

Determining All Universal Tilers

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Abstract A universal tiler is a convex polyhedron whose every cross-section tiles the plane. In this paper, we introduce a slight-rotating operation for cross-sections of polyhedra. By applying the operation to suitably chosen cross-sections, we prove that a convex polyhedron is a universal tiler if and only if it is a tetrahedron or a pentahedron with parallel facets.

Keywords Cross-section · The slight-rotating operation · Universal tiler

1 Introduction

A tiler is a polygon that can cover the plane by congruent repetitions without gaps or overlaps. The problem of determining all tilers, which is alternatively called the problem of tessellation or of plane tiling, is one of the most famous problems in discrete mathematics. It is still open to the best of our knowledge. The reader is referred to Grunbaum and Shephard's book [3] for an introduction to the theory of tessellation, and to Brass, Moser and Pach's book [2] for many further problems and results on tiling.

Considering a variant of the problem of plane tiling, Akiyama [1] found all convex polyhedra whose every development is a tiler. The key idea in his proof is to find whether the facets of a polyhedron tile the plane in a certain stamping manner. Noticing that facets are special cross-sections, we study another variant of the problem of plane tiling, namely, what kind of polyhedra are so well performed that each of its cross-sections is a tiler.

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Let \mathcal{P} be a convex polyhedron, and π a plane. Denote the intersection of π and \mathcal{P} by $C(\pi)$. We say that π intersects \mathcal{P} non-trivially if $C(\pi)$ is a non-degenerated polygon, that is, $C(\pi)$ is a polygon with at least 3 edges. We call $C(\pi)$ a *cross-section* if π crosses \mathcal{P} nontrivially. The polyhedron \mathcal{P} is said to be a *universal tiler* if every cross-section of \mathcal{P} is a tiler. The goal of this paper is to determine all universal tilers.

With the aid of Euler's formula, and Reinhardt's theorem [5], the author [7] managed to obtain the following necessary condition for the number of faces of a universal tiler by investigating specific cross-sections of any given polyhedron.

Theorem 1.1 *Any universal tiler has at most five facets. Moreover, a convex polyhedron is a universal tiler if it is a tetrahedron or a pentahedron with parallel facets.*

In light of the above theorem, the problem of determining all universal tilers turns out to be one of finding the list of pentahedral universal tilers. One of the difficulties in determining whether a pentahedron is a universal tiler is the fact that the list of pentagonal tilers is not known, although there are 14 classes of pentagonal tilers are found, see Hirschhorn and Hunt [4], and Sugimoto and Ogawa [6] for instance.

The key idea used in solving the universal tiler problem consists of two parts. The first part is to construct some technical conditions subject to which we can specify a cross-section of any given pentahedron. It is an extension of the method adopted in [7]. The other part is to introduce a *slight-rotating operation* and apply it to the specified cross-sections. Here is our main result.

Theorem 1.2 *A convex polyhedron is a universal tiler if and only if it is a tetrahedron or a pentahedron with parallel facets.*

Our proof of the above theorem has nothing to do with the knowledge of the complete list of pentagonal tilers. This paper is organized as follows. In the next section, we give necessary notions and notations on tessellations of the plane formed by a single pentagon. Section 3 is devoted to the construction of technical conditions and to the introduction of the slight-rotating operation. In Sect. 4, we complete the proof of Theorem 1.2.

2 Preliminary

In this section, we introduce some necessary notions and notations. Let $T = V^1V^2 \dots V^5$ be a pentagonal tiler. Let \mathcal{T} be a tessellation of the plane by copies of T . Denote the copies used in \mathcal{T} by $\{T_i = V_i^1V_i^2 \dots V_i^5 : i \in \Lambda\}$, where Λ is a set. Then every T_i has the same shape as T .

Let $i \in \Lambda$ and $\varepsilon > 0$. Since T is a tiler, the ε -neighborhood of the point V_i^j in \mathcal{T} must be covered without gaps or overlaps. It follows that either there is a sequence $V_i^j, V_{i_1}^{j_1}, V_{i_2}^{j_2}, \dots, V_{i_k}^{j_k}$ ($k \geq 2$) of angles arranged counter-clockwise which complete the 2π angle around the point V_i^j , or there is a sequence $V_{i_1}^{j_1}, \dots, V_{i_k}^{j_k}, V_i^j, V_{s_1}^{t_1}, \dots,$

Fig. 1 The local tessellation around the point V_i^j

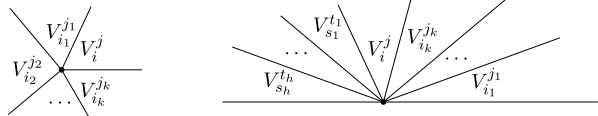
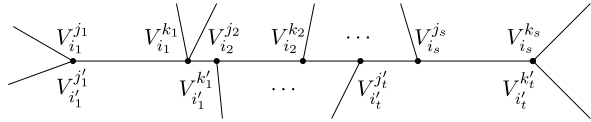


Fig. 2 The arrangement (2.3)



V_{sh}^{th} ($k + h \geq 1$) of angles arranged counter-clockwise which adds to π (see Fig. 1). In the former case, we denote the local tessellation around the point V_i^j by

$$S(V_i^j) = [V_i^j, V_{i1}^{j1}, V_{i2}^{j2}, \dots, V_{ik}^{jk}]. \tag{2.1}$$

In the latter case, we denote it by

$$S(V_i^j) = [V_i^j, V_{s1}^{t1}, \dots, V_{sh}^{th}, \pi, V_{i1}^{j1}, \dots, V_{ik}^{jk}]. \tag{2.2}$$

