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Polychromatic Colorings of Plane Graphs

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Abstract We show that the vertices of any plane graph in which every face is incident to at least g vertices can be colored by $\lfloor (3g-5)/4 \rfloor$ colors so that every color

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appears in every face. This is nearly tight, as there are plane graphs where all faces are incident to at least g vertices and that admit no vertex coloring of this type with more than $\lfloor(3g+1)/4\rfloor$ colors. We further show that the problem of determining whether a plane graph admits a vertex coloring by k colors in which all colors appear in every face is in $\mathcal{P}$ for $k=2$ and is $\mathcal{NP}$-complete for $k=3,4$. We refine this result for polychromatic 3-colorings restricted to 2-connected graphs which have face sizes from a prescribed (possibly infinite) set of integers. Thereby we find an almost complete characterization of these sets of integers (face sizes) for which the corresponding decision problem is in $\mathcal{P}$, and for the others it is $\mathcal{NP}$-complete.

Keywords Graph coloring · Planar graphs · Guarding problems

1 Introduction

Problem Statement and Main Results A *plane graph* is a graph G together with an embedding of G into the plane. We allow plane graphs to contain multi-edges and loops, i.e., we consider *multi-graphs* in general (but we will forbid loops later). When we do not allow multi-edges or loops, we use the term *simple graph*. Let $V(G)$, $E(G)$, $F(G)$ denote the set of vertices, edges, faces of G. The *size* of a face $f \in F(G)$ is the number of vertices on its boundary. For a plane graph G, let $g(G)$ denote the size of the smallest face in G. A face of size s in a plane graph is sometimes also called an s-face. A vertex k-coloring is a map $\chi : V(G) \to \{1,\dots,k\}$, and it is *proper* if for every edge $uv \in E(G)$, $\chi(u) \neq \chi(v)$. For a (not necessarily proper) vertex k-coloring $\chi : V(G) \to \{1,\dots,k\}$ of G, we say that a face $f \in F(G)$ is *polychromatic* if all k colors appear on the vertices of f. A vertex k-coloring of G is called *polychromatic* if every face (also the outer-face) of G is polychromatic. The *polychromatic number* of G, denoted by $p(G)$, is the largest number of colors k such that there is a polychromatic vertex k-coloring of G. Define $p(g) = \min\{p(G) \mid G \text{ plane graph}, g(G) = g\}$.

It is clear that for every plane graph G, $p(g(G)) \le p(G) \le g(G)$. On the other hand, $p(G) \ge 2$ for all plane graphs G with $g(G) \ge 3$. This was first proved by Bose et al. [3] (see also [17]) by using the Four-Color Theorem, and afterwards by Bose et al. [4] without using it. The following sketch of the second proof is similar in spirit to some of the ideas in the present paper.

First, we assume that G is loopless, and therefore every cycle has length at least 2. Triangulate the graph G by adding edges, resulting in a new graph H where each face (again also the outer-face) forms a triangle. The dual graph H^* of H is then 3-regular. Moreover H^* is 2-edge connected: suppose that H^* contains a cut-edge e and take a

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maximal connected component. The corresponding faces in H have some boundary which forms a cycle of length at least 2, and therefore there are in H^* at least 2 outgoing edges from this component where only one of them can be e. By Petersen's Theorem (see, e.g., [23]), there exists a perfect matching M in H^*. After deleting the edges of H corresponding to those of M, the remaining graph H' has only faces of size 4. Therefore there is no odd cycle in H', and hence H' is bipartite. Thus, there is a proper vertex 2-coloring of H', which is a polychromatic vertex 2-coloring of H and hence also of G.

If G is a plane graph with a loop around v, let G_1 be the graph inside the loop (without the loop but with v), and let G_2 the graph outside the loop (without the loop but with v). Then $p(G) = \min\{p(G_1), p(G_2)\}$ and $g(G) = \min\{g(G_1), g(G_2)\}$. Therefore it follows inductively that $p(G) \geq 2$ for every plane graph G with $g(G) \geq 3$.

In the following we will only consider *loopless* multi-graphs. As we have seen in the previous paragraph, any lower bound on $p(G)$ for loopless plane graphs is also true for graphs containing a loop. Furthermore, any upper bound construction containing a loop can be made loopless.

Our main result bounds the minimum possible polychromatic number for plane graphs G with $g(G) = g$.

Theorem 1 $p(1) = p(2) = 1,\ p(3) = p(4) = 2$, *and for* $g \geq 3$,

$$\left\lfloor \frac{3g-5}{4} \right\rfloor \leq p(g) \leq \left\lfloor \frac{3g+1}{4} \right\rfloor. \tag{1}$$

This settles a question raised in [13]. Note that the set $\{\lfloor \frac{3g-5}{4} \rfloor, \dots, \lfloor \frac{3g+1}{4} \rfloor\}$ contains two or three integers. The first case that remains open from Theorem 1 is $g = 5$.

The lower bound for $p(G)$ in (1) remains true when restricting to simple plane graphs G with $g(G) = g$. Moreover, the constructions for the upper bound in (1) are in fact simple plane graphs.

For every triangulation G, it holds that $2 \leq p(G) \leq 3$. The following simple characterization of triangulations G with $p(G) = 3$ is a consequence of an old result of Heawood [11] and will be proven in Sect. 2.

Theorem 2 *Let G be a triangulation. The following two statements are equivalent*:

(i) $p(G) = 3$, *and*
(ii) *G is Eulerian, i.e., every vertex degree in G is even.*

Theorem 2 immediately implies a polynomial-time algorithm to decide whether a triangulation admits a polychromatic 3-coloring.

For general plane graphs G, we show that the decision problem whether G is polychromatically 3-colorable is hard (and also for polychromatic 4-colorings).

Theorem 3 *The decision problem whether a plane graph is polychromatically k-colorable is*

(i) *in $\mathcal{P}$ if $k = 2$, and*
(ii) *$\mathcal{NP}$-complete for $k = 3, 4$.*

Moreover we consider the decision problem whether a 2-connected plane graph with faces of size restricted to a set of integers admits a polychromatic 3-coloring. We achieve an almost complete characterization of such sets of integers (face sizes) for which the corresponding decision problem is $\mathcal{NP}$-complete, and for the others it is in $\mathcal{P}$.

Connection to Guarding Problems Polychromatic colorings are related to a combinatorial version of *guarding problems* on graphs. In general, guarding problems ask for a small set of vertices (guards) that *see* a given input domain, for example a polygon, a terrain, or a plane graph. If we consider guarding a plane graph G, then G is guarded if every face of G is guarded. If all faces are convex, then every vertex on the boundary of a face sees the complete face. If the faces are not convex, more guards might be necessary. Certainly a guard cannot see the entire unbounded face, hence the outer-face is usually not required to be guarded. A combinatorial variant of this problem is the following: Find the smallest set of vertices S of G such that every face is incident to (at least) one of the vertices in S. Clearly each color class in a polychromatic coloring is a guarding set, that is, the vertices in each color class jointly *guard* the graph G. From now on we use "guard" in this combinatorial sense and also require the unbounded face to be guarded.

