + e^{-4\lambda\pi^2+\lambda G(x)} h\big(\lambda G(x)\big)\big)\, dx. \tag{9}$$

According to the law of total probability,

$$\mathsf{E}\,\bar{v} = 2\mathsf{E}\big(\tfrac{f_3}{f_0} \mid n(T^2) \geq 4\big) \cdot \mathsf{P}\big(n(T^2) \geq 4\big) + 2\mathsf{P}\big(n(T^2) \geq 4\big) + 2\mathsf{P}\big(n(T^2) = 3\big) + \mathsf{P}\big(n(T^2) = 2\big). \tag{10}$$

By (9), the first summand at the right-hand side of (10) is equal to the integral in (7). Further,

$$\mathsf{P}\big(n(T^2) < 2\big) + \mathsf{P}\big(n(T^2) = 2\big) + \mathsf{P}\big(n(T^2) = 3\big) + \mathsf{P}\big(n(T^2) \geq 4\big) = 1,$$

therefore

$$2\mathsf{P}\big(n(T^2) \geq 4\big) + 2\mathsf{P}\big(n(T^2) = 3\big) + \mathsf{P}\big(n(T^2) = 2\big) = 2 - \mathsf{P}\big(n(T^2) = 2\big) - 2\mathsf{P}\big(n(T^2) < 2\big),$$

so the remaining parts of the right-hand sides of (7) and (10) are equal as well. □

4 Estimates for the Measure Function

To proceed we need two statements about the caps. Both of them are proved by a fairly simple computation, so the proofs are given in the Appendix.

Lemma 4.1 *The following statements hold:*

1. *For every cap $C^+(x)$ (respectively, $C^-(x)$) there exist $a, b \geq 0$, ϕ_0, ψ_0 satisfying $a^2 + b^2 \geq 2$ and $-\pi < \phi_0, \psi_0 \leq \pi$ such that*

$$C^+(x) = \Big\{(\phi, \psi) \in T^2 : a^2 \sin^2 \frac{\phi - \phi_0}{2} + b^2 \sin^2 \tfrac{\phi-\phi_0}{2} \leq 1\Big\},$$

and, respectively,

$$C^-(x) = \big\{(\phi, \psi) \in T^2 : a^2 \sin^2 \tfrac{\phi-\phi_0}{2} + b^2 \sin^2 \frac{\phi - \phi_0}{2} \geq 1\big\}.$$

where $\gamma_1, \gamma_2 > 0$ *and do not depend on* x.

2. *For every* $a, b \geq 0$, ϕ_0, ψ_0 *satisfying* $a^2 + b^2 \geq 2$ *and* $-\pi < \phi_0, \psi_0 \leq \pi$ *the sets*

$$\big\{(\phi, \psi) \in T^2 : a^2 \sin^2 \tfrac{\phi-\phi_0}{2} + b^2 \sin^2 \tfrac{\phi-\phi_0}{2} \leq 1\big\}$$

and

$$\big\{(\phi, \psi) \in T^2 : a^2 \sin^2 \tfrac{\phi-\phi_0}{2} + b^2 \sin^2 \tfrac{\phi-\phi_0}{2} \geq 1\big\}$$

are caps.

Remark a, b, ϕ_0, ψ_0 can be now considered as functions $a(x), b(x), \phi_0(x), \psi_0(x)$ of the argument $x \in X$.

Lemma 4.2 *There exist positive constants* γ_1, γ_2 *such that for every* $x \in X$ *holds*

$$\gamma_1 < (a(x) + 1)(b(x) + 1)G(x) < \gamma_2.$$

For every $t \in \mathbb{R}$ define

$$\begin{aligned} M(t) &= \mathrm{mes}_8\{x \in X : G(x) < t\}, \\ N(t) &= \mathrm{mes}_8\{x \in X : G(x) < t \text{ and } \min(a(x), b(x)) < 100\}, \\ L(t) &= \mathrm{mes}_8\{x \in X : G(x) < t \text{ and } \min(a(x), b(x)) \geq 100\}. \end{aligned}$$

It is easily seen that $M(t) = N(t) = L(t) = 0$ for $t < 0$ and $M(t) = N(t) + L(t)$ for every $t \in \mathbb{R}$.

The main goal of the present section is to estimate $M(t)$. We estimate $N(t)$ and $L(t)$ separately in Lemmas 4.3 and 4.4.

Lemma 4.3 *There exists* $\gamma_3 > 0$ *such that*

$$N(t) < \gamma_3 t^3$$

for every $0 < t < \frac{1}{2}$.

Lemma 4.4 *There exist* $\gamma_4, \gamma_5 > 0$ *such that*

$$\gamma_4 t^3 |\ln t| < L(t) < \gamma_5 t^3 |\ln t|$$

for every $0 < t < \frac{1}{2}$.

Before the proofs we give an estimate of $M(t)$ as a corollary.