Suppose that there is a sequence $V_{i1}^{j1} V_{i1}^{k1}, V_{i2}^{j2} V_{i2}^{k2}, \dots, V_{is}^{js} V_{is}^{ks}$ of s collinear edges such that the point V_{ir}^{kr} coincides with the point V_{ir+1}^{jr+1} for each $1 \leq r \leq s - 1$, and the copies $T_{i1}, T_{i2}, \dots, T_{is}$ lie on the same side of the directed line segment $\vec{l} = V_{i1}^{j1} V_{is}^{ks}$. In this case, we denote

$$\vec{l} = V_{i1}^{j1} V_{i1}^{k1} + V_{i2}^{j2} V_{i2}^{k2} + \dots + V_{is}^{js} V_{is}^{ks}.$$

Assume also that

$$\vec{l} = V_{i'1}^{j'1} V_{i'1}^{k'1} + V_{i'2}^{j'2} V_{i'2}^{k'2} + \dots + V_{i't}^{j't} V_{i't}^{k't}$$

such that the copies $T_{i'1}, T_{i'2}, \dots, T_{i't}$ lie on the other side of \vec{l} (see Fig. 2). In this case, we say that \vec{l} is represented in \mathcal{T} , and is denoted as

$$S(V_{i1}^{j1} V_{i1}^{k1}, V_{i2}^{j2} V_{i2}^{k2}, \dots, V_{is}^{js} V_{is}^{ks}) = [V_{i'1}^{j'1} V_{i'1}^{k'1}, V_{i'2}^{j'2} V_{i'2}^{k'2}, \dots, V_{i't}^{j't} V_{i't}^{k't}]. \tag{2.3}$$

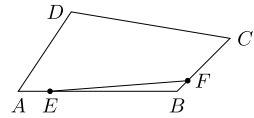
Let \mathbb{Z}, \mathbb{Z}^+ and \mathbb{N} denote the set of integers, positive integers and nonnegative integers, respectively. Let $N > 0$ and $a_1, \dots, a_5 \in \mathbb{N}$. We say that the set

$$\{a_1 V^1, a_2 V^2, \dots, a_5 V^5\} \tag{2.4}$$

is a sum- N -collection if $N = \sum_{i=1}^5 a_i V^i$. We call the set (2.4) a sum- N -collection of the angle V^j if $a_j \geq 1$. For convenience, we remove the term $a_i V^i$ from the collection (2.4) if $a_i = 0$. Denote by $R_N(V^j; T)$ the set of sum- N -collections of V^j in T . To simplify notation, we write $R_{2\pi}(V^j; T) = R(V^j; T)$. Since T is a tiler, we have

$$R(V^j; T) \neq \emptyset$$

Fig. 3 The pentagon $AEFCD$



for any $1 \leq j \leq 5$. For example, if

$$V^1 = V^2 = \frac{5\pi}{6}, \quad V^3 = \frac{2\pi}{3}, \quad V^4 = V^5 = \frac{\pi}{3},$$

then we have

$$R(V^1; T) = \{\{2V^1, V^4\}, \{2V^1, V^5\}, \{V^1, V^2, V^4\}, \{V^1, V^2, V^5\}\},$$

$$R_\pi(V^3; T) = \{\{V^3, V^4\}, \{V^3, V^5\}\}.$$

Note that the sum of any four angles of a pentagon is larger than 2π , while the sum of any three angles is larger than π . This immediately leads to the following lemma.

Lemma 2.1 *Any sum- 2π -collection (2.4) has at most three positive a_j , while any sum- π -collection (2.4) has at most two positive a_j .*

3 The Slight-Rotating Operation

In this section, we construct some technical conditions and introduce the slight-rotating operation. As will be seen, for any given pentahedron, there always exists a cross-section satisfying these conditions.

Denote by \mathcal{E} the set of convex pentahedronal universal tilers without parallel facets. To establish Theorem 1.2, it suffices to prove that $\mathcal{E} = \emptyset$. We will do this by contradiction. Throughout this paper, we assume that $\mathcal{P} \in \mathcal{E}$.

It is well-known that pentahedra have two distinct topological types. The first consists of quadrilateral-based pyramids. The other typical pentahedra have two triangular bases and three quadrilateral facets joining the corresponding sides of the two bases. In particular, we see that any pentahedron has a quadrilateral facet. Let

$$Q = ABCD$$

be a quadrilateral facet of \mathcal{P} . Let E be a point lying in the interior of the line segment AB such that E is neither A nor B . In this case, we write $E \in AB$. Set another point $F \in BC$. As illustrated in Fig. 3, we have a convex pentagon $T = AEFCD$. In [7], the author took a cross-section of \mathcal{P} by rotating T a bit along the line EF , where E is close to A , and F is near B . In this paper, we extend this method by selecting the points E and F more carefully. The following lemma will be frequently used in the proof of Lemma 3.2.

Lemma 3.1 *Let $N \geq 0$. Then the two sets*

$$\{\theta > B/2: N = a \cdot \theta + b \cdot \phi + c \cdot \psi, a \in \mathbb{Z}^+, b, c \in \mathbb{N}, \phi, \psi \in \{A, C, D\}\}$$

and

$$\{l < N : l = a \cdot CD + b \cdot DA + c \cdot AE, a, b \in \mathbb{N}, c \in \{0, \pm 1, \pm 2\}\}$$

are of finite cardinality.

The above lemma is easy to prove and we omit its proof.

