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New Examples of Oriented Matroids with Disconnected Realization Spaces

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Abstract We construct oriented matroids of rank 3 on 13 points whose realization spaces are disconnected. They are defined on smaller point-sets than the known examples with this property. Moreover, we construct one on 13 points whose realization space is a connected but non-irreducible semialgebraic variety.

Keywords Oriented matroids · Realization space

1 Oriented Matroids and Matrices

Throughout this section, we fix positive integers r and n.

Let $X=(x_1,\dots,x_n)\in\mathbb{R}^{rn}$ be a real (r,n) matrix of rank r, and $E=\{1,\dots,n\}$ be the set of labels of the columns of X. For such a matrix X, a map $\mathcal{X}_X$ can be defined as

$$\mathcal{X}_X: E^r\to\{-1,0,+1\},\qquad \mathcal{X}_X(i_1,\dots,i_r):=\operatorname{sgn}\det(x_{i_1},\dots,x_{i_r}).$$

The map $\mathcal{X}_X$ is called the *chirotope* of X. The chirotope $\mathcal{X}_X$ encodes the information regarding the combinatorial type, which is called the *oriented matroid* of X. In this case, the oriented matroid determined by $\mathcal{X}_X$ is of rank r on E.

We note some properties which the chirotope $\mathcal{X}_X$ of a matrix X satisfies.

1. $\mathcal{X}_X$ is not identically zero.
2. $\mathcal{X}_X$ is alternating, i.e. $\mathcal{X}_X(i_{\sigma(1)},\dots,i_{\sigma(r)})=\operatorname{sgn}(\sigma)\mathcal{X}_X(i_1,\dots,i_r)$ for all $i_1,\dots,i_r\in E$ and all permutations σ.
3. For all $i_1,\dots,i_r,j_1,\dots,j_r\in E$ such that $\mathcal{X}_X(j_k,i_2,\dots,i_r)\cdot\mathcal{X}_X(j_1,\dots,j_{k-1},i_1,j_{k+1},\dots,j_r)\ge 0$ for $k=1,\dots,r$, we have $\mathcal{X}_X(i_1,\dots,i_r)\cdot\mathcal{X}_X(j_1,\dots,j_r)\ge 0$.

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The third property follows from the identity

$$\begin{aligned}&\det(x_1,\dots,x_r)\cdot\det(y_1,\dots,y_r)\\&\quad=\sum_{k=1}^{r}\det(y_k,x_2,\dots,x_r)\cdot\det(y_1,\dots,y_{k-1},x_1,y_{k+1},\dots,y_r),\\&\quad\text{for all } x_1,\dots,x_r,y_1,\dots,y_r\in\mathbb{R}^r.\end{aligned}$$

Generally, an oriented matroid of rank r on E (n points) is defined by a map $\chi : E^r \to \{-1, 0, +1\}$, which satisfies the above three properties ([1]). The map χ is also called the chirotope of an oriented matroid. We use the notation $\mathcal{M}(E, \chi)$ for an oriented matroid which is on the set E and is defined by the chirotope χ.

An oriented matroid $\mathcal{M}(E, \chi)$ is called *realizable* or *constructible*, if there exists a matrix X such that $\chi = \mathcal{X}_X$. Not all oriented matroids are realizable, but we do not consider the non-realizable case in this paper.

Definition 1.1 A realization of an oriented matroid $\mathcal{M} = \mathcal{M}(E, \chi)$ is a matrix X such that $\mathcal{X}_X = \chi$ or $\mathcal{X}_X = -\chi$.

Two realizations X, X' of $\mathcal{M}$ are called linearly equivalent, if there exists a linear transformation $A \in GL(r, \mathbb{R})$ such that $X' = AX$. Here we have the equation $\mathcal{X}_{X'} = \operatorname{sgn}(\det A) \cdot \mathcal{X}_X$.

