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Rigidity of Quasicrystallic and Z^γ-Circle Patterns

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Abstract The uniqueness of the orthogonal Z^γ-circle patterns as studied by Bobenko and Agafonov is shown, given the combinatorics and some boundary conditions. Furthermore we study (infinite) rhombic embeddings in the plane which are quasicrystallic, that is, they have only finitely many different edge directions. Bicoloring the vertices of the rhombi and adding circles with centers at vertices of one of the colors and radius equal to the edge length leads to isoradial quasicrystallic circle patterns. We prove for a large class of such circle patterns which cover the whole plane that they are uniquely determined up to affine transformations by the combinatorics and the intersection angles. Combining these two results, we obtain the rigidity of large classes of quasicrystallic Z^γ-circle patterns.

Keywords Circle pattern · Rigidity · Quasicrystallic

1 Introduction

Circles, especially circle packings and circle patterns, have successfully been used over the past years to define and study discrete analogs of classical smooth objects. In particular, this approach leads to discrete holomorphic mappings, for example discrete analogs of the power functions z^γ, and to discrete holomorphic function theory. See for example [7, 8, 11, 24] for some of the contributions to the theory of circle patterns and [28] for results on circle packings.

In this article we focus on circle patterns which are characterized by a given combinatorics specifying which circles should intersect and by the corresponding intersection angles. Thus we associate to a circle pattern a pattern of kites corresponding

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to intersecting circles, see Figs. 2 and 7. A particularly suitable source for the required knowledge on circle patterns and their relations to consistency, integrability, and discrete holomorphic functions is the textbook [11] in discrete differential geometry. Our main results can roughly be summarized as follows: The combinatorics (together with a suitable embedding, intersection angles, and boundary conditions, if necessary) determines the geometry of the circle pattern. This rigidity result can be interpreted as a discrete version of Liouville's Theorem in complex analysis.

Our first result concerns the uniqueness of circle patterns which are discrete analogs of the power functions z^γ for $\gamma \in (0, 2)$ as defined in [2, 4, 7]. Here, the square grid combinatorics, the orthogonal intersection angles, and the two boundary lines of a sector uniquely determine the geometry of the pattern, see Fig. 3 for an illustration. The rigidity of orthogonal Z^γ-circle patterns was only known for rational γ, see [2].

Furthermore, we consider the case of a circle pattern which covers the whole complex plane and for which the radii of all circles are equal and the interiors of different kites are disjoint. Then the corresponding kites form a rhombic embedding. Also, assume that there are only finitely many different edge directions of the kites. Such rhombic embeddings are called *quasicrystallic* [10]. Additionally, orient an edge $\vec{e}$ and consider a line perpendicular to the edge. Move this line parallelly in positive and in negative direction along $\vec{e}$. We suppose that in both cases this moving line intersects infinitely often edges parallel to $\vec{e}$. We assume that this property is true at least for two edges with linearly independent directions. Such rhombic embeddings of the plane are for example given by Penrose tilings, see for example Fig. 7. We show that any other embedded circle pattern with the same combinatorics and intersection angles of the kites is the image of this embedding by an affine transformation.

Rigidity of some classes of infinite circle patterns of the plane have already been studied. Schramm [24] considers square grid combinatorics and orthogonal intersection angles. As an essential step of our proof we generalize his result to square grid circle patterns with regular intersection angles $\psi \in (0, \pi)$ and $(\pi - \psi)$. He [19] studies disk triangulation graphs and exterior intersection angles in $[\pi/2, \pi]$ which does not cover the class of isoradial quasicrystallic circle patterns defined above.

As observed in [10] isoradial quasicrystallic rhombic embeddings can be used to define corresponding quasicrystallic Z^γ-circle patterns, see also [11]. Examples of quasicrystallic Z^γ-circle patterns are shown in Figs. 8, 10, and 12. They have been created using software developed by Veronika Schreiber for her diploma thesis [25]. Our rigidity result for orthogonal Z^γ-circle patterns is also generalized for large classes of quasicrystallic Z^γ-circle patterns.

There is some more literature concerning rigidity for infinite planar circle *packings*, that is, configurations with non-overlapping touching circles, see [15, 22, 20, 23]. Our rigidity proofs adapt some of the ideas which have been used for packings. In particular, we apply discrete potential theory.

This paper is organized as follows. First we introduce terminology and present useful facts about circle patterns. We especially focus on regular circle patterns with square grid combinatorics. Then we recall in Sect. 4 the definition and some properties of the orthogonal Z^γ-circle patterns for $\gamma \in (0, 2)$ as studied in [1, 2, 4] and prove their rigidity. In Sect. 5 we introduce quasicrystallic circle patterns and prove

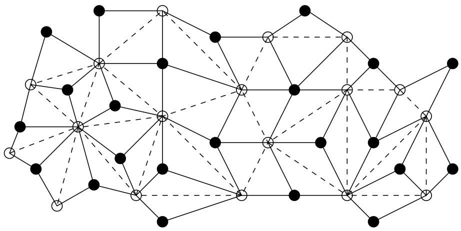

Fig. 1 An example of a b-quad-graph $\mathscr{D}$ (*black edges* and *bicolored vertices*) and its associated graph G (*dashed edges* and *white vertices*)

uniqueness for a class of these patterns. Finally, we recall the definition and some facts about quasicrystallic Z^γ-circle patterns for $\gamma \in (0,2)$ as studied in [3, 10, 11] and prove rigidity for certain classes of these patterns. A more detailed version of the results can be found in [12].

2 Circle Patterns

In this section we focus on a definition and some useful properties of circle patterns. We describe circle patterns using a planar embedded graph with quadrilateral faces (quad-graph) and intersection angles.

A *quad-graph* $\mathscr{D}$ is a strongly regular cell decomposition of a domain in $\mathbb{C}$ possibly with boundary such that all 2-cells (faces) are embedded and counterclockwise oriented. Furthermore all faces of $\mathscr{D}$ are quadrilaterals, that is, there are exactly four edges incident to each face. A quad-graph $\mathscr{D}$ is called *b-quad-graph* if its 1-skeleton, that is, the graph built by its vertices and edges, is a bipartite graph.

We always assume that the vertices of a b-quad-graph $\mathscr{D}$ are colored white and black. To these two sets of vertices we associate two planar graphs G and G^* as follows. The vertices $V(G)$ are all white vertices of $V(\mathscr{D})$. The edges $E(G)$ correspond to faces of $\mathscr{D}$, that is, two vertices of G are connected by an edge if and only if they are incident to the same face, see Fig. 1. The dual graph G^* is constructed analogously by taking as vertices $V(G^*)$ all black vertices of $\mathscr{D}$. $\mathscr{D}$ is called *simply connected* if it is the cell decomposition of a simply connected domain of $\mathbb{C}$ and if every closed chain of faces is null homotopic in $\mathscr{D}$.

For the intersection angles, we use a *labeling* $\alpha : F(\mathscr{D}) \to (0,\pi)$ of the faces of $\mathscr{D}$. By abuse of notation, α can also be understood as a function defined on $E(G)$ or on $E(G^*)$. The labeling α is called *admissible* if it satisfies the following condition at all interior black vertices $v \in V_{\text{int}}(G^*)$:

$$\sum_{f \text{ incident to } v} \alpha(f) = 2\pi. \tag{1}$$

Definition 2.1 Let $\mathscr{D}$ be a b-quad-graph with associated graph G and let $\alpha : E(G) \to (0,\pi)$ be an admissible labeling. An (*immersed planar*) *circle pattern for* $\mathscr{D}$ (*or* G) *and* α are an indexed collection $\mathscr{C} = \{C_z : z \in V(G)\}$ of circles in $\mathbb{C}$ and an indexed collection $\mathscr{K} = \{K_e : e \in E(G)\} = \{K_f : f \in F(\mathscr{D})\}$ of closed kites, which all carry the same orientation, such that the following conditions hold.

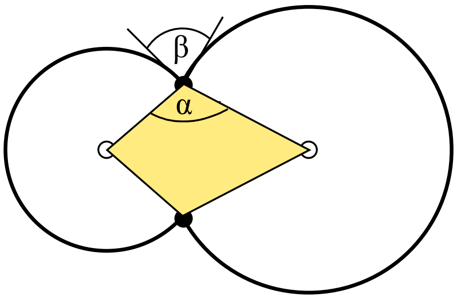

Fig. 2 The exterior intersection angle α of two intersecting circles and the associated kite built from centers and intersection points. $\beta = \pi - \alpha$ is the interior intersection angle

(1) If $z_1, z_2 \in V(G)$ are incident vertices in G, the corresponding circles C_{z_1}, C_{z_2} intersect with exterior intersection angle $\alpha([z_1, z_2])$. Furthermore, the kite $K_{[z_1,z_2]}$ is bounded by the centers of the circles C_{z_1}, C_{z_2}, the two intersection points, and the corresponding edges, as in Fig. 2. The intersection points are associated to black vertices of $V(\mathscr{D})$ or to vertices of $V(G^*)$.
(2) If two faces are incident in $\mathscr{D}$, the corresponding kites share a common edge.
(3) Let $f_1, \dots, f_n \in F(\mathscr{D})$ be the faces incident to an interior vertex $v \in V_{\text{int}}(\mathscr{D})$. Then the kites $K_{f_1}, \dots, K_{f_n}$ have mutually disjoint interiors. The union $K_{f_1} \cup \dots \cup K_{f_n}$ is homeomorphic to a closed disk and contains the point $p(v)$ corresponding to v in its interior.

The circle pattern is called *embedded* if all kites of $\mathscr{K}$ have mutually disjoint interiors. The circle pattern is called *isoradial* if all circles of $\mathscr{C}$ have the same radius.

Note that by construction and definition all circle patterns considered in this article will be planar and immersed. Therefore the notion "circle pattern" will include these properties in the following. The definition can also be adapted for other classes of circle patterns. Examples include spherical circle patterns, see [9, 21], and branch points where the circle patterns are not immersed, see [9] for the more general case of cone like singularities.

In our definition we associate a circle pattern $\mathscr{C}$ to an immersion into the plane identified with $\mathbb{C}$ of the kite pattern $\mathscr{K}$ corresponding to $\mathscr{D}$, see in particular condition (3) of the definition. By conditions (1) and (2), the edges of the kites incident to the same white vertex are of equal length. The kites may also be non-convex and can be constructed from a (suitable) given set of circles and from the combinatorics of G.

Note further that immersed circle patterns, which are locally embedded by definition, need not to be globally embedded, see for example the erf-like pattern in [24].

For our study of a circle pattern $\mathscr{C}$ we will use the radius function $r_{\mathscr{C}} = r$ which assigns to every vertex $z \in V(G)$ the radius $r_{\mathscr{C}}(z) = r(z)$ of the corresponding circle C_z. The index $\mathscr{C}$ will be dropped whenever there is no confusion likely. The following proposition specifies a condition for a radius function to originate from a circle pattern, see [9] for a proof.

Proposition 2.2 *Let G be associated to a b-quad-graph $\mathscr{D}$ and let α be an admissible labeling.*

Suppose that $\mathscr{C}$ is a circle pattern for $\mathscr{D}$ and α with radius function $r = r_{\mathscr{C}}$. Then for every interior vertex $z_0 \in V_{\text{int}}(G)$ we have

$$\left(\sum_{[z,z_0]\in E(G)} f_{\alpha([z,z_0])}\big(\log r(z) - \log r(z_0)\big) \right) - \pi = 0, \tag{2}$$

where

$$f_\theta(x) := \frac{1}{2i} \log \frac{1 - e^{x-i\theta}}{1 - e^{x+i\theta}},$$

and the branch of the logarithm is chosen such that $0 < f_\theta(x) < \pi$.

Conversely, suppose that $\mathscr{D}$ is simply connected and that $r : V(G) \to (0,\infty)$ satisfies (2) for every $z \in V_{\text{int}}(G)$. Then there is a circle pattern for G and α with radius function r. This pattern is unique up to isometries of $\mathbb{C}$.

For the special case of orthogonal circle patterns with the combinatorics of the square grid, there are also other characterizations, see for example [24].

Note that $2 f_{\alpha([z,z_0])}(\log r(z) - \log r(z_0))$ is the angle at z_0 of the kite with edge lengths $r(z)$ and $r(z_0)$ and angle $\alpha([z,z_0])$, as in Fig. 2. Equation (2) is the closing condition for the closed chain of kites which correspond to the edges incident to z_0. This corresponds to condition (3) of Definition 2.1.