In [3] it is shown that one can guard any plane graph on n vertices with no faces of size 1 or 2 by $\lfloor \frac{n}{2} \rfloor$ guards. This clearly follows from the fact that $p(G) \geq 2$ for any such graph. The authors also construct graphs on n vertices for which $\lfloor \frac{n}{2} \rfloor$ guards are necessary. Similarly, a simple consequence of Theorem 1 is the following:

Corollary 4 *Every plane graph G with $g(G) = g$ can be guarded with at most $\frac{n}{\lfloor (3g-5)/4 \rfloor} \leq \frac{4n}{3g-8}$ guards.*

Proof By Theorem 1, G admits a polychromatic $\lfloor \frac{3g-5}{4} \rfloor$-coloring. Place guards on the vertices of the smallest color class which is of size at most $\frac{n}{\lfloor \frac{3g-5}{4} \rfloor} \leq \frac{4n}{3g-8}$. Because the coloring is polychromatic, each face is incident to at least one guard, and the statement follows. □

Related Work From a result of Hoffmann and Kriegel [12] it immediately follows that the polychromatic number of any plane, bipartite, 2-connected simple graph is at least 3, see also Proposition 8. Horev and Krakovski showed in [13] that any connected plane graph G with $g(G) \geq 3$ and maximum degree at most 3, which is not K_4 or a subdivision of K_4 on 5 vertices, are polychromatically 3-colorable. In [7] it is shown that every bipartite cubic plane graph can be colored with 4 colors so that every bounded face of G is polychromatic.

A nondegenerate rectangular subdivision is a subdivision of a rectangle into a set of nonoverlapping rectangles such that no four rectangles meet in a point. Dinitz et al. [6] showed that it is possible to color the vertices of any nondegenerate rectangular subdivision S with three colors so that each rectangle in S has at least one vertex of each color. They conjectured that this is also possible with four colors. And indeed, a proof by Guenin [8] of a conjecture by Seymour [20] concerning the edge-coloring of a special class of planar graph directly implies such a 4-coloring [5].

Keszegh [15] investigates polychromatic colorings of so-called n-dimensional guillotine-partitions.

Alon et al. [2] considered two variants of polychromatic colorings for the n-dimensional hypercube Q_n. They observed that the vertices of Q_n can be colored by $d+1$ colors so that every d-dimensional subcube Q_d of Q_n is polychromatic, that is, Q_d contains a vertex of each color. Indeed, fix a vertex $v \in Q_d$ and color each vertex of Q_n at distance i from v with color $i \bmod (d+1)$. Moreover this coloring with $d+1$ colors is best possible for all d and n sufficiently large as shown in [2]. The authors also show that there is an edge-coloring of Q_n with $\lfloor \frac{(d+1)^2}{4} \rfloor$ colors such that every d-dimensional subcube is polychromatic, that is, contains an edge of each color. This bound is tight as shown by Offner [19].

Notation The (open) *neighborhood* of a vertex v in the graph G is denoted by $N_G(v)$ and contains all vertices adjacent to v, the *vertex degree* $d_G(v)$ of v is the number edges incident to v. When the graph G is clear from the context, we sometimes denote $d_G(v)$ by $d(v)$. The *minimum degree* of a vertex in G is denoted by $\delta(G)$, and the *maximum degree* by $\Delta(G)$. For a directed graph G, the *in-* and *out-neighborhood* of a vertex v are denoted by $N_G^-(v)$ and $N_G^+(v)$, respectively. Similarly we define the *in-degree* $d_G^-(v)$ and *out-degree* $d_G^+(v)$ as the numbers of incoming and outgoing edges of v. For a subset U of the vertices of $G=(V,E)$, we denote by $G[U]$ the graph induced by the vertices U, and by $G-U$ the graph $G[V(G)\setminus U]$. A graph is called 2-*connected*, 2-*edge-connected*, respectively, if after the deletion of any vertex, edge, respectively, the graph is still connected, i.e., there are no cut-vertices, cut-edges.

Organization In the next section we prove Theorem 2 and give some additional results for graphs with faces of even size only and for outerplanar graphs. In Sect. 3 we prove Theorem 1. Section 4 is dedicated to the proof of Theorem 3 and discusses some additional complexity results that restrict the face sizes of the input graph. We conclude with some remarks and open problems.

2 Simple Cases

The following two lemmas jointly prove Theorem 2.

Lemma 5 (Kempe 1879 [14], Heawood 1898 [11, 21]) *The vertices of a triangulation G are properly* 3*-colorable if and only if G is Eulerian.*

Lemma 6 *Let G be a triangulation. Then the following are equivalent*:

(i) *G is polychromatically* 3-*colorable.*
(ii) *G is properly* 3-*colorable.*

Proof A triangle is properly 3-colored if and only if its three vertices have three different colors. Also, a triangle is polychromatically 3-colored if and only if its three vertices have three different colors. Thus these two notions are equivalent for triangulations. □

Another class of plane graphs that are polychromatically 3-colorable are plane graphs with faces of even size only. To show this, we first need the following result.

Lemma 7 (Hoffmann and Kriegel [12]) *Let G be a plane*, 2-*connected*, *bipartite*, *and simple graph. Then we can add edges to G to obtain a triangulation such that every vertex-degree is even. Moreover, this triangulation can be found in polynomial time.*

Proposition 8 *Let G be a plane*, 2-*connected multi-graph with even faces only and* $g(G) \geq 4$. *Then there exists a polychromatic* 3-*coloring of G that is proper as well*, *i.e.*, *no edge is monochromatic. Moreover, such a coloring can be found in polynomial time.*

Proof We prove the statement by induction on the number of multi-edges of G. First, we assume that G is simple. Every cycle in G has even length (i.e., G is bipartite) because G is required to be 2-connected and has only even faces. The statement follows after applying Lemmas 7, 5, and 6.

Next, we assume that G has some multi-edges. Let $x, y \in V(G)$ and $e_1, e_2 \in E(G)$, where both e_1 and e_2 connect x and y. The edges e_1, e_2 build a cycle of length two, and therefore they divide the plane into two parts. Let V_1 be the vertices inside e_1, e_2 (including x, y) and V_2 the vertices outside e_1, e_2 (including x, y). Since $g(G) \geq 4$, we can conclude that $V_i \supsetneq \{x, y\}$ for $i = 1, 2$. Define $G_1 = (V_1, E(G[V_1]) \setminus \{e_2\})$ and $G_2 = (V_2, E(G[V_2]) \setminus \{e_1\})$. These two graphs are plane, 2-connected with even faces only, $g(G_1), g(G_2) \geq 4$, and each G_i contains less multi-edges than G. There exists inductively a polychromatic 3-coloring of G_i, $i = 1, 2$, such that no edge is monochromatic. In particular the coloring of G_1 and the coloring of G_2 assign distinct colors to x and y. Thus we can permute the colors of one coloring so that the colors of x and y agree in the coloring of G_1 and of G_2. This yields a 3-coloring of G which is easily seen to fulfill the condition in the statement. □

Another simple case is where the graph G is outerplanar (i.e., all vertices lie on the outer-face). Then it is easy to see that the size of the smallest face is equal to the length of the smallest cycle (i.e., the girth of G). We show that the trivial upper bound $p(G) \le g(G)$ is tight for outerplanar graphs G with $g(G) \ge 3$.