Definition 1.2 The realization space $\mathcal{R}(\mathcal{M})$ of an oriented matroid $\mathcal{M}$ is the set of all linearly equivalent classes of realizations of $\mathcal{M}$, in the quotient topology induced from $\mathbb{R}^{rn}$.

Our motivation is as follows: In 1956, Ringel asked whether the realization spaces $\mathcal{R}(\mathcal{M})$ are necessarily connected [6]. It is known that every oriented matroid on less than nine points has a contractible realization space. In 1988, Mnëv showed that $\mathcal{R}(\mathcal{M})$ can be homotopy equivalent to an arbitrary semialgebraic variety [3]. His result implies that they can have arbitrary complicated topological types. In particular, there exist oriented matroids with disconnected realization spaces. Suvorov and Richter-Gebert constructed such examples of oriented matroids of rank 3 on 14 points, in 1988 and in 1996, respectively [5, 7]. However, it is unknown which is the smallest number of points on which oriented matroids can have disconnected realization spaces. See [1] for more historical comments.

One of the main results of this paper is the following.

Theorem 1.3 *There exist oriented matroids of rank* 3 *on* 13 *points whose realization spaces are disconnected.*

Let d and p be positive integers. The solution of a finite number of polynomial equations and polynomial strict inequalities with integer coefficients on $\mathbb{R}^d$ is called an elementary semialgebraic set.

Let $f_1, \dots, f_p \in \mathbb{Z}[v_1, \dots, v_d]$ be polynomial functions on $\mathbb{R}^d$, and $V \subset \mathbb{R}^d$ be an elementary semialgebraic set. For a p-tuple $\varepsilon = (\varepsilon_1, \dots, \varepsilon_p) \in \{-, 0, +\}^p$, let

$$V_\varepsilon := \{v \in V \mid \operatorname{sgn}(f_i(v)) = \varepsilon_i \text{ for } i = 1, \dots, p\}$$

denote the corresponding subset of V. The collection of the elementary semialgebraic sets $(V_\varepsilon)_{\varepsilon\in\{-,0,+\}^p}$ is called a *partition* of V.

In the case $r = 3$, a triple $(i, j, k) \in E^3$ is called a basis of χ if $\chi(i, j, k) \neq 0$. Let $B = (i, j, k)$ be a basis of χ such that $\chi(B) = +1$. The realization space of an oriented matroid $\mathcal{M} = \mathcal{M}(E, \chi)$ of rank 3 can be given by an elementary semialgebraic set

$$\mathcal{R}(\mathcal{M}, B) := \left\{X \in \mathbb{R}^{3n} \mid x_i = e_1,\ x_j = e_2,\ x_k = e_3,\ \mathcal{X}_X = \chi\right\},$$

where e_1, e_2, e_3 are the fundamental vectors of $\mathbb{R}^3$. For another choice of basis B' of χ, we have a rational isomorphism between $\mathcal{R}(\mathcal{M}, B)$ and $\mathcal{R}(\mathcal{M}, B')$. Therefore, realization spaces of oriented matroids are semialgebraic varieties.

The universal partition theorem states that, for every partition $(V_\varepsilon)_{\varepsilon\in\{-,0,+\}^p}$ of $\mathbb{R}^d$, there exists a family of oriented matroids $(\mathcal{M}^\varepsilon)_{\varepsilon\in\{-,0,+\}^p}$ such that the collection of their realization spaces with a common basis $(\mathcal{R}(\mathcal{M}^\varepsilon, B))_{\varepsilon\in\{-,0,+\}^p}$ is stably equivalent to the family $(V_\varepsilon)_{\varepsilon\in\{-,0,+\}^p}$. See [2] or [4] for universal partition theorems.