Lemma 3.2 *There exists some $\delta_E > 0$ such that for any point $E \in AB$ with $AE < \delta_E$, there exists some $\delta_F = \delta_F(E) > 0$ such that for any point $F \in BC$ with $BF < \delta_F$, the pentagon $T = AEFCD$ satisfies*

- (i) $2AE < \min\{EF, FC, CD, DA\}$;
- (ii) $R(E; T) = R(F; T) \subseteq \{E, F, c\gamma\} : c \in \mathbb{Z}^+, \gamma \in \{A, C, D\}$;
- (iii) for any $a, b \in \mathbb{N}, c \in \{0, \pm 1, \pm 2\}$, and $l \in \{EF, 2FC\}$, we have

$$l \neq a \cdot CD + b \cdot DA + c \cdot AE.$$

Proof Choose a point $E \in AB$ and move it towards A such that the point E can be arbitrarily close to but never coincide with the point A . Similarly, choose $F \in BC$ and move it towards B such that F can be arbitrarily close to but never coincide with the point B . Let

$$x = \max\{AE, BF\}.$$

In the above transformation, the pentagon T tends to the quadrilateral Q . To be more precise, the lengths of edges vary in such a way that

$$\lim_{x \rightarrow 0} AE = 0, \quad \lim_{x \rightarrow 0} EF = AB, \quad \lim_{x \rightarrow 0} FC = BC, \tag{3.1}$$

while the sizes of angles vary as

$$\lim_{x \rightarrow 0} \angle AEF = \pi, \tag{3.2}$$

$$\lim_{x \rightarrow 0} \angle EFC = B. \tag{3.3}$$

By (3.1), there exists some δ_1 such that the condition (i) holds for any $x < \delta_1$.

By Lemma 2.1, we have

$$R(E; T) \subseteq \{aE, b\beta, c\gamma\} : a \in \mathbb{Z}^+, b, c \in \mathbb{N}, \beta, \gamma \in \{F, A, C, D\}. \tag{3.4}$$

By (3.2) and (3.3), there exists some $\delta_2 < \delta_1$ such that for any $x < \delta_2$, we have

$$2E + \alpha > 2\pi, \quad \forall \alpha \in \{A, F, C, D\}.$$

Let $x < \delta_2$. Then $a = 1$ in (3.4). By Lemma 3.1 and the limit (3.2), there exists some $\delta_3 < \delta_2$ such that for any $x < \delta_3$, we have

$$E + b\beta + c\gamma \neq 2\pi, \quad \forall b, c \in \mathbb{N}, \beta, \gamma \in \{A, C, D\}.$$

Let $x < \delta_3$. In view of (3.4), we have

$$R(E; T) \subseteq \{ \{E, bF, c\gamma\} : b \in \mathbb{Z}^+, c \in \mathbb{N}, \gamma \in \{A, C, D\} \}. \tag{3.5}$$

Note that

$$E + F = \pi + B. \tag{3.6}$$

So, every sum- 2π -collection $\{E, bF, c\gamma\}$, with $b \geq 1$, corresponds to a sum- $(\pi - B)$ -collection $\{b'F, c\gamma\}$, where $b', c \in \mathbb{N}$ and $\gamma \in \{A, C, D\}$. By Lemma 3.1 and the limit (3.3), there exists some $\delta_4 < \delta_3$ such that for any $x < \delta_4$, we have

$$b'F + c\gamma \neq \pi - B, \quad \forall b' \in \mathbb{Z}^+, c \in \mathbb{N}, \gamma \in \{A, C, D\}.$$

Let $x < \delta_4$. So $b' = 0$ and consequently $b = 1$. By (3.6), we see that $c \geq 1$. This proves that

$$R(E; T) \subseteq \{ \{E, F, c\gamma\} : c \in \mathbb{Z}^+, \gamma \in \{A, C, D\} \}. \tag{3.7}$$

Along the same lines, we have

$$R(F; T) \subseteq \{ \{aF, b\beta, c\gamma\} : a \in \mathbb{Z}^+, b, c \in \mathbb{N}, \beta, \gamma \in \{E, A, C, D\} \}. \tag{3.8}$$

By Lemma 3.1 and the limit (3.3), there exists some $\delta_5 < \delta_4$ such that for any $x < \delta_5$,

$$aF + b\beta + c\gamma \neq 2\pi, \quad \forall a \in \mathbb{Z}^+, b, c \in \mathbb{N}, \beta, \gamma \in \{A, C, D\}.$$

In view of (3.8), we deduce that

$$R(F; T) \subseteq \{ \{aF, bE, c\gamma\} : a, b \in \mathbb{Z}^+, c \in \mathbb{N}, \gamma \in \{A, C, D\} \}.$$

So any sum- 2π -collection of F is a sum- 2π -collection of E . From (3.7), we get $R(F; T) = R(E; T)$. This proves the condition (ii).

Take $\delta_E = \delta_5$. By Lemma 3.1, there exists some δ_F depending on the choice of E such that the condition (iii) holds for any $F \in BC$ with $BF < \delta_F$. This completes the proof. □

We remark that the points E and F can be chosen from any other pair of adjacent edges of Q , subject to analogous conditions. This idea will be employed in the proof of Theorem 4.1.

Fix two points $E^0 \in AB$ and $F^0 \in BC$ satisfying the conditions (i)–(iii), and consider the resulting pentagon $P^0 = A^0E^0F^0C^0D^0$, where $A^0 = A$, $C^0 = C$, and $D^0 = D$. Now we recursively define a sequence $\{P^k\}_{k \geq 1}$ of proper pentagonal cross-sections.

