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Estimating Support Functions of Random Polytopes via Orlicz Norms

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Abstract We study the expected value of support functions of random polytopes in a certain direction, where the random polytope is given by independent random vectors uniformly distributed in an isotropic convex body. All results are obtained using probabilistic estimates in terms of Orlicz norms that were not used in this connection before.

Keywords Random polytope · Support function · Orlicz norm · Mean width

1 Introduction and Notation

The study of random polytopes began with Sylvester and the famous four-point problem nearly 150 years ago. This problem asks for the probability that the convex hull of four randomly chosen points in a planar region forms a four-sided polygon and its solution depends on the probability distribution of the random points. It was the starting point for an extensive study. In their groundbreaking work [30] from 1963, Rényi and Sulanke continued it, studying expectations of various basic functionals of random polytopes. Important quantities are expectations, variances and distributions of those functionals, and their study combines convex geometry, as well as geometric analysis and geometric probability (see also [2, 29]).

In the last 30 years a tremendous effort was made to explore properties of random polytopes as they gained more and more importance due to many important applications and connections to various other fields. Those can be found not only in statistics

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(extreme points of random samples) and convex geometry (approximation of convex sets), but also in computer science in the analysis of the average complexity of algorithms [22] and optimization [5], and even in biology [33]. In 1989, Milman and Pajor revealed a deep connection to functional analysis, proving that the expected volume of a certain random simplex is closely related to the isotropic constant of a convex set. In fact, this is a fundamental quantity in convex geometry and the local theory of Banach spaces [17].

Since Gluskin's result [8], random polytopes are known to provide many examples of convex bodies (and related normed spaces) with a "pathologically bad" behavior of various parameters of a linear and geometric nature (see for instance the survey [16] and references therein). Consequently, they were also a natural candidate for a potential counterexample for the hyperplane conjecture. The isotropic constant of certain classes of random polytopes has been studied in [1, 7] and [12], showing that they do not provide a counterexample for the hyperplane conjecture.

Some other recent developments in the study of random polytopes can be found in [7] or [21], where the authors studied the relation between some parameters of a random polytope in an isotropic convex body and the isotropic constant of the body. Their results provide sharp estimates whenever $n^{1+\delta} \leq N \leq e^{\sqrt{n}}$ for some $\delta > 0$. However, their method does not cover the case where $N \sim n$, and it seems that a new approach is needed. Therefore, our paper serves this purpose, providing a new tool in the study of random polytopes where results are obtained for the range $n \leq N \leq e^{\sqrt{n}}$. More precisely, we will estimate the expected value of support functions of random polytopes for a fixed direction, using a representation of this parameter via Orlicz norms.

Even though the motivation is of a geometrical nature, the tools we use are mainly probabilistic and analytical, involving Orlicz norms and therefore spaces which naturally appear in Banach space theory. It is interesting that those spaces, as we will see, also naturally appear in the study of certain parameters of random polytopes. Hence, this interplay between convex geometry and classical Orlicz spaces is attractive both from the analytical and from the geometrical point of view.

Before stating the exact results, and to allow a better understanding, we start with some basic definitions before we go into detail. A convex body $K \subset \mathbb{R}^n$ is a compact convex set with non-empty interior. It is called symmetric if $-x \in K$ whenever $x \in K$. We will denote its volume (or Lebesgue measure) by $|\cdot|$. A convex body is said to be in isotropic position if it has volume 1 and satisfies the following two conditions:

- $\int_K x\,dx = 0$ (center of mass at 0),
- $\int_K \langle x, \theta\rangle^2\,dx = L_K^2\ \forall \theta \in S^{n-1}$,

where L_K is a constant independent of θ, which is called the isotropic constant of K. Here, $\langle\cdot,\cdot\rangle$ denotes the standard scalar product in $\mathbb{R}^n$.

We will use the notation $a \sim b$ to express that there exist two positive absolute constants c_1, c_2 such that $c_1 a \leq b \leq c_2 a$ and use $a \sim_\delta b$ in case the constants depend on some constant $\delta > 0$. Similarly, we write $a \lesssim b$ if there exists a positive absolute constant c such that $a \leq cb$. The letters $c, c', C, C', c_1, c_2, \ldots$ will denote positive absolute constants whose values may change from line to line. We will write $C(r)$ if the constant depends on some parameter $r>0$.

Let K be a convex body, and $\theta \in S^{n-1}$ a unit vector. The support function of K in the direction θ is defined by $h_K(\theta)=\max\{\langle x,\theta\rangle : x\in K\}$. The mean width of K is

$$w(K)=\int_{S^{n-1}} h_K(\theta)\,d\mu(\theta),$$

where $d\mu$ denotes the uniform probability measure on S^{n-1}.

Given an isotropic convex body K, let us consider the random polytope $K_N=\operatorname{conv}\{\pm X_1,\dots,\pm X_N\}$, where $X_1,\dots,X_N$ are independent random vectors uniformly distributed in K. It is known (see for instance [7] or [20]) that the expected value of the mean width of K_N is bounded from above by

$$\mathbb{E}w(K_N)\le C L_K\sqrt{\log N},$$

where C is a positive absolute constant. In [7] the authors showed that if $N\le e^{\sqrt{n}}$,

$$\mathbb{E}\left(\frac{|K_N|}{|B_2^n|}\right)^{\frac{1}{n}}\ge C L_K\sqrt{\log\frac{N}{n}}.$$

As a consequence, they obtained

$$\mathbb{E}w(K_N)\sim_\delta L_K\sqrt{\log N}$$

if the number of random points defining K_N verifies $n^{1+\delta}\le N\le e^{\sqrt{n}}$, $\delta>0$ a constant.

Now, let us be more precise and outline what we will prove and study in the following. First of all, by Fubini's theorem, the expected value of the mean width of K_N is the average on S^{n-1} of the expected value of the support function of K_N in the direction θ:

$$\mathbb{E}w(K_N)=\mathbb{E}\int_{S^{n-1}} h_{K_N}(\theta)\,d\mu=\int_{S^{n-1}}\mathbb{E}h_{K_N}(\theta)\,d\mu. \tag{1}$$

Initially, in this paper we are interested in estimating $\mathbb{E}h_{K_N}(\theta)=\mathbb{E}\max_{1\le i\le N}|\langle X_i,\theta\rangle|$ for a fixed direction $\theta\in S^{n-1}$, but we will also derive "high probability" (in the set of directions) results. In order to do so, we establish a completely new approach applying probabilistic estimates in connection with Orlicz norms. Those were first studied by Kwapień and Schütt in the discrete case in [14] and [15] and later extended by Gordon, Litvak, Schütt and Werner in [9] and [10] (for recent developments, see also [24, 25] and [26]). Using this method to estimate support functions of random polytopes is interesting in itself and introduces a new tool in convex geometry.

As we will see, the expected value of the mean width of a random polytope in (1) is equivalent to an average of Orlicz norms, i.e.,

$$\mathbb{E}w(K_N)\sim\int_{S^{n-1}}\big\|(1,\dots,1)\big\|_{M_\theta}\,d\mu(\theta).$$

This, in fact, is not just a nice representation, but a very interesting observation, which bears information concerning the expected value of the mean width, worth to be studied in more detail. Notice that averages of Orlicz norms naturally appear in functional analysis when studying symmetric subspaces of the classical Banach space L_1 (see [3, 14, 23], just to mention a few). To be more precise, as shown in [14], every finite-dimensional symmetric subspace of L_1 is C-isomorphic to an average of Orlicz spaces (see [28] for the corresponding result for rearrangement invariant spaces).

In Sect. 2 we will introduce the aforementioned Orlicz norm method that we will use throughout this paper to prove estimates for support functions of random polytopes.

In Sect. 3, with this approach, denoting by e_j the canonical basis vectors in $\mathbb{R}^n$, we first compute $\mathbb{E}h_{K_N}(e_j)$ when the isotropic convex body in which K_N lies is the normalized ℓ_p^n ball, i.e., in $D_p^n = \frac{B_p^n}{|B_p^n|^{\frac{1}{n}}}$. Namely, using these ideas, we prove the following:

Theorem 1 *Let $X_1, \dots, X_N$ be independent random vectors uniformly distributed in D_p^n, $1 \le p \le \infty$, with $n \le N \le e^{c'n}$, and $K_N = \operatorname{conv}\{\pm X_1, \dots, \pm X_N\}$. Then, for all $j = 1, \dots, n$,*

$$\mathbb{E}h_{K_N}(e_j) = \mathbb{E} \max_{1\le i\le N} \left|\langle X_i, e_j\rangle\right| \sim (\log N)^{\frac{1}{p}}.$$

Many properties of random variables distributed in ℓ_p^n balls have already been studied, see for instance [4, 31] and [32].

By rotational invariance in the Euclidean case, we obtain the same estimate for the expected value of the mean width of a random polytope in D_2^n, under milder conditions on the number of points N:

Corollary 2 *Let $X_1, \dots, X_N$ be independent random vectors uniformly distributed in D_2^n, with $n \le N \le e^n$, and let $K_N = \operatorname{conv}\{\pm X_1, \dots, \pm X_N\}$. Then*

$$\mathbb{E}w(K_N) \sim \sqrt{\log N}.$$

In Sect. 4 we will use our approach to give a general upper bound for $\mathbb{E}h_{K_N}(\theta)$ when K is symmetric and under some smoothness conditions on the function $h(t) = |K \cap \{\langle x, \theta\rangle = t\}|^{\frac{1}{n-1}}$. This general case will include the case when $K = D_p^n$ with $2 \le p < \infty$ and $\theta = e_j$.

As proved in [21], the expected value of the intrinsic volumes (in particular the mean width) of K_N are minimized when $K = D_2^n$. Thus, we have $\mathbb{E}w(K_N) \gtrsim \sqrt{\log N}$ and $\mathbb{E}w(K_N) \sim L_K\sqrt{\log N}$ for those bodies with the isotropic constant bounded. We prove the existence of directions such that the expected value of the support function in these directions is bounded from above by a constant times $L_K\sqrt{\log N}$ and, respectively, bounded from below by a constant times $L_K\sqrt{\log N}$. In fact, as a consequence, we estimate the measure of the set of directions verifying such estimates. It is stated in the following corollary. Notice that the constant L_K appears explicitly also in the lower bound.

Corollary 3 *Let $n \leq N \leq e^{\sqrt{n}}$, K be an isotropic convex body in $\mathbb{R}^n$, and let $X_1, \dots, X_N$ be independent random variables uniformly distributed on K. Let $K_N = \operatorname{conv}\{\pm X_1, \dots, \pm X_N\}$. For every $r > 0$, there exist positive constants $C(r), C_1(r), C_2(r)$ such that*

$$\mathbb{E}h_{K_N}(\theta) \leq C_1(r) L_K \sqrt{\log N},$$
$$\mathbb{E}h_{K_N}(\theta) \geq C_2(r) L_K \sqrt{\log N}$$

for a set of directions with measure greater than $1 - \frac{1}{N^r}$ and $\frac{C(r)\sqrt{\log N}}{N^r}$ respectively.

All the estimates we prove using our approach hold when $n \leq N \leq e^{\sqrt{n}}$. Thus, our method might provide a tool to prove $\mathbb{E}w(K_N) \sim L_K \sqrt{\log N}$ for this range of N and hence close the gap mentioned in [7], where the authors' result was restricted to the case $n^{1+\delta} \leq N \leq e^{\sqrt{n}}$, $\delta > 0$, and constants depending on δ.

2 Preliminaries

A convex function $M : [0, \infty) \to [0, \infty)$ where $M(0) = 0$ and $M(t) > 0$ for $t > 0$ is called an Orlicz function. If there is a $t_0 > 0$ such that for all $t \leq t_0$, we have $M(t) = 0$, then M is called a degenerated Orlicz function. The dual function M^* of an Orlicz function M is given by the Legendre transform

$$M^*(x) = \sup_{t \in [0,\infty)} \bigl(xt - M(t)\bigr).$$

Again, M^* is an Orlicz function, and $M^{**} = M$. For instance, taking $M(t) = \frac{1}{p} t^p$, $p \geq 1$, the dual function is given by $M^*(t) = \frac{1}{p^*} t^{p^*}$ with $\frac{1}{p^*} + \frac{1}{p} = 1$. The n-dimensional Orlicz space ℓ_M^n is $\mathbb{R}^n$ equipped with the norm

$$\|x\|_M = \inf\left\{\rho > 0 : \sum_{i=1}^n M\left(\frac{|x_i|}{\rho}\right) \leq 1\right\}.$$

In case $M(t) = t^p$, $1 \leq p < \infty$, we just have $\|\cdot\|_M = \|\cdot\|_p$. For a detailed and thorough introduction to the theory of Orlicz spaces, we refer the reader to [13] and [27].