We construct three oriented matroids $\mathcal{M}^\varepsilon$ with $\varepsilon \in \{-, 0, +\}$ of rank 3 on 13 points, whose chirotopes differ by a sign on a certain triple. These oriented matroids present a partial oriented matroid with the sign of a single base non-fixed, whose realization space is partitioned by fixing the sign of this base. The two spaces $\mathcal{R}(\mathcal{M}^-)$ and $\mathcal{R}(\mathcal{M}^+)$ are disconnected, and $\mathcal{R}(\mathcal{M}^0)$, which is a wall between the two, is connected but non-irreducible. So we also have the following.

Theorem 1.4 *There exists an oriented matroid of rank* 3 *on* 13 *points whose realization space is connected but non-irreducible.*

Remark 1.5 An oriented matroid $\mathcal{M}(E, \chi)$ is called *uniform* if it satisfies $\chi(i_1, \dots, i_r) \neq 0$ for all $i_1 < \cdots < i_r \in E$. Suvorov's example on 14 points is uniform, and the examples which we construct are non-uniform. It is still unknown whether there exists a uniform oriented matroid on less than 14 points with a disconnected realization space.

2 Construction of the Examples

Throughout this section, we set $E = \{1, \dots, 13\}$.

Let $X(s, t, u)$ be a real (3, 13) matrix with three parameters $s, t, u \in \mathbb{R}$ given by

$$\begin{aligned} X(s,t,u) &\\ &:= (x_1, \dots, x_{13}) \\ &= \begin{pmatrix} 1 & 0 & 0 & 1 & s & s & 0 & 1 & 1 & st & s+t-u-st+su & s+t-st-s^2u & s(t-u+su) \\ 0 & 1 & 0 & 1 & 0 & 1 & t & t & u & t & t-u+su & t & t-u+su \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 1-su & 1-u+su & 1-su & 1-u+su \end{pmatrix}. \end{aligned}$$

This is a consequence of the computation of the following construction sequence. Both operations "$\vee$" and "$\wedge$" can be computed in terms of the standard cross product "$\times$" in $\mathbb{R}^3$. The whole construction depends only on the choice of the three parameters $s, t, u \in \mathbb{R}$. We have

$$\begin{aligned}
x_1 &= {}^t(1,0,0), \qquad x_2 = {}^t(0,1,0),\\
x_3 &= {}^t(0,0,1), \qquad x_4 = {}^t(1,1,1),\\
x_5 &= s \cdot x_1 + x_3,\\
x_6 &= (x_1 \vee x_4) \wedge (x_2 \vee x_5),\\
x_7 &= t \cdot x_2 + x_3,\\
x_8 &= (x_1 \vee x_7) \wedge (x_2 \vee x_4),\\
x_9 &= u \cdot x_2 + x_1,\\
x_{10} &= (x_7 \vee x_9) \wedge (x_3 \vee x_6),\\
x_{11} &= (x_4 \vee x_5) \wedge (x_8 \vee x_9),\\
x_{12} &= (x_1 \vee x_{10}) \wedge (x_4 \vee x_5),\\
x_{13} &= (x_3 \vee x_6) \wedge (x_1 \vee x_{11}).
\end{aligned}$$

We set $X_0 = X(\frac{1}{2}, \frac{1}{2}, \frac{1}{3})$. The chirotope χ^ε is the alternating map such that

$$\chi^\varepsilon(i,j,k) = \begin{cases} \varepsilon & \text{if } (i,j,k) = (9,12,13),\\ \mathcal{X}_{X_0}(i,j,k) & \text{otherwise,} \end{cases}$$
$$\text{for all } (i,j,k) \in E^3 (i < j < k),$$

where $\varepsilon \in \{-, 0, +\}$.

The oriented matroid which we will study is $\mathcal{M}^\varepsilon := \mathcal{M}(E, \chi^\varepsilon)$.

Remark 2.1 We can replace X_0 with $X(\frac{1}{2}, \frac{1}{2}, u')$ where u' is chosen from $\mathbb{R}\backslash\{-1, 0, \frac{1}{2}, 1, \frac{3}{2}, 2, 3\}$. We will study the case $0 < u' < \frac{1}{2}$. If we choose u' otherwise, we can get other oriented matroids with disconnected realization spaces.