Let π be a plane which crosses \mathcal{P} nontrivially. Let l be a line in π . For any $\varepsilon > 0$, denote by π_+^ε (resp. π_-^ε) the plane obtained by rotating π around l by the angle ε (resp. $-\varepsilon$). It is clear that there exists some ε such that at least one of the planes π_+^ε and π_-^ε crosses \mathcal{P} nontrivially. Denote

$$p(\pi; l; \varepsilon) = \begin{cases} \pi_+^\varepsilon, & \text{if } \pi_+^\varepsilon \text{ crosses } \mathcal{P} \text{ nontrivially;} \\ \pi_-^\varepsilon, & \text{otherwise.} \end{cases}$$

Intuitively, the plane $p(\pi; l; \varepsilon)$ is obtained by rotating π a little along l . For simplifying notation, we use $\text{cr}(\pi; l; \varepsilon)$ to denote the intersection $C(p(\pi; l; \varepsilon))$. We say that a cross-section is *proper* if none of its vertices is a vertex of \mathcal{P} . Let $\mathcal{C}_{\mathcal{P}}$ be the set of proper pentagonal cross-sections of \mathcal{P} . Since every vertex of Q has valence 3, there exists some $\delta_0 > 0$ such that

$$\text{cr}(P^0; E^0 F^0; \varepsilon) \in \mathcal{C}_{\mathcal{P}}$$

for any $0 < \varepsilon \leq \delta_0$. Define

$$P^1 = \text{cr}(P^0; E^0 F^0; \delta_0) = A^1 E^1 F^1 C^1 D^1 \tag{3.9}$$

to be the initial cross-section. In particular, we have $E^1 = E^0$ and $F^1 = F^0$. Suppose that $P^k \in \mathcal{C}_{\mathcal{P}}$ is well defined for some $k \geq 1$. Let $e_1^k, e_2^k, \dots, e_5^k$ be the edges of P^k . It is clear that there exists some $0 < \delta_k < \delta_{k-1}$ such that for any $0 < \varepsilon \leq \delta_k$ and any edge e_j^k , we have

$$\text{cr}(P^k; e_j^k; \varepsilon) \in \mathcal{C}_{\mathcal{P}}.$$

Choosing an edge e_j^k , we can define

$$P^{k+1} = \text{cr}(P^k; e_j^k; \delta_k). \tag{3.10}$$

Then $P^{k+1} \in \mathcal{C}_{\mathcal{P}}$. Note that the cross-section P^{k+1} depends on the choices of e_j^k and δ_k , while the value of δ_k depends on P^k but is independent of the choice of e_j^k . We call the above procedure of getting P^{k+1} from P^k the *slight-rotating operation*.

Since all cross-sections P^k are proper, the slight-rotating operation has a sign-preserving property if we take δ_k small enough. Now we clarify the property. Denote by $\text{sgn}(x)$ the signum function, i.e., for any real number x ,

$$\text{sgn}(x) = \begin{cases} 1, & \text{if } x > 0; \\ 0, & \text{if } x = 0; \\ -1, & \text{if } x < 0. \end{cases}$$

Let $i \geq 0, N > 0, 1 \leq j \leq 5$ and $a_1, \dots, a_5, b_1, \dots, b_5 \in \mathbb{N}$. Let V_i^1, \dots, V_i^5 be the angles of P^i . Let

$$x_i = \text{sgn}\left(N - \sum_{l=1}^5 a_l V_i^l\right),$$

$$y_i = \text{sgn}\left(e_j^i - \sum_{l=1}^5 b_l e_l^i\right).$$

If $x_i y_i \neq 0$, then there exists some $0 < \delta \leq \delta_i$ such that for any $1 \leq j \leq 5$, the cross-section

$$P^{i+1} = \text{cr}(P^i; e_j^i; \delta)$$

satisfies $x_{i+1} = x_i$ and $y_{i+1} = y_i$. It is easy to show the above property if one regards P^{i+1} as a continuous function of the variable δ_i in the definitions (3.9) and (3.10).

With the aid of this property, we deduce that each cross-section in the sequence $\{P^k\}_{k \geq 1}$ satisfies some conditions analogous to (i)–(iii) as if the δ_i 's are chosen to be small enough. For all $k \geq 2$, we name the vertices of the pentagon P^k by

$$P^k = A^k E^k F^k C^k D^k$$

in the natural way that $E^k \in AB$ and $F^k \in BC$.

Theorem 3.3 *For any $k \geq 1$, there exists some $\delta_{k-1}^* \leq \delta_{k-1}$ such that for any edge $e_{j_1}^1$ of P^1 , any edge $e_{j_2}^2$ of P^2 , ..., and any edge $e_{j_{k-1}}^{k-1}$ of P^{k-1} , the cross-section P^k defined by (3.10) satisfies*

- (i') $2A^k E^k < \min\{E^k F^k, F^k C^k, C^k D^k, D^k A^k\}$;
- (ii') $R(E^k; P^k) = R(F^k; P^k) \subseteq \{E^k, F^k, c\gamma\} : c \in \mathbb{Z}^+, \gamma \in \{A^k, C^k, D^k\}$;
- (iii') for any $a, b \in \mathbb{N}, c \in \{0, \pm 1, \pm 2\}$ and $l \in \{E^k F^k, 2F^k C^k\}$, we have

$$l \neq a \cdot C^k D^k + b \cdot D^k A^k + c \cdot A^k E^k;$$

- (iv') if there exist some $a \in \mathbb{Z}^+, b, c \in \mathbb{N}, N \in \{\pi, 2\pi\}$, and three pairwise distinct angles $V_k^{j_1}, V_k^{j_2}, V_k^{j_3}$ of P^k such that

$$N = aV_k^{j_1} + bV_k^{j_2} + cV_k^{j_3},$$

then for any $0 \leq h \leq k$, the corresponding angles $V_h^{j_1}, V_h^{j_2}, V_h^{j_3}$ of P^h satisfy

$$N = aV_h^{j_1} + bV_h^{j_2} + cV_h^{j_3}.$$

Let δ_i^* ($i \geq 0$) be defined as in the above theorem. The next lemma will be used to exclude some cases from the set of 2π -sum-collections of the specified angle E^{k+1} .