In [10] the authors obtained the following result:

Theorem 4 ([10, Lemma 5.2]) *Let $X_1, \dots, X_N$ be iid random variables with finite first moments. For all $s \geq 0$, let*

$$M(s) = \int_0^s \int_{\{\frac{1}{t} \leq |X_1|\}} |X_1| \, d\mathbb{P} \, dt.$$

Then, for all $x = (x_i)_{i=1}^N \in \mathbb{R}^N$,

$$\mathbb{E} \max_{1 \leq i \leq N} |x_i X_i| \sim \|x\|_M.$$

Obviously, the function

$$M(s) = \int_0^s \int_{\{\frac{1}{t} \le |X_1|\}} |X_1| \, d\mathbb{P} \, dt \tag{2}$$

is non-negative and convex, since $\int_{\{\frac{1}{t} \le |X|\}} |X| \, d\mathbb{P}$ is increasing in t. Furthermore, we have $M(0) = 0$ and M is continuous. One can easily show that this Orlicz function M can also be written in the following way:

$$M(s) = \int_0^s \left(\frac{1}{t} \mathbb{P}\left(|X| \ge \frac{1}{t} \right) + \int_{\frac{1}{t}}^{\infty} \mathbb{P}\big(|X| \ge u\big) \, du \right) dt.$$

As a corollary, we obtain the following result, which is the one we use to estimate the support functions of random polytopes.

Corollary 5 *Let $X_1, \dots, X_N$ be iid random vectors in $\mathbb{R}^n$, and let $K_N = \operatorname{conv}\{\pm X_1, \dots, \pm X_N\}$. Let $\theta \in S^{n-1}$ and*

$$M_\theta(s) = \int_0^s \int_{\{\frac{1}{t} \le |\langle X_1, \theta \rangle|\}} \big|\langle X_1, \theta \rangle\big| \, d\mathbb{P} \, dt.$$

Then

$$\mathbb{E} h_{K_N}(\theta) \sim \inf\left\{ s > 0 : M_\theta\left(\frac{1}{s}\right) \le \frac{1}{N} \right\}.$$

3 Random Polytopes in Normalized ℓ_p^n-Balls

In this section we consider random polytopes $K_N = \operatorname{conv}\{\pm X_1, \dots, \pm X_N\}$, where $X_1, \dots, X_N$ are independent random vectors uniformly distributed in the normalized ℓ_p^n ball $D_p^n = \frac{B_p^n}{|B_p^n|^{\frac{1}{n}}}$. Let us recall that the volume of B_p^n equals

$$\big|B_p^n\big| = \frac{(\Gamma(1 + \frac{1}{p}))^n}{\Gamma(1 + \frac{n}{p})},$$

and so, using Stirling's formula, we have that $|B_p^n|^{1/n} \sim \frac{1}{n^{\frac{1}{p}}}$ and $\frac{|B_p^{n-1}|}{|B_p^n|} \sim n^{\frac{1}{p}}$.

We are going to estimate $\mathbb{E} h_{K_N}(e_j)$ using the Orlicz norm approach introduced in Sect. 2. In order to do so, we need to compute the Orlicz function M introduced in Corollary 5. We are doing this in the following.

Lemma 6 *Let $1 \le p < \infty$, and $M : [0, \infty) \to [0, \infty)$ be the function*

$$M(s) := M_{e_j}(s) = \int_0^s \int_{\{x \in D_p^n : |\langle x, e_j \rangle| \ge \frac{1}{t}\}} \big|\langle x, e_j \rangle\big| \, dx \, dt.$$

Then, if $s \leq \frac{1}{|B_p^n|^{\frac{1}{n}}}$,

$$M\left(\frac{1}{s}\right) = \frac{4}{p(n-1+p)} \frac{|B_p^{n-1}|}{|B_p^n|} \int_0^{\cos^{-1}(s|B_p^n|^{\frac{1}{n}})^{\frac{p}{2}}} \frac{(\sin\theta)^{2\frac{n-1}{p}+3}}{(\cos\theta)^{3-\frac{2}{p}}} \, d\theta$$
$$+ \frac{4(2-p)}{p(n-1+p)} \frac{|B_p^{n-1}|}{|B_p^n||B_p^n|^{\frac{p-2}{n}}}$$
$$\times \int_0^{\cos^{-1}(s|B_p^n|^{\frac{1}{n}})^{\frac{p}{2}}} \frac{\sin\theta}{(\cos\theta)^{1+\frac{2}{p}}}$$
$$\times \int_{(\cos\theta)^{\frac{2}{p}}|B_p^n|^{-\frac{1}{n}}}^{|B_p^n|^{-\frac{1}{n}}} r^{1-p}\left(1-|B_p^n|^{\frac{p}{n}} r^p\right)^{\frac{n-1}{p}+1} \, dr \, d\theta. \tag{3}$$

Also, if $s \leq \frac{1}{|B_p^n|^{\frac{1}{n}}}$,

$$M\left(\frac{1}{s}\right) = \frac{2}{(n-1+p)(n-1+2p)} \frac{|B_p^{n-1}|}{|B_p^n|} \frac{(1-s^p|B_p^n|^{\frac{p}{n}})^{\frac{n-1}{p}+2}}{(s^p|B_p^n|^{\frac{p}{n}})^{2-\frac{1}{p}}}$$
$$- \frac{12(p-1)}{p(n-1+p)(n-1+2p)} \frac{|B_p^{n-1}|}{|B_p^n|}$$
$$\times \int_0^{\cos^{-1}(s|B_p^n|^{\frac{1}{n}})^{\frac{p}{2}}} \frac{(\sin\theta)^{2\frac{n-1}{p}+5}}{(\cos\theta)^{5-\frac{2}{p}}} \, d\theta$$
$$- \frac{8(2-p)(p-1)}{p(n-1+2p)(n-1+p)} \frac{|B_p^{n-1}|}{|B_p^n||B_p^n|^{\frac{2p-2}{n}}}$$
$$\times \int_0^{\cos^{-1}(s|B_p^n|^{\frac{1}{n}})^{\frac{p}{2}}} \frac{\sin\theta}{(\cos\theta)^{1+\frac{2}{p}}}$$
$$\times \int_{(\cos\theta)^{\frac{2}{p}}|B_p^n|^{-\frac{1}{n}}}^{|B_p^n|^{-\frac{1}{n}}} r^{1-2p}\left(1-|B_p^n|^{\frac{p}{n}} r^p\right)^{\frac{n-1}{p}+2} \, dr \, d\theta. \tag{4}$$

Proof The $(n-1)$-dimensional volume $|D_p^n \cap \{\langle x, e_j\rangle = t\}|$ equals

$$\frac{|B_p^{n-1}||B_p^n|^{1/n}}{|B_p^n|}\left(1-|B_p^n|^{p/n}t^p\right)^{\frac{n-1}{p}} \mathbb{1}_{[-|B_p^n|^{-1/n},|B_p^n|^{-1/n}]}(t).$$

By Fubini's theorem we have that if $s \geq |B_p^n|^{1/n}$,

$$M(s) = 2\frac{|B_p^{n-1}||B_p^n|^{\frac{1}{n}}}{|B_p^n|}\int_{|B_p^n|^{1/n}}^{s}\int_{\frac{1}{t}}^{|B_p^n|^{-1/n}} r\big(1-|B_p^n|^{p/n}r^p\big)^{\frac{n-1}{p}}\,dr\,dt$$

$$= 2\frac{|B_p^{n-1}||B_p^n|^{\frac{1}{n}}}{|B_p^n|}\int_{|B_p^n|^{1/n}}^{s}\int_{\frac{1}{t}}^{|B_p^n|^{-1/n}} \frac{r^{p-1}}{r^{p-2}}\big(1-|B_p^n|^{p/n}r^p\big)^{\frac{n-1}{p}}\,dr\,dt.$$

Otherwise M is 0. Integration by parts yields

$$M(s) = 2\frac{|B_p^{n-1}||B_p^n|^{\frac{1}{n}}}{|B_p^n||B_p^n|^{\frac{p}{n}}}\int_{|B_p^n|^{\frac{1}{n}}}^{s}\Bigg[\frac{(\frac{1}{t})^{2-p}(1-\frac{|B_p^n|^{\frac{p}{n}}}{t^p})^{\frac{n-1}{p}+1}}{n-1+p}$$

$$+\int_{\frac{1}{t}}^{|B_p^n|^{-\frac{1}{n}}}(2-p)r^{1-p}\frac{(1-|B_p^n|^{\frac{p}{n}}r^p)^{\frac{n-1}{p}+1}}{n-1+p}\,dr\Bigg]dt.$$

Now, making the change of variables

$$\frac{|B_p^n|^{\frac{1}{n}}}{t} = (\cos\theta)^{\frac{2}{p}} \quad\Longrightarrow\quad \frac{dt}{d\theta} = |B_p^n|^{\frac{1}{n}}\frac{2}{p}\frac{\sin\theta}{(\cos\theta)^{1+\frac{2}{p}}},$$

we obtain

$$M(s) = \frac{4}{p(n-1+p)}\frac{|B_p^{n-1}|}{|B_p^n|}\int_0^{\cos^{-1}(s^{-1}|B_p^n|^{\frac{1}{n}})^{\frac{p}{2}}}\frac{(\sin\theta)^{2\frac{n-1}{p}+3}}{(\cos\theta)^{3-\frac{2}{p}}}\,d\theta$$

$$+\frac{4(2-p)}{p(n-1+p)}\frac{|B_p^{n-1}|}{|B_p^n||B_p^n|^{\frac{p-2}{n}}}$$

$$\times\int_0^{\cos^{-1}(s^{-1}|B_p^n|^{\frac{1}{n}})^{\frac{p}{2}}}\frac{\sin\theta}{(\cos\theta)^{1+\frac{2}{p}}}$$

$$\times\int_{(\cos\theta)^{\frac{2}{p}}|B_p^n|^{-\frac{1}{n}}}^{|B_p^n|^{-\frac{1}{n}}} r^{1-p}\big(1-|B_p^n|^{\frac{p}{n}}r^p\big)^{\frac{n-1}{p}+1}\,dr\,d\theta.$$

Therefore,

$$M\left(\frac{1}{s}\right) = \frac{4}{p(n-1+p)}\frac{|B_p^{n-1}|}{|B_p^n|}\int_0^{\cos^{-1}(s|B_p^n|^{\frac{1}{n}})^{\frac{p}{2}}}\frac{(\sin\theta)^{2\frac{n-1}{p}+3}}{(\cos\theta)^{3-\frac{2}{p}}}\,d\theta$$

$$+\frac{4(2-p)}{p(n-1+p)}\frac{|B_p^{n-1}|}{|B_p^n||B_p^n|^{\frac{p-2}{n}}}$$

$$\times \int_0^{\cos^{-1}(s|B_p^n|^{\frac{1}{n}})^{\frac{p}{2}}} \frac{\sin\theta}{(\cos\theta)^{1+\frac{2}{p}}}$$

$$\times \int_{(\cos\theta)^{\frac{2}{p}}|B_p^n|^{-\frac{1}{n}}}^{|B_p^n|^{-\frac{1}{n}}} r^{1-p}\big(1-|B_p^n|^{\frac{p}{n}}r^p\big)^{\frac{n-1}{p}+1}\,dr\,d\theta$$

if $s \leq \frac{1}{|B_p^n|^{\frac{1}{n}}}$ and 0 otherwise, which is the expression in (3). The first term in the previous sum equals

$$\frac{4}{p(n-1+p)}\frac{|B_p^{n-1}|}{|B_p^n|}\int_0^{\cos^{-1}(s|B_p^n|^{\frac{1}{n}})^{\frac{p}{2}}} \frac{(\sin\theta)^{2\frac{n-1}{p}+3}(\cos\theta)}{(\cos\theta)^{4-\frac{2}{p}}}\,d\theta,$$

and integration by parts yields that this equals

$$\frac{2}{p(n-1+p)(\frac{n-1}{p}+2)}\frac{|B_p^{n-1}|}{|B_p^n|}$$

$$\times\left(\frac{(1-s^p|B_p^n|^{\frac{p}{n}})^{\frac{n-1}{p}+2}}{(s^p|B_p^n|^{\frac{p}{n}})^{2-\frac{1}{p}}}-\left(4-\frac{2}{p}\right)\int_0^{\cos^{-1}(s|B_p^n|^{\frac{1}{n}})^{\frac{p}{2}}} \frac{(\sin\theta)^{2\frac{n-1}{p}+5}}{(\cos\theta)^{5-\frac{2}{p}}}\,d\theta\right).$$