Proposition 9 *Let G be an outerplanar graph with $g = g(G) \ge 3$. Then there exists a polychromatic coloring of G with g colors that is also proper, i.e., no edge is monochromatic.*

Proof We prove this result by induction on the number of faces. If we have only one face, then the graph G is a tree, and clearly we can polychromatically color every tree with $|V(G)| = g(G)$ colors so that no edge is monochromatic. Let us assume now that G has more than one face. Obviously, it is sufficient to find a g-coloring of the vertices of G such that all bounded faces are polychromatic and no edge is monochromatic. The outer-face will by the outerplanarity automatically be polychromatic since all vertices lie on the outer-face. Also we can assume without loss of generality that G is connected and has no cut-vertex. Otherwise color the 2-connected components separately and combine the coloring (maybe rename the colors in each component correspondingly).

It is well known that the dual graph G^* without the outer-face forms a forest; and since G is 2-connected, G^* is connected, and so G^* forms a tree. Every tree has at least two leafs. Choose f_0 as a face corresponding to a leaf in the tree with maximal size. Let G' be the graph obtained from G after deleting all vertices incident to only f_0 and the outer-face. Then G' is an outerplanar graph with $g(G') = g(G)$ and has one fewer face than G. By the induction hypothesis we can color G' polychromatically with g colors so that no edge is monochromatic.

Finally, add f_0 again to G'. There is exactly one edge $e_0 \in E(G')$ which is on the boundary of the face f_0, i.e., e_0 is the edge between f_0 and its parent. The intersection of the vertices of f_0 and $V(G')$ are exactly the endpoints z_1, z_2 of e_0. For simplicity, assume that z_1 has color 1 and z_2 has color 2. Let $z_3, \dots, z_k$ be the other vertices of f_0 such that $z_1, z_2, z_3, \dots, z_k$ is the clockwise or counterclockwise order in that face. Extend the coloring of f_0 to $1, 2, \dots, g, g-1, g, g-1, \dots$. The face f_0 will then be polychromatic (because $k \ge g$), and no edge of f_0 will be monochromatic (because $g \ge 3$). □

The graph G' from Fig. 1 shows an outerplanar graph with $g(G') = 2$ which is not polychromatically 2-colorable.

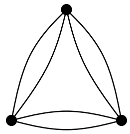


Fig. 1 Graph G' with $g(G') = 2$ and $p(G') = 1$

3 The Proof of Theorem 1

In this section we prove Theorem 1. We begin by proving the theorem for $1 \leq g \leq 4$ in Sect. 3.1. We then tackle the general case. First, we make vertices responsible for certain faces by assigning them to these faces. Section 3.2 shows how to compute an appropriate assignment. Using this assignment, we can relate our problem to the existence of certain edge-colorings. In Sect. 3.3 we establish the necessary results for these edge-colorings. Finally, we prove the general lower and upper bounds in Sects. 3.4 and 3.5.

Note that the function $p(g)$ is monotone, i.e., for $g \leq g'$, we have $p(g) \leq p(g')$.

3.1 Theorem 1 for $1 \leq g \leq 4$

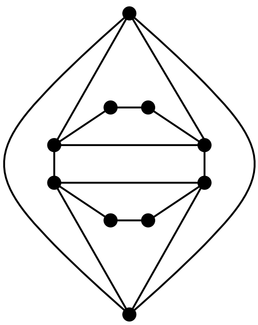
If $g(G) = 1$, then G contains only one vertex, and therefore $p(1) \leq 1$. If $g(G) = 2$, then G contains either multi-edges or only two vertices. The graph G' depicted in Fig. 1 shows that also $p(2) \leq 1$.

For all graphs G with $g(G) \geq 3$, we have already seen in Sect. 1 that $p(G) \geq 2$. A planar embedding of K_4 is a non-Eulerian triangulation and has therefore $g(K_4) = 3$ and $p(K_4) = 2$ (this is a trivial special case of Theorem 2).

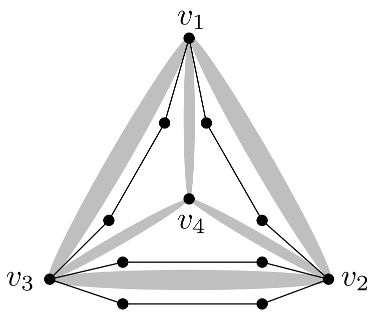
For $g = 4$, consider Figs. 2(a) and (b), which illustrate the construction of a graph G. The graph G equals the *forcing graph* (see Fig. 2(b)) where each of the six shaded edges $v_i v_j$ is replaced by a copy of the base graph (see Fig. 2(a)) by identifying the vertices x and y with the vertices v_i and v_j. Clearly, $g(G) = 4$. It is easy to check that the following holds.

Observation 10 *In any polychromatic* 3*-coloring of a base graph* (*see Fig.* 2(*a*)) *the vertices v_i and v_j are colored with distinct colors.*

Thus from the fact that K_4, the graph underlying the forcing graph, is not properly 3-colorable it follows that $p(G) \leq 2$.



(a) Base graph for $g = 4$.



(b) Forcing graph.

Fig. 2 Graph G with $p(G) = 2$ and $g(G) = 4$

3.2 Assigning Vertices to Faces

Lemma 11 *Let G be a plane graph, let $\emptyset \neq F' \subseteq F(G), \emptyset \neq V' \subseteq V(G)$, and let $i(V', F')$ denote the number of incidences between F' and V'. Then* $i(V', F') \leq 2|F'| + 2|V'| - 3$.

Proof Define the incidence graph H of $V' \subseteq V(G)$ and $F' \subseteq F(G)$ by $V(H) = F' \cup V'$ and $fv \in E(H)$ for $v \in V'$ and $f \in F'$ if and only if v is incident to the boundary of f in G. It is easy to see that H is planar, simple, and bipartite. From Euler's Formula and the fact that H is simple and triangle-free it follows that H contains at most $2V(H) - 4$ edges, provided that H contains at least three vertices. In this case we conclude that $i(V', F') = |E(H)| \leq 2(|V'| + |F'|) - 4$. On the other hand, if $|V(H)| = 2$ and H contains one edge, then $i(V', F') = 2(|V'| + |F'|) - 3 = 1$. □

The following result is well known (see, for example, [16], Theorem 2.4.2). For completeness, we include a short proof.

Lemma 12 *Let $A \in \{0, 1\}^{m\times n}$ be a matrix, $A = (a_{i,j})_{i\in\{1,\dots,m\}, j\in\{1,\dots,n\}}$. The following two statements are equivalent*:

(i) *There is a matrix $C \in \{0, 1\}^{m\times n}$, $C \leq A$ (that is, $c_{i,j} \leq a_{i,j}$ for all $i \in \{1, \dots, m\}$ and all $j \in \{1, \dots, n\}$) such that every row in C contains at least q 1's and every column in C contains at most r 1's.*

(ii) *For every $M \subseteq \{1, \dots, m\}$ and every $N \subseteq \{1, \dots, n\}$,* $\sum_{i\in M, j\in\{1,\dots,n\}\setminus N} a_{i,j} \geq q|M| - r|N|$.