Lemma 3.4 *Let $\mathcal{P} \in \mathcal{E}$ and $k \in \mathbb{Z}^+$. Suppose that*

$$\{E^k, F^k, A^k\} \in R(E^k; P^k). \tag{3.11}$$

Then we have

$$\{E^{k+1}, F^{k+1}, A^{k+1}\} \notin R(E^{k+1}; \text{cr}(P^k; C^k D^k; \delta_k^*)). \tag{3.12}$$

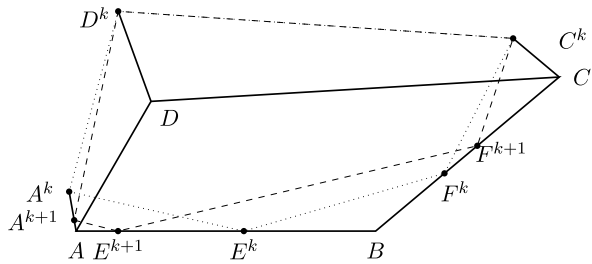
Similarly, if

$$\{E^k, F^k, C^k\} \in R(E^k; P^k),$$

then we have

$$\{E^{k+1}, F^{k+1}, C^{k+1}\} \notin R(E^{k+1}; \text{cr}(P^k; D^k A^k; \delta_k^*)). \tag{3.13}$$

Fig. 4 The parallel relation $F^k F^{k+1} C^k \parallel D^k A^k A^{k+1}$



Proof By (3.11), we have

$$F^k C^k \parallel D^k A^k. \tag{3.14}$$

Denote $\text{cr}(P^k; C^k D^k; \delta_k^*) = A^{k+1} E^{k+1} F^{k+1} C^k D^k$. If (3.12) is false, then we have

$$F^{k+1} C^k \parallel D^k A^{k+1}. \tag{3.15}$$

It is clear that the points F^k and F^{k+1} are distinct, while the points A^k and A^{k+1} are also distinct. Hence by (3.14) and (3.15), we find parallel facets $F^k F^{k+1} C^k \parallel D^k A^k A^{k+1}$ (see Fig. 4). But $\mathcal{P} \in \mathcal{E}$, thus we obtain a contradiction. The relation (3.13) can be proved similarly. This completes the proof. \square

4 The Main Result

In this section, we complete the proof of Theorem 1.2.

Let $k \in \mathbb{Z}^+$. Suppose that \mathcal{T}_k is a tessellation of the plane by copies of P^k . Denote the copies used in \mathcal{T}_k by

$$\{P_i^k = A_i^k E_i^k F_i^k C_i^k D_i^k : i \in \Lambda_k\},$$

where Λ_k is a set. Note that each copy in \mathcal{T}^k is arranged counter-clockwise either in the order

$$A_i^k, E_i^k, F_i^k, C_i^k, D_i^k, \tag{4.1}$$

or in the order

$$A_j^k, D_j^k, C_j^k, F_j^k, E_j^k.$$

Denote by \mathcal{I}_k the set of indices $i \in \Lambda_k$ such that the vertices of P_i^k are arranged counter-clockwise in the order (4.1). Without loss of generality, we can always suppose that $1 \in \mathcal{I}_k$.

A quadrilateral is said to be cyclic if all its vertices lie on the same circle. Recall that Q is a quadrilateral facet of \mathcal{P} .

Theorem 4.1 *The facet Q is either a parallelogram or a cyclic quadrilateral.*



Fig. 5 The local tessellations of A_1 and E_1

Proof Assume, to the contrary, that Q is neither a parallelogram nor cyclic. Without loss of generality, we can suppose that C is one of the largest angles of Q . Consider the cross-section

$$P^1 = \text{cr}(P^0; E^0F^0; \delta_1^*).$$

For convenience, we rewrite the copies of P^1 as $P_i^1 = A_iE_iF_iC_iD_i$. Since P^1 is a tiler, we can express $S(E_1) = [E_1, \beta_j, X]$, where $\beta_j \in \{A_j, E_j, F_j, C_j, D_j\}$, and X is a sequence of angles. By the condition (ii'), we have $\beta_j \neq E_j$.

Assume that $\beta_j = A_j$. By (i'), the sequence X contains no angle A_i . By (ii'), we deduce that $S(E_1) = [E_1, A_j, F_k]$ for some $k \in \Lambda_1$ (see Fig. 5). It follows that $E^1 + A^1 + F^1 = 2\pi$. By the condition (iv'), we obtain $C + D = \pi$. So Q is a parallelogram, which is a contradiction.