In the construction sequence, we need no assumption on the collinearity of x_9, x_{12}, x_{13}. Hence every realization of $\mathcal{M}^\varepsilon$ is linearly equivalent to a matrix $X(s,t,u)$ for certain s, t, u, up to multiplication on each column with positive scalar.

Moreover, we have the rational isomorphism

$$\mathcal{R}^*(\chi^\varepsilon) \times (0,\infty)^{12} \cong \mathcal{R}(\mathcal{M}^\varepsilon),$$

where $\mathcal{R}^*(\chi^\varepsilon) := \{(s,t,u) \in \mathbb{R}^3 \mid \mathcal{X}_{X(s,t,u)} = \chi^\varepsilon\}$. Thus we have only to prove that the set $\mathcal{R}^*(\chi^\varepsilon)$ is disconnected (resp. non-irreducible) to show that the realization space $\mathcal{R}(\mathcal{M}^\varepsilon)$ is disconnected (resp. non-irreducible).

The equation $\mathcal{X}_{X(s,t,u)} = \chi^\varepsilon$ means that

$$\operatorname{sgn}\det(x_i, x_j, x_k) = \chi^\varepsilon(i,j,k), \quad \text{for all } (i,j,k) \in E^3. \tag{1}$$

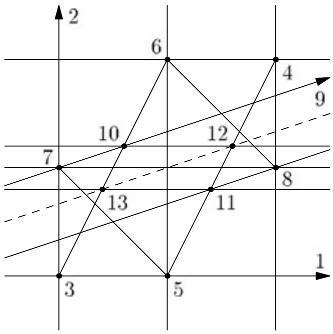

Fig. 1 Column vectors of X_0

We write some of them which give the equations on the parameters s, t, u. Note that for all $(i, j, k) \in E^3(\{i, j, k\} \neq \{9, 12, 13\})$, the sign is given by

$$\chi^\varepsilon(i, j, k) = \operatorname{sgn}\det(x_i, x_j, x_k)|_{s=t=1/2, u=1/3}.$$

From the equation $\operatorname{sgn}\det(x_2, x_3, x_5) = \operatorname{sgn}(s) = \operatorname{sgn}(1/2) = +1$, we get $s > 0$. Similarly, we get $\det(x_2, x_5, x_4) = 1 - s > 0$; therefore,

$$0 < s < 1. \tag{2}$$

From the equations $\det(x_1, x_7, x_3) = t > 0$, $\det(x_1, x_4, x_7) = 1 - t > 0$, we get

$$0 < t < 1. \tag{3}$$

Moreover, we have the inequalities

$$\det(x_1, x_9, x_3) = u > 0, \tag{4}$$

$$\det(x_4, x_7, x_9) = 1 - t - u > 0, \tag{5}$$

$$\det(x_3, x_9, x_8) = t - u > 0, \tag{6}$$

$$\det(x_5, x_{13}, x_7) = s\big(t^2 - (1 - s)u\big) > 0, \tag{7}$$

$$\det(x_6, x_{12}, x_8) = (1 - s)\big((1 - t)^2 - su\big) > 0. \tag{8}$$

From the equation $\det(x_9, x_{12}, x_{13}) = u(1 - 2s)(1 - 2t + tu - su)$, we get

$$\operatorname{sgn}\big(u(1 - 2s)(1 - 2t + tu - su)\big) = \varepsilon. \tag{9}$$

Conversely, if we have Eqs. (2)–(9), then we get (1).

We can interpret a (3, 13) matrix as the set of vectors $\{x_1, \dots, x_{13}\} \subset \mathbb{R}^3$. After we normalize the last coordinate for x_i $(i \in E\backslash\{1, 2, 9\})$, we can visualize the matrix on the affine plane $\{(x, y, 1) \in \mathbb{R}^3\} \cong \mathbb{R}^2$. Figure 1 shows the affine image of X_0. See Figs. 2, 3 for realizations of $\mathcal{M}^\varepsilon$.