The integral inside the second term equals

$$\int_{(\cos\theta)^{\frac{2}{p}}|B_p^n|^{-\frac{1}{n}}}^{|B_p^n|^{-\frac{1}{n}}} r^{1-p}\big(1-|B_p^n|^{\frac{p}{n}}r^p\big)^{\frac{n-1}{p}+1}\,dr$$

$$=\int_{(\cos\theta)^{\frac{2}{p}}|B_p^n|^{-\frac{1}{n}}}^{|B_p^n|^{-\frac{1}{n}}} r^{2-2p}r^{p-1}\big(1-|B_p^n|^{\frac{p}{n}}r^p\big)^{\frac{n-1}{p}+1}\,dr,$$

and, integrating by parts, this equals

$$\frac{1}{p(\frac{n-1}{p}+2)|B_p^n|^{\frac{p}{n}}}\left(\frac{(\sin\theta)^{2\frac{n-1}{p}+4}|B_p^n|^{\frac{2p-2}{n}}}{(\cos\theta)^{4-\frac{4}{p}}}\right.$$

$$\left.-2(p-1)\int_{(\cos\theta)^{\frac{2}{p}}|B_p^n|^{-\frac{1}{n}}}^{|B_p^n|^{-\frac{1}{n}}} r^{1-2p}\big(1-|B_p^n|^{\frac{p}{n}}r^p\big)^{\frac{n-1}{p}+2}\,dr\right),$$

and so, the second term above equals

$$\frac{4(2-p)}{p(n-1+2p)(n-1+p)}\frac{|B_p^{n-1}|}{|B_p^n|}\int_0^{\cos^{-1}(s|B_p^n|^{\frac{1}{n}})^{\frac{p}{2}}} \frac{(\sin\theta)^{2\frac{n-1}{p}+5}}{(\cos\theta)^{5-\frac{2}{p}}}\,d\theta$$

$$-\frac{8(2-p)(p-1)}{p(n-1+2p)(n-1+p)}\frac{|B_p^{n-1}|}{|B_p^n||B_p^n|^{\frac{2p-2}{n}}}$$

$$\times \int_0^{\cos^{-1}(s|B_p^n|^{\frac{1}{n}})^{\frac{p}{2}}} \frac{\sin\theta}{(\cos\theta)^{1+\frac{2}{p}}}$$

$$\times \int_{(\cos\theta)^{\frac{2}{p}}|B_p^n|^{-\frac{1}{n}}}^{|B_p^n|^{-\frac{1}{n}}} r^{1-2p}\big(1-|B_p^n|^{\frac{p}{n}}r^p\big)^{\frac{n-1}{p}+2}\,dr\,d\theta.$$

Thus, adding the two terms we have that if $s \le \frac{1}{|B_p^n|^{\frac{1}{n}}}$,

$$\begin{aligned}
M\left(\frac{1}{s}\right) &= \frac{2}{(n-1+p)(n-1+2p)}\frac{|B_p^{n-1}|}{|B_p^n|}\frac{(1-s^p|B_p^n|^{\frac{p}{n}})^{\frac{n-1}{p}+2}}{(s^p|B_p^n|^{\frac{p}{n}})^{2-\frac{1}{p}}}\\
&\quad - \frac{12(p-1)}{p(n-1+p)(n-1+2p)}\frac{|B_p^{n-1}|}{|B_p^n|}\int_0^{\cos^{-1}(s|B_p^n|^{\frac{1}{n}})^{\frac{p}{2}}} \frac{(\sin\theta)^{2\frac{n-1}{p}+5}}{(\cos\theta)^{5-\frac{2}{p}}}\,d\theta\\
&\quad - \frac{8(2-p)(p-1)}{p(n-1+2p)(n-1+p)}\frac{|B_p^{n-1}|}{|B_p^n||B_p^n|^{\frac{2p-2}{n}}}\\
&\quad \times \int_0^{\cos^{-1}(s|B_p^n|^{\frac{1}{n}})^{\frac{p}{2}}} \frac{\sin\theta}{(\cos\theta)^{1+\frac{2}{p}}}\\
&\quad \times \int_{(\cos\theta)^{\frac{2}{p}}|B_p^n|^{-\frac{1}{n}}}^{|B_p^n|^{-\frac{1}{n}}} r^{1-2p}\big(1-|B_p^n|^{\frac{p}{n}}r^p\big)^{\frac{n-1}{p}+2}\,dr\,d\theta,
\end{aligned}$$

which is the expression in (4). □

Now we are going to prove Theorem 1. It will be a consequence of the next two propositions, where we will prove the upper and lower bound for $\mathbb{E}h_{K_N}(e_j)$ respectively.

Proposition 7 *For every $n, N \in \mathbb{N}$, with $n \le N$, and every $1 \le p < \infty$, we have that if $X_1, \dots, X_N$ are independent random vectors uniformly distributed in D_p^n, then*

$$\mathbb{E}\max_{1\le i\le N}\big|\langle X_i, e_j\rangle\big| \lesssim (\log N)^{\frac{1}{p}}$$

for all $j = 1, \dots, n$.

Remark 1 Notice that for $p = 2$, this result is similar to the analogous one for Gaussian random vectors.

Proof If $p \ge 2$ and $s \le \frac{1}{|B_p^n|^{\frac{1}{n}}}$, the second term in the expression of $M(\frac{1}{s})$ given by (3) is negative, and so

$$M\left(\frac{1}{s}\right) \le \frac{4}{p(n-1+p)}\frac{|B_p^{n-1}|}{|B_p^n|}\int_0^{\cos^{-1}(s|B_p^n|^{\frac{1}{n}})^{\frac{p}{2}}} \frac{(\sin\theta)^{\frac{2(n-1)}{p}+3}}{(\cos\theta)^{3-\frac{2}{p}}}\,d\theta.$$

Integration by parts gives

$$M\left(\frac{1}{s}\right) \le \frac{4|B_p^{n-1}|}{p(n-1+p)|B_p^n|}\left[\frac{(\sin\theta)^{\frac{2(n-1)}{p}+2}}{(2-\frac{2}{p})(\cos\theta)^{2-\frac{2}{p}}}\Bigg|_0^{\cos^{-1}(s|B_p^n|^{\frac{1}{n}})^{\frac{p}{2}}}\right.$$

$$\left. - \frac{\frac{2(n-1)}{p}+2}{2-\frac{2}{p}}\int_0^{\cos^{-1}(s|B_p^n|^{\frac{1}{n}})^{\frac{p}{2}}} \frac{(\sin\theta)^{\frac{2(n-1)+1}{p}+1}}{(\cos\theta)^{1-\frac{2}{p}}}\,d\theta\right]$$

$$\le \frac{2|B_p^{n-1}|}{(p-1)(n-1+p)|B_p^n|}\frac{(1-s^p|B_p^n|^{\frac{p}{n}})^{\frac{n-1}{p}+1}}{(s|B_p^n|^{\frac{1}{n}})^{p-1}}$$

$$= \frac{2|B_p^{n-1}|}{(p-1)(n-1+p)|B_p^n|}\frac{1}{(s|B_p^n|^{\frac{1}{n}})^{p-1}}e^{\frac{n-1+p}{p}\log(1-s^p|B_p^n|^{\frac{p}{n}})}.$$

Take $s_0 = \frac{1}{2^{\frac{1}{p}}|B_p^n|^{\frac{1}{n}}}\min\{\alpha(\frac{p}{n-1+p})(\log N),1\}^{\frac{1}{p}}$, $\alpha>0$ to be specified later. Since $s_0^p|B_p^n|^{\frac{p}{n}} \le \frac{1}{2}$, there exists a constant c such that

$$M\left(\frac{1}{s_0}\right) \le \frac{2|B_p^{n-1}|}{(p-1)(n-1+p)|B_p^n|}\frac{1}{(s_0|B_p^n|^{\frac{1}{n}})^{p-1}}e^{-cs_0^p|B_p^n|^{\frac{p}{n}}\frac{n-1+p}{p}}.$$

Take $\alpha=\frac{2}{c}$. If the minimum in the definition of s_0 is $\frac{1}{2}$, then trivially we have

$$\mathbb{E}\max_{1\le i\le N}\left|\langle X_i,e_1\rangle\right| \le \frac{1}{|B_p^n|^{\frac{1}{n}}}.$$

If not, then

$$M\left(\frac{1}{s_0}\right) \le \frac{2|B_p^{n-1}|e^{-\log N}}{(p-1)(n-1+p)|B_p^n|(\frac{1}{c}\frac{p}{n-1+p}\log N)^{\frac{p-1}{p}}}.$$

Since $|B_p^{n-1}|/|B_p^n| \sim n^{1/p}$, we get

$$\begin{aligned} M\left(\frac{1}{s_0}\right) &\le \frac{Cn^{1/p}}{p^2(n-1+p)^{\frac{1}{p}}}\frac{1}{(\log N)^{1-\frac{1}{p}}N} \\ &= \frac{C}{p^2(1+\frac{p-1}{n})^{\frac{1}{p}}}\frac{1}{(\log N)^{1-\frac{1}{p}}N} \le \frac{1}{N} \end{aligned}$$

when $N \ge N_0$ for some sufficiently large $N_0\in\mathbb{N}$. Altogether, for $p\ge 2$, we obtain

$$\mathbb{E}h_{K_N}(e_1) = \mathbb{E}\max_{1\le i\le N}\left|\langle X_i,e_1\rangle\right| \le \frac{C}{|B_p^n|^{\frac{1}{n}}}\min\left\{\left(\frac{p}{n-1+p}\right)(\log N),1\right\}^{\frac{1}{p}},$$

where C is an absolute positive constant. This minimum is 1 if and only if $\log N \geq 1 + \frac{n-1}{p}$. In this case the upper bound we obtain is $\frac{C}{|B_p^n|^{\frac{1}{n}}} \sim Cn^{\frac{1}{p}}$. Since $n-1 \leq p\log N$, we have that the upper bound $Cn^{\frac{1}{p}} \leq C(\log N)^{\frac{1}{p}}$. If the minimum is not 1, since $|B_p^n|^{\frac{1}{n}} \sim \frac{1}{n^{\frac{1}{p}}}$, we also obtain an upper bound of the order $(\log N)^{\frac{1}{p}}$.

If $p \in [1,2]$, we use that in the representation of $M(\frac{1}{s})$ given by (4) only the first term is positive and so

$$M\left(\frac{1}{s}\right) \leq \frac{2}{(n-1+p)(n-1+2p)} \frac{|B_p^{n-1}|}{|B_p^n|} \frac{(1-s^p|B_p^n|^{\frac{p}{n}})^{\frac{n-1}{p}+2}}{(s^p|B_p^n|^{\frac{p}{n}})^{2-\frac{1}{p}}}$$

$$= \frac{2}{(n-1+p)(n-1+2p)} \frac{|B_p^{n-1}|}{|B_p^n|} \frac{e^{\frac{n-1+2p}{p}\log(1-s^p|B_p^n|^{\frac{p}{n}})}}{(s|B_p^n|^{\frac{1}{n}})^{2p-1}}.$$

Take $s_0 = \frac{1}{2^{\frac{1}{p}}|B_p^n|^{\frac{1}{n}}} \min\{\alpha(\frac{p}{n-1+2p})(\log N), 1\}^{\frac{1}{p}}$, $\alpha > 0$ to be specified later. Since $s_0^p|B_p^n|^{\frac{p}{n}} \leq \frac{1}{2}$, there exists a constant such that

$$M\left(\frac{1}{s_0}\right) \leq \frac{2}{(n-1+p)(n-1+2p)} \frac{|B_p^{n-1}|}{|B_p^n|} \frac{1}{(s_0|B_p^n|^{\frac{1}{n}})^{2p-1}} e^{-cs_0^p|B_p^n|^{\frac{p}{n}}\frac{n-1+2p}{p}}.$$

Take $\alpha = \frac{2}{c}$. If the minimum in the definition of s_0 is $\frac{1}{2}$, then trivially we have

$$\mathbb{E} \max_{1\leq i\leq N} \left|\langle X_i, e_1\rangle\right| \leq \frac{1}{|B_p^n|^{\frac{1}{n}}}.$$

If not, then

$$M\left(\frac{1}{s_0}\right) \leq \frac{2e^{-\log N}}{(n-1+p)(n-1+2p)} \frac{|B_p^{n-1}|}{|B_p^n|} \frac{1}{(\frac{1}{c}\frac{p}{n-1+2p}\log N)^{\frac{2p-1}{p}}}.$$