Below we can suppose that $\beta_j \in \{F_j, C_j, D_j\}$. In this case, the condition (i') implies $S(A_1) = [A_1, Y, \pi]$, where Y is a sequence of angles (see Fig. 5). By (i') and (ii'), we deduce that Y contains no angle A_i . Thus there exist some $b \in \mathbb{Z}^+$ and $\beta^1 \in \{C^1, D^1\}$ such that $A^1 + b\beta^1 = \pi$. By (iv'), we find that $A + b\beta = \pi$ for some $\beta \in \{C, D\}$. Since Q is neither a parallelogram nor cyclic, we see that $b \geq 2$. But C is one of the largest angles, so $\beta = D$. Namely,

$$A + bD = \pi. \tag{4.2}$$

Consider $E' \in AD$ and $F' \in CD$ subject to the conditions corresponding to (i')–(iv'). Since C is one of the largest angles, we derive

$$A + b'B = \pi \tag{4.3}$$

for some $b' \geq 2$. Adding (4.2) and (4.3) yields

$$2\pi = 2A + bD + b'B \geq 2(A + B + D) = 2(2\pi - C),$$

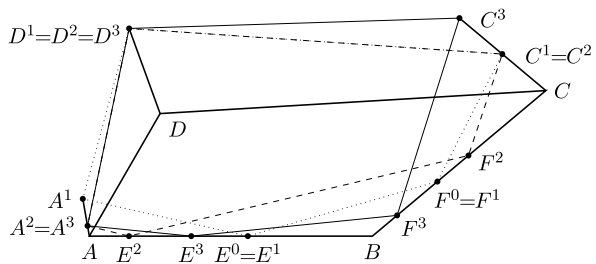
namely $C \geq \pi$, hence we obtain a contradiction. This completes the proof. □

Theorem 4.2 *The facet Q is cyclic.*

Proof Assume to the contrary that Q is a non-cyclic facet. By Theorem 4.1, it is a parallelogram. Without loss of generality, we can suppose that C is one of the smallest angles of Q , and that $A + D = \pi$. Since Q is non-cyclic, we deduce that the angles C and D have distinct sizes. Therefore, the condition (ii') implies that

$$R(E^0; P^0) \subseteq \{\{E^0, F^0, A^0\}, \{E^0, F^0, C^0\}\}.$$

Fig. 6 The cross-sections P^1 (dotted), P^2 (dashed) and P^3 (thin)



Consider the cross-sections (see Fig. 6)

$$P^1 = \text{cr}(P^0; E^0 F^0; \delta_2^*) = A^1 E^1 F^1 C^1 D^1, \tag{4.4}$$

$$P^2 = \text{cr}(P^1; C^1 D^1; \delta_2^*) = A^2 E^2 F^2 C^2 D^2, \tag{4.5}$$

$$P^3 = \text{cr}(P^2; D^2 A^2; \delta_2^*) = A^3 E^3 F^3 C^3 D^3. \tag{4.6}$$

By Lemma 3.4 and (iv'), we see that $R(E^3; P^3) = \emptyset$. But the cross-section P^3 tiles the plane, hence we obtain a contradiction. This completes the proof. \square

To proceed further, we need the following technical lemma.

Lemma 4.3 *Let $k \geq 1$. Suppose that*

$$R(E^k; P^k) = R(F^k; P^k) = \{\{E^k, F^k, D^k\}\}. \tag{4.7}$$

Then any edge $F_i^k C_i^k$ is not represented. Moreover, any line segment $F_i^k C_i^k + C_j^k F_j^k$ or $E_i^k A_i^k + C_j^k F_j^k$ (if it exists) is not represented.

Proof For convenience, rewrite the copies $\{P_i^k : i \in \Lambda_k\}$ as $P_i^k = A_i E_i F_i C_i D_i$. Assume to the contrary that $F_i C_i$ is represented. By (4.7), there is no point F_j ($j \neq i$) lying on the edge $F_i C_i$, and there is at most one point E_j lying on $F_i C_i$. Therefore, we have

$$F^k C^k = a_1 \cdot C^k D^k + b_1 \cdot D^k A^k + c_1 \cdot A^k E^k$$

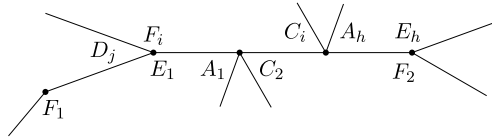
for some $a_1, b_1 \in \mathbb{N}$ and $c_1 \in \{0, 1\}$. This contradicts the condition (iii'). Hence $F_i C_i$ is not represented. For the same reason, any line segment $F_i C_i + C_j F_j$ (if it exists) is not represented.

Suppose that $1 \in \mathcal{I}_k$ and the line segment $E_1 A_1 + C_2 F_2$ is represented in \mathcal{T}_k . Then $2 \in \mathcal{I}_k$ (see Fig. 7). We claim that there exist some $i, j \in \Lambda_k$ such that

$$S(E_1) = [E_1, F_i, D_j]. \tag{4.8}$$

In fact, by (4.7), there is at most one copy P_i^k such that the vertex E_i lying on the line segment $E_1 F_2$, where $i \neq 1, 2$. Also, there is at most one point F_j ($j \neq 1, 2$) lying on the line segment $E_1 F_2$. If the sequence $S(E_1 F_2)$ contains an edge $E_i F_i$, then the

Fig. 7 The tessellation of the ε -neighborhood of the line segment E_1F_2



condition (ii') implies (4.8). Otherwise, by (iii'), we can deduce that $S(E_1F_2)$ contains an edge F_iC_i . Thus, the condition (i') yields $S(E_1A_1, C_2F_2) = [F_iC_i, A_hE_h]$ for some i and h . In particular, the relation (4.8) holds. This proves the claim.

