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Ehrhart Polynomials of Matroid Polytopes and Polymatroids

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Abstract We investigate properties of Ehrhart polynomials for matroid polytopes, independence matroid polytopes, and polymatroids. In the first half of the paper we prove that, for fixed rank, Ehrhart polynomials of matroid polytopes and polymatroids are computable in polynomial time. The proof relies on the geometry of these polytopes as well as a new refined analysis of the evaluation of Todd polynomials. In the second half we discuss two conjectures about the h^*-vector and the coefficients of Ehrhart polynomials of matroid polytopes; we provide theoretical and computational evidence for their validity.

Keywords Matroid · Matroid polytopes · Polymatroids · Ehrhart polynomials · Volume computation · Rational generating functions · h^*-vector · Unimodality · Ehrhart series

1 Introduction

Recall that a *matroid* M is a finite collection $\mathcal{F}$ of subsets of $[n] = \{1, 2, \dots, n\}$ called *independent sets*, such that the following properties are satisfied: (1) $\emptyset \in \mathcal{F}$; (2) if $X \in \mathcal{F}$ and $Y \subseteq X$, then $Y \in \mathcal{F}$; and (3) if $U, V \in \mathcal{F}$ and $|U| = |V| + 1$ there exists $x \in U \setminus V$ such that $V \cup x \in \mathcal{F}$. In this paper we investigate convex polyhedra associated with matroids.

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One of the reasons matroids have become fundamental objects in pure and applied combinatorics is their many equivalent axiomatizations. For instance, for a matroid M on n elements with independent sets $\mathcal{F}$ the *rank function* is a function $\varphi\colon 2^{[n]} \to \mathbb{Z}$ where $\varphi(A) := \max\{|X| \mid X \subseteq A,\ X \in \mathcal{F}\}$. Conversely, a function $\varphi\colon 2^{[n]} \to \mathbb{Z}$ is the rank function of a matroid on $[n]$ if and only if the following are satisfied: (1) $0 \le \varphi(X) \le |X|$, (2) $X \subseteq Y \Longrightarrow \varphi(X) \le \varphi(Y)$, and (3) $\varphi(X \cup Y) + \varphi(X \cap Y) \le \varphi(X) + \varphi(Y)$. Similarly, recall that a matroid M can be defined by its *bases*, which are the inclusion-maximal independent sets. The bases of a matroid M can be recovered by its rank function φ. For the reader we recommend [32] or [41] for excellent introductions to the theory of matroids.

Now we introduce the main object of this paper. Let $\mathcal{B}$ be the set of bases of a matroid M. If $B = \{\sigma_1, \dots, \sigma_r\} \in \mathcal{B}$, we define the *incidence vector of B* as $\mathbf{e}_B := \sum_{i=1}^{r} \mathbf{e}_{\sigma_i}$, where $\mathbf{e}_j$ is the standard elementary jth vector in $\mathbb{R}^n$. The *matroid polytope* of M is defined as $\mathcal{P}(M) := \operatorname{conv}\{\mathbf{e}_B \mid B \in \mathcal{B}\}$, where $\operatorname{conv}(\cdot)$ denotes the convex hull. This is different from the well-known *independence matroid polytope*, $\mathcal{P}^{\mathcal{I}}(M) := \operatorname{conv}\{\mathbf{e}_I \mid I \subseteq B \in \mathcal{B}\}$, the convex hull of the incidence vectors of all the independent sets. We can see that $\mathcal{P}(M) \subseteq \mathcal{P}^{\mathcal{I}}(M)$ and $\mathcal{P}(M)$ is a face of $\mathcal{P}^{\mathcal{I}}(M)$ lying in the hyperplane $\sum_{i=1}^{n} x_i = \operatorname{rank}(M)$, where $\operatorname{rank}(M)$ is the cardinality of any basis of M.