Since $|B_p^{n-1}|/|B_p^n| \sim n^{1/p}$ and $p \in [1,2]$, we get

$$M\left(\frac{1}{s_0}\right) \leq \frac{C}{(\log N)^{2-\frac{1}{p}}N} \leq \frac{1}{N}$$

when $N \geq N_0$ for some sufficiently large $N_0 \in \mathbb{N}$. Altogether, for $1 \leq p \leq 2$, we obtain

$$\mathbb{E}h_{K_N}(e_1) = \mathbb{E} \max_{1\leq i\leq N} \left|\langle X_i, e_1\rangle\right| \leq \frac{C}{|B_p^n|^{\frac{1}{n}}} \min\left\{\frac{\log N}{n}, 1\right\}^{\frac{1}{p}} \leq C(\log N)^{\frac{1}{p}},$$

where C is an absolute positive constant. $\square$

In order to prove the lower bound for $\mathbb{E}h_{K_N}(e_j)$, we need the two following technical results:

Lemma 8 *Let $\alpha, \beta \in \mathbb{R} \setminus \{-1\}$. Then we have*

$$\int \sin^{\alpha}(\theta)\cos^{\beta}(\theta)\,d\theta = \frac{\sin^{\alpha+1}(\theta)\cos^{\beta+1}(\theta)}{\alpha+1} + \frac{\alpha+\beta+2}{\alpha+1}\int \sin^{\alpha+2}(\theta)\cos^{\beta}(\theta)\,d\theta.$$

Proof We consider $\int \sin^{\alpha+2}(\theta)\cos^{\beta}(\theta)\,d\theta$. Integration by parts yields

$$\int \sin^{\alpha+2}(\theta)\cos^{\beta}(\theta)\,d\theta = -\frac{\sin^{\alpha+1}(\theta)\cos^{\beta+1}(\theta)}{\beta+1} + \frac{\alpha+1}{\beta+1}\int \sin^{\alpha}(\theta)\cos^{\beta+2}(\theta)\,d\theta.$$

Since $\cos^{\beta+2}(\theta) = \cos^{\beta}(\theta)(1-\sin^2(\theta))$, we obtain

$$\begin{aligned}
&\int \sin^{\alpha+2}(\theta)\cos^{\beta}(\theta)\,d\theta \\
&\quad = -\frac{\sin^{\alpha+1}(\theta)\cos^{\beta+1}(\theta)}{\beta+1} + \frac{\alpha+1}{\beta+1}\int \sin^{\alpha}(\theta)\cos^{\beta}(\theta)\,d\theta \\
&\qquad - \frac{\alpha+1}{\beta+1}\int \sin^{\alpha+2}(\theta)\cos^{\beta}(\theta)\,d\theta.
\end{aligned}$$

Thus,

$$\begin{aligned}
&\frac{\alpha+\beta+2}{\beta+1}\int \sin^{\alpha+2}(\theta)\cos^{\beta}(\theta)\,d\theta \\
&\quad = -\frac{\sin^{\alpha+1}(\theta)\cos^{\beta+1}(\theta)}{\beta+1} + \frac{\alpha+1}{\beta+1}\int \sin^{\alpha}(\theta)\cos^{\beta}(\theta)\,d\theta,
\end{aligned}$$

and so

$$\begin{aligned}
\int \sin^{\alpha}(\theta)\cos^{\beta}(\theta)\,d\theta &= \frac{\sin^{\alpha+1}(\theta)\cos^{\beta+1}(\theta)}{\alpha+1} \\
&\quad + \frac{\alpha+\beta+2}{\alpha+1}\int \sin^{\alpha+2}(\theta)\cos^{\beta}(\theta)\,d\theta.
\end{aligned}$$

□

As a corollary, we obtain the kth iteration of Lemma 8.

Corollary 9 *Let $\alpha, \beta \in \mathbb{R} \setminus \{-1\}$. Then, for any $k \in \mathbb{N}$, we have*

$$\begin{aligned}
&\int \sin^{\alpha}(\theta)\cos^{\beta}(\theta)\,d\theta \\
&\quad = \frac{\sin^{\alpha+1}(\theta)\cos^{\beta+1}(\theta)}{\alpha+1} + \frac{\alpha+\beta+2}{(\alpha+1)(\alpha+3)}\sin^{\alpha+3}(\theta)\cos^{\beta+1}(\theta) \\
&\qquad + \frac{(\alpha+\beta+2)(\alpha+\beta+4)}{(\alpha+1)(\alpha+3)(\alpha+5)}\sin^{\alpha+5}(\theta)\cos^{\beta+1}(\theta) + \cdots
\end{aligned}$$

$$+\frac{(\alpha+\beta+2)\cdots(\alpha+\beta+2k)}{(\alpha+1)\cdots(\alpha+2k+1)}\sin^{\alpha+2k+1}(\theta)\cos^{\beta+1}(\theta)$$
$$+\frac{(\alpha+\beta+2)\cdots(\alpha+\beta+2k+2)}{(\alpha+1)\cdots(\alpha+2k+1)}\int\sin^{\alpha+2k+2}(\theta)\cos^{\beta}(\theta)\,d\theta.$$

We will now prove the lower estimate.

Proposition 10 *There exists a positive absolute constant c' such that for every $n,N\in\mathbb{N}$, with $n\le N\le e^{c'n}$, and every $1\le p<\infty$, we have that if $X_1,\dots,X_N$ are independent random vectors uniformly distributed on D_p^n, then*

$$\mathbb{E}\max_{1\le i\le N}\left|\langle X_i,e_j\rangle\right|\gtrsim(\log N)^{\frac{1}{p}}$$

for all $j=1,\dots,n$.

Proof We start with the case $1<p\le 2$ where we use the recursion formula. Since $1<p\le 2$, using the representation of M in (3), we have that

$$M\left(\frac{1}{s}\right)\ge\frac{4}{p(n-1+p)}\frac{|B_p^{n-1}|}{|B_p^n|}\int_0^{\cos^{-1}(s^{-1}|B_p^n|^{\frac{1}{n}})^{\frac{p}{2}}}(\sin\theta)^{2\frac{n-1}{p}+3}(\cos\theta)^{\frac{2}{p}-3}\,d\theta.$$

Using Corollary 9 with $\alpha=\frac{2n}{p}-\frac{2}{p}+3$ and $\beta=\frac{2}{p}-3$, we have $-1\le\beta+1<0$, and for any $k\in\mathbb{N}$, we get

$$M\left(\frac{1}{s}\right)\ge\frac{4}{p(n-1+p)}\frac{|B_p^{n-1}|}{|B_p^n|}\Bigg[\frac{(\cos\theta)^{\beta+1}}{\alpha+1}\Bigg\{(\sin\theta)^{\alpha+1}+\frac{\alpha+\beta+2}{\alpha+3}(\sin\theta)^{\alpha+3}+\cdots$$
$$+\frac{(\alpha+\beta+2)\cdots(\alpha+\beta+2k)}{(\alpha+3)\cdots(\alpha+2k+1)}(\sin\theta)^{\alpha+2k+1}\Bigg\}\Bigg|_0^{\cos^{-1}((s|B_p^n|^{\frac{1}{n}})^{\frac{p}{2}})}\Bigg].$$

Since $\beta+1=\frac{2}{p}(1-p)$, we get

$$M\left(\frac{1}{s}\right)\ge\frac{4}{p(n-1+p)}\frac{|B_p^{n-1}|}{|B_p^n|(\alpha+1)(s|B_p^n|^{\frac{1}{n}})^{p-1}}$$
$$\times\left[\left(1-s^p|B_p^n|^{\frac{p}{n}}\right)^{\frac{\alpha+1}{2}}+\frac{\alpha+\beta+2}{\alpha+3}\left(1-s^p|B_p^n|^{\frac{p}{n}}\right)^{\frac{\alpha+3}{2}}+\cdots\right.$$
$$\left.+\frac{(\alpha+\beta+2)\cdots(\alpha+\beta+2k)}{(\alpha+3)\cdots(\alpha+2k+1)}\left(1-s^p|B_p^n|^{\frac{p}{n}}\right)^{\frac{\alpha+2k+1}{2}}\right]$$
$$\ge\frac{4}{p(n-1+p)}\frac{|B_p^{n-1}|(1-s^p|B_p^n|^{\frac{p}{n}})^{\frac{\alpha+2k+1}{2}}}{|B_p^n|(\alpha+1)(s|B_p^n|^{\frac{1}{n}})^{p-1}}$$
$$\times\left[1+\left(1-\frac{1-\beta}{\alpha+3}\right)+\left(1-\frac{1-\beta}{\alpha+3}\right)\left(1-\frac{1-\beta}{\alpha+5}\right)+\cdots\right.$$

$$+\left(1-\frac{1-\beta}{\alpha+3}\right)\left(1-\frac{1-\beta}{\alpha+5}\right)\cdots\left(1-\frac{1-\beta}{\alpha+2k+1}\right)\Bigg]$$

$$\geq \frac{4(k+1)}{p(n-1+p)}\frac{|B_p^{n-1}|(1-s^p|B_p^n|^{\frac{p}{n}})^{\frac{\alpha+2k+1}{2}}}{|B_p^n|(\alpha+1)(s|B_p^n|^{\frac{1}{n}})^{p-1}}\left(1-\frac{1-\beta}{\alpha+2k+1}\right)^k.$$

So this yields

$$M\left(\frac{1}{s}\right)\geq \frac{2(k+1)}{p(n-1+p)}\frac{|B_p^{n-1}|(1-s^p|B_p^n|^{\frac{p}{n}})^{\frac{n-1}{p}+k+2}}{|B_p^n|(\frac{n-1}{p}+2)(s|B_p^n|^{\frac{1}{n}})^{p-1}}\left(1-\frac{2-\frac{1}{p}}{\frac{n-1}{p}+k+1}\right)^k.$$

If we choose $k=n$ and take into account that $1<p\leq 2$, we get

$$M\left(\frac{1}{s}\right)\geq C\frac{|B_p^{n-1}|}{|B_p^n|}\frac{e^{\frac{n-1+np}{p}\log(1-s^p|B_p^n|^{\frac{p}{n}})}}{(n-1+2p)(s|B_p^n|^{\frac{1}{n}})^{p-1}}.$$

We take $s_0=\frac{\gamma^{\frac{1}{p}}(\log N)^{\frac{1}{p}}}{|B_p^n|^{\frac{1}{n}}n^{\frac{1}{p}}}$ with γ a constant to be chosen later. Then, since $N\leq e^n$, we obtain

$$M\left(\frac{1}{s_0}\right)\geq C\frac{|B_p^{n-1}|}{|B_p^n|}\frac{e^{-c_1\gamma\log N}}{(n-1+2p)(\gamma\frac{\log N}{n})^{1-\frac{1}{p}}}\geq\frac{C'}{N^{c_1\gamma}(\gamma\log N)^{1-\frac{1}{p}}}.$$

Choosing γ small enough, so that $c_1\gamma<1$, we get

$$M\left(\frac{1}{s_0}\right)\geq\frac{1}{N}$$

if $N\geq N_0$ for some $N_0\in\mathbb{N}$ large enough. Therefore, there exists an absolute positive constant c such that

$$\mathbb{E}\max_{1\leq i\leq N}\left|\langle X_i,e_j\rangle\right|\geq c(\log N)^{\frac{1}{p}}.$$

Now, let us consider the easier case where $p=1$. In this case, we have

$$M\left(\frac{1}{s}\right)=\frac{2}{n(n+1)}\frac{|B_1^{n-1}|}{|B_1^n|}\frac{(1-s|B_1^n|^{\frac{1}{n}})^{n+1}}{s|B_1^n|^{\frac{1}{n}}}.$$

If we now choose $s_0=\alpha\log N$, where α is a constant to be chosen later, we obtain

$$M\left(\frac{1}{s_0}\right)\geq\frac{C}{N^{c\alpha}\log N},$$

and so, choosing α a constant small enough so that $c\alpha<1$, we obtain that

$$M\left(\frac{1}{s_0}\right)\geq\frac{1}{N}$$

whenever $N \geq N_0$. Therefore, if $p = 1$, there exists an absolute positive constant c such that

$$\mathbb{E} \max_{1 \leq i \leq N} \left| \langle X_i, e_j \rangle \right| \geq c(\log N).$$