Proof Define a network with vertices $s, t, r_1, \dots, r_m, c_1, \dots, c_n$ as follows. Connect the source s with all vertices r_i with edges having capacity q, connect r_i with c_j with edges having capacity $a_{i,j}$, and connect all c_j to the sink t with edges having capacity r. If condition (i) holds, then we can also assume that there exists such a matrix C where in every row there are exactly q 1's. Thus there exists a flow of value mq if and only if (i) holds. It is easy to show that all cuts have size at least qm if and only if condition (ii) holds. This implies the statement by using the MaxFlow–MinCut Theorem. □

Corollary 13 *Let G be a plane graph with $g(G) = g$. For each face $f \in F(G)$, we can assign $g - 2$ vertices that lie on its boundary such that no vertex is assigned to more than two faces.*

Proof Let $A = (a_{f,v})_{f\in F, v\in V} \in \{0, 1\}^{|F|\times|V|}$ be the face–vertex incidence matrix of G. That is, $a_{f,v} = 1$ if and only if vertex v is contained in face f. We want to show that there is a matrix $C \in \{0, 1\}^{|F|\times|V|}$ such that $C \leq A$, in every row of C there are at least $g - 2$ 1's, and in every column of C there are at most two 1's.

By Lemma 12 with $q = g - 2$ and $r = 2$, it is sufficient to show that for all $F' \subseteq F$ and $V' \subseteq V$, $\sum_{f\in F', v\in V\setminus V'} a_{f,v} \geq (g - 2)|F'| - 2|V'|$.

Henceforth we obtain

$$\sum_{f\in F',v\in V\setminus V'} a_{f,v} = \sum_{f\in F',v\in V} a_{f,v} - \sum_{f\in F',v\in V'} a_{f,v}$$
$$\geq g|F'| - \sum_{f\in F',v\in V'} a_{f,v}$$
$$\geq g|F'| - 2|F'| - 2|V'|,$$

where the last inequality follows from Lemma 11 in case both V' and F' are non-empty and is trivial if at least one of them is empty. □

Alternative Proof of Corollary 13 The following lemma and its short proof have been suggested by one of the anonymous referees. The lemma easily implies Corollary 13. Nevertheless we think that our original proof yields a more general result, which is applied later in Theorem 30.

Lemma 14 *Every plane graph G admits a vertex–face assignment such that*

(i) *every vertex is assigned to at most two faces*,
(ii) *every face is assigned to all but at most two of the vertices incident to it*, *and*
(iii) *the outer-face is assigned to all of its vertices*.

Sketch of a Proof We can assume that G is 2-edge connected and consider an ear-decomposition of G. The ear-decomposition gives a construction of G as follows: start with the cycle C bounding the outer-face and successively add paths lying in a face of the current plane graph. A vertex–face assignment can be constructed simultaneously with this procedure. Initially add the cycle C, splitting the plane into two faces f_0 (the outer-face) and f_1 (the interior of the cycle). Meanwhile, assign all vertices of C to f_0 and f_1. Whenever a path P is inserted into a face $f_i, i \geq 1$, and splits it into two faces f_{i_1} and f_{i_2}, update the assignment such that the assigned vertices of f_i incident to only f_{i_j} are reassigned to f_{i_j} for $j \in \{1, 2\}$ and the new vertices of P are assigned to both faces f_{i_1}, f_{i_2}. The (possibly coinciding) endpoints can be assigned in a good way: if both faces f_{i_1}, f_{i_2} currently have all but three vertices assigned to, then assign one endpoint of P to f_{i_1} and the other to f_{i_2}. If f_{i_j} for some $j \in \{1, 2\}$ currently has all but four vertices assigned to, then assign both endpoints to f_{i_j}. □

3.3 Polychromatic Edge-Colorings

Similar to the case of vertex-colorings, we define a polychromatic edge k-coloring $\chi : E(G) \to \{1, \dots, k\}$ of G. A vertex $v \in V(G)$ is called *polychromatic* if all k-colors appear on the edges incident to v. An edge-coloring of G is *polychromatic* if every vertex $v \in V(G)$ is polychromatic.

Recently, we learned that our result about polychromatic edge-colorings were obtained independently by Gupta [9].

Proposition 15 (See also [9]) *For every integer $d > 0$ and every multi-graph G with minimum degree at least d, there is a polychromatic edge-coloring of G with $\lfloor \frac{3d+1}{4} \rfloor$ colors.*

Before we prove Proposition 15 we state three lemmas. For completeness, we include their short proofs.

Lemma 16 [1, 10] *Let r be a positive integer. It is possible to color the edges of any bipartite multi-graph G by r colors $\{1, \dots, r\}$ such that for every vertex v of G, the numbers of edges of each color incident with v are nearly equal. That is, for every $i \in \{1, \dots, r\}$, the number of edges of color i incident with v is either $\lfloor d(v)/r \rfloor$ or $\lceil d(v)/r \rceil$.*

Proof First split vertices of G, if needed, to make its maximum degree at most r: As long as there is a vertex v of G of degree $d > r$, modify it using the following procedure. Define $k = \lceil d/r \rceil$ and replace v by k new vertices $v_1, v_2, \dots, v_k$, called its descendants. Let $vu_1, vu_2, \dots, vu_d$ be an arbitrary enumeration of all edges of G incident with v. For each i, $1 \le i \le k$, let the edges incident with the new vertex v_i be the edges $v_i u_j$ for all j satisfying $(i-1)r < j \le \min\{d, ir\}$. This process terminates with a bipartite graph in which all degrees are at most r. By König's Theorem (see, e.g., [23]) the edges of this graph can be properly colored by the r colors $\{1, \dots, r\}$. By collapsing all descendants of each vertex v back, keeping the colors of the edges, we obtain a coloring $f : E(G) \to \{1, \dots, r\}$ of the edges of the original graph G by r colors satisfying the assertion of the claim. □

The following two lemmas are well known.

Lemma 17 *Every multi-graph G contains a spanning bipartite graph $B \subseteq G$ with $d_B(v) \ge \lceil \frac{d_G(v)}{2} \rceil$ for every $v \in V(G)$.*

Proof Let B be a maximum edge-cut in G with respect to the number of edges. Assume that there is a vertex $v \in V(G)$ with $d_B(v) < \lceil \frac{d_G(v)}{2} \rceil$. If we then swap v to the other bipartite set, this would yield another edge-cut with more edges, contradicting the maximality. □

Lemma 18 *Every multi-graph G has an orientation of its edges such that $d^+(v) \ge \lfloor \frac{d(v)}{2} \rfloor$ for all $v \in V(G)$.*

Proof We may assume that G is connected. If all degrees in G are even, we simply orient it along an Eulerian cycle. Otherwise, define a new graph G' which consists of all vertices of G and a new vertex x and connect all odd degree vertices of G to x. Then all vertices in G' have even degrees, and therefore there is an Eulerian cycle in G'. Orient the edges along such an Eulerian cycle and delete the vertex x. Every vertex $v \in V(G)$ with even degree has then exactly $d(v)/2$ outgoing edges. Each vertex $v \in V(G)$ with odd degree has either $(d(v)+1)/2$ or $(d(v)-1)/2$ outgoing edges. □

Proof of Proposition 15 By Lemma 17 there is a spanning bipartite subgraph H of G satisfying $\delta(H) \geq \lceil \frac{d}{2} \rceil$. Let A_1 and A_2 denote its vertex classes. Applying Lemma 16 to H with $r = p = \lfloor \frac{3d+1}{4} \rfloor$ results in an edge-coloring χ with the following two properties.