Polymatroids are closely related to matroid polytopes and independence matroid polytopes. We first recall some basic definitions (see [41]). A function $\psi : 2^{[n]} \longrightarrow \mathbb{R}$ is *submodular* if $\psi(X \cap Y) + \psi(X \cup Y) \le \psi(X) + \psi(Y)$ for all $X, Y \subseteq [n]$. A function $\psi : 2^{[n]} \longrightarrow \mathbb{R}$ is *nondecreasing* if $\psi(X) \le \psi(Y)$ for all $X \subseteq Y \subseteq [n]$. We say ψ is a *polymatroid rank function* if it is submodular, nondecreasing, and $\psi(\emptyset) = 0$. For example, the rank function of a matroid is a polymatroid rank function. The *polymatroid* determined by a polymatroid rank function ψ is the convex polyhedron (see Theorem 18.2.2 in [41]) in $\mathbb{R}^n$ given by

$$\mathcal{P}(\psi) := \left\{ \mathbf{x} \in \mathbb{R}^n \;\middle|\; \sum_{i \in A} x_i \le \psi(A)\ \forall A \subseteq [n],\ \mathbf{x} \ge \mathbf{0} \right\}.$$

Independence matroid polytopes are a special class of polymatroids. Indeed, if φ is a rank function on some matroid M, then $\mathcal{P}^{\mathcal{I}}(M) = \mathcal{P}(\varphi)$ [18]. Moreover, the matroid polytope $\mathcal{P}(M)$ is the face of $\mathcal{P}(\varphi)$ lying in the hyperplane $\sum_{i=1}^{n} x_i = \varphi([n])$. Matroid polytopes and polymatroids appear in combinatorial optimization [35], algebraic combinatorics [20], and algebraic geometry [22]. The main theme of this paper is the study of the volumes and Ehrhart functions of matroid polytopes, independence matroid polytopes, and polymatroids (from now on we often refer to all three families as matroid polytopes).

To state our main results recall that given an integer $k > 0$ and a polytope $\mathcal{P} \subseteq \mathbb{R}^n$ we define $k\mathcal{P} := \{k\boldsymbol{\alpha} \mid \boldsymbol{\alpha} \in \mathcal{P}\}$ and the function $i(\mathcal{P}, k) := \#(k\mathcal{P} \cap \mathbb{Z}^n)$, where we define $i(\mathcal{P}, 0) := 1$. It is well known that for integral polytopes, as in the case of matroid polytopes, $i(\mathcal{P}, k)$ is a polynomial, called the *Ehrhart polynomial* of $\mathcal{P}$. Moreover the leading coefficient of the Ehrhart polynomial is the *normalized volume* of $\mathcal{P}$, where a unit is the volume of the fundamental domain of the affine lattice spanned by $\mathcal{P}$ [37]. Our first theorem states:

Theorem 1 *Let r be a fixed integer. Then there exist algorithms whose input data consists of a number n and an evaluation oracle for*

(a) *a rank function φ of a matroid M on n elements satisfying $\varphi(A) \leq r$ for all A, or*
(b) *an integral polymatroid rank function ψ satisfying $\psi(A) \leq r$ for all A*

that compute in time polynomial in n the Ehrhart polynomial (in particular, the volume) of the matroid polytope $\mathcal{P}(M)$, the independence matroid polytope $\mathcal{P}^{\mathcal{I}}(M)$, and the polymatroid $\mathcal{P}(\psi)$, respectively.

The computation of volumes is one of the most fundamental geometric operations and it has been investigated by several authors from the algorithmic point of view. Although there are a few cases for which the volume can be computed efficiently (e.g., for convex polytopes in fixed dimension), it has been proved that computing the volume of polytopes of varying dimension is $\#P$-hard [9, 17, 26, 30]. Moreover it was proved that even approximating the volume is hard [19]. Clearly, computing Ehrhart polynomials is a harder problem still. To our knowledge there were only two previously known families of varying-dimension polytopes with a polynomial there is efficient computation of the volume. These two families are simplices or simple polytopes with number of vertices (this follows from Lawrence's volume formula [30]). Note that for simplices, it is at least NP-hard to compute the whole list of coefficients of the Ehrhart polynomial, while recently [2] presented a polynomial time algorithm to compute any fixed number of the highest coefficients of the Ehrhart polynomial of a simplex of varying dimension. Theorem 1 provides another interesting family of varying-dimension polytopes whose volumes and Ehrhart polynomials can be computed efficiently. The proof of Theorem 1, presented in Sect. 2, relies on the geometry of tangent cones at vertices of our polytopes as well as a new, refined analysis of the evaluation of Todd polynomials in the context of the computational theory of rational generating functions developed by [1–4, 11, 12, 40, 42]. A nice introduction to these topics can be found at [5].