Corollary 4.5 *There exist positive constants* γ_6, γ_7 *such that*

$$\gamma_6 t^3 |\ln t| < M(t) < \gamma_7 t^3 |\ln t|$$

for every $0 < t < \frac{1}{2}$.

Proof of Lemma 4.3 Introduce the functions

$$N_1(t) = \mathrm{mes}_8\{x \in X : G(x) < t \text{ and } a(x) < 100\},$$
$$N_2(t) = \mathrm{mes}_8\{x \in X : G(x) < t \text{ and } b(x) < 100\}.$$

Obviously, $N_1(t) = N_2(t)$ and $N(t) \le N_1(t) + N_2(t)$.
Suppose

$$0 < t \le \frac{\gamma_1}{1000\pi}.$$

Let $a(x) < 100$ and $G(x) < t$. Lemma 4.2 implies

$$b(x) \ge \frac{\gamma_1}{(a+1)G(x)} - 1 > \frac{\gamma_1}{200t}.$$

By Lemma 4.1, cap $C^+(x)$ is described by the inequality

$$a(x)^2 \sin^2 \frac{\phi - \phi_0}{2} + b(x)^2 \sin^2 \frac{\psi - \psi_0}{2} \le 1.$$

From the last inequality follows that every point of $C^+(x)$ with coordinates (ϕ, ψ) satisfies

$$\left| \sin \frac{\psi - \psi_0}{2} \right| \le \frac{1}{b} < \frac{200t}{\gamma_1}.$$

Hence

$$C^+(x) \subset \left\{ (\phi, \psi) \in T^2 : \left| \sin \frac{\psi - \psi_0}{2} \right| < \frac{200t}{\gamma_1} \right\} = S(t, x).$$

A set $S \subset T^2$ is called a *strip* if there exist $\psi_1 \in (-\pi, \pi]$ and $d \in (-1, 1)$ such that

$$S = \left\{ (\phi, \psi) \in T^2 : \cos(\psi - \psi_1) \le d \right\}.$$

The *centerline* of S is the line $\psi = \psi_1$, and $2 \arccos d$ is the *width* of S.
$S(t, x)$ is obviously a strip of width

$$w(t) = 4 \arcsin \frac{200t}{\gamma_1} < \frac{1000\pi t}{\gamma_1}$$

and the centerline of $S(t, x)$ is described by the equation $\psi = \psi_0$.

Let

$$k = k(t) = \left\lceil \frac{2\pi}{w(t)} \right\rceil.$$

Consider k strips $S_1, S_2, \dots S_k \subset T^2$ of width $2w(t)$ each such that S_j has centerline $\psi = -\pi + \frac{2\pi j}{k}$.

It is obvious that $S(t,x) \subset S_j$, where j is the nearest integer to $\frac{k(\psi_0+\pi)}{2\pi}$ and $S_0 = S_k$.

Let $x = (x_1, x_2, x_3, x_4)$, where $x_i \in T^2$. Obviously, every $x_i \in \partial C^+(x)$, therefore $x \in S_j^4$.

Finally,

$$\begin{aligned} N_1(t) &= \mathrm{mes}_8\{x \in X : G(x) < t \text{ and } a(x) < 100\} \\ &\le \mathrm{mes}_8\Big(\bigcup_{j=1}^{k}(t) S_j^4\Big) \le k(t)(4\pi w(t))^4 \le (4\pi)^5 w(t)^3 \le \gamma_3' t^3. \end{aligned}$$

Then

$$N(t) \le 2N_1(t) \le 2\gamma_3' t^3.$$

The case

$$0 < t \le \frac{\gamma_1}{1000\pi}$$

is proved completely.

Suppose

$$\frac{\gamma_1}{1000\pi} < t < \frac{1}{2}.$$

Obviously,

$$N(t) \le \mathrm{mes}_8\big((T^2)^4\big) = 256\pi^8.$$

Then

$$N(t) < 256\pi^8 \Big(\frac{1000\pi}{\gamma_1}\Big)^3 t^3,$$

and Lemma 4.3 is now proved completely. □

Proof of Theorem 2.1 Suppose $\min(a(x), b(x)) \ge 100$. Assume

$$x = (x_1, x_2, x_3, x_4), \quad \text{where} \quad x_i = (\phi_i, \psi_i) \in T^2 \quad \text{for} \quad i = 1, 2, 3, 4.$$

Let

$$\alpha(x) = \frac{1}{a(x)}, \quad \beta(x) = \frac{1}{b(x)}.$$

By assumptions, $0 < \alpha(x), \beta(x) < \frac{1}{100}$.