Now, let us treat the case $2 \leq p$. We will assume that $p - 1 \leq c \frac{n}{\alpha \log N}$, where α is a constant that will be determined later, and c is an absolute constant small enough. We will also assume that $p \leq N^{\frac{1}{4}}$. We have seen that the second term in (3) equals

$$\begin{aligned} &\frac{4(2-p)}{p(n-1+2p)(n-1+p)} \frac{|B_p^{n-1}|}{|B_p^n|} \int_0^{\cos^{-1}(s|B_p^n|^{\frac{1}{n}})^{\frac{p}{2}}} \frac{(\sin \theta)^{2\frac{n-1}{p}+5}}{(\cos \theta)^{5-\frac{2}{p}}} \, d\theta \\ &\quad - \frac{8(2-p)(p-1)}{p(n-1+2p)(n-1+p)} \frac{|B_p^{n-1}|}{|B_p^n||B_p^n|^{\frac{2p-2}{n}}} \\ &\quad \times \int_0^{\cos^{-1}(s|B_p^n|^{\frac{1}{n}})^{\frac{p}{2}}} \frac{\sin \theta}{(\cos \theta)^{1+\frac{2}{p}}} \\ &\quad \times \int_{(\cos\theta)^{\frac{2}{p}}|B_p^n|^{-\frac{1}{n}}}^{|B_p^n|^{-\frac{1}{n}}} r^{1-2p} \big(1 - |B_p^n|^{\frac{p}{n}} r^p\big)^{\frac{n-1}{p}+2} \, dr \, d\theta, \end{aligned}$$

and so if $p \geq 2$, the second term in the expression (3) defining $M(\frac{1}{s})$ is greater than or equal to

$$\frac{4(2-p)}{p(n-1+2p)(n-1+p)} \frac{|B_p^{n-1}|}{|B_p^n|} \int_0^{\cos^{-1}(s|B_p^n|^{\frac{1}{n}})^{\frac{p}{2}}} \frac{(\sin \theta)^{2\frac{n-1}{p}+5}}{(\cos \theta)^{5-\frac{2}{p}}} \, d\theta.$$

Integration by parts yields that this quantity equals

$$\begin{aligned} &\frac{4(2-p)}{p(n-1+p)(n-1+2p)} \frac{|B_p^{n-1}|}{|B_p^n|} \\ &\quad \times \left(\frac{(1 - s^p |B_p^n|^{\frac{p}{n}})^{\frac{n-1}{p}+2}}{(4 - \frac{2}{p})(s|B_p^n|^{\frac{1}{n}})^{2p-1}} - \frac{2\frac{n-1}{p}+4}{4-\frac{2}{p}} \int_0^{\cos^{-1}(s|B_p^n|^{\frac{1}{n}})^{\frac{p}{2}}} \frac{(\sin \theta)^{2\frac{n-1}{p}+3}}{(\cos \theta)^{3-\frac{2}{p}}} \, d\theta \right). \end{aligned}$$

Thus, putting this together with the first term, we have that if $p \geq 2$,

$$\begin{aligned} M\left(\frac{1}{s}\right) \geq &\frac{12(p-1)}{p(2p-1)(n-1+p)} \frac{|B_p^{n-1}|}{|B_p^n|} \int_0^{\cos^{-1}(s|B_p^n|^{\frac{1}{n}})^{\frac{p}{2}}} \frac{(\sin \theta)^{2\frac{n-1}{p}+3}}{(\cos \theta)^{3-\frac{2}{p}}} \, d\theta \\ &- \frac{2(p-2)}{(n-1+p)(n-1+2p)(2p-1)} \frac{|B_p^{n-1}|}{|B_p^n|} \frac{(1 - s^p |B_p^n|^{\frac{p}{n}})^{\frac{n-1}{p}+2}}{(s|B_p^n|^{\frac{1}{n}})^{2p-1}}. \end{aligned}$$

Using integration by parts, the first term in the previous expression equals

$$\frac{6}{(2p-1)(n-1+p)}\frac{|B_p^{n-1}|}{|B_p^n|}\frac{(1-s^p|B_p^n|^{\frac{p}{n}})^{\frac{n-1}{p}+1}}{(s|B_p^n|^{\frac{1}{n}})^{p-1}}$$
$$-\frac{12}{p(2p-1)}\frac{|B_p^{n-1}|}{|B_p^n|}\int_0^{\cos^{-1}(s|B_p^n|^{\frac{1}{n}})^{\frac{p}{2}}}\frac{(\sin\theta)^{2\frac{n-1}{p}+1}}{(\cos\theta)^{1-\frac{2}{p}}}\,d\theta.$$

Using the recursion formula in Corollary 9, we obtain that for any $k\in\mathbb{N}$, this quantity equals

$$\frac{6}{(2p-1)(n-1+p)}\frac{|B_p^{n-1}|}{|B_p^n|}\frac{(1-s^p|B_p^n|^{\frac{p}{n}})^{\frac{n-1}{p}+1}}{(s|B_p^n|^{\frac{1}{n}})^{p-1}}$$
$$-\frac{6}{(2p-1)(n-1+p)}\frac{|B_p^{n-1}|}{|B_p^n|}\big(s|B_p^n|^{\frac{1}{n}}\big)\big(1-s^p|B_p^n|^{\frac{p}{n}}\big)^{\frac{n-1}{p}+1}$$
$$-\frac{6(2\frac{n}{p}+2)}{(2p-1)(n-1+p)(2\frac{n-1}{p}+4)}\frac{|B_p^{n-1}|}{|B_p^n|}\big(s|B_p^n|^{\frac{1}{n}}\big)\big(1-s^p|B_p^n|^{\frac{p}{n}}\big)^{\frac{n-1}{p}+2}$$
$$-\frac{6(2\frac{n}{p}+2)(2\frac{n}{p}+4)}{(2p-1)(n-1+p)(2\frac{n-1}{p}+4)(2\frac{n-1}{p}+6)}\frac{|B_p^{n-1}|}{|B_p^n|}$$
$$\times\big(s|B_p^n|^{\frac{1}{n}}\big)\big(1-s^p|B_p^n|^{\frac{p}{n}}\big)^{\frac{n-1}{p}+3}-\cdots$$
$$-\frac{6(2\frac{n}{p}+2)(2\frac{n}{p}+4)\cdots(2\frac{n}{p}+2k-2)}{(2p-1)(n-1+p)(2\frac{n-1}{p}+4)(2\frac{n-1}{p}+6)\cdots(2\frac{n-1}{p}+2k)}\frac{|B_p^{n-1}|}{|B_p^n|}$$
$$\times\big(s|B_p^n|^{\frac{1}{n}}\big)\big(1-s^p|B_p^n|^{\frac{p}{n}}\big)^{\frac{n-1}{p}+k}$$
$$-\frac{6(2\frac{n}{p}+2)(2\frac{n}{p}+4)\cdots(2\frac{n}{p}+2k-2)(2\frac{n-1}{p}+2k)}{(2p-1)(n-1+p)(2\frac{n-1}{p}+4)(2\frac{n-1}{p}+6)\cdots(2\frac{n-1}{p}+2k)}\frac{|B_p^{n-1}|}{|B_p^n|}$$
$$\times\int_0^{\cos^{-1}(s|B_p^n|^{\frac{1}{n}})^{\frac{p}{2}}}\frac{(\sin\theta)^{2\frac{n-1}{p}+2k+1}(\cos\theta)}{(\cos\theta)^{2-\frac{2}{p}}}\,d\theta.$$

Estimating the cosine in the denominator inside the integral by the value at its extreme point, we obtain that this quantity is greater than

$$\frac{6}{(2p-1)(n-1+p)}\frac{|B_p^{n-1}|}{|B_p^n|}\frac{1}{(s|B_p^n|^{\frac{1}{n}})^{p-1}}$$
$$\times\bigg(\big(1-s^p|B_p^n|^{\frac{p}{n}}\big)^{\frac{n-1}{p}+1}-s^p|B_p^n|^{\frac{p}{n}}\big(1-s^p|B_p^n|^{\frac{p}{n}}\big)^{\frac{n-1}{p}+1}$$

$$
\begin{aligned}
&- \frac{(2\frac{n}{p}+2)}{(2\frac{n-1}{p}+4)} s^p |B_p^n|^{\frac{p}{n}} \left(1 - s^p |B_p^n|^{\frac{p}{n}}\right)^{\frac{n-1}{p}+2} \\
&- \frac{(2\frac{n}{p}+2)(2\frac{n}{p}+4)}{(2\frac{n-1}{p}+4)(2\frac{n-1}{p}+6)} s^p |B_p^n|^{\frac{p}{n}} \left(1 - s^p |B_p^n|^{\frac{p}{n}}\right)^{\frac{n-1}{p}+3} - \cdots \\
&- \frac{(2\frac{n}{p}+2)(2\frac{n}{p}+4)\cdots(2\frac{n}{p}+2k-2)}{(2\frac{n-1}{p}+4)(2\frac{n-1}{p}+6)\cdots(2\frac{n-1}{p}+2k)} s^p |B_p^n|^{\frac{p}{n}} \left(1 - s^p |B_p^n|^{\frac{p}{n}}\right)^{\frac{n-1}{p}+k} \\
&- \frac{(2\frac{n}{p}+2)(2\frac{n}{p}+4)\cdots(2\frac{n}{p}+2k-2)(2\frac{n}{p}+2k)}{(2\frac{n-1}{p}+4)(2\frac{n-1}{p}+6)\cdots(2\frac{n-1}{p}+2k+2)} \left(1 - s^p |B_p^n|^{\frac{p}{n}}\right)^{\frac{n-1}{p}+k+1} \Bigg).
\end{aligned}
$$

Since for every m, we have that $\frac{2\frac{n}{p}+2m}{2\frac{n-1}{p}+2m+2} = 1 - \frac{2-\frac{2}{p}}{2\frac{n-1}{p}+2m+2} \leq 1$, this expression is greater than

$$
\begin{aligned}
&\frac{6|B_p^{n-1}|(1 - s^p |B_p^n|^{\frac{p}{n}})^{\frac{n-1}{p}+k+1}}{(2p-1)(n-1+p)|B_p^n|(s|B_p^n|^{\frac{1}{n}})^{p-1}} \\
&\quad \times \left(1 - \frac{(2\frac{n}{p}+2)(2\frac{n}{p}+4)\cdots(2\frac{n}{p}+2k-2)(2\frac{n}{p}+2k)}{(2\frac{n-1}{p}+4)(2\frac{n-1}{p}+6)\cdots(2\frac{n-1}{p}+2k+2)}\right) \\
&\geq \frac{6}{(2p-1)(n-1+p)} \frac{|B_p^{n-1}|}{|B_p^n|} \frac{(1 - s^p |B_p^n|^{\frac{p}{n}})^{\frac{n-1}{p}+k+1}}{(s|B_p^n|^{\frac{1}{n}})^{p-1}} \\
&\quad \times \left(1 - \left(1 - \frac{p-1}{n-1+(k+1)p}\right)^k\right) \\
&= \frac{6}{(2p-1)(n-1+p)} \frac{|B_p^{n-1}|}{|B_p^n|} \frac{(1 - s^p |B_p^n|^{\frac{p}{n}})^{\frac{n-1}{p}+k+1}}{(s|B_p^n|^{\frac{1}{n}})^{p-1}} \left(1 - e^{k\log(1-\frac{p-1}{n-1+(k+1)p})}\right).
\end{aligned}
$$

Hence,

$$
\begin{aligned}
M\left(\frac{1}{s}\right) &\geq \frac{6}{(2p-1)(n-1+p)} \frac{|B_p^{n-1}|}{|B_p^n|} \frac{(1 - s^p |B_p^n|^{\frac{p}{n}})^{\frac{n-1}{p}+k+1}}{(s|B_p^n|^{\frac{1}{n}})^{p-1}} \\
&\quad \times \left(1 - e^{k\log(1-\frac{p-1}{n-1+(k+1)p})}\right) \\
&\quad - \frac{2(p-2)}{(n-1+p)(n-1+2p)(2p-1)} \frac{|B_p^{n-1}|}{|B_p^n|} \frac{(1 - s^p |B_p^n|^{\frac{p}{n}})^{\frac{n-1}{p}+2}}{(s|B_p^n|^{\frac{1}{n}})^{2p-1}} \\
&= \frac{1}{(2p-1)(n-1+p)} \frac{|B_p^{n-1}|}{|B_p^n|} \frac{(1 - s^p |B_p^n|^{\frac{p}{n}})^{\frac{n-1}{p}+2}}{(s|B_p^n|^{\frac{1}{n}})^{p-1}}
\end{aligned}
$$

$$\times\left(6\left(1-s^p|B_p^n|^{\frac{p}{n}}\right)^{k-1}\left(1-e^{k\log(1-\frac{p-1}{n-1+(k+1)p})}\right)\right.$$
$$\left.-\frac{2(p-2)}{(n-1+2p)(s^p|B_p^n|^{\frac{p}{n}})}\right).$$