(i) Every vertex v with $d_H(v) \geq p$ is polychromatic. Indeed v is incident with at least $\lfloor d_H(v)/p \rfloor \geq 1$ edges of each of the p colors.

(ii) For every vertex u with $d_H(u) < p$, each color appears at most once on edges incident to u since $\lceil d_H(u)/p \rceil = 1$. In other words, all edges incident with u have distinct colors. Extend the edge-coloring χ to G as follows: orient the edges of both $G[A_1]$ and $G[A_2]$ according to Lemma 18. Hence $d^+_{G[A_i]}(v) \geq \lfloor \frac{d-d_H(v)}{2} \rfloor$ for $i = 1, 2$ and all $v \in A_i \subset V(G)$. For each vertex $v \in A_i$, color the edges oriented from v to its outneighbors in $G[A_i]$ with the colors not appearing at the edges of H incident to v (if there are any such colors). Thus, the edges incident with any vertex $v \in V(G)$ are finally colored with $\min\{d_H(v) + \lfloor \frac{d-d_H(v)}{2} \rfloor, p\} \geq \lceil \frac{d}{2} \rceil + \lfloor \frac{\lfloor \frac{d}{2} \rfloor}{2} \rfloor = \lfloor \frac{3d+1}{4} \rfloor = p$ distinct colors, where the inequality follows from the fact that $d_H(v) \geq \lceil \frac{d}{2} \rceil$. This completes the proof.

□

3.4 The Lower Bound of Theorem 1

We now prove that $p(g) \geq \lfloor \frac{3g-5}{4} \rfloor$ for all $g \geq 5$.

Let $G = (V, E)$ be a plane graph with $g(G) = g$. By Corollary 13 we can assign $g - 2$ vertices from its boundary to every face of G such that no vertex is assigned to more than two faces of G. Define an auxiliary multi-graph H with $V(H) = F(G) \cup \{x, y\}$, where $F(G)$ is the set of faces of G, and x, y are two additional vertices. For every vertex $v \in V(G)$, define an edge of H, which we call the v-edge, as follows. If v is assigned to two distinct faces f_1 and f_2, then the v-edge is $f_1 f_2$. If it is assigned only to one face f, the v-edge is fx, and if it is not assigned to any face, then the v-edge is xy. In addition, add $g-2$ (multi)edges to H connecting x and y to ensure that all degrees in H are at least $g - 2$. Thus, H is a loopless multi-graph with minimum degree at least $g - 2$. By Proposition 15 with $d = g - 2$ we can color the edges of H with $p = \lfloor \frac{3(g-2)+1}{4} \rfloor = \lfloor \frac{3g-5}{4} \rfloor$ colors so that every vertex $f \in V(H)$ is incident with edges of all p colors.

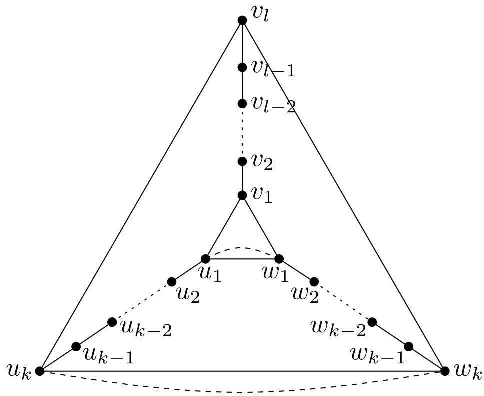
Define a vertex-coloring of G by coloring every vertex $v \in V(G)$ by the same color as that of the v-edge. This clearly gives a coloring in which every face $f \in F(G)$ is polychromatic, as needed.

The above proof is constructive, i.e., one can find in polynomial time a polychromatic coloring of G with $\lfloor \frac{3g-5}{4} \rfloor$ colors.

3.5 The Upper Bound of Theorem 1

We next show that $p(g) \leq \lfloor \frac{3g+1}{4} \rfloor$ for all $g \geq 3$. Define the graph G_g as depicted in Fig. 3. For g even, set $k = l = \frac{g}{2}$, and for g odd, set $k = \frac{g+1}{2}$ and $l = k - 1$. Inside the small triangle and outside the big triangle add a path of $g-2$ new vertices as indicated by the dashed arcs. Then $g(G_g) = g$. Note that the vertices of the three faces of G_g

Fig. 3 Graph G_g with $g(G_g)=g$ and $p(G_g) \le \lfloor \frac{3g+1}{4} \rfloor$



that contain no dashed arcs are $\{u_1, u_2, \ldots, u_k, w_1, w_2, \ldots, w_k, v_1, v_2, \ldots, v_l\}$, and none of these vertices lies in all three faces. This implies:

Observation 19 *In every polychromatic coloring of G_g, every color appears on at least two vertices in the set $\{u_1, u_2, \ldots, u_k, w_1, w_2, \ldots, w_k, v_1, v_2, \ldots, v_l\}$.*

Since each such vertex is incident to two faces, we have:

$$2p(G_g) \le 2k + l = \begin{cases} 3k & \text{if } g \text{ is even,} \\ 3k - 1 & \text{if } g \text{ is odd,} \end{cases}$$

$$= \begin{cases} \frac{3g}{2} & \text{if } g \text{ is even,} \\ \frac{3g+1}{2} & \text{if } g \text{ is odd.} \end{cases}$$

In both cases we thus have $p(G_g) \le \left\lfloor \frac{3g+1}{4} \right\rfloor$.

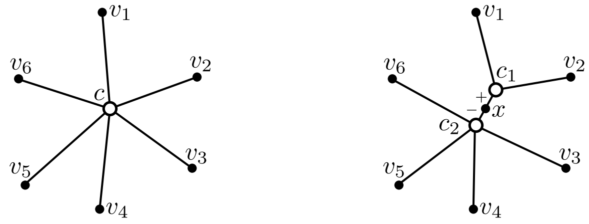
4 Complexity of Polychromatic Colorability

It can be checked in polynomial time whether a vertex k-coloring is polychromatic, and therefore the corresponding decision problem is in $\mathcal{NP}$. Every plane graph is polychromatically 1-colorable, which implies that the decision problem for $k = 1$ is trivial, in the sense that the answer for every instance is always "Yes."

Next, we turn our focus to polychromatic 2-colorings and prove Theorem 3(i). At this point, it is worth to remind ourselves that every plane graph G with $p(G) < 2$ contains a face of size at most two.

Proposition 20 *There is a polynomial-time algorithm to decide whether a given plane graph is polychromatically 2-colorable.*

Proof We call a CNF-formula F planar* if its literal-clause incidence graph H is planar. Note that this differs from the common notion of a planar CNF-formula, where one assumes that the literal-clause incidence graph H, together with a cycle connect-



(a) Clause c. (b) Clauses c_1 and c_2.

Fig. 4 Reducing PLANAR-NAE-SAT to PLANAR-NAE-3-SAT

ing the positive literals and together with edges between the corresponding positive and negative literals, is required to be planar.