By (4.8), the edge E_1F_1 is represented. It follows that there is at most one point E_i ($i \neq 1$) lying on the edge E_1F_1 , and there is no point F_j ($j \neq 1$) lying on E_1F_1 . So there exist some $a_2, b_2 \in \mathbb{N}$ and $c_2 \in \{0, 1\}$ such that

$$E^k F^k = a_2 \cdot C^k D^k + b_2 \cdot D^k A^k + c_2 \cdot A^k E^k,$$

contradicting the condition (iii'). This completes the proof. □

Now we are in a position to complete the proof of the main result.

Proof of Theorem 1.2 It suffices to show that $\mathcal{E} = \emptyset$. Assume, to the contrary, that $\mathcal{P} \in \mathcal{E}$. By Theorem 4.2, we see that \mathcal{Q} is cyclic. Suppose that D is one of the smallest angles of \mathcal{Q} . By the condition (ii'), we have

$$R(E^0; P^0) \subseteq \{ \{E^0, F^0, A^0\}, \{E^0, F^0, C^0\}, \{E^0, F^0, D^0\} \}. \tag{4.9}$$

Consider the cross-sections P^1, P^2, P^3 defined by (4.4)–(4.6). By (4.9), Lemma 3.4 and the condition (iv'), we see that $R(E^3; P^3) \subseteq \{ \{E^3, F^3, D^3\} \}$. Since P^3 is a tiler, by (ii'), we have

$$R(E^3; P^3) = R(F^3; P^3) = \{ \{E^3, F^3, D^3\} \}. \tag{4.10}$$

By (4.10) and (iv'), we deduce that

$$R_\pi(A^3; P^3) \subseteq \{ \{A^3, C^3\} \} \cup \{ \{bA^3, aD^3\} : b \in \mathbb{Z}^+, a \in \mathbb{N} \}. \tag{4.11}$$

Consider the tessellations of the plane by copies of P^3 . For convenience, rewrite

$$\mathcal{T} = \mathcal{T}_3, \quad P_i^3 = A_i E_i F_i C_i D_i, \quad \Lambda_3 = \Lambda, \quad \mathcal{I}_3 = \mathcal{I}.$$

By (4.10) and (i'), there exist an $m \in \mathbb{Z}^+$ and some $i_1, i_2, \dots, i_m \in \Lambda_3$ such that

$$S(A_1) = [A_1, \alpha_{i_1}, \alpha_{i_2}, \dots, \alpha_{i_m}, \pi], \tag{4.12}$$

where $\alpha_{i_j} \in \{A_{i_j}, C_{i_j}, D_{i_j}\}$. Let $\varepsilon > 0$. Considering the tessellation of the ε -neighborhood of the point E_{i_j} , we see that

$$\alpha_{i_j} \neq A_{i_j}, \quad \forall 1 \leq j \leq m. \tag{4.13}$$

Set $i_1 = 2$. Assume that $m = 1$. If $\alpha_2 = D_2$, then $\{E^3, F^3, C^3\} \in R(E^3; P^3)$, contradicting (4.10). Therefore, by (4.13), the expression (4.12) reduces to $S(A_1) =$

Fig. 8 The tessellation for the case of $m = 1$

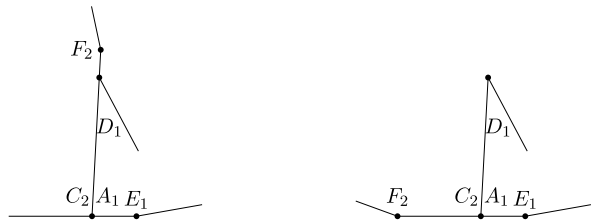
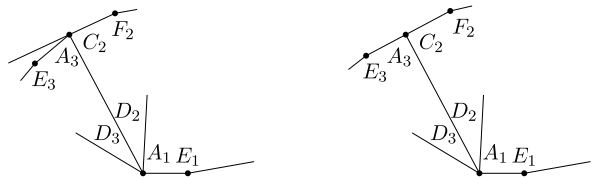


Fig. 9 The tessellation for the case of $S(D_2C_2) = [D_3A_3]$



$[A_1, C_2, \pi]$ (see Fig. 8). In this case, if $2 \notin \mathcal{I}$, then the point F_2 lies on the line determined by D_1A_1 . By (4.10), we find that the edge F_2C_2 is represented, contradicting Lemma 4.3. So $2 \in \mathcal{I}$. By (4.10), the line segment $E_1A_1 + C_2F_2$ is represented, also contradicting Lemma 4.3.

Below we can suppose that $m \geq 2$. Set $i_2 = 3$. Then the expression (4.12) reduces to

$$S(A_1) = [A_1, D_2, D_3, \dots, \pi]. \tag{4.14}$$

Assume that $2 \in \mathcal{I}$. By Lemma 4.3, the edge F_2C_2 is not represented. So there exist an $s \in \mathbb{Z}^+$ and some $j_1, j_2, \dots, j_s \in \Lambda_3$ such that

$$S(C_2) = [C_2, \pi, \beta_{j_1}, \beta_{j_2}, \dots, \beta_{j_s}], \tag{4.15}$$

where β_i is an angle of the copy T_i . Therefore the edge D_2C_2 is represented. In view of (4.10), (4.14), and (4.15), the line segment D_2C_2 contains neither a point E_i , nor a point F_i , for some i . Moreover, by (4.10), no point A_j lies in the interior of D_2C_2 , since otherwise the ε -neighborhood of the point E_j cannot be tiled. For the same reason, there is no point, say, C_j , lying in the interior of D_2C_2 . Therefore,