Lemma 4.2 easily implies that there exist $\gamma_1', \gamma_2' > 0$ such that

$$\gamma_1'\alpha(x)\beta(x) < G(x) < \gamma_2'\alpha(x)\beta(x). \tag{11}$$

Since $x_i = (\phi_i, \psi_i) \in \partial C^+(x)$ for $i = 1, 2, 3, 4$, then

$$\frac{\sin^2 \frac{\phi_i - \phi_0}{2}}{\alpha^2} + \frac{\sin^2 \frac{\psi_i - \psi_0}{2}}{\beta^2} = 1.$$

Therefore, we can define parameters $-\pi < \theta_i \le \pi$ for $i = 1, 2, 3, 4$ such that

$$\sin \frac{\phi_i - \phi_0}{2} = \alpha \cos \theta_i \quad \text{and} \quad \sin \frac{\psi_i - \psi_0}{2} = \beta \sin \theta_i.$$

It is not hard to see that every point $x \in X$ parametrized by 8 numbers

$$(\alpha, \beta, \phi_0, \psi_0, \theta_1, \theta_2, \theta_3, \theta_4)$$

can be uniquely parametrized by another 8 numbers

$$(\phi_1, \psi_1, \phi_2, \psi_2, \phi_3, \psi_3, \phi_4, \psi_4).$$

Since there are two parametrizations of a point $x \in X$, consider a Jacobi matrix between these parametrizations. The elements are computed as follows:

$$\begin{aligned}
&\frac{\partial \phi_i}{\partial \phi_0} = 1, \quad \frac{\partial \psi_i}{\partial \phi_0} = 0;\\
&\frac{\partial \phi_i}{\partial \psi_0} = 0, \quad \frac{\partial \psi_i}{\partial \psi_0} = 1;\\
&\frac{\partial \phi_i}{\partial \alpha} = \frac{2\cos\theta_i}{\cos\frac{\phi_i - \phi_0}{2}}; \quad \frac{\partial \psi_i}{\partial \alpha} = 0;\\
&\frac{\partial \phi_i}{\partial \beta} = 0; \quad \frac{\partial \psi_i}{\partial \beta} = \frac{2\sin\theta_i}{\cos\frac{\psi_i - \psi_0}{2}};\\
&\frac{\partial \phi_i}{\partial \theta_i} = \frac{-2\alpha\sin\theta_i}{\cos\frac{\phi_i - \phi_0}{2}}; \quad \frac{\partial \psi_i}{\partial \theta_i} = \frac{2\beta\cos\theta_i}{\cos\frac{\psi_i - \psi_0}{2}}; \quad \frac{\partial \phi_i}{\partial \theta_j} = \frac{\partial \psi_i}{\partial \theta_j} = 0.
\end{aligned}$$

Therefore

$$J = \left| \frac{D(\phi_1, \psi_1, \phi_2, \psi_2, \phi_3, \psi_3, \phi_4, \psi_4)}{D(\phi_0, \psi_0, \alpha, \beta, \theta_1, \theta_2, \theta_3, \theta_4)} \right|$$

$$= \begin{vmatrix} 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ \frac{2\cos\theta_1}{\cos\frac{\phi_1-\phi_0}{2}} & 0 & \frac{2\cos\theta_2}{\cos\frac{\phi_2-\phi_0}{2}} & 0 & \frac{2\cos\theta_3}{\cos\frac{\phi_3-\phi_0}{2}} & 0 & \frac{2\cos\theta_4}{\cos\frac{\phi_4-\phi_0}{2}} & 0 \\ 0 & \frac{2\sin\theta_1}{\cos\frac{\psi_1-\psi_0}{2}} & 0 & \frac{2\sin\theta_2}{\cos\frac{\psi_2-\psi_0}{2}} & 0 & \frac{2\sin\theta_3}{\cos\frac{\psi_3-\psi_0}{2}} & 0 & \frac{2\sin\theta_4}{\cos\frac{\psi_4-\psi_0}{2}} \\ \frac{-2\alpha\sin\theta_1}{\cos\frac{\phi_1-\phi_0}{2}} & \frac{2\beta\cos\theta_1}{\cos\frac{\psi_1-\psi_0}{2}} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{-2\alpha\sin\theta_2}{\cos\frac{\phi_2-\phi_0}{2}} & \frac{2\beta\cos\theta_2}{\cos\frac{\psi_2-\psi_0}{2}} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{-2\alpha\sin\theta_3}{\cos\frac{\phi_3-\phi_0}{2}} & \frac{2\beta\cos\theta_3}{\cos\frac{\psi_3-\psi_0}{2}} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{-2\alpha\sin\theta_4}{\cos\frac{\phi_4-\phi_0}{2}} & \frac{2\beta\cos\theta_4}{\cos\frac{\psi_4-\psi_0}{2}} \end{vmatrix}.$$

Direct computation shows that

$$J = \sum_{(i\,j\,k\,l)} 64\,\mathrm{sign}\,(i\,j\,k\,l)\,\alpha^2\beta^2 \cdot \frac{1}{\prod_{m=1}^4 \cos\frac{\phi_m-\phi_0}{2}\cos\frac{\psi_m-\psi_0}{2}} \\ \times \cos^2\theta_i \cos\theta_j \sin^2\theta_k \sin\theta_l \cos\frac{\phi_j-\phi_0}{2}\cos\frac{\psi_l-\psi_0}{2},$$

where $(i\,j\,k\,l)$ runs through all permutations of $(1\,2\,3\,4)$.