For further use we mention some properties of f_θ, see [27, Lemma 2.2].

Lemma 2.3 (i) *The derivative of f_θ is $f_\theta'(x) = \frac{\sin\theta}{2(\cosh x - \cos\theta)} > 0$.*

(ii) *The function f_θ satisfies the functional equation $f_\theta(x) + f_\theta(-x) = \pi - \theta$.*

If there exists an isoradial circle pattern, we can obtain another circle pattern from a given radius function.

Lemma 2.4 *Let G be associated to a b-quad-graph $\mathscr{D}$ and let α be an admissible labeling. Suppose that there exists an isoradial circle pattern for G and α. Let r be the radius function of a circle pattern for G and α. Then there is a circle pattern $\mathscr{C}$ for G and α with radius function $r_{\mathscr{C}} = 1/r$.*

Proof By Lemma 2.3(ii) the function $1/r$ satisfies condition (2) of Proposition 2.2 for all interior vertices $z_0 \in V_{\text{int}}(G)$. In particular,

$$\sum_{[z,z_0]\in E(G)} f_{\alpha([z,z_0])}\left(\log \frac{1}{r(z)} - \log \frac{1}{r(z_0)}\right)$$

$$= \sum_{[z,z_0]\in E(G)} \big(\pi - \alpha\big([z,z_0]\big)\big) - \sum_{[z,z_0]\in E(G)} f_{\alpha([z,z_0])}\left(\log \frac{r(z)}{r(z_0)}\right) = 2\pi - \pi = \pi.$$

Here we have also used that $\sum_{[z,z_0]\in E(G)}(\pi - \alpha([z,z_0])) = 2\pi$ since there is an isoradial circle pattern for G and α and the assumption that r is the radius function of a circle pattern for G and α. □

Let G be associated to a b-quad-graph $\mathscr{D}$ and let α be an admissible labeling. Suppose that $\mathscr{C}_1$ and $\mathscr{C}_2$ are circle patterns for $\mathscr{D}$ and α with radius functions $r_1 = r_{\mathscr{C}_1}$ and $r_2 = r_{\mathscr{C}_2}$ respectively. Define a *comparison function* $w : V(\mathscr{D}) \to \mathbb{C}$ by

$$\begin{cases} w(y) = r_2(y)/r_1(y) & \text{for } y \in V(G), \\ w(x) = \mathrm{e}^{i\delta(x)} \in \mathbb{S}^1 & \text{for } x \in V(G^*). \end{cases} \tag{3}$$

Here $\delta(x) \in \mathbb{R}$ and $w(x) = \mathrm{e}^{i\delta(x)}$ are defined to be the rotation angle and the rotation respectively of the edge-star at $x \in V(G^*)$ when changing from the circle pattern $\mathscr{C}_1$ to $\mathscr{C}_2$. Note that $w(y)$ is the scaling factor of the circle corresponding to $y \in V(G)$. Then w satisfies the following *Hirota Equation* for all faces $f \in F(\mathscr{D})$.

$$w(x_0)w(y_0)a_0 - w(x_1)w(y_0)a_1 - w(x_1)w(y_1)a_0 + w(x_0)w(y_1)a_1 = 0. \tag{4}$$

Here $x_0, x_1 \in V(G^*)$ and $y_0, y_1 \in V(G)$ are the black and white vertices incident to f and $a_0 = x_0 - y_0$ and $a_1 = x_1 - y_0$ are the directed edges. Thus equation (4) is the closing condition for the kite corresponding to the face f. Furthermore, the Hirota Equation is 3D-consistent; see Sects. 10 and 11 of [10] or [11] for more details. This property will be used in Sect. 5.

3 SG-Circle Patterns

In this paper we are particularly interested in the special case of regular circle patterns with square grid combinatorics. First, we fix some notation. Let *SGD* be the regular square grid cell decomposition of the complex plane, that is, the vertices are $V(SGD) = \mathbb{Z} + i\mathbb{Z}$ and the edges are given by pairs of vertices $[z, z']$ with $z, z' \in V(SGD)$ and $|z - z'| = 1$. The 2-cells are squares $\{z + a + ib : a, b \in [0, 1]\}$ for $z \in V(SGD)$. As *SGD* is a b-quad-graph, the vertices

$$V(SG) = \{n + im \in \mathbb{Z} + i\mathbb{Z} : n + m = 0 \ (\mathrm{mod}\, 2)\}$$

are assumed to be colored white. As above, *SG* and its dual SG^* are defined as the associated graphs to *SGD*. Furthermore, $SG(n, v)$ with $n \in \mathbb{N}$ and $v \in V(SG)$ denotes the subgraph of all vertices with combinatorial distance at most n from v in *SG*.

Let $\psi \in (0, \pi)$ be a fixed angle. Define the following regular labeling α_ψ on $E(SG)$. Let $[z_1, z_2]$ be an edge connecting the vertices $z_1, z_2 \in V(SG)$. Without loss of generality, we assume that $\mathrm{Re}(z_1) < \mathrm{Re}(z_2)$. Then

$$\alpha_\psi([z_1, z_2]) = \begin{cases} \psi & \text{if } \mathrm{Im}(z_1) < \mathrm{Im}(z_2), \\ \pi - \psi & \text{if } \mathrm{Im}(z_1) > \mathrm{Im}(z_2). \end{cases} \tag{5}$$

If G is a subgraph of *SG*, a circle pattern for G and α_ψ is called *SG-circle pattern*. The choice $\psi = \pi/2$ leads to orthogonal *SG*-circle patterns as considered by Schramm in [24].

Theorem 3.1 (Rigidity of *SG*-circle patterns) *Suppose that $\mathscr{C}$ is an embedded circle pattern for SG and α_ψ. Then $\mathscr{C}$ is the image of a regular isoradial circle pattern for SG and α_ψ under a similarity.*

The proof is a suitable adaption of the corresponding proof for orthogonal *SG*-circle patterns given by Schramm using suitable Möbius invariants, see [24, Theorem 7.1] or [12]. This adaption needs the following generalization of the Ring Lemma of [22], which is also useful in the following.

Lemma 3.2 *Let r be the radius function of an embedded circle pattern for $SG(3,0)$ and α_ψ. There is a constant $C = C(\psi) > 0$, independent of r, such that for $k = 0, 1, 2, 3$ there holds*

$$\frac{r(i^k(1+i))}{r(0)} > C.$$

Proof Assume the contrary. Then there is a sequence of embedded circle patterns for $SG(3,0)$ and α_ψ such that $r_n(0) = 1$ and $r_n(i^k(1+i)) \to 0$ as $n \to \infty$ for some $k \in \{0,1,2,3\}$. Without loss of generality we assume that $k = 0$. We also may assume that the circle C_0 corresponding to the vertex $0 \in V(SG)$ and the intersection point corresponding to $1 \in V(SG^*)$ are fixed for the whole sequence. Then there is a subsequence such that all the circles converge to circles or lines, that is, converge in the Riemann sphere $\hat{\mathbb{C}} \cong \mathbb{S}^2$. Now (2) implies that there exist some kites which intersect in the limit in their interiors. But this is a contradiction to the embeddedness of the sequence. □

If the number of surrounding generations is big enough, there is the following useful estimation on the quotient of radii of incident vertices.

Lemma 3.3 *There is an absolute constant $C > 0$ such that the following holds.*

Let G be a subgraph of SG and let $\mathscr{C}$ be an embedded circle pattern for G and α_ψ with radius function r. Let $v \in V(G)$ be a vertex and suppose that $SG(n, v) \subset G$, that is, G contains n generations of SG around v, for some $n \geq 3$. Then for all vertices w incident to v there holds

$$1 - \frac{C}{n} \leq \frac{r(w)}{r(v)} \leq 1 + \frac{C}{n}. \tag{6}$$

The proof is an adaption of the corresponding proof for hexagonal circle packings given by Aharonov in [5, 6] and uses Lemma 3.2, see [12] for more details. The necessary results on discrete potential theory can be found in the appendix of [13] and in [12].

4 Uniqueness of Orthogonal Z^γ-Circle Patterns

An orthogonal circle pattern with the combinatorics of the square grid associated to the map z^γ was introduced by Bobenko in [7]. Further development of the theory is due to Agafonov and Bobenko [1, 2, 4].

4.1 Definition and Useful Properties

In the following we briefly summarize the definition and some known facts about orthogonal Z^γ-circle patterns, see [1, 2, 4] for more details.

Definition 4.1 Let $D \subset \mathbb{Z}^2$. A map $f : D \to \mathbb{C}$ is called *discrete conformal* if all its elementary quadrilaterals are conformal squares, i.e. their cross-ratios are equal to -1:

$$\begin{aligned} &q(f_{n,m}, f_{n+1,m}, f_{n+1,m+1}, f_{n,m+1}) \\ &\quad := \frac{(f_{n,m} - f_{n+1,m})(f_{n+1,m+1} - f_{n,m+1})}{(f_{n+1,m} - f_{n+1,m+1})(f_{n,m+1} - f_{n,m})} = -1. \end{aligned} \tag{7}$$

Here and below we abbreviate $f_{n,m} = f(n,m)$.

A discrete conformal map $f_{n,m}$ is called *embedded* if the interiors of different elementary quadrilaterals $(f_{n,m}, f_{n+1,m}, f_{n+1,m+1}, f_{n,m+1})$ are disjoint.

Note that the definition of a discrete conformal map is Möbius invariant and is motivated by the following characterization for smooth mappings: A smooth map $f : \mathbb{C} \subset D \to \mathbb{C}$ is called *conformal* (holomorphic or antiholomorphic) if and only if for all $z = x + iy \in D$ there holds

$$\lim_{\varepsilon \to 0} q\bigl(f(x,y), f(x+\varepsilon, y), f(x+\varepsilon, y+\varepsilon), f(x, y+\varepsilon)\bigr) = -1.$$

In order to construct an embedded discrete analog of z^γ the following approach is used. Equation (7) can be supplemented with the nonautonomous constraint

$$\begin{aligned} \gamma f_{n,m} = 2n &\frac{(f_{n+1,m} - f_{n,m})(f_{n,m} - f_{n-1,m})}{(f_{n+1,m} - f_{n-1,m})} \\ &+ 2m \frac{(f_{n,m+1} - f_{n,m})(f_{n,m} - f_{n,m-1})}{(f_{n,m+1} - f_{n,m-1})}. \end{aligned} \tag{8}$$

This constraint, as well as its compatibility with (7), is derived from some monodromy problem; see [4]. We assume that $0 < \gamma < 2$ and denote

$$\mathbb{Z}^2_+ = \bigl\{(n,m) \in \mathbb{Z}^2 : n, m \geq 0\bigr\}.$$

The asymptotics of the constraint (8) for $n, m \to \infty$ and the properties $z^\gamma(\mathbb{R}_+) = \mathbb{R}_+$ and $z^\gamma(i\mathbb{R}_+) = e^{\gamma\pi i/2}\mathbb{R}_+$ of the holomorphic mapping z^γ motivate the following definition of the discrete analog.

Definition 4.2 For $0 < \gamma < 2$, the discrete conformal map $Z^\gamma : \mathbb{Z}^2_+ \to \mathbb{C}$ is the solution of (7) and (8) with the initial conditions

$$Z^\gamma(0,0) = 0, \qquad Z^\gamma(1,0) = 1, \qquad Z^\gamma(0,1) = e^{\gamma\pi i/2}.$$

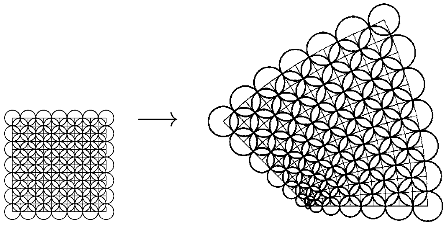

Fig. 3 Illustration of the discrete conformal map $Z^{3/2}$ and the orthogonal $Z^{3/2}$-circle pattern (*right*)

From this definition, the properties $Z^\gamma(n,0) \in \mathbb{R}_+$ and $Z^\gamma(0,m) \in e^{\gamma\pi i/2}\mathbb{R}_+$ are obvious for all $n, m \in \mathbb{N}$. Furthermore, the discrete conformal map Z^γ from Definition 4.2 determines an *SG*-circle pattern. Indeed, by Proposition 1 of [4] all edges at the vertex $f_{n,m}$ with $n+m=0 \pmod 2$ have the same length and all angles between neighboring edges at the vertex $f_{n,m}$ with $n+m=1 \pmod 2$ are equal to $\pi/2$. Thus, all elementary quadrilaterals $(f_{n,m}, f_{n+1,m}, f_{n+1,m+1}, f_{n,m+1})$ build orthogonal kites and for any $(n,m) \in \mathbb{Z}_+^2$ with $n+m=0 \pmod 2$ the points $f_{n+1,m}, f_{n,m+1}, f_{n-1,m}, f_{n,m-1}$ lie on a circle with center $f_{n,m}$. Therefore, we consider the sublattice $\{(n,m) \in \mathbb{Z}_+^2 : n+m=0 \pmod 2\}$ and denote by $\mathbb{V}$ the quadrant

$$\mathbb{V} = \big\{z = N + iM : N, M \in \mathbb{Z}, M \geq |N|\big\} \subset \mathbb{Z}^2,$$

where $N = (n-m)/2$, $M = (n+m)/2$. Two vertices $z_1, z_2 \in \mathbb{V}$ are connected by an edge if and only if $|z_1 - z_2| = 1$.