$$S(D_2C_2) \in \{[D_3A_3], [D_3C_3]\}.$$

So $j_s = 3$. If $S(D_2C_2) = [D_3A_3]$, then either the edge A_3E_3 is represented (when $s \geq 2$), which is impossible by the conditions (i') and (4.10); or the line segment $E_3A_3 + C_2F_2$ is represented (when $s = 1$), which contradicts Lemma 4.3. See Fig. 9. In the other case, $S(D_2C_2) = [D_3C_3]$ (see Fig. 10). Then either F_3C_3 is represented (when $s \geq 2$), or $F_2C_2 + C_3F_3$ is represented (when $s = 1$). Both of them contradict Lemma 4.3.

Hence we have $2 \notin \mathcal{I}$. Considering the tessellation of the ε -neighborhood of the point E_2 , we find that

$$S(A_2) = [A_2, \pi, \gamma_{h_1}, \gamma_{h_2}, \dots, \gamma_{h_t}], \tag{4.16}$$

Fig. 10 The tessellation for the case of $S(D_2C_2) = [D_3C_3]$

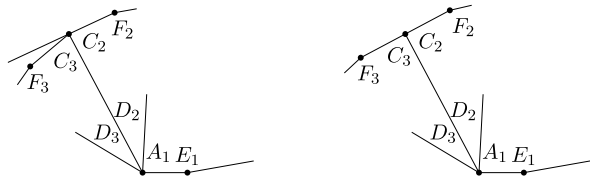


Fig. 11 The tessellation for the case of $S(D_2A_2) = [D_3A_3]$

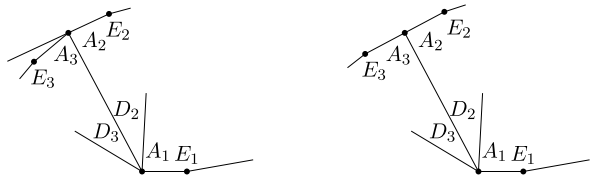


Fig. 12 The tessellation for the case of $S(D_2A_2) = [D_3C_3]$

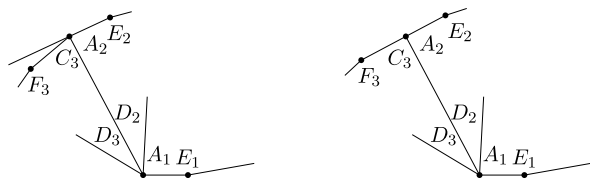
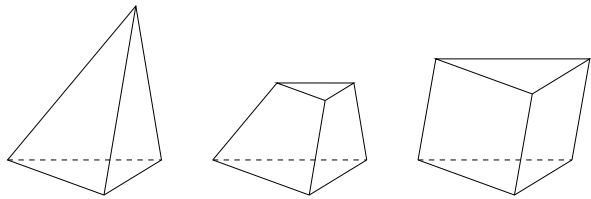


Fig. 13 Three classes of universal tilers



where $t \geq 1$ and γ_{h_i} is an angle of the copy T_{h_i} . Therefore D_2A_2 is represented. For the same reason as in the case of $2 \in \mathcal{I}$, we deduce that

$$S(D_2A_2) \in \{[D_3A_3], [D_3C_3]\}.$$

If $S(D_2A_2) = [D_3A_3]$, then either A_3E_3 is represented (when $t \geq 2$), or $E_2A_2 + A_3E_3$ is represented (when $t = 1$). See Fig. 11. Both of them are absurd by the condition (i'). In the other case $S(D_2A_2) = [D_3C_3]$. Then either C_3F_3 is represented (when $t \geq 2$), or $F_3C_3 + A_2E_2$ is represented. See Fig. 12. Both of them contradict Lemma 4.3.

To sum up, the cross-section P^3 does not tile the plane. This implies that $\mathcal{E} = \emptyset$, and that completes the proof. \square

We conclude this paper by giving a graphical illustration of our main result. Let P be a pentahedral universal tiler. Since any quadrilateral-based pyramid has no parallel facets, we see that P has two triangular bases. Moreover, the parallel facets of P must be its two triangular bases. It turns out that either all quadrilateral facets of P are trapezoids, or P is a triangular prism. In the former case, the pentahedron P can be obtained from cutting a tetrahedron by a plane parallel to one of its facets. In conclusion, universal tilers have three classes as illustrated in Fig. 13.

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References

1. Akiyama, J.: Tile-makers and semi-tile-makers. *Am. Math. Mon.* **114**, 602–609 (2007)
2. Brass, P., Moser, W.O.J., Pach, J.: *Research Problems in Discrete Geometry*. Springer, Berlin (2005)
3. Grünbaum, B., Shephard, G.C.: *Tilings and Patterns*. Freeman, New York (1986)
4. Hirschhorn, M.D., Hunt, D.C.: Equilateral convex pentagons which tile the plane. *J. Comb. Theory, Ser. A* **39**, 1–18 (1985)
5. Reinhardt, K.: *Über die Zerlegung der Ebene in Polygone*, Dissertation, Universität Frankfurt (1918)
6. Sugimoto, T., Ogawa, T.: Properties of tilings by convex pentagons. *Forma* **21**, 113–128 (2006)
7. Wang, D.G.L.: On universal tilers. [arXiv:1109.0813](https://arxiv.org/abs/1109.0813) [math.CO]