From (11) easily follows that

$$\mathrm{mes}_8\bigl\{x \in (T^2)^4 : \alpha(x)\beta(x) < \tfrac{t}{\gamma_2'} \text{ and} \max(\alpha(x),\beta(x)) < \tfrac{1}{100}\bigr\} \le L(t) \\ \le \mathrm{mes}_8\bigl\{x \in (T^2)^4 : \alpha(x)\beta(x) < \tfrac{t}{\gamma_1'} \text{ and} \max(\alpha(x),\beta(x)) < \tfrac{1}{100}\bigr\}.$$

Therefore

$$\int_{\substack{\max(\alpha,\beta)<\frac{1}{100}\\ \alpha\beta<\frac{t}{\gamma_2'}}} d\phi_1\,d\psi_1\,d\phi_2\,d\psi_2\,d\phi_3\,d\psi_3\,d\phi_4\,d\psi_4 \le L(t) \\ \le \int_{\substack{\max(\alpha,\beta)<\frac{1}{100}\\ \alpha\beta<\frac{t}{\gamma_1'}}} d\phi_1\,d\psi_1\,d\phi_2\,d\psi_2\,d\phi_3\,d\psi_3\,d\phi_4\,d\psi_4.$$

In variables $(\alpha,\beta,\phi_0,\psi_0,\theta_1,\theta_2,\theta_3,\theta_4)$ the last inequality can be written as follows:

$$\int\limits_{\substack{\max(\alpha,\beta)<\frac{1}{100}\\ \alpha\beta<\frac{t}{\gamma_2'}}}\int\limits_{\substack{\phi_0,\psi_0\in(-\pi,\pi]\\ \theta_{1,2,3,4}\in(-\pi,\pi]}} |J|\,d\alpha\,d\beta\,d\phi_0\,d\psi_0\,d\theta_1\,d\theta_2\,d\theta_3\,d\theta_4 \le L(t)$$

$$\le \int\limits_{\substack{\max(\alpha,\beta)<\frac{1}{100}\\ \alpha\beta<\frac{t}{\gamma_1'}}}\int\limits_{\substack{\phi_0,\psi_0\in(-\pi,\pi]\\ \theta_{1,2,3,4}\in(-\pi,\pi]}} |J|\,d\alpha\,d\beta\,d\phi_0\,d\psi_0\,d\theta_1\,d\theta_2\,d\theta_3\,d\theta_4.$$

Let

$$J_1 = \frac{J}{\alpha^2\beta^2} = \sum_{(i\ j\ k\ l)} 64\,\mathrm{sign}\,(i\ j\ k\ l)\cdot\frac{1}{\prod_{m=1}^4 \cos\frac{\phi_m-\phi_0}{2}\cos\frac{\psi_m-\psi_0}{2}}$$
$$\times\cos^2\theta_i\cos\theta_j\sin^2\theta_k\sin\theta_l\cos\frac{\phi_j-\phi_0}{2}\cos\frac{\psi_l-\psi_0}{2}. \tag{12}$$

Then J_1 can be considered as a function $J_1(\alpha,\beta,\phi_0,\psi_0,\theta_1,\theta_2,\theta_3,\theta_4)$.

Obviously, with $\gamma=\gamma_1'$ or γ_2'

$$\int\limits_{\substack{\max(\alpha,\beta)<\frac{1}{100}\\ \alpha\beta<\frac{t}{\gamma}}}\int\limits_{\substack{\phi_0,\psi_0\in(-\pi,\pi]\\ \theta_{1,2,3,4}\in(-\pi,\pi]}} |J|\,d\alpha\,d\beta\,d\phi_0\,d\psi_0\,d\theta_1\,d\theta_2\,d\theta_3\,d\theta_4$$
$$= \int\limits_{\substack{\max(\alpha,\beta)<\frac{1}{100}\\ \alpha\beta<\frac{t}{\gamma}}} \alpha^2\beta^2\,d\alpha\,d\beta \int\limits_{\substack{\phi_0,\psi_0\in(-\pi,\pi]\\ \theta_{1,2,3,4}\in(-\pi,\pi]}} |J_1|\,d\phi_0\,d\psi_0\,d\theta_1\,d\theta_2\,d\theta_3\,d\theta_4. \tag{13}$$

Since $\max(\alpha,\beta)<\frac{1}{100}$, then

$$\left|\sin\tfrac{\phi_m-\phi_0}{2}\right| < \tfrac{1}{100} \quad\text{and}\quad \left|\sin\tfrac{\psi_m-\psi_0}{2}\right| < \tfrac{1}{100}$$

for $m=1,2,3,4$.