Theorem 4.3 [1, 2, 4]

(i) *If $R(z)$ denotes the radius function corresponding to the discrete conformal map Z^γ for some $0 < \gamma < 2$, then it holds that*

$$(\gamma - 1)\big(R(z)^2 - R(z-i)R(z+1)\big) \geq 0 \tag{9}$$

for all $z \in \mathbb{V} \setminus \{\pm N + iN | N \in \mathbb{N}\}$.

(ii) *For $0 < \gamma < 2$, the discrete conformal maps Z^γ given by Definition 4.2 are embedded. Consequently, the corresponding circle patterns are also embedded.*

In the following section and in Sect. 6 we continue to use the notation of this section. In particular the radius function is denoted by R and we have the normalization $R(0) = 1$.

4.2 Uniqueness of the Orthogonal Z^γ-Circle Patterns

This section is devoted to the proof of following uniqueness result.

Theorem 4.4 (Rigidity of orthogonal Z^γ-circle patterns) *For $\gamma \in (0,2)$ the infinite orthogonal embedded circle pattern corresponding to Z^γ is the unique embedded orthogonal circle pattern (up to global scaling) with the following two properties.*

(i) *The union of the corresponding kites of the Z^γ-circle pattern covers the infinite sector $\{z = \rho e^{i\beta} \in \mathbb{C} : \rho \geq 0,\ \beta \in [0, \gamma\pi/2]\}$ with angle $\gamma\pi/2$.*

(ii) *The centers of the boundary circles lie on the boundary half lines* $\mathbb{R}_+$ *and* $e^{i\gamma\pi/2}\mathbb{R}_+$.

The rigidity of orthogonal Z^γ-circle patterns was only known for rational γ, see [2]. Note that we only consider globally embedded circle patterns.

Our proof uses results of discrete potential theory or of the theory of random walks which can be found in standard textbooks, for example by Doyle and Snell [14] or by Woess [29]. We recall some basic terminology and notation and cite adapted versions of a few theorems which will be useful for our argumentation.

By abuse of notation, we denote by $\mathbb{Z}^2$ the points $(a,b)\in\mathbb{R}^2\cong\mathbb{C}$ with $a,b\in\mathbb{Z}$ as well as the graph with vertices at these points and edges $e=[z_1,z_2]$ if $|z_1-z_2|=1$. The meaning will be clear from the context.

Consider the network $(\mathbb{Z}^2, c)$ with *conductances* $c(e)>0$ and *resistances* $1/c(e)$ on the undirected edges $e\in E(\mathbb{Z}^2)$. Then a *transition probability function* p is given by

$$p(z_1,z_2) := \begin{cases} c([z_1,z_2])/(\sum_{e=[z_1,z]\in E(\mathbb{Z}^2)} c(e)) & \text{if } [z_1,z_2]\in E(\mathbb{Z}^2), \\ 0 & \text{otherwise.} \end{cases}$$

The stochastic process on $\mathbb{Z}^2$ given by this probability function p is a reversible random walk or a reversible Markov chain on $\mathbb{Z}^2$. The *simple random walk* on $\mathbb{Z}^2$ is given by specifying $c(e)=1$ for all edges which leads to $p(z_1,z_2)=1/4$ if $[z_1,z_2]\in E(\mathbb{Z}^2)$.

Denote by p_{esc} the probability that a random walk starting at any point will never return to this point. The network $(\mathbb{Z}^2, c)$ is called *recurrent* if $p_{\text{esc}}=0$ (and *transient* otherwise). Note that $p_{\text{esc}}=1/R_{\text{eff}}$, where R_{eff} denotes the effective resistance from a point to infinity.

Theorem 4.5 (i) *The simple random walk on* $\mathbb{Z}^2$ *is recurrent.*

(ii) *Let* $(\mathbb{Z}^2, c_1)$ *and* $(\mathbb{Z}^2, c_2)$ *be two networks with conductances* $c_1(e)>0$ *and* $c_2(e)>0$ *on the edges. If* $c_2(e)\le c_1(e)$ *for all edges* $e\in E(\mathbb{Z}^2)$, *then the recurrence of* $(\mathbb{Z}^2, c_1)$ *implies the recurrence of* $(\mathbb{Z}^2, c_2)$.

A proof can for example be found in [14, Chaps. 5, 7, and 8] or [29, Sects. 1.A, 1.B, and Corollary (2.14)].

A function $f:\mathbb{Z}^2\to\mathbb{R}$ is called *superharmonic* (*subharmonic*) with respect to the probability function p or with respect to the conductances c if for every vertex $v\in\mathbb{Z}^2$ we have $\sum_{w\in\mathbb{Z}^2} p(v,w)f(w)\le f(v)$ $(\sum_{w\in\mathbb{Z}^2} p(v,w)f(w)\ge f(v))$.

The following proposition shows that the quotient of the radius functions of two orthogonal *SG*-circle patterns is subharmonic with respect to suitably chosen conductances. As the statement is a special case of Proposition 6.5 below, we omit the proof.

Proposition 4.6 *Consider two orthogonal circle patterns for* $SG(1,v_0)$. *Denote the radii by* ρ_j *and* r_j *respectively, where* ρ_0 *and* r_0 *denote the radii of the inner circles* *associated to* v_0. *Then*

$$\sum_{j=1}^{4} c_j \frac{r_j}{\rho_j} \geq \sum_{j=1}^{4} c_j \frac{r_0}{\rho_0} \quad \textit{and} \quad \sum_{j=1}^{4} c_j \frac{\rho_j}{r_j} \geq \sum_{j=1}^{4} c_j \frac{\rho_0}{r_0}, \tag{10}$$

where $c_j = 1/((\rho_j/\rho_0) + (\rho_0/\rho_j))$.

Our proof of rigidity is based on the following property of superharmonic functions on recurrent networks.

Theorem 4.7 ([29, Theorem (1.16)]) *A network is recurrent if and only if all non-negative superharmonic functions are constant.*

Proof of Theorem 4.4 Let $\gamma \in (0, 2)$ and denote by $R : \mathbb{V} \to \mathbb{R}_+$ the radius function of the embedded Z^γ-circle pattern with $R(0) = 1$. Let $r : \mathbb{V} \to \mathbb{R}_+$ denote the radius function of an embedded orthogonal circle pattern with the same combinatorics and the same boundary conditions (orthogonal boundary circles to the half lines $\mathbb{R}_+$ and $e^{i\gamma\pi/2}\mathbb{R}_+$). Without loss of generality we assume the same normalization $r(0) = 1$. This can always be achieved by a suitable scaling. In the following, we will show that the radius functions R and r take the same values on all of $\mathbb{V}$. This implies that the circle patterns coincide.

As both circle patterns are embedded, Lemma 3.3 implies that for some constant $A > 0$ and $n \geq 3$

$$1 - \frac{A}{n} \leq \frac{r(z_j^{(n)})}{r(z_0^{(n)})} \leq 1 + \frac{A}{n} \tag{11}$$

holds for all radii $r(z_0^{(n)})$ for vertices $z_0^{(n)}$ of the nth generation away from the origin and their incident vertices $z_j^{(n)}$. Here, vertices $z \in \mathbb{V}$ belong to the nth generation if their combinatorial distance in $\mathbb{V}$ to the origin is n. For estimation (11) we have also used that the reflection of the circle pattern in one of the boundary lines $\mathbb{R}_+$ or $e^{i\gamma\pi/2}\mathbb{R}_+$ also leads to an embedded orthogonal *SG*-circle pattern. The same reasoning applies to the radii of the Z^γ-circle pattern, so

$$1 - \frac{A}{n} \leq \frac{R(z_j^{(n)})}{R(z_0^{(n)})} \leq 1 + \frac{A}{n} \tag{12}$$

for $n \geq 3$ with the same constant A. Estimations (11) and (12), the boundary conditions and a suitable adaptation of the Ring Lemma 5.11 for circles of generation two and three from the origin imply that there is a constant $K > 0$ such that

$$\frac{1}{K} \leq \frac{r(z_j)}{r(z_0)} \leq K \quad \text{and} \quad \frac{1}{K} \leq \frac{R(z_j)}{R(z_0)} \leq K \tag{13}$$

for all incident vertices z_j and z_0.

We now consider two undirected networks $(\mathbb{Z}^2, C)$ and $(\mathbb{Z}^2, \tilde{C})$ as follows. On the edges of $\mathbb{V}$ we define two conductance functions C and $\tilde{C}$ by

$$C(e) = C(R(z_j), R(z_k)) := \left(\frac{R(z_j)}{R(z_k)} + \frac{R(z_k)}{R(z_j)}\right)^{-1},$$

$$\tilde{C}(e) = \tilde{C}(r(z_j), r(z_k)) := \left(\frac{r(z_j)}{r(z_k)} + \frac{r(z_k)}{r(z_j)}\right)^{-1},$$

where the edge $e = [z_j, z_k]$ connects the vertices $z_j, z_k \in \mathbb{V}$. Estimations (13) imply that both positive functions $C > 0$ and $\tilde{C} > 0$ are uniformly bounded away from 0 (and from infinity). These two conductance networks on $\mathbb{V} \subset \mathbb{Z}^2$ can be continued to all of $\mathbb{Z}^2$ by reflection in the lines $\{|M| = |N|\}$. From Theorem 4.5 we deduce that both networks $(\mathbb{Z}^2, C)$ and $(\mathbb{Z}^2, \tilde{C})$ are recurrent.

Consider the following positive functions on $\mathbb{V}$

$$f_1(z) = r(z)/R(z) > 0 \quad \text{and} \quad f_2(z) = R(z)/r(z) = 1/f_1(z) > 0.$$

By Proposition 4.6 these functions are subharmonic. Using the boundary conditions of the circle patterns, this remains true if f_1 and f_2 are continued to all of $\mathbb{Z}^2$ using reflection. Consequently, $M - f_1$ and $M - f_2$ are superharmonic for all constants $M \in \mathbb{R}$. If f_1 or f_2 is bounded from above, we get a positive superharmonic function using the upper bound. Then Theorem 4.7 implies that both functions are constant. Thus $r \equiv R$ and consequently both circle patterns coincide.

Denote by $M_1(n)$ and $M_2(n)$ the maximum of f_1 and f_2, respectively, for the set of vertices of the nth generation about the origin. As f_1 and f_2 are subharmonic, they assume their maxima on the boundary. Therefore the functions M_1 and M_2 are monotonically increasing. The estimations (11) and (12) imply that the quotients of any two radii of one circle pattern in the nth generation are bounded from above for $n \geq 3$, as two vertices in the nth generation can be connected by at most $4n$ edges using only vertices of the nth and $n+1$st generation. So their quotient is bounded by e^{4A} for both radius functions r and R. Note that with the normalization $r(0) = 1 = R(0)$, the maxima M_1 and M_2 are bounded from below by 1. Thus their product

$$M_1(n)M_2(n) = f_1\big(z_{M_1}^{(n)}\big) f_2\big(z_{M_2}^{(n)}\big) = \frac{R(z_{M_1}^{(n)})}{r(z_{M_1}^{(n)})} \frac{r(z_{M_2}^{(n)})}{R(z_{M_2}^{(n)})} \leq \mathrm{e}^{8A}$$

is bounded from above. Here $z_{M_1}^{(n)}$ and $z_{M_2}^{(n)}$ denote the vertices of the nth generation where f_1 and f_2 assume their maxima, respectively. Therefore M_1 and M_2 are also bounded. This finishes the proof of uniqueness. □

5 Uniqueness of Isoradial Quasicrystallic Circle Patterns

The uniqueness result of Theorem 3.1 can be generalized for some classes of quasicrystallic circle patterns. On this basis we will then generalize Theorem 4.4 for some classes of quasicrystallic Z^γ-circle pattern.