Proof of Theorem 1.3 We prove that $\mathcal{R}^*(\chi^-)$ and $\mathcal{R}^*(\chi^+)$ are disconnected. From Eqs. (2)–(9), we obtain

$$\mathcal{R}^*(\chi^-) = \left\{ (s, t, u) \in \mathbb{R}^3 \,\middle|\, \begin{array}{c} 0 < s < 1,\ 0 < u < t < 1 - u, \\ (1 - t)^2 - su > 0,\ t^2 - (1 - s)u > 0, \\ (1 - 2s)(1 - 2t + tu - su) < 0 \end{array} \right\},$$

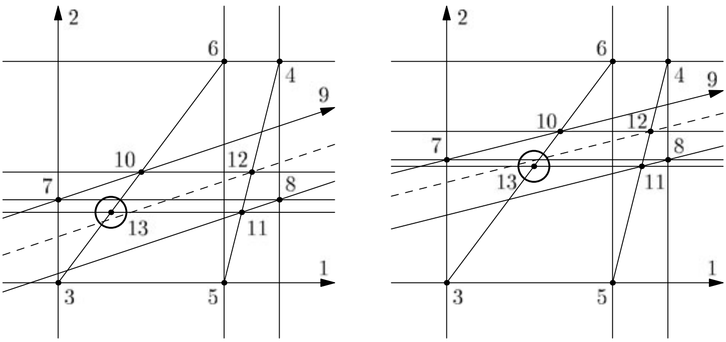

Fig. 2 Realization of $\mathcal{M}^-$ (*on the left*) and that of $\mathcal{M}^+$ (*on the right*)

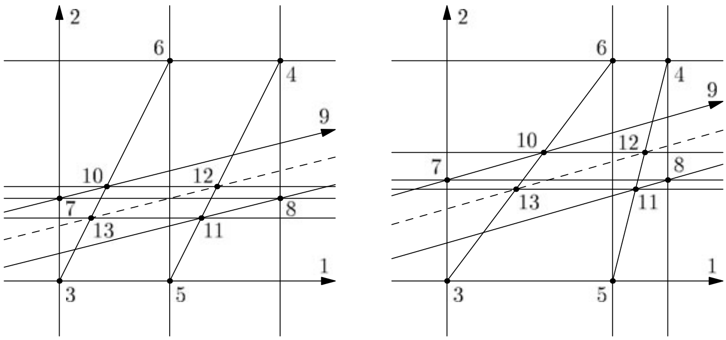

Fig. 3 Realizations of $\mathcal{M}^0$

$$\mathcal{R}^*(\chi^+) = \left\{ (s,t,u) \in \mathbb{R}^3 \;\middle|\; \begin{array}{c} 0 < s < 1,\ 0 < u < t < 1-u, \\ (1-t)^2 - su > 0,\ t^2 - (1-s)u > 0, \\ (1-2s)(1-2t+tu-su) > 0 \end{array} \right\}.$$

First, we show that $\mathcal{R}^*(\chi^-)$ is disconnected; more precisely, that it consists of two connected components. We do this by proving the next proposition.

Proposition 2.2

$$\mathcal{R}^*(\chi^-) = \left\{ (s,t,u) \in \mathbb{R}^3 \;\middle|\; \begin{array}{l} 0 < s < 1/2 \\ 1/2 < t < 1 \end{array},\ 0 < u < \min\left\{ 1-t, \frac{(1-t)^2}{s}, \frac{2t-1}{t-s} \right\} \right\}$$
$$\cup \left\{ (s,t,u) \in \mathbb{R}^3 \;\middle|\; \begin{array}{l} 1/2 < s < 1 \\ 0 < t < 1/2 \end{array},\ 0 < u < \min\left\{ t, \frac{t^2}{1-s}, \frac{1-2t}{s-t} \right\} \right\}.$$