Hence

$$\cos\tfrac{\phi_m-\phi_0}{2} > \tfrac{4999}{5000} \quad\text{and}\quad \cos\tfrac{\psi_m-\psi_0}{2} > \tfrac{4999}{5000}.$$

Consequently,

$$\Big|\cos^2\theta_i\cos\theta_j\sin^2\theta_k\sin\theta_l\cos\tfrac{\phi_j-\phi_0}{2}\cos\tfrac{\psi_l-\psi_0}{2}$$
$$-\cos^2\theta_i\cos\theta_j\sin^2\theta_k\sin\theta_l\Big| \le \tfrac{2}{5000}.$$

Applying (12), we obtain the following sequence of inequalities which are independent from α, β:

$$J_1 \ge 64\Big|\sum_{(i\,j\,k\,l)} \operatorname{sign}(i\;j\;k\;l)\;\cdot\cos^2\theta_i\cos\theta_j\sin^2\theta_k\sin\theta_l\cos\tfrac{\phi_j-\phi_0}{2}\cos\frac{\psi_l-\psi_0}{2}\Big|$$
$$\ge 64\Big|\sum_{(i\,j\,k\,l)} \operatorname{sign}(i\;j\;k\;l)\;\cdot\cos^2\theta_i\cos\theta_j\sin^2\theta_k\sin\theta_l\Big| - 64\cdot 24\cdot\frac{2}{5000}.$$

Let $\theta_m = (m-2)\pi/2$. Then

$$\Big|\sum_{(i\,j\,k\,l)} \operatorname{sign}(i\;j\;k\;l)\;\cdot\cos^2\theta_i\cos\theta_j\sin^2\theta_k\sin\theta_l\Big| = 4.$$

Consequently, there exists some neighbourhood of point $(-\frac{\pi}{2}, 0, \frac{\pi}{2}, \pi)$ in coordinates $(\theta_1, \theta_2, \theta_3, \theta_4)$ such that $|J_1| > 64$ in this neighbourhood, and the neighbourhood is independent from α, β, ϕ_0, ψ_0.

Therefore, there exist positive constants γ_4', γ_5' independent of α, β and satisfying

$$\gamma_4' < \int\limits_{\substack{\phi_0,\psi_0\in(-\pi,\pi]\\ \theta_{1,2,3,4}\in(-\pi,\pi]}} |J_1|\,d\phi_0\,d\psi_0\,d\theta_1\,d\theta_2\,d\theta_3\,d\theta_4 < \gamma_5' \tag{14}$$

for every $0 < \alpha, \beta < \frac{1}{100}$.

Inequalities (13) and (14) together imply

$$\gamma_4'\cdot\int\limits_{\substack{\max(\alpha,\beta)<\frac{1}{100}\\ \alpha\beta<\frac{t}{\gamma_2'}}} \alpha^2\beta^2\,d\alpha\,d\beta < L(t) < \gamma_5'\cdot\int\limits_{\substack{\max(\alpha,\beta)<\frac{1}{100}\\ \alpha\beta<\frac{t}{\gamma_1'}}} \alpha^2\beta^2\,d\alpha\,d\beta. \tag{15}$$

If $\tau < \frac{1}{10000}$ then

$$\int\limits_{\substack{\max(\alpha,\beta)<\frac{1}{100}\\ \alpha\beta<\tau}} \alpha^2\beta^2\,d\,\alpha\,d\beta = \tfrac{\tau^3}{9} - \tfrac{2\ln 100}{9}\tau^3 + \tfrac{1}{9}\tau^3|\ln\tau|.$$

Consequently, as $t\to 0$, the main terms in left and right parts of (15) have order $t^3|\ln t|$. Therefore, $\frac{L(t)}{t^3|\ln t|}$ is bounded from above and below in some interval $(0,\varepsilon)$ by two positive constants.

In the segment $[\varepsilon, \frac{1}{2}]$ the functions $L(t)$ and $t^3|\ln t|$ are continuous and positive. Consequently, the quotient $\frac{L(t)}{t^3|\ln t|}$ in this segment is also bounded from above and below by two positive constants (but, probably, not the same as in the previous paragraph). However, combining the cases $t\in(0,\varepsilon)$ and $t\in[\varepsilon,\frac{1}{2}]$ allows to conclude that $\frac{L(t)}{t^3|\ln t|}$ is bounded from above and below by some positive constants in the whole interval $(0, \frac{1}{2}]$. Thus Lemma 4.4 is proved. □

5 Proofs of Main Results

Now proceed with the proofs of Theorems 2.1, 2.2 and 2.3.