We take

$$s_0=\frac{\alpha^{\frac{1}{p}}(p-1)^{\frac{1}{p}}}{|B_p^n|^{\frac{1}{n}}n^{\frac{1}{p}}}(\log N)^{\frac{1}{p}}.$$

Then,

$$\frac{2(p-2)}{(n-1+2p)(s_0^p|B_p^n|^{\frac{p}{n}})}\le\frac{2}{(1-\frac{1}{n}+\frac{2p}{n})\alpha\log N}\le\frac{2.1}{\alpha\log N}$$

if $n\ge n_0$. On the other hand, choosing k so that $k+1=\frac{2n}{\alpha(p-1)\log N}$, we have

$$\begin{aligned}
&6\left(1-s_0^p|B_p^n|^{\frac{p}{n}}\right)^{k-1}\left(1-e^{k\log\left(1-\frac{p-1}{n-1+(k+1)p}\right)}\right)\\
&\quad\ge 6\left(1-\frac{\alpha(p-1)\log N}{n}\right)^{\frac{2n}{\alpha(p-1)\log N}}\left(1-e^{\left(\frac{2n}{(p-1)\alpha\log N}-1\right)\log\left(1-\frac{p-1}{n-1+\frac{p}{p-1}\frac{2n}{\alpha\log N}}\right)}\right)\\
&\quad\ge 6e^{-1}\left(1-e^{\left(\frac{2n}{(p-1)\alpha\log N}-1\right)\log\left(1-\frac{p-1}{n-1+\frac{p}{p-1}\frac{2n}{\alpha\log N}}\right)}\right),
\end{aligned}$$

where the last inequality holds because of our assumptions on p. This last quantity is greater than

$$\begin{aligned}
&6e^{-1}\left(1-e^{-\left(\frac{2n}{(p-1)\alpha\log N}-1\right)\frac{p-1}{n-1+\frac{p}{p-1}\frac{2n}{\alpha\log N}}}\right)=6e^{-1}\left(1-e^{-\frac{2}{\alpha\log N}\frac{1-\frac{(p-1)\alpha\log N}{n}}{1-\frac{1}{n}+\frac{p}{p-1}\frac{1}{2\alpha\log N}}}\right)\\
&\quad\ge 6e^{-1}\left(1-e^{-\frac{2(1-c)}{\alpha\log N}}\right)\ge\frac{6e^{-1}(1-c)}{\alpha\log N}
\end{aligned}$$

if $N\ge N_0$. Taking c small enough so that $6e^{-1}(1-c)>2.1$, we have that

$$M\left(\frac{1}{s_0}\right)\ge\frac{C}{p^2N^{c_1\alpha}(\alpha\log N)^{2-\frac{2}{p}}}\ge\frac{C}{N^{c_1\alpha+\frac{1}{2}}(\alpha\log N)^{2-\frac{2}{p}}},$$

since we are assuming that $p\le N^{\frac{1}{4}}$. Taking α such that $c_1\alpha+\frac{1}{2}<1$, we obtain

$$M\left(\frac{1}{s_0}\right)\ge\frac{1}{N}$$

if $N\ge N_1$ and $n\ge n_0$ for some n_0,N_1 big enough. Therefore,

$$\mathbb{E}\max_{1\le i\le N}\left|\langle X_i,e_1\rangle\right|\ge\tilde{C}\left(\log\frac{N}{p^{\frac{1}{4}}}\right)^{\frac{1}{p}}\ge C(\log N)^{\frac{1}{p}},$$

where $N\ge N_0$, and C is a positive absolute constant.

Now we consider the case $p \geq c\frac{n}{\log N}$ or $p \geq N^{1/4}$. In that case we choose

$$s_0 = \frac{1}{2|B_p^n|^{\frac{1}{n}}}.$$

Then

$$\begin{aligned}
M\left(\frac{1}{s_0}\right) &= 2\frac{|B_p^{n-1}||B_p^n|^{\frac{1}{n}}}{|B_p^n|}\int_{|B_p^n|^{\frac{1}{n}}}^{2|B_p^n|^{\frac{1}{n}}}\int_{\frac{1}{t}}^{|B_p^n|^{-\frac{1}{n}}} r\left(1-|B_p^n|^{\frac{p}{n}}r^p\right)^{\frac{n-1}{p}}\,dr\,dt \\
&\geq 2\frac{|B_p^{n-1}||B_p^n|^{\frac{1}{n}}}{|B_p^n|}\int_{\frac{7}{4}|B_p^n|^{\frac{1}{n}}}^{2|B_p^n|^{\frac{1}{n}}}\int_{\frac{1}{t}}^{\frac{2}{3}|B_p^n|^{-\frac{1}{n}}} r\left(1-|B_p^n|^{\frac{p}{n}}r^p\right)^{\frac{n-1}{p}}\,dr\,dt \\
&\geq 2\frac{|B_p^{n-1}||B_p^n|^{\frac{1}{n}}}{|B_p^n|}\int_{\frac{7}{4}|B_p^n|^{\frac{1}{n}}}^{2|B_p^n|^{\frac{1}{n}}}\left(\frac{2}{3|B_p^n|^{\frac{1}{n}}}-\frac{1}{t}\right)\frac{1}{t}\left(1-\frac{1}{(\frac{3}{2})^p}\right)^{\frac{n-1}{p}}dt \\
&\geq \frac{1}{42}\frac{|B_p^{n-1}|}{|B_p^n|}\left(1-\frac{1}{(\frac{3}{2})^p}\right)^{\frac{n-1}{p}} \geq C_1 n^{\frac{1}{p}} e^{\frac{n-1}{p}\log(1-\frac{1}{(\frac{3}{2})^p})} \\
&\geq C_1 n^{\frac{1}{p}} e^{-c_2\frac{n-1}{p(3/2)^p}}.
\end{aligned}$$

We want the latter expression to be greater or equal to N^{-1}, i.e.,

$$C_1 n^{\frac{1}{p}} e^{-c_2\frac{n-1}{p(3/2)^p}} \geq \frac{1}{N},$$

which is equivalent to

$$\log N + \log(C_1) + \frac{1}{p}\log(n) \geq c_2\frac{n-1}{p(\frac{3}{2})^p}.$$

To obtain this, it is enough to show

$$\log N \geq c_2\frac{n-1}{p(\frac{3}{2})^p},$$

and since $p \geq c\frac{n}{\log N}$ and $N \leq e^{c'n}$, to obtain the latter inequality, it is enough to have

$$\log N \geq c_2\frac{n-1}{p(\frac{3}{2})^{\frac{c}{c'}}}.$$

But

$$c_2\frac{n-1}{p(\frac{3}{2})^{\frac{c}{c'}}} \leq c_2\frac{n-1}{c\frac{n}{\log N}(\frac{3}{2})^{\frac{c}{c'}}} \leq c_2\frac{\log N}{c(\frac{3}{2})^{\frac{c}{c'}}} \leq \log N$$

if c' is small enough. So we obtain the estimate. If $p \geq N^{\frac{1}{4}}$, we immediately obtain

$$C_1 n^{\frac{1}{p}} e^{-c_2 \frac{n-1}{p(3/2)^p}} \geq C > \frac{1}{N}$$

for $N \geq N_0$. Therefore, in these two cases, we obtain the estimate

$$\mathbb{E} h_{K_N}(e_j) \sim \frac{1}{|B_p^n|^{\frac{1}{n}}} \sim n^{\frac{1}{p}} \gtrsim (\log N)^{\frac{1}{p}}.$$

□

Remark 2 In the case $p = \infty$ it is very easy to check that

$$\inf\left\{s > 0 : M\left(\frac{1}{s}\right) \leq \frac{1}{N}\right\} = \frac{1 + \frac{1}{N} - \sqrt{\frac{2}{N} + \frac{1}{N^2}}}{2} \sim 1,$$

and so $\mathbb{E} h_{K_N}(e_j) \sim 1$.

4 General Results

Using our approach, we will now prove more general bounds for symmetric isotropic convex bodies. In the first theorem we assume some mild technical conditions which are verified by the ℓ_p^n balls ($p \geq 2$). In this way we recover the upper estimates proved in the previous section.

Since $\mathbb{E} h_{K_N}(\theta) \sim \inf\{s > 0 : M_\theta(\frac{1}{s}) \leq \frac{1}{N}\}$, it seems natural to study for which value of s

$$\int_{S^{n-1}} M_\theta\left(\frac{1}{s}\right) d\mu(\theta) = \frac{1}{N}.$$

As one could expect, this value of s is of the order $L_K \sqrt{\log N}$. As a consequence of Chebychev's inequality, we will obtain probability estimates for the set of directions verifying $\mathbb{E} h_{K_N}(\theta) \leq C L_K \sqrt{\log N}$ or $\mathbb{E} h_{K_N}(\theta) \geq C L_K \sqrt{\log N}$.

Theorem 11 *Let K be a symmetric and isotropic convex body, $n \leq N$, $\theta \in S^{n-1}$, and $X_1, \dots, X_N$ be independent random vectors uniformly distributed in K. Define $h(t) = |K \cap \{\langle x, \theta\rangle = t\}|^{\frac{1}{n-1}}$. Assume that h is twice differentiable and that $h'(t) \neq 0$ for all $t \in (0, h_K(\theta))$. Assume also that $-h'(t)/t$ is increasing and that $h(h_K(\theta)) = 0$. Then,*

$$\mathbb{E} \max_{1 \leq i \leq N} \left|\langle X_i, \theta\rangle\right| \leq C h^{-1}\left(h(0)\left(1 - \alpha \frac{\log N}{n}\right)\right),$$

where α, C, $\alpha > C$ are positive absolute constants.

Proof First of all, notice that h is a concave function. Then, using Theorem 4, we get

$$M\left(\frac{1}{s}\right) = \int_{\frac{1}{h_K(\theta)}}^{\frac{1}{s}} 2 \int_{\frac{1}{t}}^{h_K(\theta)} r h(r)^{n-1}\, dr\, dt = 2 \int_{\frac{1}{h_K(\theta)}}^{\frac{1}{s}} \int_{\frac{1}{t}}^{h_K(\theta)} \frac{r}{h'(r)} h'(r) h(r)^{n-1}\, dr\, dt.$$

Integration by parts yields

$$M\left(\frac{1}{s}\right)=2\int_{\frac{1}{h_K(\theta)}}^{\frac{1}{s}}-\frac{\frac{1}{t}h(\frac{1}{t})^n}{nh'(\frac{1}{t})}\,dt-\int_{\frac{1}{h_K(\theta)}}^{\frac{1}{s}}\int_{\frac{1}{t}}^{h_K(\theta)}h(r)^n\frac{h'(r)-rh''(r)}{nh'(r)^2}\,dr\,dt.$$

Since $h'(t)-th''(t)\geq 0$, we have

$$\begin{aligned}M\left(\frac{1}{s}\right)&\leq 2\int_{\frac{1}{h_K(\theta)}}^{\frac{1}{s}}-\frac{\frac{1}{t}h(\frac{1}{t})^n}{nh'(\frac{1}{t})}\,dt\\&=-\frac{2}{n}\int_s^{h_K(\theta)}\frac{h(u)^n}{uh'(u)}\,du=-\frac{2}{n}\int_s^{h_K(\theta)}\frac{h'(u)h(u)^n}{uh'(u)^2}\,du.\end{aligned}$$

Again we use integration by parts and get

$$\begin{aligned}M\left(\frac{1}{s}\right)&\leq\frac{2h(s)^{n+1}}{n(n+1)sh'(s)^2}-\frac{2}{n(n+1)}\int_s^{h_K(\theta)}\frac{h(u)^{n+1}(h'(u)+2uh''(u))}{u^2h'(u)^3}\,du\\&\leq\frac{2sh(s)^{n+1}}{n(n+1)s^2h'(s)^2}.\end{aligned}$$

Furthermore, since we have $h'(t)-th''(t)\geq 0$, we get

$$-sh'(s)=\left|sh'(s)\right|=\int_0^s-h'(t)-th''(t)\,dt\geq -2\int_0^s h'(t)\,dt=2\big(h(0)-h(s)\big).$$

Thus,

$$M\left(\frac{1}{s}\right)\leq\frac{sh(s)^{n-1}}{2n(n+1)\big(\frac{h(0)}{h(s)}-1\big)^2}=\frac{se^{(n-1)\log\frac{h(s)}{h(0)}}|K\cap\theta^{\perp}|h(s)^2}{2n(n+1)\big(1-\frac{h(s)}{h(0)}\big)^2h(0)^2}.$$