In the second part of the paper, developed in Sect. 3, we investigate algebraic properties of the Ehrhart functions of matroid polytopes: The *Ehrhart series* of a polytope $\mathcal{P}$ is the infinite series $\sum_{k=0}^{\infty} i(\mathcal{P},k)t^k$. We recall the following classic result about Ehrhart series (see e.g., [24, 37]). Let $\mathcal{P} \subseteq \mathbb{R}^n$ be an integral convex polytope of dimension d. Then it is known that its Ehrhart series is a rational function of the form

$$\sum_{k=0}^{\infty} i(\mathcal{P},k)t^k = \frac{h_0^* + h_1^* t + \cdots + h_{d-1}^* t^{d-1} + h_d^* t^d}{(1-t)^{d+1}}. \tag{1}$$

The numerator is often called the *h^*-polynomial* of $\mathcal{P}$ (some authors also call it the *Ehrhart h-polynomial*), and we define the coefficients of the polynomial in the numerator of Lemma 1, $h_0^* + h_1^* t + \cdots + h_{d-1}^* t^{d-1} + h_d^* t^d$, as the *$h^*$-vector* of $\mathcal{P}$, which we write as $\mathbf{h}^*(\mathcal{P}) := (h_0^*, h_1^*, \ldots, h_{d-1}^*, h_d^*)$.

A vector $(c_0, \ldots, c_d)$ is *unimodal* if there exists an index p, $0 \leq p \leq d$, such that $c_{i-1} \leq c_i$ for $i \leq p$ and $c_j \geq c_{j+1}$ for $j \geq p$. Due to its algebraic implications, several authors have studied the unimodality of h^*-vectors (see [24] and [37] and references therein).

Suppose, as before, that $\mathcal{P} \subseteq \mathbb{R}^n$, and each vertex of $\mathcal{P}$ has integral (or rational) vertices. Let $Y_1, Y_2, \dots, Y_n$ and T be indeterminates over a field K. Letting $q \geq 1$ we define $A(\mathcal{P})_q$ as the vector space over K which is spanned by the monomials $Y_1^{\alpha_1}, Y_2^{\alpha_2}, \dots, Y_n^{\alpha_n} T^q$ such that $(\alpha_1, \alpha_2, \dots, \alpha_n) \in q\mathcal{P} \cap \mathbb{Z}^n$. Since $\mathcal{P}$ is convex it follows that $A(\mathcal{P})_q A(\mathcal{P})_p \subseteq A(\mathcal{P})_{q+p}$ for all p, q, and thus the *Ehrhart ring* of $\mathcal{P}$, $A(\mathcal{P}) = \bigoplus_{q=0}^{\infty} A(\mathcal{P})_q$, is a graded algebra [24, 37].

It is well known that if the Ehrhart ring of an integral polytope $\mathcal{P}$, $A(\mathcal{P})$, is Gorenstein, then $\mathbf{h}^*(\mathcal{P})$ is unimodal and symmetric [24, 37]. Nevertheless, the vector $\mathbf{h}^*(\mathcal{P})$ can be unimodal even when the Ehrhart ring $A(\mathcal{P})$ is not Gorenstein. For matroid polytopes, their Ehrhart ring is indeed often not Gorenstein. For instance, De Negri and Hibi [16] proved explicitly when the Ehrhart ring of a uniform matroid polytope is Gorenstein or not. Two fascinating facts, uncovered through experimentation, are that all h^*-vectors seen thus far are unimodal, even for the cases when their Ehrhart rings are not Gorenstein. In addition, when we computed the explicit Ehrhart polynomials of matroid polytopes we observe their coefficients are always positive. We conjecture:

Conjecture 2 *Let $\mathcal{P}(M)$ be the matroid polytope of a matroid M.*

(A) *The h^*-vector of $\mathcal{P}(M)$ is unimodal.*
(B) *The coefficients of the Ehrhart polynomial of $\mathcal{P}(M)$ are positive.*