A vertex-coloring of a plane graph is 2-polychromatic if no face is monochromatic. We can associate with one color the logic predicate "true" and with the other color "false" and interpret the vertices as boolean variables. Then we add a clause-vertex to each face and connect it to its incident variable-vertices. By this we get a planar* CNF-formula (where all variables occur only as positive ones).

Deciding whether the plane graph is polychromatically 2-colorable is equivalent to deciding whether the corresponding planar* CNF-formula is not-all-equal satisfiable (PLANAR*-NAE-SAT).

In [18] it is shown that PLANAR-NAE-3-SAT is in $\mathcal{P}$ by a reduction to PLANAR-MAX-CUT. The reduction in fact holds also for PLANAR*-NAE-SAT. A well-known reduction works to shorten the clauses of a planar (and planar*) formula to length 3, whilst preserving not-all-equal satisfiability and planarity. We briefly sketch this reduction which is illustrated in Fig. 4. A clause c of length $k > 3$ is replaced by two clauses c_1 and c_2 of lengths 3 and $k-1$, respectively. A new variable x occurs positive in c_1 and negative in c_2. Placing the new variable and clauses as in Fig. 4 preserves planarity and not-all-equal satisfiability. □

In the following we want to show hardness results for polychromatic 3- and 4-colorability, by constructing reductions from proper 3-colorability of plane graphs. We start by proving Theorem 3(ii) for $k = 3$.

Proposition 21 *To decide whether a given plane simple graph is polychromatically 3-colorable is $\mathcal{NP}$-hard.*

Proof It has been shown in [22] that deciding whether a plane simple graph is properly 3-colorable is $\mathcal{NP}$-hard. Given a plane simple graph G, we construct in polynomial time a plane simple graph G' such that G is properly 3-colorable if and only if G' is polychromatically 3-colorable.

For every edge $uv \in E(G)$, we add a new vertex y_{uv} inside one of the two faces and connect it with u and v. Thus every edge $uv \in E(G)$ is now contained in a triangle. Furthermore, for every face $f \in F(G)$, select a vertex x incident to f. Then add a new vertex x_f into the interior of f and connect x and x_f by an edge. The resulting graph G' is simple. In every polychromatic 3-coloring of G', the edges $E(G)$ are not monochromatic, and every proper 3-coloring of G can be extended to a polychromatic 3-coloring of G' by using the extra vertices x_f. Thus G' is polychromatically 3-colorable if and only if G is properly 3-colorable. □

We will refine Proposition 21 by restricting on plane graphs with only faces of given sizes. To do so we will restrict on 2-connected graphs. One reason is that a graph G is properly k-colorable if and only if all its 2-connected components are properly k-colorable and the maximal 2-connected components (block-cutvertex graph) of a graph can be computed in polynomial time by using a depth-first-search. Thus it follows that proper k-colorability is also $\mathcal{NP}$-hard restricted on 2-connected graphs for $k \geq 3$. Another reason is that any face in a 2-connected plane graph is a cycle and therefore there are no artifacts such as dangling paths.

Let L denote some set of positive integers. We define the following two decision problems.

L-PLANE-PROPER-k-COLORABILITY:
Given: A plane 2-connected graph G where the size of each face of G is in L.
Question: Does there exist a proper k-coloring of $V(G)$?

L-PLANE-POLY-k-COLORABILITY:
Given: A plane 2-connected graph G where the size of each face of G is in L.
Question: Does there exist a polychromatic k-coloring of $V(G)$?

In case we do not impose any restriction on the sizes of the faces in G, we omit the set L.

Let f be a face of a plane graph G and $L \subseteq \mathbb{N}$. We say a plane graph G' is an *L-extension* of f if G' is a plane graph containing G and some new vertices $V' \neq \emptyset$ and some new edges $E' \neq \emptyset$ (thus also some new faces) such that

(i) the new vertices V' and the new edges E' are contained in the interior of f,
(ii) every new edge of E' is incident to at most one old vertex $v \in V(f)$, and
(iii) the size of any new face is contained in L.

An extension is called *2-degenerate* if there is an order $v_1, v_2, \ldots, v_k$ of the new vertices V' such that the $d_{G'[V(G)\cup\{v_1,\ldots,v_i\})]}(v_i) \leq 2$ for all $i \in \{1, \ldots, k\}$.

Observation 22 *Let G' be a 2-degenerate extension of f of G. Any proper 3-coloring of G can be extended to a proper 3-coloring of G' (i.e., it preserves proper-3-colorability).*

Lemma 23 *Let $k \geq 3$. Every k-face f of a plane 2-connected graph G has a $\{3,4,5\}$-extension G' in G that is 2-degenerate and 2-connected.*

Proof The statement is trivial for $k = 3, 4,$ or 5. Therefore assume that $k \geq 6$ and that the statement is true for every smaller k. Let $x_1, \ldots, x_k$ be the vertices of f in clockwise order. Let f_0 be the graph obtained from f by adding a vertex y and connecting y with x_1 and x_4. Then f_0 has a 5-face and a $(k-1)$-face. By induction assumption we can extend the $(k-1)$-face to 3-, 4-, and 5-faces so that the extension is 2-degenerate and 2-connected. Together this yields a $\{3,4,5\}$-extension that is 2-degenerate and 2-connected. □

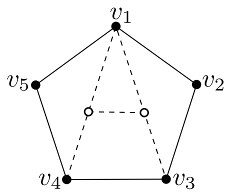
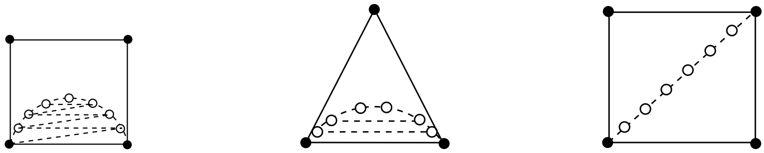


Fig. 5 Fill graph for 5-faces



(a) $\{3,11\}$-ext. of 4-face (b) $\{4,9\}$-ext. of 3-face (c) $\{9\}$-ext. of 4-face

Fig. 6 2-degenerate extensions of faces

Lemma 24 *Every* 5*-face* f *of a plane* 2*-connected graph* G *has a* $\{3,4\}$*-extension* G' *that is* 2*-connected, and moreover* G *is properly* 3*-colorable if and only if* G' *is properly* 3*-colorable.*

Proof First note that every 5-face forms a 5-cycle due to the assumption that G is 2-connected. We extend each 5-face f by the construction depicted in Fig. 5. Specifically, let f be a 5-face, and let $v_1, v_2, \dots, v_5$ be the five vertices of f. We add two copies of P_2 (the path of length two) with vertices u, v, w, $P' : u', v', w'$ and $P'' : u'', v'', w''$ by identifying both u' and u'' with v_1, w' with v_3, and w'' with v_4. Further we connect v' with v''. This yields the $\{3,4\}$-extension G' of G which is 2-connected. It is easy to check that every proper 3-coloring χ of the 5-face has an extension to a proper 3-coloring of G': We can assume that $\chi(v_1) \neq \chi(v_4)$. Color v' with $\chi(v_4)$, and color v'' with the third color not appearing on any of the neighbors of v''. □