Choosing

$$s_0=h^{-1}\left(h(0)\left(1-\alpha\frac{\log N}{n}\right)\right),$$

we have that there exists a positive constant c_1 such that

$$M\left(\frac{1}{s_0}\right)\leq C\frac{s_0|K\cap\theta^{\perp}|}{N^{c_1\alpha}\alpha^2(\log N)^2}.$$

Since K is isotropic, $s_0\leq(n+1)L_K$. Therefore,

$$M\left(\frac{1}{s_0}\right)\leq C\frac{nL_K|K\cap\theta^{\perp}|}{N^{c_1\alpha}\alpha^2(\log N)^2}.$$

By Hensley's result (see [11]), $L_K\sim\frac{1}{|K\cap\theta^{\perp}|}$, and because $n\leq N$, we have

$$M\left(\frac{1}{s_0}\right)\leq\frac{CN}{N^{c_1\alpha}\alpha^2(\log N)^2}=\frac{C}{N^{c_1\alpha-1}\alpha^2(\log N)^2}.$$

Taking α so that $c_1\alpha > 2$, we have $M(\frac{1}{s_0}) \leq \frac{1}{N}$ for $N \geq N_0$ for some $N_0 \in \mathbb{N}$ big enough. $\square$

With the method, introduced in Sect. 2, we are also able to prove the following general result, which will lead us to estimates of the support function for some directions of random polytopes in symmetric isotropic convex bodies:

Theorem 12 *Let $n \leq N \leq e^{\sqrt{n}}$, K be a symmetric isotropic convex body in $\mathbb{R}^n$, and let $X_1, \dots, X_N$ be independent random variables uniformly distributed in K. Then,*

$$\int_{S^{n-1}} M_\theta\left(\frac{1}{C_1 L_K \sqrt{\log N}}\right) d\mu(\theta) \leq \frac{1}{N},$$

and

$$\int_{S^{n-1}} M_\theta\left(\frac{1}{C_2 L_K \sqrt{\log N}}\right) d\mu(\theta) \geq \frac{1}{N},$$

where C_1, C_2 are positive absolute constants.

Consequently, if $\tilde{s}$ is chosen such that

$$\int_{S^{n-1}} M_\theta\left(\frac{1}{\tilde{s}}\right) d\mu(\theta) = \frac{1}{N},$$

then $\tilde{s} \sim L_K \sqrt{\log N}$.

In order to prove this theorem, we need the following proposition:

Proposition 13 *Let K be a symmetric convex body in $\mathbb{R}^n$ of volume 1. Let $s > 0$, $\theta \in S^{n-1}$, and M_θ be the Orlicz function associated to the random variable $\langle X, \theta\rangle$, where X is uniformly distributed in K. Then,*

$$\int_{S^{n-1}} M_\theta\left(\frac{1}{s}\right) d\mu(\theta) = \int_K M_{\langle\theta,e_1\rangle}\left(\frac{\|x\|_2}{s}\right) dx, \tag{5}$$

where $M_{\langle\theta,e_1\rangle}$ is the Orlicz function associated to the random variable $\langle\theta, e_1\rangle$ with θ uniformly distributed on S^{n-1}. For any $s \leq \|x\|_2$,

$$M_{\langle\theta,e_1\rangle}\left(\frac{\|x\|_2}{s}\right) = \frac{2w_{n-1}}{nw_n} \int_0^{\cos^{-1}(\frac{s}{\|x\|_2})} \frac{\sin^n y}{\cos^2 y} dy, \tag{6}$$

and 0 otherwise.

Proof Using the definition of M_θ, we obtain

$$\begin{aligned}
&\int_{S^{n-1}} M_\theta\left(\frac{1}{s}\right) d\mu(\theta) \\
&\quad = \int_{S^{n-1}} \int_0^{\frac{1}{s}} \int_K \mathbb{1}_{\{|\langle x,\theta\rangle| \geq \frac{1}{t}\}}(x,\theta,t) \left|\langle x,\theta\rangle\right| dx\, dt\, d\mu(\theta)
\end{aligned}$$

$$= \int_K \int_0^{\frac{1}{s}} \int_{S^{n-1}} \mathbb{1}_{\{|\langle x,\theta\rangle|\geq \frac{1}{t}\}}(x,\theta,t)\big|\langle x,\theta\rangle\big|\, d\mu(\theta)\, dt\, dx$$

$$= \int_K \int_0^{\frac{\|x\|_2}{s}} \int_{S^{n-1}} \mathbb{1}_{\{|\langle \frac{x}{\|x\|_2},\theta\rangle|\geq \frac{1}{u}\}}(x,\theta,u)\left|\left\langle \frac{x}{\|x\|_2},\theta\right\rangle\right| d\mu(\theta)\, du\, dx,$$

where the last equality is obtained by the change of variable $t = \frac{u}{\|x\|_2}$. Hence, by the rotational invariance of S^{n-1},

$$\int_{S^{n-1}} M_\theta\left(\frac{1}{s}\right) d\mu(\theta) = \int_K M_{\langle\theta,e_1\rangle}\left(\frac{\|x\|_2}{s}\right) dx.$$

Now, let us compute $M_{\langle\theta,e_1\rangle}$. For any $s > 1$, otherwise the function is 0, we have

$$\begin{aligned} M_{\langle\theta,e_1\rangle}(s) &= \int_1^s \int_{S^{n-1}\cap\{\langle e_1,\theta\rangle\geq\frac{1}{t}\}} d\mu(\theta)\, dt \\ &= 2\int_1^s \frac{(n-1)w_{n-1}}{nw_n} \int_{\frac{1}{t}}^1 r\left(1-r^2\right)^{\frac{n-3}{2}} dr\, dt \\ &= \frac{2w_{n-1}}{nw_n} \int_1^s \left(1-\frac{1}{t^2}\right)^{\frac{n-1}{2}} dt. \end{aligned}$$

The change of variables $\frac{1}{t} = \cos y$ yields

$$M_{\langle\theta,e_1\rangle}(s) = \frac{2w_{n-1}}{nw_n} \int_0^{\cos^{-1}(\frac{1}{s})} \frac{\sin^n y}{\cos^2 y}\, dy. \qquad \square$$

Given that the expected mean width of K_N is minimized when $K = D_2^n$, it is natural to expect that given s, the average $\int_{S^{n-1}} M_\theta(\frac{1}{s})\, d\mu(\theta)$ would also be minimized when $K = D_2^n$. We prove it, using this representation, in the following:

Corollary 14 *Let K be a symmetric convex body in $\mathbb{R}^n$ of volume 1, and let $s > 0$. Then*

$$\int_{S^{n-1}} M_\theta\left(\frac{1}{s}\right) d\mu(\theta) \geq \int_{S^{n-1}} M_{D_2^n,\theta}\left(\frac{1}{s}\right) d\mu(\theta) = M_{D_2^n,e_1}\left(\frac{1}{s}\right),$$

where $M_{D_2^n,\theta}$ denotes the Orlicz function associated to D_2^n.

Proof By (5) and the facts that $M_{\langle\theta,e_1\rangle}$ is increasing and $|K| = |D_2^n| = 1$ we have that if r_n is the radius of D_2^n,

$$\begin{aligned} \int_{S^{n-1}} M_\theta\left(\frac{1}{s}\right) d\mu(\theta) &= \int_K M_{\langle\theta,e_1\rangle}\left(\frac{\|x\|_2}{s}\right) dx \\ &= \int_{K\cap D_2^n} M_{\langle\theta,e_1\rangle}\left(\frac{\|x\|_2}{s}\right) dx + \int_{K\setminus D_2^n} M_{\langle\theta,e_1\rangle}\left(\frac{\|x\|_2}{s}\right) dx \end{aligned}$$

$$
\begin{aligned}
&\geq \int_{K\cap D_2^n} M_{\langle\theta,e_1\rangle}\left(\frac{\|x\|_2}{s}\right)dx + \left|K\backslash D_2^n\right| M_{\langle\theta,e_1\rangle}\left(\frac{r_n}{s}\right)\\
&= \int_{K\cap D_2^n} M_{\langle\theta,e_1\rangle}\left(\frac{\|x\|_2}{s}\right)dx + \left|D_2^n\backslash K\right| M_{\langle\theta,e_1\rangle}\left(\frac{r_n}{s}\right)\\
&\geq \int_{K\cap D_2^n} M_{\langle\theta,e_1\rangle}\left(\frac{\|x\|_2}{s}\right)dx + \int_{D_2^n\backslash K} M_{\langle\theta,e_1\rangle}\left(\frac{\|x\|_2}{s}\right)dx\\
&= \int_{D_2^n} M_{\langle\theta,e_1\rangle}\left(\frac{\|x\|_2}{s}\right)dx = \int_{S^{n-1}} M_{D_2^n,\theta}\left(\frac{1}{s}\right)d\mu(\theta). \qquad \square
\end{aligned}
$$

Now, we give the proof of Theorem 12:

Proof By (6), if $\|x\|_2 \geq s$, we have

$$
M_{\langle\theta,e_1\rangle}\left(\frac{\|x\|_2}{s}\right) = \frac{2w_{n-1}}{nw_n}\int_0^{\cos^{-1}(\frac{s}{\|x\|_2})} \frac{\sin^n y}{\cos^2 y}\,dy.
$$

Integration by parts yields

$$
\begin{aligned}
&M_{\langle\theta,e_1\rangle}\left(\frac{\|x\|_2}{s}\right)\\
&\quad= \frac{2w_{n-1}}{nw_n}\left[\frac{(\sin y)^{n-1}}{\cos y}\bigg|_0^{\cos^{-1}(\frac{s}{\|x\|_2})} - (n-1)\int_0^{\cos^{-1}(\frac{s}{\|x\|_2})} (\sin y)^{n-2}\,dy\right]\\
&\quad= \frac{2w_{n-1}}{nw_n}\left[\frac{\|x\|_2}{s}\left(1-\frac{s^2}{\|x\|_2^2}\right)^{\frac{n-1}{2}} - (n-1)\int_0^{\cos^{-1}(\frac{s}{\|x\|_2})} (\sin y)^{n-2}\,dy\right].
\end{aligned}
$$

We start with the upper bound where we will use Paouris' result about the concentration of mass on isotropic convex bodies from [18]. First of all, we have

$$
M_{\langle\theta,e_1\rangle}\left(\frac{\|x\|_2}{s}\right) \leq \frac{2w_{n-1}}{nw_n}\frac{\|x\|_2}{s}\left(1-\frac{s^2}{\|x\|_2^2}\right)^{\frac{n-1}{2}}.
$$

From (5) and since $M_{\langle\theta,e_1\rangle}(\frac{\|x\|_2}{s}) = 0$ for $s > \|x\|_2$, we get

$$
\begin{aligned}
\int_{S^{n-1}} M_\theta\left(\frac{1}{s}\right)d\mu(\theta) &\leq \int_{K\backslash sB_2^n} \frac{2w_{n-1}}{nw_n}\frac{\|x\|_2}{s}\left(1-\frac{s^2}{\|x\|_2^2}\right)^{\frac{n-1}{2}} dx\\
&\leq \int_{K\backslash sB_2^n} \frac{2w_{n-1}}{nw_n}\frac{\|x\|_2}{s} e^{-\frac{n-1}{2}\frac{s^2}{\|x\|_2^2}}\,dx\\
&\leq \int_K \frac{2w_{n-1}}{nw_n}\frac{\|x\|_2}{s} e^{-\frac{n-1}{2}\frac{s^2}{\|x\|_2^2}}\,dx.
\end{aligned}
$$

We choose $s_0=\sqrt{\alpha}L_K\sqrt{\log N}$, with $\alpha>0$ a constant to be chosen later. Then, if $N\le e^{\sqrt{n}}$,

$$\begin{aligned}
\int_{S^{n-1}} M_\theta\left(\frac{1}{s_0}\right)d\mu(\theta) &\le \frac{2w_{n-1}}{nw_n}\frac{1}{\sqrt{\alpha}L_K\sqrt{\log N}}\int_K \|x\|_2 e^{-\frac{c_1\alpha n L_K^2}{\|x\|_2^2}\log N}dx\\
&= \frac{2w_{n-1}}{nw_n}\frac{1}{\sqrt{\alpha}L_K\sqrt{\log N}}\left[\int_{K\cap\gamma\sqrt{n}L_K B_2^n}\|x\|_2 e^{-\frac{c_1\alpha n L_K^2}{\|x\|_2^2}\log N}dx\right.\\
&\quad\left.+\int_{K\setminus\gamma\sqrt{n}L_K B_2^n}\|x\|_2 e^{-\frac{c_1\alpha n L_K^2}{\|x\|_2^2}\log N}dx\right]\\
&\le \frac{2w_{n-1}}{nw_n}\frac{1}{\sqrt{\alpha}L_K\sqrt{\log N}}\left[\frac{\gamma\sqrt{n}L_K}{N^{\frac{c_1\alpha}{\gamma^2}}}+nL_K e^{-c_1\sqrt{n}\gamma}\right]\\
&\le \frac{C}{\sqrt{\alpha}\sqrt{\log N}}\left[\frac{\gamma}{N^{\frac{c_1\alpha}{\gamma^2}}}+\frac{\sqrt{n}}{N^{c_1\gamma}}\right]\\
&\le \frac{C}{\sqrt{\alpha}\sqrt{\log N}}\left[\frac{\gamma}{N^{\frac{c_1\alpha}{\gamma^2}}}+\frac{1}{N^{c_1\gamma-\frac{1}{2}}}\right].
\end{aligned}$$

We choose $\gamma>0$ such that $c_1\gamma-\frac{1}{2}>1$ and then $\alpha>0$ so that $\frac{c_1\alpha}{\gamma^2}>1$. Then,

$$\int_{S^{n-1}} M_\theta\left(\frac{1}{s_0}\right)d\mu(\theta)\le\frac{C}{\sqrt{\alpha}\sqrt{\log N}}\left[\frac{\gamma}{N^{\frac{c_1\alpha}{\gamma^2}}}+\frac{1}{N^{c_1\gamma-\frac{1}{2}}}\right]\le\frac{1}{N}$$

for $N\le e^{\sqrt{n}}$ and $N\ge N_0$.