Proof of Theorem 2.1 It is obvious that

$$\int\limits_{(T^2)^4} \lambda^4 e^{-\lambda G(x)}\,dx < \int\limits_{(T^2)^4} \lambda^4 \left(e^{-\lambda G(x)} + e^{-\lambda(4\pi^2-\lambda G(x))}\right)\,dx$$
$$< 2\int\limits_{(T^2)^4} \lambda^4 e^{-\lambda G(x)}\,dx.$$

Then, by Lemma 3.2,

$$\mathsf{E}\, f_3 = O^*\Big(\int\limits_{(T^2)^4} \lambda^4 e^{-\lambda G(x)}\,dx\Big).$$

According to [4], the identity

$$\int\limits_{(T^2)^4} \lambda^4 e^{-\lambda G(x)}\,dx = \int\limits_{\mathbb{R}} \lambda^4 e^{-\lambda t}\,dM(t)$$

holds, where the right-hand side is a Stieltjes integral.

Since $G(x)$ is the measure of $C^+(x)$, then $0 < G(x) \le 2\pi^2$ holds for every $x \in X$. Therefore, $M(t)$ is a constant for $t \le 0$ and for $t \ge 2\pi^2$. Hence

$$\int\limits_{(T^2)^4} \lambda^4 e^{-\lambda G(x)}\,dx = \int\limits_0^{2\pi^2} \lambda^4 e^{-\lambda t}\,dM(t).$$

Thus

$$\mathsf{E}\, f_3 = O^*\Big(\int\limits_0^{2\pi^2} \lambda^4 e^{-\lambda t}\,dM(t)\Big). \tag{16}$$

Since $M(t)$ is non-decreasing, $e^{-\lambda t}$ is decreasing and continuous, then integration by parts is possible and gives

$$\int_0^{2\pi^2} \lambda^4 e^{-\lambda t}\, dM(t) = \lambda^4 M(2\pi^2) e^{-2\lambda\pi^2} + \lambda^5 \int_{\frac{1}{2}}^{2\pi^2} e^{-\lambda t} M(t)\, dt + \lambda^5 \int_0^{\frac{1}{2}} e^{-\lambda t} M(t)\, dt. \tag{17}$$

Obviously, as $\lambda \to \infty$,

$$\lambda^4 M(2\pi^2) e^{-2\lambda\pi^2} = o(1), \quad \lambda^5 \int_{\frac{1}{2}}^{2\pi^2} e^{-\lambda t} M(t)\, dt = o(1),$$

$$\lambda^5 \int_0^{\frac{1}{2}} e^{-\lambda t} M(t)\, dt = O^*\Bigl(\lambda^5 \int_0^{\frac{1}{2}} e^{-\lambda t} t^3 |\ln t|\, dt\Bigr). \tag{18}$$

Let $u = e^{-\lambda t}$, then

$$t = -\frac{\ln u}{\lambda} \quad \text{and} \quad dt = -\frac{du}{\lambda u}.$$

Therefore

$$\int_0^{\frac{1}{2}} e^{-\lambda t} t^3 |\ln t|\, dt = \int_{e^{\frac{-\lambda}{2}}}^{1} \frac{|\ln^3 u|}{\lambda^4} \cdot (\ln\lambda - \ln(-\ln u))\, du = O^*(\lambda^{-4}\ln\lambda).$$

Hence

$$\lambda^5 \int_0^{\frac{1}{2}} e^{-\lambda t} M(t)\, dt = O^*(\lambda \ln \lambda). \tag{19}$$

Substitution of (18) and (19) into (17) gives

$$\int_0^{2\pi^2} \lambda^4 e^{-\lambda t}\, dM(t) = O^*(\lambda \ln \lambda).$$

Thus, according to (16),

$$\mathsf{E}\, f_3 = O^*(\lambda \ln \lambda),$$

which is the statement of Theorem 2.1. □

Proof of Theorem 2.2 From Dehn–Sommerville equations for a simplicial 4-polytope follows that

$$f_2 = 2f_3 + r_2 \quad \text{and} \quad f_1 = f_3 - f_0 + r_1,$$

where random variables r_1 and r_2 are errors of degenerate cases, i.e. $r_1 = r_2 = 0$ almost surely if $n(T^2) > 4$ and $r_1, r_2 < 10$ almost surely. Since

$$\lim_{\lambda \to \infty} \mathsf{P}(n(T^2) \le 4) = 0,$$

then

$$\mathsf{E}\, r_1 = o(1), \quad \mathsf{E}\, r_2 = o(1), \quad \mathsf{E}\, f_3 = O^*(\lambda \ln \lambda)$$

as $\lambda \to \infty$ by Theorem 2.1. Also,

$$\mathsf{E}\, f_0 = \mathsf{E}\, n(T^2) = 4\lambda\pi^2$$

since $n(T^2)$ is distributed as Pois $(4\lambda\pi^2)$.