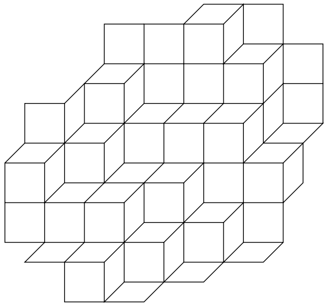

Fig. 4 An example of a quad-surface $\Omega_{\mathscr{D}} \subset \mathbb{Z}^3$

5.1 Quasicrystallic Circle Patterns and Connection to $\mathbb{Z}^d$

Definition 5.1 A *rhombic embedding in* $\mathbb{C}$ of a b-quad-graph $\mathscr{D}$ is an embedding with the property that each face of $\mathscr{D}$ is mapped to a rhombus. Given a rhombic embedding of $\mathscr{D}$, consider for each directed edge $\vec{e} \in \vec{E}(\mathscr{D})$ the vector of its embedding as a complex number with length one. Half of the number of different values of these directions is called the *dimension* d of the rhombic embedding. If d is finite, the rhombic embedding is called *quasicrystallic*.

Adding circles with centers in the white vertices of the rhombic embedding and radius equal to the edge length reveals the one-to-one correspondence to embedded isoradial circle patterns.

Given a b-quad-graph $\mathscr{D}$, assume that there exists a quasicrystallic rhombic embedding of $\mathscr{D}$ and a corresponding isoradial circle pattern $\mathscr{C}_1$. Take the intersection angles α from this rhombic embedding. Then any circle pattern $\mathscr{C}_2$ for $\mathscr{D}$ and α—not necessarily isoradial—is called a *quasicrystallic circle pattern*. The comparison function of the isoradial circle pattern $\mathscr{C}_1$ for $\mathscr{D}$ and the quasicrystallic circle pattern $\mathscr{C}_2$ is also called *comparison function for* $\mathscr{C}_2$.

In the following we will often identify the b-quad-graph $\mathscr{D}$ with a rhombic embedding of $\mathscr{D}$.

Remark 5.2 The notion "quasicrystallic" is not uniquely defined in literature. Here we adopt the definition given in [10]. Naturally, this property only makes sense for infinite graphs or sequences of graphs with growing number of vertices and edges.

Any rhombic embedding of a b-quad-graph $\mathscr{D}$ can be seen as a sort of projection of a certain two-dimensional subcomplex (*quad-surface*) $\Omega_{\mathscr{D}}$ of the multi-dimensional lattice $\mathbb{Z}^d$ (or of a multi-dimensional lattice $\mathcal{L}$ which is isomorphic to $\mathbb{Z}^d$). An illustrating example is given in Fig. 4.

The quad-surface $\Omega_{\mathscr{D}}$ in $\mathbb{Z}^d$ can be constructed in the following way. Denote the set of the different edge directions of $\mathscr{D}$ by $\mathcal{A} = \{\pm a_1, \ldots, \pm a_d\} \subset \mathbb{S}^1$. We suppose that $d > 1$ and that any two non-opposite elements of $\mathcal{A}$ are linearly independent over $\mathbb{R}$. Let $\mathbf{e}_1, \ldots, \mathbf{e}_d$ denote the standard orthonormal basis of $\mathbb{R}^d$. Fix a white vertex $x_0 \in V(\mathscr{D})$ and the origin of $\mathbb{R}^d$. Add the edges of $\{\pm\mathbf{e}_1, \ldots, \pm\mathbf{e}_d\}$ at the origin which correspond to the edges of $\{\pm a_1, \ldots, \pm a_d\}$ incident to x_0 in $\mathscr{D}$, together with their endpoints. Successively continue the construction at the new endpoints. Also, add two-dimensional facets (quadrilateral faces) of $\mathbb{Z}^d$ corresponding to the quadrilateral faces of $\mathscr{D}$, spanned by incident edges.

A quad-surface $\Omega_{\mathscr{D}}$ in $\mathbb{Z}^d$ corresponding to a quasicrystallic rhombic embedding can be characterized using the following monotonicity property. For a proof see [10, Sect. 6].

Lemma 5.3 (Monotonicity criterium) *A quad-surface $\Omega_{\mathscr{D}}$ in $\mathbb{Z}^d$ corresponds to a quasicrystallic rhombic embedding if and only if any two points of $\Omega_{\mathscr{D}}$ can be connected by a path in $\Omega_{\mathscr{D}}$ with all directed edges lying in one d-dimensional octant, that is, all directed edges of this path are elements of one of the 2^d subsets of $\{\pm\mathbf{e}_1, \dots, \pm\mathbf{e}_d\}$ containing d linearly independent vectors.*

An important class of examples of rhombic embeddings of b-quad-graphs can be constructed using ideas of the grid projection method for quasiperiodic tilings of the plane; see for example [16, 17, 26].

Example 5.4 (Quasicrystallic rhombic embedding obtained from a plane) Let E be a two-dimensional plane in $\mathbb{R}^d$. Let $\mathbf{e}_1, \dots, \mathbf{e}_d$ denote the standard orthonormal basis of $\mathbb{R}^d$ and let $\mathbf{t} \in E$. We assume that E does not contain any of the segments $s_j = \{\mathbf{t} + \lambda\mathbf{e}_j : \lambda \in [0, 1]\}$ for $j = 1, \dots, d$. If E contains two different segments s_{j_1} and s_{j_2}, the following construction only leads to the standard square grid pattern $\mathbb{Z}^2$. If E contains exactly one segment s_j, the construction may be adapted for the remaining dimensions (excluding $\mathbf{e}_j$). We further assume that the orthogonal projections onto E of the two-dimensional facets $E_{j_1,j_2} = \{\lambda_1\mathbf{e}_{j_1} + \lambda_2\mathbf{e}_{j_2} : \lambda_1, \lambda_2 \in [0, 1]\}$ for $1 \le j_1 < j_2 \le d$ are non-degenerate parallelograms. Then we can choose positive numbers $c_1, \dots, c_d$ such that the orthogonal projections $P_E(c_j\mathbf{e}_j)$ have length 1.

Consider around each vertex $\mathbf{p}$ of the lattice $\mathcal{L} = c_1\mathbb{Z} \times \cdots \times c_d\mathbb{Z}$ the hypercuboid $V = [-c_1/2, c_1/2] \times \cdots \times [-c_d/2, c_d/2]$, that is, the Voronoi cell $\mathbf{p} + V$. These translations of V cover $\mathbb{R}^d$. If E intersects the interior of the Voronoi cell of a lattice point (i.e. $(\mathbf{p} + V)^\circ \cap E \neq \emptyset$ for $\mathbf{p} \in \mathcal{L}$), then this point belongs to $\Omega^{\mathcal{L}}(E)$. Undirected edges correspond to intersections of E with the interior of a $(d-1)$-dimensional facet bounding two Voronoi cells. Thus we get a connected graph in $\mathcal{L}$. An intersection of E with the interior of a translated $(d-2)$-dimensional facet of V corresponds to a rectangular two-dimensional face of the lattice. By construction, the orthogonal projection of this graph onto E results in a planar connected graph whose faces are all of even degree (=number of incident edges or of incident vertices). A face of degree bigger than 4 corresponds to an intersection of E with the translation of a $(d-k)$-dimensional facet of V for some $k \ge 3$. Consider the vertices and edges of such a face and the corresponding points and edges in the lattice $\mathcal{L}$. These points lie on a combinatorial k-dimensional hypercuboid contained in $\mathcal{L}$. By construction, it is easy to see that there are two points of the k-dimensional hypercuboid which are each incident to k of the given vertices. Choose a point with least distance from E and add it to the surface. Adding edges to neighboring vertices splits the face of degree $2k$ into k faces of degree 4.

Thus we obtain an infinite monotone two-dimensional quad-surface $\Omega^{\mathcal{L}}(E)$ which projects to an infinite rhombic embedding covering the whole plane E. Parts of such rhombic embeddings are shown in Figs. 7, 9, and 11.

5.2 Quasicrystallic Circle Patterns and Integrability

Let $\mathscr{D}$ be a quasicrystallic rhombic embedding of a b-quad-graph. The quad-surface $\Omega_{\mathscr{D}}$ in $\mathbb{Z}^d$ is important due to its connection with integrability. See also [11] for a more detailed presentation and a deepened study of integrability and consistency.

In particular, a function defined on the vertices of $\Omega_{\mathscr{D}}$ which satisfies some 3D-consistent equation on all faces of $\Omega_{\mathscr{D}}$ can be uniquely extended to the *brick*

$$\Pi(\Omega_{\mathscr{D}}) := \big\{\mathbf{n} = (n_1, \dots, n_d) \in \mathbb{Z}^d : a_k(\Omega_{\mathscr{D}}) \le n_k \le b_k(\Omega_{\mathscr{D}}),\ k = 1, \dots, d\big\},$$

where $a_k(\Omega_{\mathscr{D}}) = \min_{\mathbf{n}\in V(\Omega_{\mathscr{D}})} n_k$ and $b_k(\Omega_{\mathscr{D}}) = \max_{\mathbf{n}\in V(\Omega_{\mathscr{D}})} n_k$. Note that $\Pi(\Omega_{\mathscr{D}})$ is the hull of $\Omega_{\mathscr{D}}$. A proof may be found in [10, Sect. 6].

Let $\mathscr{D}$ be a quasicrystallic rhombic embedding and let $\mathscr{C}_2$ be a quasicrystallic circle pattern with the same combinatorics and the same intersection angles. Denote the comparison function for $\mathscr{C}_2$ by w as in (3). Since the Hirota equation (4) is 3D-consistent (see Sects. 10 and 11 of [10]) w considered as a function on $V(\Omega_{\mathscr{D}})$ can uniquely be extended to the brick $\Pi(\Omega_{\mathscr{D}})$ such that (4) holds on all two-dimensional facets. Additionally, w and its extension are real valued on white points of $V(\Omega_{\mathscr{D}})$ and have values in $\mathbb{S}^1$ for black points of $V(\Omega_{\mathscr{D}})$. This can easily be deduced from the Hirota equation (4).

Lemma 5.5 *Let $\mathscr{D}$ and $\mathscr{D}'$ be two simply connected finite rhombic embeddings of b-quad-graphs with the same set of edge directions. Assume that $\mathscr{D}$ and $\mathscr{D}'$ agree on all boundary faces, that is, all 2-cells which intersect the boundary in an edge or a point. Let $\mathscr{C}$ be an* (*embedded*) *circle pattern for $\mathscr{D}$ and the labeling given by the rhombic embedding. Then there is an* (*embedded*) *circle pattern $\mathscr{C}'$ for $\mathscr{D}'$ and the corresponding labeling which agrees with $\mathscr{C}$ for all boundary circles.*

Proof Consider the monotone quad-surfaces $\Omega_{\mathscr{D}}$ and $\Omega'_{\mathscr{D}}$. Without loss of generality, we can assume that $\Omega_{\mathscr{D}}$ and $\Omega'_{\mathscr{D}}$ have the same boundary faces in $\mathbb{Z}^d$. Thus both define the same brick $\Pi(\Omega_{\mathscr{D}}) = \Pi(\Omega'_{\mathscr{D}}) =: \Pi$. Given the circle pattern $\mathscr{C}$, define the comparison function w for $\mathscr{C}$ on $V(\Omega_{\mathscr{D}})$ by (3). Extend w to the brick Π such that condition (4) holds for all two-dimensional facets. Consider w on $\Omega'_{\mathscr{D}}$ and build the corresponding pattern $\mathscr{C}'$, such that the points on the boundary agree with those of the given circle pattern $\mathscr{C}$. Equation (4) guarantees that all faces of $\Omega'_{\mathscr{D}}$ are mapped to closed kites. Due to the combinatorics, the chain of kites is closed around each vertex. Since the boundary kites of $\mathscr{C}'$ are given by $\mathscr{C}$ which is an immersed circle pattern, at every interior white point the angles of the kites sum up to 2π. Thus $\mathscr{C}'$ is an immersed circle pattern.