To prove the lower bound, we use the recursion formula (9). For $\|x\|_2\ge s$ and any $k\in\mathbb{N}$,

$$\begin{aligned}
&M_{\langle\theta,e_1\rangle}\left(\frac{\|x\|_2}{s}\right)\\
&\ge\frac{2w_{n-1}}{nw_n}\frac{\|x\|_2}{s(n+1)}\left[\left(1-\frac{s^2}{\|x\|_2^2}\right)^{\frac{n+1}{2}}+\left(1-\frac{3}{n+3}\right)\left(1-\frac{s^2}{\|x\|_2^2}\right)^{\frac{n+3}{2}}+\cdots\right.\\
&\quad\left.+\left(1-\frac{3}{n+3}\right)\cdots\left(1-\frac{3}{n+2k+1}\right)\left(1-\frac{s^2}{\|x\|_2^2}\right)^{\frac{n+2k+1}{2}}\right]\\
&\ge\frac{2w_{n-1}}{nw_n}\frac{\|x\|_2(k+1)}{s(n+1)}\left(1-\frac{3}{n+2k+1}\right)^k\left(1-\frac{s^2}{\|x\|_2^2}\right)^{\frac{n+2k+1}{2}}.
\end{aligned}$$

Taking $k=n$, we have

$$\begin{aligned}
M_{\langle\theta,e_1\rangle}\left(\frac{\|x\|_2}{s}\right)&\ge\frac{2w_{n-1}}{nw_n}\frac{\|x\|_2}{s}\left(1-\frac{3}{n+2k+1}\right)^n\left(1-\frac{s^2}{\|x\|_2^2}\right)^{\frac{3n+1}{2}}\\
&\ge\frac{Cw_{n-1}}{nw_n}\frac{\|x\|_2}{s}\left(1-\frac{s^2}{\|x\|_2^2}\right)^{\frac{3n+1}{2}}.
\end{aligned}$$

Thus,

$$\int_{S^{n-1}} M_\theta\left(\frac{1}{s}\right) d\mu(\theta) \geq \int_{K\setminus sB_2^n} \frac{Cw_{n-1}}{nw_n} \frac{\|x\|_2}{s}\left(1-\frac{s^2}{\|x\|_2^2}\right)^{\frac{3n+1}{2}} dx$$

$$\geq \int_{K\setminus 2sB_2^n} \frac{Cw_{n-1}}{nw_n} \frac{\|x\|_2}{s}\left(1-\frac{s^2}{\|x\|_2^2}\right)^{\frac{3n+1}{2}} dx$$

$$\geq \int_{K\setminus 2sB_2^n} \frac{Cw_{n-1}}{nw_n} \frac{\|x\|_2}{s} e^{-c_4 n \frac{s^2}{\|x\|_2^2}} dx.$$

Take $s_1 = \sqrt{\beta} L_K \sqrt{\log N}$, $\beta > 0$ a constant to be chosen later. Then

$$\int_{S^{n-1}} M_\theta\left(\frac{1}{s_1}\right) d\mu(\theta) \geq \int_{K\setminus 2\sqrt{\beta}L_K\sqrt{\log N}B_2^n} \frac{Cw_{n-1}}{nw_n} \frac{\|x\|_2}{\sqrt{\beta}L_K\sqrt{\log N}} e^{-c_4 n \frac{\beta L_K^2 \log N}{\|x\|_2^2}} dx.$$

Using the small ball probability result proved in [19], we get that there exists a constant $c_5 > 0$ such that

$$\left|K \setminus c_5\sqrt{n} L_K B_2^n\right| \geq \frac{1}{2}$$

for $N \leq e^n$. Therefore,

$$\int_{K\setminus 2\sqrt{\beta}L_K\sqrt{\log N}B_2^n} \frac{Cw_{n-1}}{nw_n} \frac{\|x\|_2}{\sqrt{\beta}L_K\sqrt{\log N}} e^{-c_4 n \frac{\beta L_K^2 \log N}{\|x\|_2^2}} dx$$

$$\geq \int_{K\setminus c_5\sqrt{n}L_K B_2^n} \frac{Cw_{n-1}}{nw_n} \frac{\|x\|_2}{\sqrt{\beta}L_K\sqrt{\log N}} e^{-c_4 n \frac{\beta L_K^2 \log N}{\|x\|_2^2}} dx$$

$$\geq \frac{C'}{\sqrt{\beta}\sqrt{\log N}} e^{-c_6\beta \log N} |K \setminus c_5\sqrt{n} L_K B_2^n|$$

$$\geq \frac{C''}{N^{c_6\beta}\sqrt{\beta}\sqrt{\log N}},$$

where the inequality before the last one holds because $\|x\|_2^2 \geq c_5^2 n L_K^2$. We take β small enough, so that $c_6\beta < 1$ and $2\sqrt{\beta}\sqrt{\log N} \leq c_5\sqrt{n}$. Then

$$\frac{C''}{N^{c_6\beta}\sqrt{\beta}\sqrt{\log N}} \geq \frac{1}{N}$$

for $N \geq N_0$ and $N \leq e^n$. Hence,

$$\int_{S^{n-1}} M_\theta\left(\frac{1}{s_1}\right) d\mu(\theta) \geq \frac{1}{N}. \qquad \square$$

Obviously, the theorem implies that there are directions $\theta_1, \theta_2 \in S^{n-1}$ such that the expectation of the support function in those directions is bounded from above and below respectively by a constant times $L_K\sqrt{\log N}$. In Corollary 3 we give estimates for the measure of the set of directions verifying such estimates. However, we do not think that the estimate we give for the measure of the set of directions verifying the lower bound is optimal.

Proof of Corollary 3 To prove that the upper bound is true for most directions, we proceed as in the proof of Theorem 12. We choose s_0 like there and α, γ so that $c_1\gamma-\frac{1}{2}>2(r+1)$ and $\frac{c_1\alpha}{\gamma^2}>2(r+1)$ and obtain

$$\int_{S^{n-1}} M_\theta\left(\frac{1}{s_0}\right)d\mu(\theta)\le\frac{1}{N^{r+1}}.$$

Then, by Chebychev's inequality,

$$\frac{1}{N^{r+1}}\ge\int_{S^{n-1}} M_\theta\left(\frac{1}{s_0}\right)d\mu(\theta)\ge\frac{1}{N}\mu\left\{\theta\in S^{n-1}:M_\theta\left(\frac{1}{s_0}\right)>\frac{1}{N}\right\}.$$

Thus,

$$\mu\left\{\theta\in S^{n-1}:M_\theta\left(\frac{1}{s_0}\right)\le\frac{1}{N}\right\}\ge 1-\frac{1}{N^r},$$

and so

$$\mu\left\{\theta\in S^{n-1}:\mathbb{E}h_{K_N}(\theta)\le C_1(r)L_K\sqrt{\log N}\right\}\ge 1-\frac{1}{N^r}.$$

To prove the probability estimate for the lower bound, we can assume that $r<1$. We proceed as in Theorem 12. We choose s_1 like there and take β small enough so that $c_6\beta<r$. We obtain

$$\int_{S^{n-1}} M_\theta\left(\frac{1}{s_1}\right)d\mu(\theta)>\frac{1}{N^r}.$$

Then, for any decreasing, positive and concave function f, we get

$$f\left(\int_{S^{n-1}} M_\theta\left(\frac{1}{s_1}\right)d\mu(\theta)\right)<f\left(\frac{1}{N^r}\right).$$

Using Jensen's inequality, from this we obtain

$$\begin{aligned} f\left(\frac{1}{N^r}\right)&\ge\int_{S^{n-1}} f\left(M_\theta\left(\frac{1}{s_1}\right)\right)d\mu(\theta)\\ &\ge f\left(\frac{1}{N}\right)\mu\left\{\theta\in S^{n-1}:f\left(M_\theta\left(\frac{1}{s_1}\right)\right)>f\left(\frac{1}{N}\right)\right\}\\ &=f\left(\frac{1}{N}\right)\mu\left\{\theta\in S^{n-1}:M_\theta\left(\frac{1}{s_1}\right)<\frac{1}{N}\right\}. \end{aligned}$$

Thus,

$$\mu\left\{\theta\in S^{n-1}:M_\theta\left(\frac{1}{s_1}\right)<\frac{1}{N}\right\}\le\frac{f(\frac{1}{N^r})}{f(\frac{1}{N})},$$

and therefore,

$$\mu\left\{\theta \in S^{n-1} : M_\theta\left(\frac{1}{s_1}\right) \geq \frac{1}{N}\right\} \geq 1 - \frac{f(\frac{1}{N^r})}{f(\frac{1}{N})}.$$

This means that

$$\mu\left\{\theta \in S^{n-1} : \mathbb{E}h_{K_N}(\theta) \geq cs_1\right\} \geq 1 - \frac{f(\frac{1}{N^r})}{f(\frac{1}{N})}.$$

We choose $f(t) = -at + a\max_{\theta\in S^{n-1}} M_\theta(\frac{1}{s_1})$, $a > 0$. Then

$$\frac{f(\frac{1}{N^r})}{f(\frac{1}{N})} = \frac{-\frac{1}{N^r} + \max_{\theta\in S^{n-1}} M_\theta(\frac{1}{s_1})}{-\frac{1}{N} + \max_{\theta\in S^{n-1}} M_\theta(\frac{1}{s_1})},$$

and thus

$$1 - \frac{f(\frac{1}{N^r})}{f(\frac{1}{N})} = \frac{\frac{1}{N^r} - \frac{1}{N}}{\max_{\theta\in S^{n-1}} M_\theta(\frac{1}{s_1}) - \frac{1}{N}}.$$

From Hölder's inequality we obtain

$$\begin{aligned} M_\theta\left(\frac{1}{s_1}\right) &= \int_0^{\frac{1}{s_1}} \int_{K\cap\{|\langle x,\theta\rangle| \geq \frac{1}{t}\}} |\langle x,\theta\rangle|\,dx\,dt \\ &\leq \int_0^{\frac{1}{s_1}} \int_K |\langle x,\theta\rangle|\,dx\,dt \\ &\leq \int_0^{\frac{1}{s_1}} L_K\,dt = \frac{L_K}{s_1}. \end{aligned}$$

Because of our choice of s_1, we get

$$M_\theta\left(\frac{1}{s_1}\right) \leq \frac{C(r)}{\sqrt{\log N}}.$$

Therefore,

$$1 - \frac{f(\frac{1}{N^r})}{f(\frac{1}{N})} \geq \frac{\frac{1}{N^r} - \frac{1}{N}}{\frac{C(r)}{\sqrt{\log N}} - \frac{1}{N}} \geq \frac{C'(r)\sqrt{\log N}}{N^r}.$$

This yields

$$\mu\left\{\theta \in S^{n-1} : \mathbb{E}h_{K_N}(\theta) \geq C_2(r)L_K\sqrt{\log N}\right\} \geq \frac{C(r)\sqrt{\log N}}{N^r}. \qquad \square$$

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