Proof There are two cases:

$$(1-2s)(1-2t+tu-su) < 0 \Leftrightarrow \begin{cases} 1-2s > 0,\ 1-2t+tu-su < 0, \\ \text{or} \\ 1-2s < 0,\ 1-2t+tu-su > 0. \end{cases}$$

Note that

$$(2-u)(2t-1) = -2(1-2t+tu-su) + u(1-2s), \tag{10}$$

$$t^2 - (1-s)u = -(1-2t+tu-su) + (1-t)(1-t-u), \tag{11}$$

$$(1-t)^2 - su = (1-2t+tu-su) + t(t-u). \tag{12}$$

($\subset$) For the case $1-2s>0$ and $1-2t+tu-su<0$, the inequality $2t-1>0$ follows from Eq. (10). Since we have $0<s<1/2<t<1$, we get

$$\begin{cases} 1-2t+tu-su<0, \\ (1-t)^2-su>0, \\ 1-t-u>0 \end{cases} \Leftrightarrow u < \min\left\{1-t, \frac{(1-t)^2}{s}, \frac{2t-1}{t-s}\right\}. \tag{13}$$

For the other case $1-2s<0$, similarly, we get $1-2t>0$ from Eq. (10). Since we have $0<t<1/2<s<1$, we get

$$\begin{cases} 1-2t+tu-su>0, \\ t^2-(1-s)u>0, \\ t-u>0 \end{cases} \Leftrightarrow u < \min\left\{t, \frac{t^2}{1-s}, \frac{1-2t}{s-t}\right\}. \tag{14}$$

($\supset$) For the component $0<s<1/2<t<1$, the inequalities $1-2t+tu-su<0$, $(1-t)^2-su>0$, $1-t-u>0$ follow from (13). Thus we get $t^2-(1-s)u>0$ from Eq. (11). The inequality $u<t$ holds because $t>1/2$ and $u<1-t$.

For the other component $0<t<1/2<s<1$, similarly, we get the inequalities $1-2t+tu-su>0, t^2-(1-s)u>0, t-u>0$ from (14), and $(1-t)^2-su>0$ from Eq. (12). Last, we get $u<1-t$ from $t<1/2$ and $u<t$. □

For the set $\mathcal{R}^*(\chi^+)$, we have the following proposition.

Proposition 2.3

$$\mathcal{R}^*(\chi^+) = \left\{(s,t,u)\in\mathbb{R}^3 \,\middle|\, \begin{array}{l} 0<s<1/2, 0<u<1/2, \\ (1-u)^2-(1-s)u>0, \end{array} \sqrt{(1-s)u} < t < \frac{1-su}{2-u} \right\}$$
$$\cup \left\{(s,t,u)\in\mathbb{R}^3 \,\middle|\, \begin{array}{l} 1/2<s<1, 0<u<1/2, \\ (1-u)^2-su>0, \end{array} \frac{1-su}{2-u} < t < 1-\sqrt{su} \right\}.$$

The proof is similar to that of Proposition 2.2 and is omitted.

Proof of Theorem 1.4 We show that $\mathcal{R}^*(\chi^0)$ consists of two irreducible components whose intersection is not empty. From Eqs. (2)–(9), we get

$$\mathcal{R}^*(\chi^0) = \left\{(s,t,u)\in\mathbb{R}^3 \,\middle|\, \begin{array}{c} 0<s<1,\ 0<u<t<1-u, \\ (1-t)^2-su>0,\ t^2-(1-s)u>0, \\ (1-2s)(1-2t+tu-su)=0 \end{array} \right\}.$$