Furthermore, $\mathscr{C}'$ is embedded if $\mathscr{C}$ is, because $\mathscr{C}'$ is an immersed circle pattern, and $\mathscr{C}'$ and $\mathscr{C}$ agree for all boundary kites. □

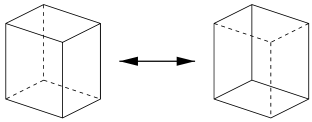

Fig. 5 A flip of a three-dimensional cube. Only the faces bounded by *solid edges* are part of the quad-surface in $\mathbb{Z}^d$

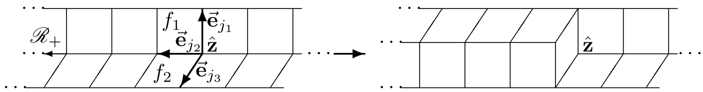

Fig. 6 An example of a flip for an infinite strip

5.3 Local Changes of Rhombic Embeddings

Let $\mathscr{D}$ be a rhombic embedding of a finite simply connected b-quad-graph and let $\Omega_{\mathscr{D}}$ be the corresponding quad-surface in $\mathbb{Z}^d$. Let $\hat{\mathbf{z}} \in V_{\text{int}}(\Omega_{\mathscr{D}})$ be an interior vertex with exactly three incident two-dimensional facets of $\Omega_{\mathscr{D}}$. Consider the three-dimensional cube with these boundary facets. Replace the three given facets with the three other two-dimensional facets of this cube. This procedure is called a *flip*; see Fig. 5 for an illustration.

A vertex $\mathbf{z} \in \mathbb{Z}^d$ *can be reached with flips from* $\Omega_{\mathscr{D}}$ if $\mathbf{z}$ is contained in a quad-surface obtained from $\Omega_{\mathscr{D}}$ by a suitable sequence of flips. The set of all vertices which can be reached with flips, including $V(\Omega_{\mathscr{D}})$, will be denoted by $\mathcal{F}(\Omega_{\mathscr{D}})$.

Remark 5.6 The quad-surface $\Omega^{\mathcal{L}}(E)$ can also be generalized using a finite sequence of flips. Such an infinite rhombic embedding will be called a *plane based quasicrystallic rhombic embedding*.

As a generalization of simple flips we define *flips for simply or doubly infinite strips* of the following form. See Fig. 6 for an illustration.

Let $\Omega_{\mathscr{D}} \subset \mathbb{Z}^d$ be a simply connected monotone quad-surface. Let $\hat{\mathbf{z}} \in V_w(\Omega_{\mathscr{D}})$ be a white vertex. Let $\mathbf{e}_{j_1}, \mathbf{e}_{j_2}, \mathbf{e}_{j_3}$ be three different edges incident to $\hat{\mathbf{z}}$ such that there are two-dimensional faces f_1, f_2 of $\Omega_{\mathscr{D}}$ incident to $\mathbf{e}_{j_1}$ and $\mathbf{e}_{j_2}$, and to $\mathbf{e}_{j_2}$ and $\mathbf{e}_{j_3}$, respectively. Let $\alpha_1 = \alpha(f_1)$ and $\alpha_2 = \alpha(f_2)$ be the intersection angles associated to these faces. Let α_3 be the intersection angles associated to the two-dimensional facet of $\mathbb{Z}^d$ incident to $\mathbf{e}_{j_1}$ and $\mathbf{e}_{j_3}$. Then $\sum_{i=1}^{2} \alpha_i + (\pi - \alpha_3) = 2\pi$ or $\sum_{i=1}^{3} \alpha_i = \pi$. In the first case, consider the half-axis $\mathscr{R}_+ = \{\hat{\mathbf{z}} + \lambda \vec{\mathbf{e}}_{j_2} : \lambda \geq 0\}$, where $\vec{\mathbf{e}}_{j_2}$ is the vector corresponding to the edge $\mathbf{e}_{j_2}$ and pointing away from $\hat{\mathbf{z}}$ as in Fig. 6. In the second case, consider the other half-axis $\mathscr{R}_- = \{\hat{\mathbf{z}} + \lambda \vec{\mathbf{e}}_{j_2} : \lambda \leq 0\}$. In both cases we may also consider the whole axis $\mathscr{R} = \{\hat{\mathbf{z}} + \lambda \vec{\mathbf{e}}_{j_2} : \lambda \in \mathbb{R}\}$. Assume that the translations of f_1 and f_2 along these (half-)axis, that is, the faces $f_j + n\vec{\mathbf{e}}_{j_2} + \hat{\mathbf{z}}$ for $j = 1, 2$ and $n \in \mathbb{N}$, $(-n) \in \mathbb{N}$, or $n \in \mathbb{Z}$ respectively, are contained in $\Omega_{\mathscr{D}}$. We only consider the case of the positive half-axis $\mathscr{R}_+$ further. (For $\mathscr{R}_-$ the argumentation is analogous and the case of the whole axis $\mathscr{R}$ is a simple consequence.) Replace each face $f_1 + n\vec{\mathbf{e}}_{j_2} + \hat{\mathbf{z}}$ by its translate $f_1 + n\vec{\mathbf{e}}_{j_2} + \hat{\mathbf{z}} + \vec{\mathbf{e}}_{j_3}$ for $n \in \mathbb{N}_0$ and similarly $f_2 + n\vec{\mathbf{e}}_{j_2} + \hat{\mathbf{z}}$ by $f_2 + n\vec{\mathbf{e}}_{j_2} + \hat{\mathbf{z}} + \vec{\mathbf{e}}_{j_1}$ for $n \in \mathbb{N}_0$, where $\vec{\mathbf{e}}_{j_1}$ and $\vec{\mathbf{e}}_{j_3}$ are the vectors corresponding to the edges $\mathbf{e}_{j_1}$ and $\mathbf{e}_{j_3}$ respectively and pointing away from $\hat{\mathbf{z}}$. Adding the face incident to $\hat{\mathbf{z}}$, $\mathbf{e}_{j_1}$, and $\mathbf{e}_{j_3}$, we obtain a different, but still monotone simply connected quad-surface.

The definition of an infinite flip for a black vertex $\hat{\mathbf{z}} \in V_b(\Omega_{\mathscr{D}})$ is very similar. The only difference is the relation between the intersection angles: $\sum_{i=1}^{2} \alpha_i + (\pi - \alpha_3) = \pi$ or $\sum_{i=1}^{3} \alpha_i = 2\pi$.

Lemma 5.7 *Let $\Omega_{\mathscr{D}} \subset \mathbb{Z}^d$ be a simply connected monotone quad-surface and let $\Omega'_{\mathscr{D}}$ be the simply connected monotone quad-surface obtained from $\Omega_{\mathscr{D}}$ after performing a flip for a simply infinite strip. Let $\mathscr{C}$ be a circle pattern for $\mathscr{D}$ and the corresponding labeling and let $\mathscr{C}'$ be the corresponding circle pattern after performing the corresponding infinite flip as for $\Omega'_{\mathscr{D}}$. Then the resulting circle pattern $\mathscr{C}'$ is embedded if the original one $\mathscr{C}$ is.*

The proof is based on similar arguments as the proof of Lemma 5.5 and is therefore left to the reader.

5.4 Uniqueness of Isoradial Quasicrystallic Circle Patterns

Let $\mathscr{E}$ be the family of all infinite quasicrystallic rhombic embeddings of connected and simply connected b-quad-graphs $\mathscr{D}$ which cover the entire complex plane and such that the brick $\Pi(\Omega_{\mathscr{D}})$ of the corresponding quad-surface $\Omega_{\mathscr{D}}$ contains a $\mathbb{Z}^2$-sublattice, that is, there are at least two different indices j_1, j_2 such that $\min_{\mathbf{n}\in V(\Omega_{\mathscr{D}})} n_{j_k} = -\infty$ and $\max_{\mathbf{n}\in V(\Omega_{\mathscr{D}})} n_{j_k} = \infty$ for $k = 1, 2$. Note that $\mathscr{E}$ contains in particular the plane based rhombic embeddings, like the Penrose tilings, for which $\Pi(\Omega_{\mathscr{D}}) = \mathbb{Z}^d$. Now we use the uniqueness of *SG*-circle patterns of Theorem 3.1 in order to establish the uniqueness of the circle patterns of $\mathscr{E}$.

Theorem 5.8 (Rigidity of quasicrystallic isoradial circle patterns) *Let $\mathscr{D} \in \mathscr{E}$ be an infinite quasicrystallic rhombic embedding with associated graph G and corresponding labeling α. Let $\mathscr{C}$ be an embedded circle pattern for G and α. Then $\mathscr{C}$ is the image of the isoradial circle pattern corresponding to $\mathscr{D}$ under a similarity of the complex plane.*

Proof Let $\mathscr{D} \in \mathscr{E}$ be an infinite quasicrystallic rhombic embeddings with associated graph G and corresponding labeling α. Let $\mathscr{C}$ be an embedded circle pattern for G and α. Consider the comparison function w for $\mathscr{C}$ defined by (3) on $\Omega_{\mathscr{D}}$ and extend it to $\Pi(\Omega_{\mathscr{D}})$. Let $\hat{\mathbf{z}} \in V(\Omega_{\mathscr{D}})$. By assumption on $\mathscr{D}$, there is a $\mathbb{Z}^2$-sublattice $\Omega(\hat{\mathbf{z}})$ with $\hat{\mathbf{z}} \in V(\Omega(\hat{\mathbf{z}}))$ which is contained in $\Pi(\Omega_{\mathscr{D}})$. Furthermore, we can perform flips for $\Omega_{\mathscr{D}}$ and corresponding flips for the circle pattern $\mathscr{C}$ such that the resulting quad-surface Ω' contains an arbitrary number of generations of the lattice $\Omega(\hat{\mathbf{z}})$ about $\hat{\mathbf{z}}$. As the corresponding circle pattern $\mathscr{C}'$ is embedded by Lemma 5.7 and as the number of generations about z can be chosen arbitrarily large, we deduce from the Rigidity Theorem 3.1 that the radius function is constant on $\Omega(\hat{\mathbf{z}})$. More precisely, the extension of w is constant on white and black vertices of $\Omega(\hat{\mathbf{z}})$ respectively.

This argumentation is valid for all vertices $\hat{\mathbf{z}} \in V(\Omega_{\mathscr{D}})$ and all $\mathbb{Z}^2$-sublattice $\Omega(\hat{\mathbf{z}})$ with $\hat{\mathbf{z}} \in V(\Omega(\hat{\mathbf{z}}))$ which are contained in $\Pi(\Omega_{\mathscr{D}})$. Therefore the extension of w is constant, on white and black vertices respectively, on all $\mathbb{Z}^2$-sublattices which are contained in $\Pi(\Omega_{\mathscr{D}})$. Due to our assumptions on the combinatorics of $\mathscr{D}$, the radius function which is w restricted to white vertices has to be constant on the whole brick $\Pi(\Omega_{\mathscr{D}})$. This implies in particular that $\mathscr{C}$ is an isoradial circle pattern and thus is the image of the isoradial circle pattern corresponding to $\mathscr{D}$ under a similarity of the complex plane. □

Definition 5.9 Let $\mathscr{D}$ be a rhombic embedding of a simply connected b-quad-graph. The *combinatorial K-environment* of a point $z \in V(\mathscr{D})$ is the subgraph corresponding to all vertices which have combinatorial distance at most K from z in $\mathscr{D}$.

Corollary 5.10 *Let $\mathscr{D} \in \mathscr{E}$ be an infinite quasicrystallic rhombic embedding with corresponding labeling α. Let $v_0 \in V_w(\mathscr{D})$ be a white vertex. Then there are a constant $n_0 = n_0(\mathscr{D}) \in \mathbb{N}$ and a sequence $s_n(v_0, \mathscr{D})$ decreasing to 0 for $n \to \infty$ such that the following holds.*

For $n \in \mathbb{N}$, $n \geq n_0$, let $\mathscr{D}_{2n}(v_0)$ be the rhombic embedding corresponding to the $2n$-environment of v_0. Let $G_n(v_0)$ be the associated graph. Let $\mathscr{C}_n$ be an embedded circle pattern for $G_n(v_0)$ and the labeling α taken from $\mathscr{D}$ with radius function r_n. Then there holds

$$\left| \frac{r_n(v_0)}{r_n(v_1)} - 1 \right| \leq s_n(v_0, \mathscr{D}) \tag{14}$$

for all vertices $v_1 \in V(G_n(v_0))$ incident to v_0.

We omit the proof which is very similar to the proof of the Hexagonal Packing Lemma of [22]. The following lemma is similar to Lemma 3.2 and corresponds to the Ring Lemma of [22].