Lemma 25 *Let* G *be a plane* 2*-connected graph.*

(i) *Let* $s \geq 4$. *Every* 4*-face of* G *has a* 2*-degenerate* $\{3, s\}$*-extension* G' *such that* G' *is* 2*-connected as well.*
(ii) *Let* $t \geq 5$ *odd. Every* 3*-face and every* 4*-face has a* 2*-degenerate* $\{t\}$*-extension* G' *such that* G' *is* 2*-connected.*

Proof

(i) For $s = 11$, the extension is drawn in Fig. 6(a), and it should be clear how to obtain a similar construction for arbitrary s.
(ii) In Fig. 6(b) an extension of a 3-face into 4-faces and 9-faces is shown. The 4-faces can be extended into 9-faces as shown in Fig. 6(c). Together this gives the extensions for the case $t = 9$. Again the general case should be clear. □

This leads to the following complete characterization of the complexity of L-PLANE-PROPER-3-COLORABILITY:

Corollary 26 *L-PLANE-PROPER-3-COLORABILITY*

(i) *... is in $\mathcal{P}$ for $L = \{2, 3\}$.*
(ii) *... is trivial, provided that L contains only even numbers.*
(iii) *... is $\mathcal{NP}$-complete, provided that there is $t \in L$ with $t \geq 5$ odd.*
(iv) *... is $\mathcal{NP}$-complete, provided that $3 \in L$ and there is $s \in L$ with $s \geq 4$.*

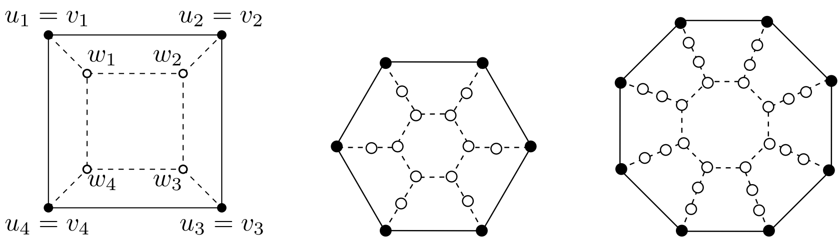
Proof First observe that we can assume that G contains no face of size two, since deleting one edge from a 2-face neither changes the size of any other face of G nor yields any cut-vertex.

(i) The only case left is $L = \{3\}$, i.e., triangulations. Theorem 2 provides a polynomial-time checkable criterion for 3-colorability of triangulations.
(ii) The graphs are bipartite because any cycle has even length. Therefore there is a proper 2-coloring which is also a proper 3-coloring.
(iii), (iv) Using Lemmas 23, 24, and 25, we can extend every plane 2-connected graph to a graph only having faces of the given size such that the proper 3-colorability and 2-connectedness are preserved. Thus the restricted proper 3-colorability problem on plane, 2-connected graphs is as hard as the nonrestricted one. □

Note here that every proper 3-coloring of an odd face is a polychromatic 3-coloring as well. For even faces, some special care has to be taken.

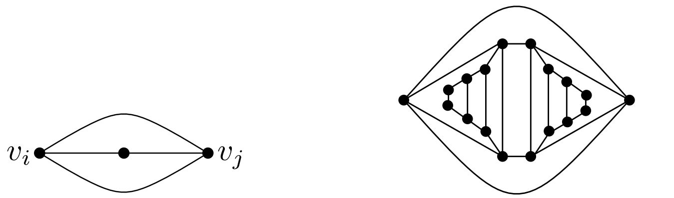
Lemma 27 *Let $s \geq 4$ even, and let C be an s-cycle embedded in the plane. Then there exists an $\{s\}$-extension C' of C such that any proper 3-coloring of C can be extended to a 3-coloring of C' such that every bounded face is polychromatic. Moreover, C' is 2-connected as well.*

Proof First, we consider the case $s = 4$. We "fill" C by substituting it with a copy of the graph in Fig. 7(a). Let v_1, v_2, v_3, v_4 be the four consecutive vertices of C. We identify v_i with the copy of the vertex u_i for $i \in \{1, \ldots, 4\}$. The resulting subgraph is poly-



(a) 4-face (b) 6-face (c) 8-face

Fig. 7 Fill graphs



(a) Base graph with 3-faces

(b) Base graph with 4- and 9-faces

Fig. 8 Gadgets for the reduction

chromatically 3-colorable if f is properly 3-colorable. To see this, we fix a proper 3-coloring χ of f. Suppose first that all three colors appear on the vertices of f. Without loss of generality we can assume that $\chi(v_1) = 1,\ \chi(v_2) = 2,\ \chi(v_3) = 3$, and $\chi(v_4) = 2$. Then, for instance, coloring the copies of w_1 by 3, w_2 by 2, w_3 by 1, and w_4 by 2 extends χ to a 3-coloring of the new vertices in f such that each of the five new faces inside f is polychromatic. Suppose now that only two distinct colors appear on the vertices of C, say $\chi(v_1) = \chi(v_3) = 1$ and $\chi(v_2) = \chi(v_4) = 2$. We can extend χ to a polychromatic 3-coloring including the new vertices in C as follows. Color w_1 by 3, w_2 by 2, w_3 by 3, and w_4 by 1. Again the five new faces inside C are polychromatic.

The case $s \geq 6$ is even simpler, and we will only sketch it here. We use a similar construction as for the previous case (see Fig. 7(b), (c) for the cases $s = 6$ and $s = 8$). The claim is now that every proper 3-coloring can be extend to a polychromatic 3-coloring inside that face. The new faces incident to the original boundary have a nonmonochromatic edge already colored. For each such face f, we can assign one incident vertex x_f that is not incident to the middle face and all these vertices are distinct. Color the vertex x_f such that the face f will be polychromatic and color the middle face also polychromatic. □

Proposition 28 *L-PLANE-POLY-3-COLORABILITY*

(i) *... is in $\mathcal{P}$ for $L = \{2, 3\}$.*
(ii) *... is trivial if L contains only even numbers.*
(iii) *... is $\mathcal{NP}$-complete for $L \supseteq \{3, s\},\ s \geq 4$.*
(iv) *... is $\mathcal{NP}$-complete for $L \supseteq \{4, t\},\ t \geq 5$ odd.*
(v) *... is trivial if $L \subseteq \{6, \ldots\}$.*

Proof If $g(G) < 3$, then G is certainly not polychromatically 3-colorable. Thus we can assume that $g(G) \geq 3$.

(i) Theorem 2 gives a polynomial-time checkable criterion for graphs with 3-faces only.
(ii) Because G is bipartite, we have $g(G) \geq 4$, and therefore G is polychromatically 3-colorable by Proposition 8.
(iii) If $s \geq 5$ is odd, then we substitute each edge with a copy of the base graph Fig. 8(a) but start with a graph which contains only s-faces. By Corollary 26(iii)

the proper 3-coloring problem restricted to such graphs is $\mathcal{NP}$-hard. Each proper 3-coloring of the s-faces is also a polychromatic 3-coloring, and therefore the old graph is properly 3-colorable if and only if the new graph is polychromatically 3-colorable.

If s is even, then we start with a graph G with 3- and s-faces only and substitute each edge with a copy of the base graph as in Fig. 8(a) and extend each s-face as described in Lemma 27. Then it holds that the new graph is polychromatically 3-colorable if and only if G is properly 3-colorable.