We have proved both parts of this conjecture in many instances. A class of matroids that we considered are the uniform matroids; recall that the *uniform matroid* on n elements of rank r is the collection of all r-subsets of n. Using computers, we were able to verify Conjecture 2 for all uniform matroids up to 75 elements as well as for a wide variety of nonuniform matroids, see [15]. We include this information here just for the 28 famous matroids presented in [32]. Results in [25], with some additional careful calculations, imply that Conjecture 2 is true for all rank 2 uniform matroids. Regarding part (A) of the conjecture, we were also able to prove *partial unimodality* for uniform matroids of rank 3. Concretely we obtain:

Theorem 3

(1) *Conjecture 2 is true for all uniform matroids up to* 75 *elements and all uniform matroids of rank* 2. *It is also true for all matroids listed in* [15].
(2) *Let $\mathcal{P}(U^{3,n})$ be the matroid polytope of a uniform matroid of rank* 3 *on n elements, and let I be a nonnegative integer. Then there exists $n(I) \in \mathbb{N}$ such that for all $n \geq n(I)$ the h^*-vector of $\mathcal{P}(U^{3,n})$, $(h_0^*, \dots, h_n^*)$, is nondecreasing from index* 0 *to I. That is, $h_0^* \leq h_1^* \leq \cdots \leq h_I^*$.*

2 Computing the Ehrhart Polynomials

2.1 Preliminaries on Rational Generating Functions

Generating functions are crucial to proving our main results. For a good reference for the basic concepts used here see [5]. Let $\mathcal{P} \subseteq \mathbb{R}^n$ be a rational polyhedron.

The *multivariate generating function* of $\mathcal{P}$ is defined as the formal Laurent series in $\mathbb{Z}[[z_1,\dots,z_n,z_1^{-1},\dots,z_n^{-1}]]$

$$\tilde{g}_{\mathcal{P}}(\mathbf{z}) = \sum_{\boldsymbol{\alpha}\in\mathcal{P}\cap\mathbb{Z}^n} \mathbf{z}^{\alpha},$$

where we use the multiexponent notation $\mathbf{z}^{\alpha} = \prod_{i=1}^{n} z_i^{\alpha_i}$. If $\mathcal{P}$ is bounded, $\tilde{g}_{\mathcal{P}}$ is a Laurent polynomial, which we consider as a rational function $g_{\mathcal{P}}$. If $\mathcal{P}$ is not bounded but is pointed (i.e., $\mathcal{P}$ does not contain a straight line), there is a nonempty open subset $U \subseteq \mathbb{C}^n$ such that the series converges absolutely and uniformly on every compact subset of U to a rational function $g_{\mathcal{P}}$ (see [3] and references therein). If $\mathcal{P}$ contains a straight line, we set $g_{\mathcal{P}} = 0$. The rational function $g_{\mathcal{P}} \in \mathbb{Q}(z_1,\dots,z_n)$ defined in this way is called the *multivariate rational generating function* of $\mathcal{P}\cap\mathbb{Z}^n$. Barvinok [1] proved that in polynomial time, when the dimension of a polyhedron is fixed, $g_{\mathcal{P}}$ can be represented as a short sum of rational functions

$$g_{\mathcal{P}}(\mathbf{z}) = \sum_{i\in I} \varepsilon_i \frac{\mathbf{z}^{\mathbf{a}_i}}{\prod_{j=1}^{n}(1-\mathbf{z}^{\mathbf{b}_{ij}})},$$

where $\varepsilon_i \in \{-1, 1\}$.

Our first contribution is to show that in the case of matroid polytopes of fixed rank, this still holds even when their dimension grows. Let $\mathbf{v}$ be a vertex of $\mathcal{P}$. Define the *tangent cone* or *supporting cone* of $\mathbf{v}$ to be

$$\mathcal{C}_{\mathcal{P}}(\mathbf{v}) := \{\, \mathbf{v}+\mathbf{w} \mid \mathbf{v}+\varepsilon\mathbf{w}\in\mathcal{P} \text{ for some } \varepsilon > 0 \,\}.$$

We rely on the following result, which connects the rational generating function of a rational polyhedron to those of the tangent cones of its vertices. This result was independently discovered by Brion [10] and Lawrence [31]. A proof can also be found in [3] and [7].

Lemma 4 (Brion–Lawrence's Theorem) *Let $\mathcal{P}$ be a rational polyhedron and $V(\mathcal{P})$ be the set of vertices of $\mathcal{P}$. Then,*

$$g_{\mathcal{P}}(\mathbf{z}) = \sum_{\mathbf{v}\in V(\mathcal{P})} g_{\mathcal{C}_{\mathcal{P}}(\mathbf{v})}(\mathbf{z}),$$

where $\mathcal{C}_{\mathcal{P}}(\mathbf{v})$ is the tangent cone of $\mathbf{v}$.