Lemma 5.11 *Let $\mathscr{D}$ be a quasicrystallic rhombic embedding with associated graph G and labeling α. Denote by $\alpha_{\min} = \min\{\alpha(e) : e \in E(G)\}$ the smallest intersection angle. Let $n_0 \in \mathbb{N}$ be such that $(n_0 - 3)\alpha_{\min} > \pi$. Let $v_0 \in V_w(\mathscr{D})$ be a white vertex. Assume that $\mathscr{D}$ contains a $(2n_0)$-environment about v_0. Then there is a constant $C = C(\mathscr{D}) > 0$ such that the following holds.*

Let r be the radius function of an embedded circle pattern for $\mathscr{D}$ and α and let v_1 be a vertex incident to v_0 in G. Then

$$\frac{r(v_1)}{r(v_0)} > C.$$

Proof Suppose that there is a vertex v_1 incident to v_0 and a sequence of embedded circle patterns for $\mathscr{D}$ and α with radius functions r_n such that $r_n(v_0) = 1$ and $r_n(v_1) \to 0$ as $n \to \infty$. Without loss of generality we may assume that the circle C_0 corresponding to the vertex v_0 and the intersection point corresponding to one fixed black vertex w_0 incident to v_0 and v_1 in $\mathscr{D}$ are fixed for the whole sequence. Then there is a subsequence such that all the circles converge to circles or lines, that is, converge in the Riemann sphere $\hat{\mathbb{C}} \cong \mathbb{S}^2$.

Equation (2) implies that there are at least two circles corresponding to vertices incident to v_0 whose radii do not converge to 0. If the limit is finite, there are at least two circles whose radii do not converge to 0 corresponding to vertices incident to this vertex of the first generation and so on.

Consider the kites which contain in the limit the intersection point corresponding to w_0 and apply the above argument at most n_0 times. Then by our assumption on n_0 we obtain two kites whose interiors intersect in the limit. This is a contradiction to the embeddedness of the sequence. □

If $\mathscr{D}$ is a plane based quasicrystallic rhombic embedding we also obtain an analog of the Rodin–Sullivan Conjecture, see [6, 18, 22].

Corollary 5.12 *Let $\mathscr{D}$ be a plane based quasicrystallic rhombic embedding. There are absolute constants $C = C(\mathscr{D}) > 0$ and $n_0 = n_0(\mathscr{D}) \in \mathbb{N}$, depending only on $\mathscr{D}$, such that for all white vertices $v_0 \in V_w(\mathscr{D})$ and all $n \geq n_0$ there holds*

$$s_n(v_0, \mathscr{D}) \leq s_n(\mathscr{D}) \leq C/n. \tag{15}$$

Proof Let $v_0 \in V_w(\mathscr{D})$ be any white vertex. If $n \geq n_0(\mathscr{D})$ is big enough, for each $\mathbb{Z}^2$-sublattice $\Omega(\hat{\mathbf{v}}_0)$ the set $\mathcal{F}(\Omega_{\mathscr{D}})$ contains a $\lfloor B(\mathscr{D})n \rfloor$-environment of $\hat{\mathbf{v}}_0$ in $\Omega(\hat{\mathbf{v}}_0)$, where the constant $B(\mathscr{D})$ depends only on the construction parameters of $\mathscr{D}$. Here $\lfloor p \rfloor$ denotes the largest integer smaller than $p \in \mathbb{R}$. Therefore we can choose $s_n(\mathscr{D})$ to be the maximum of $s_{\lfloor B(\mathscr{D})n \rfloor}(\mathscr{D}(\hat{\mathbf{v}}_0))$ for all possible regular rhombic embeddings $\mathscr{D}(\hat{\mathbf{v}}_0)$ corresponding to $\mathbb{Z}^2$-sublattice $\Omega(\hat{\mathbf{v}}_0)$. Now the claim follows from Corollary 3.3 for *SG*-circle patterns. □

6 Uniqueness of Quasicrystallic Z^γ-Circle Patterns

In this section we consider quasicrystallic Z^γ-circle patterns as defined in [10] and then prove their rigidity. The proofs are based on the results of the previous sections and on similar arguments as for orthogonal Z^γ-circle patterns.

6.1 Definition and Useful Properties

Let $\psi \in (0, \pi)$ be a fixed angle. Recall the definition of the labeling α_ψ in (5). We consider the following generalization of Definition 4.2.

Definition 6.1 [3] For $0 < \gamma < 2$, the discrete map $Z^\gamma : \mathbb{Z}^2_+ \to \mathbb{C}$ is the solution of

$$\begin{aligned} &q(f_{n,m}, f_{n+1,m}, f_{n+1,m+1}, f_{n,m+1}) \\ &\quad := \frac{(f_{n,m} - f_{n+1,m})(f_{n+1,m+1} - f_{n,m+1})}{(f_{n+1,m} - f_{n+1,m+1})(f_{n,m+1} - f_{n,m})} = \mathrm{e}^{2i(\psi-\pi)} \end{aligned} \tag{16}$$

and (8) with the initial conditions

$$Z^\gamma(0,0) = 0, \qquad Z^\gamma(1,0) = 1, \qquad Z^\gamma(0,1) = \mathrm{e}^{\gamma(\pi-\psi)i}.$$

As in the orthogonal case one can again associate a circle pattern (for a quadrant of SG corresponding to $\mathbb{Z}^2_+$ and related to $\mathbb{V}$ and α_ψ) to the map Z^γ, see [3] for more details. Furthermore, the following results generalize the orthogonal case.

Theorem 6.2 [3] (i) *If $R(z)$ denotes the radius function corresponding to the discrete conformal map Z^γ for some $0<\gamma<2$, then*

$$(\gamma-1)\bigl(R(z)^2 - R(z-i)R(z+1) - \cos\psi\, R(z)\bigl(R(z-i)-R(z+1)\bigr)\bigr) \geq 0 \quad (17)$$

for all $z \in \mathbb{V}\setminus\{\pm N + iN \mid N \in \mathbb{N}\}$.

(ii) *For $0<\gamma<2$, the discrete conformal maps Z^γ given by Definition 4.2 are embedded. Consequently, the corresponding circle patterns are also embedded.*

Definition 6.1 can be generalized further. As explained in Sect. 13 of [10], discrete analogs of the power function z^γ can also be defined for quasicrystallic rhombic embeddings $\mathscr{D}$ instead of $\mathbb{Z}^2_+$ (or $\mathbb{Z}^2$). In particular, let $\mathcal{A}=\{\pm a_1,\dots,\pm a_d\}\subset\mathbb{S}^1$ be the set of edge directions. Suppose that $d>1$ and that any two non-opposite elements of $\mathcal{A}$ are linearly independent over $\mathbb{R}$. For $0<\gamma<2$ define the following values of the comparison function w on the coordinate semi-axes of $\mathbb{Z}^d_+$:

$$w(ne_k) = \begin{cases} 1 & \text{if } n=0,\\ a_k^{\gamma-1} = \mathrm{e}^{(\gamma-1)\log a_k} & \text{if } n \text{ is odd},\\ \prod_{m=1}^{n/2} \frac{m-1+\frac{\gamma}{2}}{m-\frac{\gamma}{2}} & \text{if } n\geq 2 \text{ and } n \text{ is even.}\end{cases} \quad (18)$$

The value of the logarithm $\log a_k$ is chosen as follows. Without loss of generality, we assume a circular order of the points of $\mathcal{A}$ on the positively oriented unit circle $\mathbb{S}^1$ is $a_1,\dots,a_d,-a_1,\dots,-a_d$. Set $a_{k+d}=-a_k$ for $k=1,\dots,d$ and define a_m for all $m\in\mathbb{Z}$ by $2d$-periodicity. To each $a_m=\mathrm{e}^{i\theta_m}\in\mathbb{S}^1$ assign a certain value of the argument $\theta_m\in\mathbb{R}$: choose θ_1 arbitrarily and then use the rule

$$\theta_{m+1}-\theta_m \in (0,\pi) \quad \text{for all } m\in\mathbb{Z}.$$

Clearly we then have $\theta_{m+d}=\theta_m+\pi$. The points a_m supplied with the arguments θ_m can be considered as belonging to the Riemann surface of the logarithmic function (i.e. a branched covering of the complex plane). Now, the branch of the logarithm is chosen such that

$$\log(a_l)\in[i\theta_m, i\theta_{m+d-1}], \quad l=m,\dots,m+d-1.$$

Using the Hirota equation (4), this function w can be extended to the whole sector $\mathbb{Z}^d_+$. Using suitable branches of the logarithm, w may also be extended to other sectors or to a branched covering of $\mathbb{Z}^d$. A logarithmic branching over the origin is necessary if γ is not rational.

Figure 8 shows an example of such a quasicrystallic Z^γ-circle pattern.

Note that for $d=2$ the boundary conditions given in (18) lead to the circle patterns specified in Definition 6.1. Therefore, by Theorem 6.2(ii), the circle patterns

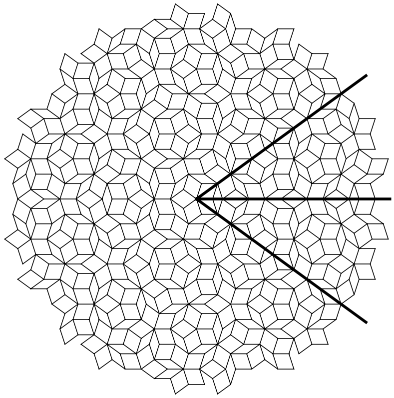

Fig. 7 A quasicrystallic rhombic embedding with five-fold rotation symmetry

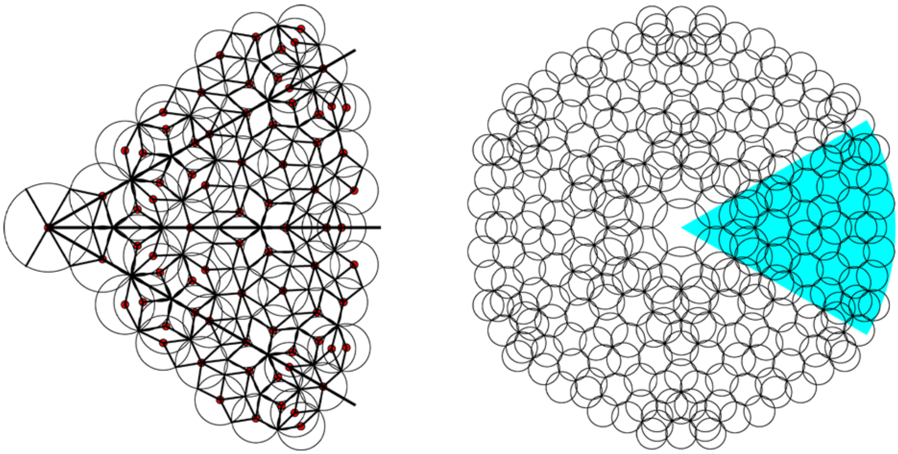

Fig. 8 Example of a quasicrystallic $Z^{5/6}$-circle pattern from the rhombic embedding of Fig. 7: Construction (*left*) and corresponding circle pattern (*right*)

corresponding to the restriction of w to quad-surfaces $\mathbb{Z}_+^2 \subset \mathbb{Z}_+^d$ which are spanned by two coordinate semi-axis are embedded. We can apply finite and infinite flips to obtain other monotone quad-surfaces corresponding to rhombic embeddings. In particular, we obtain restrictions to $\mathbb{Z}_+^d$ of the plane based quad-surfaces constructed in Example 5.4 and Remark 5.6. Lemmas 5.5 and 5.7 imply that these lead again to embedded circle patterns. Thus we have proven

Theorem 6.3 (Embeddedness of quasicrystallic Z^γ-circle patterns) *Let $\Omega \subset \mathbb{Z}_+^d$ be a simply connected monotone quad-surface. Then the circle pattern given by the function w with initial values* (18) *is embedded.*

Example 6.4 (Construction of the examples in Figs. 8, 10, and 12) The pictures in Figs. 8, 10 and 12 have been obtained using a computer program implemented by Veronika Schreiber for her master thesis [25].