Here we have the decomposition

$$\mathcal{R}^*(\chi^0) = \left\{(s,t,u)\in\mathbb{R}^3 \,\middle|\, 0<t<1,\ 0<u<2t^2,\ u<2(1-t)^2,\ 1-2s=0 \right\}$$
$$\cup \left\{(s,t,u)\in\mathbb{R}^3 \,\middle|\, \begin{array}{l} 0<s<1,\ 0<u<1/2,\ (1-u)^2-su>0, \\ (1-u)^2-(1-s)u>0,\ 1-2t+tu-su=0 \end{array} \right\}.$$

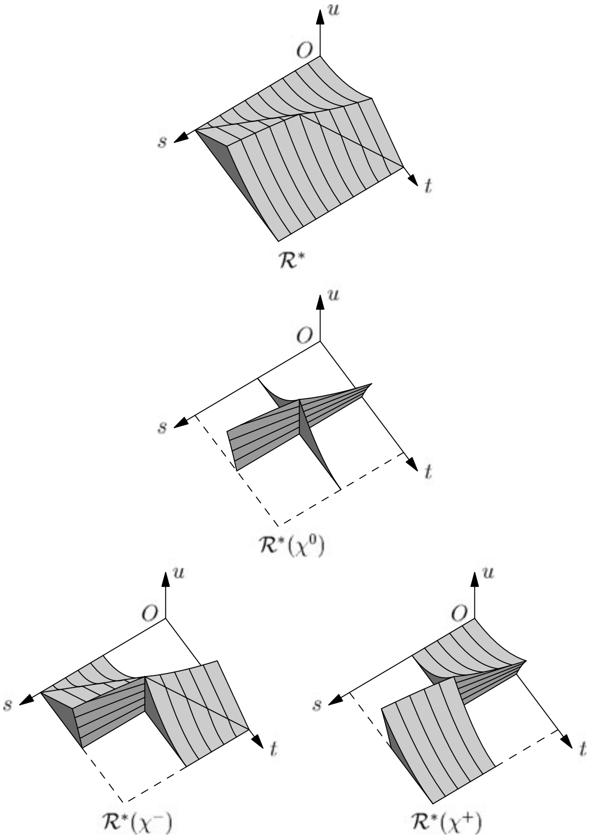

Fig. 4 $\mathcal{R}^*$ (*on the top*) and its partition $(\mathcal{R}^*(\chi^\varepsilon))_{\varepsilon\in\{-,0,+\}}$

The intersection of the two irreducible components is the set

$$\left\{(s,t,u)\in\mathbb{R}^3 \,\middle|\, s=t=\frac{1}{2}, 0<u<\frac{1}{2}\right\} \cong \left\{X\left(\frac{1}{2},\frac{1}{2},u\right) \,\middle|\, 0<u<\frac{1}{2}\right\}.$$

The proof is also similar to that of Proposition 2.2 and is omitted. □

Figure 3 shows two realizations of $\mathcal{M}^0$. On the left, it shows the affine image of $X(\frac{1}{2},\frac{3}{8},\frac{1}{4})$, on the irreducible component $1-2s=0$. On the right, it shows the image of $X(\frac{3}{4},\frac{11}{24},\frac{2}{7})$, on the other component $1-2t+tu-su=0$. These images can be deformed continuously to each other via $X(\frac{1}{2},\frac{1}{2},u)$ $(0<u<\frac{1}{2})$.

We set

$$\mathcal{R}^* := \left\{(s,t,u)\in\mathbb{R}^3 \,\middle|\, \begin{array}{c} 0<s<1,\ 0<u<t<1-u, \\ (1-t)^2-su>0,\ t^2-(1-s)u>0 \end{array}\right\}.$$

The set $\mathcal{R}^*\times(0,\infty)^{12}$ is rationally isomorphic to a realization space of a partial oriented matroid with the sign $\chi(9,12,13)$ non-fixed. The collection of semialgebraic sets $(\mathcal{R}^*(\chi^\varepsilon))_{\varepsilon\in\{-,0,+\}}$ is a partition of $\mathcal{R}^*$. Figure 4 illustrates this partition in 3-space.

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