Thus, we can write the multivariate generating function of $\mathcal{P}$ by writing all multivariate generating functions of all the tangent cones of the vertices of $\mathcal{P}$. Moreover, the map that assigns a rational polyhedron $\mathcal{P}$ its multivariate rational generating function $g_{\mathcal{P}}(\mathbf{z})$ is a *valuation*, i.e., a finitely additive measure, so it satisfies the equation

$$g_{\mathcal{P}_1\cup\mathcal{P}_2}(\mathbf{z}) = g_{\mathcal{P}_1}(\mathbf{z}) + g_{\mathcal{P}_2}(\mathbf{z}) - g_{\mathcal{P}_1\cap\mathcal{P}_2}(\mathbf{z}),$$

for arbitrary rational polytopes $\mathcal{P}_1$ and $\mathcal{P}_2$, the so-called *inclusion–exclusion principle*. This allows us to break a polyhedron $\mathcal{P}$ into pieces $\mathcal{P}_1$ and $\mathcal{P}_2$ and to compute the multivariate rational generating functions for the pieces (and their intersection) separately in order to get the generating function $g_{\mathcal{P}}$. More generally, let us denote by $[\mathcal{P}]$ the *indicator function* of $\mathcal{P}$, i.e., the function

$$[\mathcal{P}]\colon \mathbb{R}^n \to \mathbb{R}, \quad [\mathcal{P}](\mathbf{x}) = \begin{cases} 1 & \text{if } \mathbf{x} \in \mathcal{P}, \\ 0 & \text{otherwise.} \end{cases}$$

Let $\sum_{i\in I} \varepsilon_i [\mathcal{P}_i] = 0$ be an arbitrary linear identity of indicator functions of rational polyhedra (with rational coefficients ε_i); the valuation property now implies that it carries over to a linear identity $\sum_{i\in I} \varepsilon_i\, g_{\mathcal{P}_i}(\mathbf{z}) = 0$ of rational generating functions.

Now let $\mathcal{C}$ be one of the tangent cones of $\mathcal{P}$, and let $\mathcal{T}$ be a triangulation of $\mathcal{C}$, given by its simplicial cones of maximal dimension. Let $\hat{\mathcal{T}}$ denote the set of all (lower-dimensional) intersecting proper faces of the cones $\mathcal{C}_i \in \mathcal{T}$. Then we can assign an integer coefficient ϵ_i to every cone $\mathcal{C}_i \in \hat{\mathcal{T}}$, such that the following identity holds:

$$[\mathcal{C}] = \sum_{\mathcal{C}_i \in \mathcal{T}} [\mathcal{C}_i] + \sum_{\mathcal{C}_i \in \hat{\mathcal{T}}} \varepsilon_i\, [\mathcal{C}_i].$$

This identity immediately carries over to an identity of multivariate rational generating functions,

$$g_{\mathcal{C}}(\mathbf{z}) = \sum_{\mathcal{C}_i \in \mathcal{T}} g_{\mathcal{C}_i}(\mathbf{z}) + \sum_{\mathcal{C}_i \in \hat{\mathcal{T}}} \varepsilon_i\, g_{\mathcal{C}_i}(\mathbf{z}). \tag{2}$$

Hence, the problem of computing rational generating functions of a polyhedron is reduced to the case of simplicial cones.

In the following (Sects. 2.2 and 2.3), we study the tangent cones of the matroid polytopes and polymatroids and introduce algorithms that construct triangulations for them. Then, in Sect. 2.4, we construct a short multivariate generating function using an *efficient* variant of identity (2) and Brion–Lawrence's Theorem. Finally, in Sect. 2.5, we compute the Ehrhart polynomial.

2.2 On the Tangent Cones of Matroid Polytopes

Our goal is to compute the multivariate generating function of matroid polytopes and independence matroid polytopes with fixed rank (later, in Sect. 2.3, we will deal with the case of polymatroids), and to do this we will use a crucial property of adjacent vertices. To illustrate our techniques we will use a running example throughout this section.