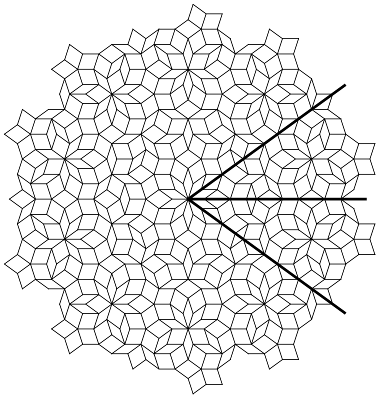

Fig. 9 A quasicrystallic rhombic embedding with ten-fold rotation symmetry

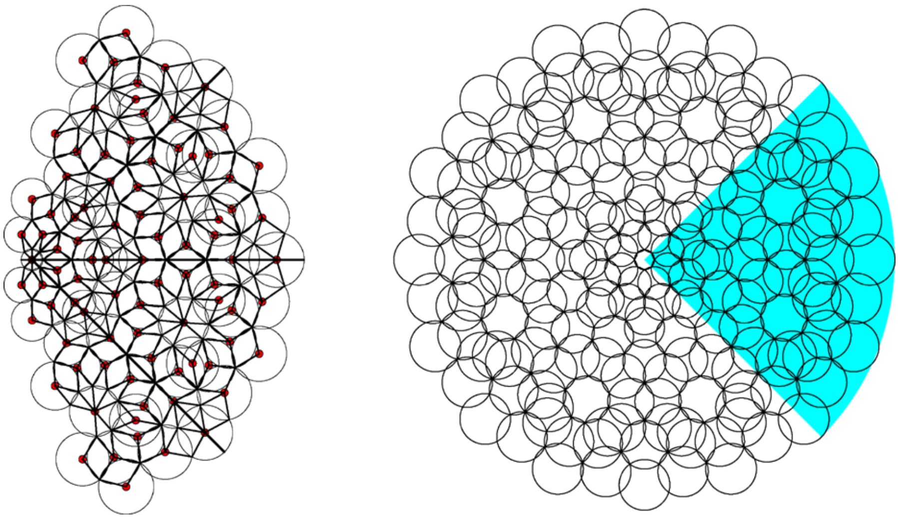

Fig. 10 Example of a quasicrystallic $Z^{5/4}$-circle pattern from the rhombic embedding of Fig. 9: Construction (*left*) and corresponding circle pattern (*right*)

The rhombic embeddings in Figs. 7 and 9 can be obtained by the construction method explained in Example 5.4 using the affine planes $E_1 = \{x = t_1 + \lambda_1 u_1 + \lambda_2 u_2, \lambda_1, \lambda_2 \in \mathbb{R}\}$ and $E_2 = \{x = t_2 + \lambda_1 u_1 + \lambda_2 u_2, \lambda_1, \lambda_2 \in \mathbb{R}\}$ respectively, where u_1, u_2, t_1, t_2 are defined as follows. $\{u_1, u_2, u_3, u_4, u_5\}$ is an orthonormal basis such that the matrix

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

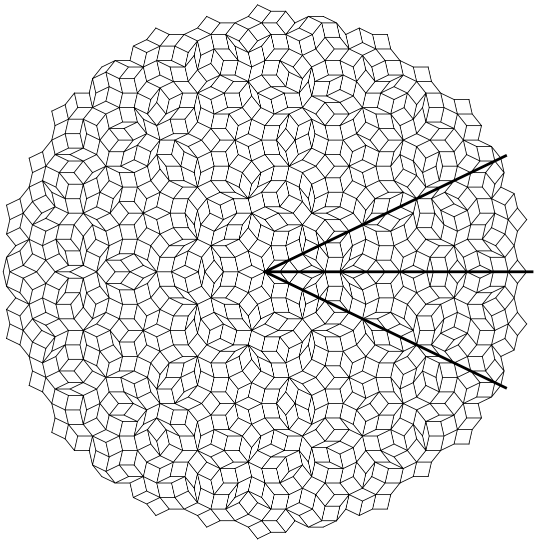

Fig. 11 A quasicrystallic rhombic embedding with seven-fold rotation symmetry

takes the form

$$\begin{pmatrix} \cos(2\pi/5) & -\sin(2\pi/5) & 0 & 0 & 0 \\ \sin(2\pi/5) & \cos(2\pi/5) & 0 & 0 & 0 \\ 0 & 0 & \cos(4\pi/5) & -\sin(4\pi/5) & 0 \\ 0 & 0 & \sin(4\pi/5) & \cos(4\pi/5) & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

with respect to this basis. The translation vectors are $t_1 = (-0.2\ \ -0.2\ \ -0.2\ \ -0.2\ \ -0.2)^T$ and $t_2 = (-0.5\ \ -0.5\ \ -0.5\ \ -0.5\ \ -0.5)^T$ respectively. Here v^T is the transpose of the vector v. Translations in direction of the vector $(1\ 1\ 1\ 1\ 1)^T$ generally lead to rotationally symmetric rhombic embeddings.

For the remaining construction it is important that the rhombic embeddings have rotational symmetry. This is the reason for the rotational symmetry of the quasicrystallic Z^γ-circle patterns.

Now consider the two sectors indicated by lines in Figs. 7 and 9 or more precisely the corresponding 5-dimensional octant in $\mathbb{Z}^5$. Choose an exponent γ of the power function z^γ. In order to construct an image circle pattern which closes up as in Figs. 8(right) and 10(right) we need to choose an integer $p \geq 3$ and take $\gamma = 5/p$. Given these ingredients, define the values of the comparison function w on the coordinate axes of the octant in $\mathbb{Z}^5$ according to (18) and calculate the missing values for the octant using the Hirota equation (4). Taking the values on the quad-surface corresponding to the original rhombic embedding, we can construct a sector of the desired quasicrystallic circle pattern, see Figs. 8(left) and 10(left). The closed circle patterns in Figs. 8(right) and 10(right) are obtained using the rotational symmetry of the quasicrystallic Z^γ-circle patterns.

The construction of the examples in Figs. 11 and 12 is similar using a suitable affine plane in $\mathbb{Z}^7$.

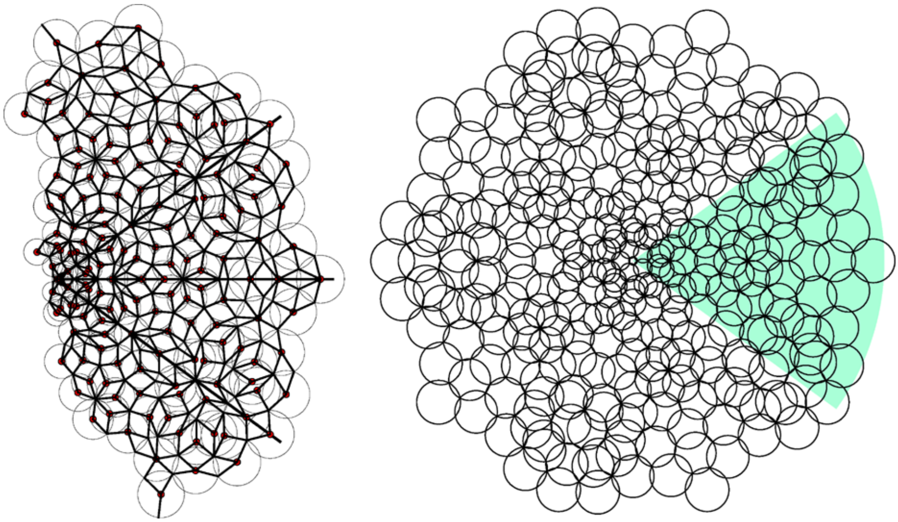

Fig. 12 Example of a quasicrystallic $Z^{7/5}$-circle pattern from the rhombic embedding of Fig. 11: Construction (*left*) and corresponding circle pattern (*right*)

6.2 Uniqueness of Quasicrystallic Z^γ-Circle Patterns

In this section we prove uniqueness of quasicrystallic Z^γ-circle patterns using the same arguments as for the orthogonal case.

We begin with a generalization of Proposition 4.6. Note that we need a (geometric) restriction which ensures that the kites corresponding to intersecting circles are convex. Unfortunately, flips (finite or infinite) may destroy convexity.

Proposition 6.5 *Let $\mathscr{D}$ be a quasicrystallic rhombic embedding with associated graph G. Let α be the labeling corresponding to $\mathscr{D}$. Let v_0 be an interior vertex of G with incident vertices $v_1, \dots, v_m$. Consider two circle patterns for G and α with radius functions r and ρ. Denote $r_j = r(v_j)$ and $\rho_j = \rho(v_j)$ for $j = 0, 1, \dots, m$ and suppose that $r_j \ge r_0 \cos\alpha_j$ and $\rho_j \ge \rho_0 \cos\alpha_j$ for $j = 1, \dots, m$, where $\alpha_j = \alpha([v_0, v_j])$. Then*

$$\sum_{j=1}^{4} c_j \frac{r_j}{\rho_j} \ge \sum_{j=1}^{4} c_j \frac{r_0}{\rho_0} \quad \textit{and} \quad \sum_{j=1}^{4} c_j \frac{\rho_j}{r_j} \ge \sum_{j=1}^{4} c_j \frac{\rho_0}{r_0}, \tag{19}$$

where $c_j = \sin\alpha_j / ((\rho_j/\rho_0) + (\rho_0/\rho_j) - 2\cos\alpha_j)$ for $j = 1, \dots, m$.

Proof

The proof is based on a Taylor expansion of $f_\alpha(x + \log y)$ about $y = 1$.

$$f_\alpha(x + \log y) = f_\alpha(x) + f'_\alpha(x)(y-1) - \frac{\sin\alpha(\mathrm{e}^{x+\log\xi} - \cos\alpha)}{2\xi^2(\cosh(x+\log\xi) - \cos\alpha)^2}(y-1)^2$$

by Lemma 2.3(i) with $\xi = t + (1-t)y$ for some suitable $t \in (0,1)$ depending on x and y. Equation (2) for the two circle patterns implies

$$\begin{aligned}
\pi &= \sum_{j=1}^{m} f_{\alpha_j}\left(\log \frac{r_j}{r_0}\right) = \sum_{j=1}^{m} f_{\alpha_j}\left(\log \frac{\rho_j}{\rho_0} + \log \frac{r_j \rho_0}{r_0 \rho_j}\right) \\
&= \underbrace{\sum_{j=1}^{m} f_{\alpha_j}\left(\log \frac{\rho_j}{\rho_0}\right)}_{=\pi} + \sum_{j=1}^{m} f'_{\alpha_j}\left(\log \frac{\rho_j}{\rho_0}\right)\left(\frac{r_j \rho_0}{r_0 \rho_j} - 1\right) \\
&\quad - \sum_{j=1}^{m} \frac{(\frac{\rho_j}{\rho_0}\xi_j - \cos\alpha_j)\sin\alpha_j}{2\xi_j^2(\cosh(\rho_j/\rho_0 + \log\xi_j) - \cos\alpha_j)^2}\left(\frac{r_j \rho_0}{r_0 \rho_j} - 1\right)^2,
\end{aligned}$$

where $\xi_j = t_j + (1-t_j)\frac{r_j \rho_0}{r_0 \rho_j} > 0$ with suitable $t_j \in (0,1)$ for $j = 1, \ldots, m$. Furthermore

$$\frac{\rho_j}{\rho_0}\xi_j - \cos\alpha_j = t_j \frac{\rho_j}{\rho_0} + (1-t_j)\frac{r_j}{r_0} - \cos\alpha_j \geq 0.$$

by our assumption. Thus

$$\sum_{j=1}^{m} f'_{\alpha_j}\left(\log \frac{\rho_j}{\rho_0}\right)\left(\frac{r_j \rho_0}{r_0 \rho_j} - 1\right) \geq 0.$$

This implies the first claim since $f'_{\alpha_j}(\log \frac{\rho_j}{\rho_0}) = c_j$ by Lemma 2.3(i).

The second claim follows from the fact that $1/\rho$ and $1/r$ are also radius functions of circle patterns for G and α by Lemma 2.4. Also, the coefficients c_j are invariant under the transformation $\rho \mapsto 1/\rho$. □

The following lemma specifies for which parameters the kites of the Z^γ-circle patterns are convex. For the cases excluded below, there exist non-convex kites, because already the kite built from the circle centered at the origin is non-convex.

Lemma 6.6 *If $\psi \geq \pi/2$ and $0 < \gamma < 2$ or if $\psi < \pi/2$ and $(\pi - 2\psi)/(\pi - \psi) \leq \gamma \leq \pi/(\pi - \psi)$, all kites in the Z^γ-circle pattern given by Definition 6.1 are convex.*

The proof is technical. A brief version is presented in the Appendix. More details can be found in [12].