Finally,

$$\begin{aligned} \mathsf{E}\, f_2 &= 2\mathsf{E}\, f_3 + \mathsf{E}\, r_2 = O^*(\lambda \ln \lambda), \\ \mathsf{E}\, f_1 &= \mathsf{E}\, f_3 - \mathsf{E}\, f_0 + \mathsf{E}\, r_1 = O^*(\lambda \ln \lambda), \end{aligned}$$

and Theorem 2.2 is proved. □

Proof of Theorem 2.3 Notice that

$$h\bigl(\lambda G(x)\bigr) < \frac{1}{4} \quad \text{and} \quad e^{-4\lambda\pi^2 + \lambda G(x)} \le e^{-2\lambda\pi^2}.$$

These inequalities imply the estimate

$$\int\limits_{(T^2)^4} e^{-4\lambda\pi^2 + \lambda G(x)} h\bigl(\lambda G(x)\bigr)\, dx \le 64\pi^8 e^{-2\lambda\pi^2} = o(1)$$

as $\lambda \to \infty$.

Further,

$$2\lambda\pi^2 \leq 4\lambda\pi^2 - \lambda G(x) < 4\lambda\pi^2.$$

Therefore, from (6) follows

$$h\big(4\lambda\pi^2 - \lambda G(x)\big) = O^*(\lambda^{-1}).$$

Consequently,

$$\int\limits_{(T^2)^4} \lambda^4 e^{-\lambda G(x)} h\big(4\lambda\pi^2 - \lambda G(x)\big)\, dx = O^*\Big(\lambda^3 \int\limits_{(T^2)^4} e^{-\lambda G(x)}\, dx\Big) O^*(\ln\lambda),$$

as the integral in the middle part was estimated in the proof of Theorem 2.1.

Obviously,

$$\mathsf{P}\big(n(T^2) = 2\big) = o(1), \quad \mathsf{P}\big(n(T^2) < 2\big) = o(1).$$

Now Theorem 2.3 easily follows from (7) because every summand in this identity was estimated. □

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Appendix: Structure of Caps

This section is devoted to obtaining analytical description and measure estimates for the caps.

Proof of Lemma 4.1 Suppose

$$p(x) = \big(\xi_1, \xi_2, \xi_3, \xi_4\big) \in \mathbb{E}^4 : a_1\xi_1 + a_2\xi_2 + b_1\xi_3 + b_2\xi_4 = c\big\}$$

where $c \geq 0$.

Then

$$\begin{aligned}\partial C^+(x) &= \partial C^-(x) \\ &= \big\{(\phi, \psi) \in T^2 : a_1\cos\phi + a_2\sin\phi + b_1\cos\psi + b_2\sin\psi = c\big\}.\end{aligned}$$

The equation for $\partial C^+(x)$ can be rewritten as

$$a'\cos(\phi - \phi_0) + b'\cos(\psi - \psi_0) = c,$$

where $a' = \sqrt{a_1^2 + a_2^2}$ and $b' = \sqrt{b_1^2 + b_2^2}$.

Since

$$\cos(\phi-\phi_0)=1-2\sin^2\tfrac{\phi-\phi_0}{2}$$

and

$$\cos(\psi-\psi_0)=1-2\sin^2\tfrac{\psi-\psi_0}{2},$$

the previous equation is equivalent to

$$a'\sin^2\tfrac{\phi-\phi_0}{2}+b'\sin^2\tfrac{\psi-\psi_0}{2}=\tfrac{a'+b'-c}{2}.$$

The set $\partial C^+(x)$ contains infinitely many points, therefore

$$0\le c<a'+b'.$$

If $c=0$ then $p(x)$ passes through the origin and, therefore, divides T^2 into equal parts. Consequently,

$$\begin{aligned}&\mathrm{mes}_2\left(\left\{(\phi,\psi)\in T^2: a'\sin^2\tfrac{\phi-\phi_0}{2}+b'\sin^2\tfrac{\psi-\psi_0}{2}\le\tfrac{a'+b'}{2}\right\}\right)\\&\quad=mes_2\left(\left\{(\phi,\psi)\in T^2: a'\sin^2\tfrac{\phi-\phi_0}{2}+b'\sin^2\tfrac{\psi-\psi_0}{2}\ge\tfrac{a'+b'}{2}\right\}\right).\end{aligned}$$

Therefore, for $c>0$

$$\begin{aligned}&\mathrm{mes}_2\left(\left\{(\phi,\psi)\in T^2: a'\sin^2\tfrac{\phi-\phi_0}{2}+b'\sin^2\tfrac{\psi-\psi_0}{2}\le\tfrac{a'+b'-c}{2}\right\}\right)\\&\quad<\mathrm{mes}_2\left(\left\{(\phi,\psi)\in T^2: a'\sin^2\tfrac{\phi-\phi_0}{2}+b'\sin^2\tfrac{\psi-\psi_0}{2}\ge\tfrac{a'+b'-c}{2}\right\}\right).\end{aligned}$$