(iv) Start with a graph G containing only t-faces. We can modify our base graph as indicated in Fig. 8(b) so that we only have 4-faces and t-faces (and the outer face). By substituting every edge from the input graph G with this new gadget we get G'. The new graph G' has only 4- and t-faces, and in every polychromatic 3-coloring of G' the vertices corresponding to the endpoints of edges in G are colored with different colors (Observation 1). Moreover, there exists a 3-coloring of the base graph where v_i, v_j have different colors and all bounded faces are polychromatic. Because t is odd, every proper 3-coloring of G can be extended to a polychromatic 3-coloring of G'. Applying Corollary 26(iv) shows the $\mathcal{NP}$-hardness.

(v) Theorem 1 implies that all these graphs are polychromatically 3-colorable. □

This result covers all cases except where 5 is the smallest number in L. If $p(5) \geq 3$, which we do not know at the moment, then also $\{5, \dots\}$-PLANE-POLY-3-COLORABILITY is trivial.

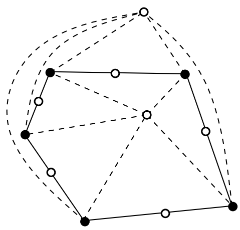
Also note that our base graphs for the Cases (iii) and (iv) contain multi-edges and at the moment we do not know whether the results carry over if we restrict to simple graphs.

Finally we prove Theorem 3(ii) for $k = 4$.

Proposition 29 *$\{4\}$-PLANE-POLY-4-COLORABILITY is $\mathcal{NP}$-complete also restricted on simple graphs.*

Proof Again, we reduce from PLANE-PROPER-3-COLORABILITY. Let G be a simple plane graph. We add a new vertex x_{uv} on each edge $uv \in E(G)$ and replace the edge uv by a path of length two with vertices u, x_{uv}, v. For each face $f \in F(G)$, we add a vertex v_f, place it into the interior of f, and connect v_f to the vertices of f as encountered when traversing the boundary of f in either direction. This yields a new plane simple graph G' where all faces have size exactly 4. See Fig. 9 for an example.

Fig. 9 Constructing G'



We claim that G is properly 3-colorable if and only if G' is polychromatically 4-colorable. If G is properly 3-colorable with colors 1, 2, and 3, then we extend this coloring χ in G' so that each vertex v_f corresponding to a face f in G gets color 4 and the vertex x_{uv} with neighbors u and v gets the color $\{1,2,3\} \setminus \{\chi(u), \chi(v)\}$. In this way each face of G' is polychromatic, and therefore the whole coloring χ is polychromatic.

Now let us fix a polychromatic 4-coloring χ' of G'. Let v_f be any vertex of G' corresponding to a face f of G. Without loss of generality suppose that v_f has color 4. Then for each edge $uv \in E(G)$ which is incident to f, the vertices $u, x_{uv}, v \in V(G')$ have to get the colors 1, 2, or 3. Henceforth for every face g of G that shares an edge with f, the vertex v_g gets color 4 as well. Since the dual graph G^* is connected, color 4 "propagates" from face to face, and $\chi'(v_{f'}) = 4$ for every face f' of G. Also color 4 appears at no other vertex of G'. Now the coloring restricted to the vertices in G uses only three colors and has to be proper because every 4-face f with vertices u, x_{uv}, v, v_f of G' can only be polychromatic if all of its four vertices are colored with distinct colors, and in particular u and v get distinct colors. □

5 Concluding Remarks and Open Problems

One could consider polychromatic edge-colorings of plane graphs, rather than polychromatic vertex-colorings. Here the situation is simpler, and a direct application of Proposition 15 to the dual of a given plane graph in which every face contains at least g edges implies that the edges of any such plane graph can be colored by $\lfloor \frac{3g+1}{4} \rfloor$ colors so that every color appears in every face. This is tight, as shown by the plane graph consisting of two vertices with three internally vertex disjoint paths P_1, P_2, P_3 between them, with P_1, P_2 of length $\lceil g/2 \rceil$ and P_3 of length $\lfloor g/2 \rfloor$.

Theorem 1 can be (slightly) strengthened; it actually implies that for every positive integer g, the vertices of any plane graph G (with no assumption on $g(G)$) can be colored by $\lfloor \frac{3g-5}{4} \rfloor$ colors, so that every face of size at least g contains vertices of all colors. This follows, for example, by applying Theorem 1 to the graph G' obtained from G as follows: For every face f of G of size $s < g$, add a set of g vertices that form a path inside f to two distinct vertices on the boundary of f.

The proof of Theorem 1 can be easily extended to colorings of graphs embedded on surfaces of higher genus. In fact, one can consider polychromatic colorings of general hypergraphs. Call a vertex-coloring of a hypergraph $H = (V, E)$ polychromatic if all colors appear in every hyperedge. The polychromatic number of H is the maximum k such that there is a polychromatic vertex-coloring of H with k colors. A close look at the proof of Theorem 1 shows that it actually gives the following.

Theorem 30 *For every constant c, there is a constant $b(c)$ so that the following holds: Let $H = (V, E)$ be a hypergraph in which the number of incidences between any set $V' \subseteq V$ and $E' \subseteq E$ is at most $2(|V'| + |E'|) + c$. Suppose, further, that each hyperedge of H is of cardinality at least g. Then the polychromatic number of H is at least $\frac{3}{4}g - b(c)$.*

The assumption about the incidences replaces the assertion of Lemma 11, and the rest of the proof is essentially identical to that of Theorem 1.

If G is a graph embedded on a surface of Euler characteristic χ, then the argument in the proof of Lemma 11 and Euler's formula for the corresponding surface imply that the number of incidences between any set V' of vertices of G and any set F' of its faces with $|F'| \geq 2$ is at most $2(|V'| + |F'| - \chi)$ (and it is $|V'|$ if $|F'| = 1$ and 0 if $F' = \emptyset$). We can thus apply the theorem above to the hypergraph whose vertices are the vertices of G and whose edges are the sets of vertices of the faces of G, and conclude that if every face has at least g vertices, then there is a vertex-coloring with $\frac{3}{4}g - b(\chi)$ colors such that every color appears in every face.

Open Problem 1 *Determine $p(g)$ exactly for every positive integer g. The first open case is $g = 5$, where it is known that $2 \leq p(5) \leq 4$.*

Let us remark here that the statement $p(5) = 4$ implies the Four Color Theorem. This fact follows easily by a reduction from polychromatic 4-colorability to proper 4-colorability, similar in spirit to the reduction used in the proof of Theorem 3.

The graph G in Fig. 2 showing that $p(4) \leq 2$ contains multi-edges, and we do not know if there is also an example without multi-edges. Define $p'(g) = \min\{p(G) \mid G \text{ plane simple graph}, g(G) = g\}$. The same bounds as in (1) are true for $p'(g)$.

Open Problem 2 *For which values g does $p(g) = p'(g)$ hold?*

We proved the decision problem PLANE-POLY-k-COLORABILITY to be hard for $k = 3, 4$, and we are interested in the cases with $k \geq 5$.

Open Problem 3 *Is the problem PLANE-POLY-k-COLORABILITY $\mathcal{NP}$-complete for every fixed $k \geq 5$?*

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