Theorem 6.7 (Rigidity of Z^γ-circle patterns from Definition 7) *If $\psi \geq \pi/2$ and $0 < \gamma < 2$ or if $\psi < \pi/2$ and $(\pi - 2\psi)/(\pi - \psi) \leq \gamma \leq \pi/(\pi - \psi)$ then the Z^γ-circle pattern given by Definition 6.1 is the unique embedded SG-circle pattern for $\mathbb{Z}^2_+$ and α_ψ (up to global scaling) with the following properties.*

(i) *The infinite sector $\{z = \rho e^{i\beta} \in \mathbb{C} : \rho \geq 0,\ \beta \in [0, \gamma(\pi - \psi)]\}$ with angle $\gamma(\pi - \psi)$ is covered by the union of the corresponding kites of the circle pattern.*
(ii) *The centers of the boundary circles lie on the boundary half lines.*
(iii) *All kites corresponding to intersecting circles are convex.*

The proof is a straight forward adaption of the proof of Theorem 4.4 (orthogonal case) using Proposition 6.5 and Lemma 6.6.

Theorem 6.8 *Let* $\psi \in (0,\pi)$ *and* $\gamma \in (0,2) \cap [\frac{\pi-2\psi}{\pi-\psi}, \frac{\pi}{\pi-\psi}]$. *Define* Z^γ*-circle patterns on all four sectors* $\mathbb{Z}_\pm \times \mathbb{Z}_\pm$ *according to Definition 6.1 and glue these patterns to a circle pattern* $\mathcal{C}_\gamma$ *on a cone with cone angle* $2\pi\gamma$. *Then any embedded circle pattern with the same combinatorics and intersection angles which covers the same cone with one center of circle placed at the apex and which has only convex kites coincides with* $\mathcal{C}_\gamma$ (*up to scaling and rotation about the apex of the cone*).

The proof is a simple generalization of the proof of Theorem 6.7 to circle patterns on a cone and combinatorics of $\mathbb{Z}^2$ instead of $\mathbb{Z}^2_+$.

For the generalized quasicrystallic Z^γ-circle patterns (remind the construction explained in Sect. 5.1 or Sect. 13 of [10]) we obtain the following theorem as a direct consequence of the previous theorem and Lemmas 5.5 and 5.7 on local changes of quasicrystallic circle patterns.

Theorem 6.9 (Rigidity of quasicrystallic Z^γ-circle patterns I) *Let* $\mathcal{D}$ *be a quasicrystallic rhombic embedding of a b-quad-graph covering the whole plane. Let* $\mathcal{A} = \{\pm a_1, \ldots, \pm a_d\} \subset \mathbb{S}^1$ *be the edge directions*, *where* $d > 1$ *and any two non-opposite elements of* $\mathcal{A}$ *are linearly independent over* $\mathbb{R}$. *Denote by* $\psi_{\min}$ *the minimum of the undirected angles between any two elements of* $\mathcal{A}$. *Let* $\gamma \in (0,2)$ *with* $(\pi - 2\psi_{\min})/(\pi - \psi_{\min}) \le \gamma \le \pi/(\pi - \psi_{\min})$. *Assume that the origin is a white vertex of* $\mathcal{D}$. *Then a quasicrystallic* Z^γ*-circle pattern* $\mathcal{C}_\gamma$ *corresponding to* $\mathcal{D}$ *and embedded on a cone with cone angle* $2\pi\gamma$ *can be defined using the definition of the comparison function* w *on the* $2d$ *sectors of* $\mathbb{Z}^d$ *which contain the quad-surface* $\Omega_{\mathcal{D}}$; *see* (18) *and the remarks below. Assume further that the brick* $\Pi(\Omega_{\mathcal{D}})$ *contains the whole lattice* $\mathbb{Z}^d$.

Let $\mathcal{C}$ *be an embedded circle pattern with the same combinatorics and the same intersection angles which covers the same cone with one center of circle placed at the apex. Extend the comparison function* w *for* $\mathcal{C}$ *from* $\Omega_{\mathcal{D}}$ *to* $\mathbb{Z}^d$. *For each* $\mathbb{Z}^2$*-sublattice which contains two coordinate axes suppose that the corresponding circle pattern built according to this comparison function has only convex kites. Then* $\mathcal{C}$ *coincides with* $\mathcal{C}_\gamma$ *up to scaling and rotation about the apex of the cone.*

Note that the assumption on the convexity of the kites is only a restriction for a (small) neighborhood of the origin. This is due to Lemma 3.3 which implies that the ratio of the radii is almost one and thus the corresponding angles are almost the same in the isoradial case if the combinatorial distance to the origin is big enough.

If all intersection angles of the labeling α are larger than $\pi/2$, then the kites of any corresponding circle pattern are convex. Examples of such rhombic embeddings are suitable regular hexagonal patterns. Hexagonal circle patterns and in particular analogs of the holomorphic mappings z^γ have been studied by Bobenko and Hoffmann in [8].

Theorem 6.10 (Rigidity of quasicrystallic Z^γ-circle patterns II) *Let* $\mathcal{D}$ *be a quasicrystallic rhombic embedding of a b-quad-graph. Assume that the corresponding*

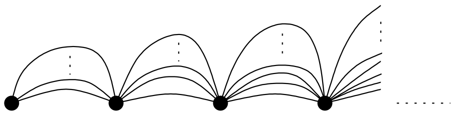

Fig. 13 An illustration of the network obtained from G by shortening

labeling $\alpha : F(\mathscr{D}) \to [\pi/2, \pi)$ has only values larger than $\pi/2$. Assume further that the origin is a white vertex. Let $\gamma \in (0, 2)$.

Define a quasicrystallic Z^γ-circle pattern $\mathscr{C}_\gamma$ for $\mathscr{D}$ and α which is embedded on a cone with cone angle $2\pi\gamma$ using the definition of the comparison function on the $2d$ sectors of $\mathbb{Z}^d$ whose union contains the quad-surface $\Omega_{\mathscr{D}}$; see (18). In particular, the circle corresponding to the origin is centered at the apex of the cone.

Let $\mathscr{C}$ be an embedded circle pattern for $\mathscr{D}$ and α which covers the same cone. Suppose that the circle corresponding to the origin is centered at the apex and that $\mathscr{C}$ has only convex kites. Then $\mathscr{C}$ coincides with $\mathscr{C}_\gamma$ up to scaling and rotation about the apex of the cone.

In this case that all intersection angles are larger than $\pi/2$ the proof of Theorem 4.4 can be directly adapted using Proposition 6.5 and the following lemma.

Lemma 6.11 *Let $\mathscr{D}$ be a quasicrystallic rhombic embedding of a b-quad-graph which covers the whole plane $\mathbb{C}$. Let G be the associated infinite graph built from white vertices. Then the simple random walk on G is recurrent.*

Proof Recall that for the simple random walk all edges have conductance $c(e) = 1$ and also resistance $r(e) = 1/c(e) = 1$. Our aim is to prove that the effective resistance R_{eff} of the network G with these unit conductances is infinite. If two incident vertices of G are identified, that is, the conductance of the connecting edge is increased to ∞ and the resistance decreased to 0, then the effective resistance of the new network is certainly smaller. More generally, the procedure of identifying a set of vertices of G is called *shortening* and reduces the effective resistance; see [14, Sect. 2.2.2] or [29, Theorem 2.19]. Therefore, it is sufficient to show that we have infinite effective resistance $R'_{\text{eff}} = \infty$ for a network G' which is obtained from G by shortening.

Without loss of generality we assume that all edges of $\mathscr{D}$ have length one. Since $\mathscr{D}$ is quasicrystallic rhombic embedding the areas of the rhombi are uniformly bounded. Denote the lower bound by $C_1 > 0$. Let $v_0 \in V(G)$ be any vertex and set $\varrho_k = 4k$ for $k \in \mathbb{N}$. Denote by $V_k \subset V(G)$ the set of vertices which are contained in the annulus $A_k = \{z \in \mathbb{C} : \varrho_{k-1} \le |z - v_0| < \varrho_k\}$ for $k \ge 1$. Then $V(G) = \cup_{k=1}^\infty V_k$. Identify the vertices of each V_k to one new vertex v_k. Then by construction, v_k is only incident to v_{k-1} for $k \ge 2$ and to v_{k+1} for $k \ge 1$; see Fig. 13. Denote by $|E_k|$ the number of edges which are incident to v_k and v_{k+1} for $k \ge 1$. Then the effective resistance of the shortened network is $R'_{\text{eff}} = \sum_{k=1}^\infty 1/|E_k|$. Furthermore $|E_k| \le$ area of $\{z \in \mathbb{C} : \varrho_k - 2 \le |z| < \varrho_k + 2\}/C_1$. Now a simple estimation shows that $R'_{\text{eff}} = \infty$. □

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Appendix: Proof of Lemma 6.6

First note that kites with intersection angle larger than $\pi/2$ are always convex. The remaining claims on convexity are consequences of Proposition 3 of [3], namely

$$(N+M)\big(R(z)^2 - R(z+1)R(z-i) - \cos\psi R(z)\big(R(z-i) - R(z+1)\big)\big) \times \big(R(z+i) + R(z+1)\big) + (M-N)\big(R(z)^2 - R(z+i)R(z+1) - \cos\psi R(z)\big(R(z+i) - R(z+1)\big)\big)\big(R(z+1) + R(z-i)\big) = 0 \qquad (20)$$

for $z \in \mathbb{V} \setminus \{N + iN | N \in \mathbb{N}\}$, and the results of Theorem 6.2.

In particular, for $\pi - \psi \le \pi/2$ the kites with intersection angle $\psi \le \pi/2$ at black vertices are always convex. This follows from (20) and inequality (17) by simple calculations. This shows the claim for $\psi \ge \pi/2$.

If $\psi < \pi/2$ and $(\pi - 2\psi)/(\pi - \psi) \le \gamma \le \pi/(\pi - \psi)$, inequality (17) only implies that for all kites with white vertices z and $z + i$ and intersection angle ψ the angle at the point corresponding to z for $0 < \gamma < 1$ and to $z + i$ for $1 < \gamma < 2$ respectively is smaller than π. This excludes some types of non-convex kites, but not all.

For fixed $n > 0$, let Γ_n be the piecewise linear curve formed by the segments $[f_{n,m}, f_{n,m+1}]$ for $m \ge 0$. By Theorem 6.2 and its proof in [3], these curves are embedded without self-intersections and the vector $\mathbf{v}_n(m) = f_{n,m+1} - f_{n,m}$ rotates clockwise for $0 < \gamma < 1$ and counterclockwise for $1 < \gamma < 2$ along Γ_n.

Without loss of generality, we only consider the case $1 < \gamma < 2$ further. For $0 < \gamma < 1$ the proof is very similar. First, consider a kite on the symmetry axis, that is, with white vertices corresponding to iK and $i(K+1)$. Then the assumption $\gamma(\pi - \psi) \le \pi$ and the properties of the curves Γ_n imply that the kites on the symmetry axis are convex. Next, we consider the intersection angles α and β of the line in direction of the vector $\mathbf{v}_n(m)$ with the oriented half lines $\mathbb{R}^+$ and $e^{i\gamma(\pi-\psi)/2}\mathbb{R}^+$ respectively. We additionally assume that n is odd. As the kites on the symmetry axis are convex and $\gamma(\pi - \psi) \le \pi$, we deduce $\alpha \le \psi$ and $\pi/2 - \psi \le \beta \le \pi/2$.

Finally, consider a kite with white vertex $f_{n,m}$ for odd n and $m < n$. We estimate the angles α_1 and α_2 at this vertex of the kites containing the points $f_{n,m-1}$, $f_{n,m}$, $f_{n+1,m}$ and $f_{n+1,m}$, $f_{n,m}$, $f_{n,m+1}$ respectively. Note that these kites both have intersection angles ψ. Using the above estimations on α and β and the properties of the curve Γ_n we obtain $\pi - 2\psi \le \alpha_1$ and $\alpha_2 \le \pi$. Therefore the angle at $f_{n-1,m-1}$ of the kite containing the points $f_{n,m-1}$, $f_{n,m}$, $f_{n+1,m}$ is $2\pi - 2\psi - \alpha_1 \le \pi$. So this kite is convex. Furthermore, we deduce that the kite containing the points $f_{n+1,m}$, $f_{n,m}$, $f_{n,m+1}$ is also convex. Consequently all kites containing the white vertex $f_{n,m}$ are convex.

Thus the Z^γ-circle pattern with $\psi < \pi/2$ and $(\pi - 2\psi)/(\pi - \psi) \le \gamma \le \pi/(\pi - \psi)$ only contains convex kites.

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