Since $\mathrm{mes}_2(C^+(x))\le\mathrm{mes}_2(C^-(x))$,

$$C^+(x)=\left\{(\phi,\psi)\in T^2: a'\sin^2\tfrac{\phi-\phi_0}{2}+b'\sin^2\tfrac{\psi-\psi_0}{2}\le\tfrac{a'+b'-c}{2}\right\}.$$

Let

$$a=\sqrt{\tfrac{2a'}{a'+b'-c}}\quad\text{and}\quad b=\sqrt{\tfrac{2b'}{a'+b'-c}}.$$

Then

$$C^+(x)=\left\{(\phi,\psi)\in T^2: a^2\sin^2\tfrac{\phi-\phi_0}{2}+b^2\sin^2\tfrac{\psi-\psi_0}{2}\le 1\right\}$$

and, respectively,

$$C^-(x)=\left\{(\phi,\psi)\in T^2: a^2\sin^2\tfrac{\phi-\phi_0}{2}+b^2\sin^2\tfrac{\psi-\psi_0}{2}\ge 1\right\},$$

hence statement 1 of Lemma 4.1.

All the computations are obviously invertible, and performing them in the inverse order gives statement 2 of Lemma 4.1.

Proof of Lemma 4.2 Without loss of generality assume $\phi_0 = \psi_0 = 0$.

Consider the case $a = 0$. Then

$$C^+(x) = \left\{(\phi, \psi) \in T^2 : b^2 \sin^2 \tfrac{\phi}{2} \le 1\right\},$$

or, equivalently,

$$C^+(x) = \left\{(\phi, \psi) \in T^2 : |\psi| \le 2 \arcsin \tfrac{1}{b}\right\}.$$

Consequently,

$$(a+1)(b+1)G(x) = 8\pi(b+1) \arcsin \frac{1}{b}.$$

Since $\frac{1}{b} < \arcsin \frac{1}{b} < \frac{\pi}{2} \cdot \frac{1}{b}$ and $b \ge \sqrt{2}$,

$$8\pi < (a+1)(b+1)G(x) < 4\pi^2\Bigl(1 + \frac{\sqrt{2}}{2}\Bigr),$$

and the case $a = 0$ is completely proved. The case $b = 0$ is similar.

Now suppose $a > 0$ and $b > 0$. Since

$$\frac{|\phi|}{\pi} \le \Bigl|\sin \tfrac{\phi}{2}\Bigr| \le \frac{|\phi|}{2} \quad \text{and} \quad \tfrac{|\psi|}{\pi} \le \Bigl|\sin \tfrac{\psi}{2}\Bigr| \le \tfrac{|\psi|}{2},$$

then the following inclusions hold:

$$C^+(x) \subset \left\{(\phi, \psi) \in T^2 : a^2\bigl(\tfrac{\phi}{2}\bigr)^2 \le \tfrac{1}{2} \text{ and } b^2\Bigl(\frac{\psi}{2}\Bigr)^2 \le \tfrac{1}{2}\right\},$$
$$C^+(x) \supset \left\{(\phi, \psi) \in T^2 : a^2\bigl(\tfrac{\phi}{\pi}\bigr)^2 \le 1 \text{ and } b^2\bigl(\tfrac{\psi}{\pi}\bigr)^2 \le 1\right\}.$$

Therefore

$$\min\Bigl(2\pi, \tfrac{2\sqrt{2}}{a}\Bigr) \cdot \min\Bigl(2\pi, \tfrac{2\sqrt{2}}{b}\Bigr) \le G(x) \le \min\Bigl(2\pi, \tfrac{2\pi}{a}\Bigr) \cdot \min\Bigl(2\pi, \frac{2\pi}{b}\Bigr).$$

It is easy to check that

$$\min\Bigl(2\pi, \tfrac{2\sqrt{2}}{a}\Bigr) = \frac{1}{\max\bigl(\frac{1}{2\pi}, \frac{a}{2\sqrt{2}}\bigr)} \ge \frac{1}{\frac{1}{2\pi} + \frac{a}{2\sqrt{2}}} = \frac{2\pi\sqrt{2}}{\pi a + \sqrt{2}},$$
$$\min\Bigl(2\pi, \tfrac{2\pi}{a}\Bigr) \le \frac{2}{\frac{1}{2\pi} + \frac{a}{2\pi}} = \frac{4\pi}{a+1},$$

and, similarly,

$$\min\left(2\pi, \tfrac{2\sqrt{2}}{b}\right) \geq \tfrac{2\pi\sqrt{2}}{\pi a+\sqrt{2}},$$
$$\min\left(2\pi, \tfrac{\pi\sqrt{2}}{b}\right) \leq \tfrac{4\pi}{b+1}.$$

Finally,

$$8 \leq \frac{2\pi\sqrt{2}(a+1)}{\pi a+\sqrt{2}} \cdot \frac{2\pi\sqrt{2}(b+1)}{\pi b+\sqrt{2}} \leq (a+1)(b+1)G(x) \leq 16\pi^2,$$

which completes the proof of Lemma 4.2. □

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