Example 5 (Matroid on K_4) Let K_4 be the complete graph on four vertices. Label the $\binom{4}{2} = 6$ edges with $\{1, \ldots, 6\}$ as in Fig. 1. Every graph induces a matroid on its edges where the bases are all spanning trees (spanning forests for disconnected graphs) [41]. Let $M(K_4)$ be the matroid on the elements $\{1, \ldots, 6\}$ with bases as all spanning trees of K_4. The rank of $M(K_4)$ is the size of any spanning tree of K_4, thus the rank of $M(K_4)$ is 3. The 16 bases of $M(K_4)$ are: $\{3,5,6\}$, $\{3,4,6\}$, $\{3,4,5\}$, $\{2,5,6\}$, $\{2,4,6\}$, $\{2,4,5\}$, $\{2,3,5\}$, $\{2,3,4\}$, $\{1,5,6\}$, $\{1,4,6\}$, $\{1,4,5\}$, $\{1,3,6\}$, $\{1,3,4\}$, $\{1,2,6\}$, $\{1,2,5\}$, $\{1,2,3\}$.

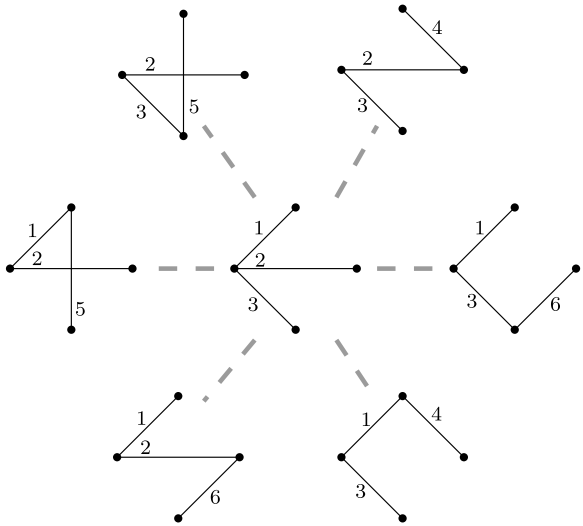

Fig. 1 {2, 3, 5}, {2, 3, 4}, {1, 3, 6}, {1, 3, 4}, {1, 2, 6} and {1, 2, 5} are spanning trees of K_4 that differ from {1, 2, 3} by adding one edge and removing one edge

Lemma 6 (See Theorem 4.1 in [22], Theorem 5.1 and Corollary 5.5 in [39]) *Let M be a matroid.*

(A) *Two vertices $\mathbf{e}_{B_1}$ and $\mathbf{e}_{B_2}$ are adjacent in $\mathcal{P}(M)$ if and only if $\mathbf{e}_{B_1} - \mathbf{e}_{B_2} = \mathbf{e}_i - \mathbf{e}_j$ for some i, j.*

(B) *If two vertices $\mathbf{e}_{I_1}$ and $\mathbf{e}_{I_2}$ are adjacent in $\mathcal{P}^{\mathcal{I}}(M)$, then $\mathbf{e}_{I_1} - \mathbf{e}_{I_2} \in \{\mathbf{e}_i - \mathbf{e}_j, \mathbf{e}_i, -\mathbf{e}_j\}$ for some i, j. Moreover if $\mathbf{v}$ is a vertex of $\mathcal{P}^{\mathcal{I}}(M)$, then all adjacent vertices of $\mathbf{v}$ can be computed in polynomial time in n, even if the matroid M is only presented by an evaluation oracle of its rank function φ.*

Let M be a matroid on n elements with fixed rank r. Then the number of vertices of $\mathcal{P}(M)$ is a polynomial in n of degree r. We can see this since the number of vertices is equal to the number of bases of M, and the number of bases is bounded by $\binom{n}{r}$, a polynomial in n of degree r. Clearly the number of vertices of $\mathcal{P}^{\mathcal{I}}(M)$ is also polynomial in n. It is also clear that, when the rank r is fixed, all vertices of either polytope can be enumerated in polynomial time in n, even when the matroid is only presented by an evaluation oracle for its rank function φ.

Throughout this section we shall discuss polyhedral cones $\mathcal{C}$ with extremal rays $\{\mathbf{r}_1, \dots, \mathbf{r}_l\}$ such that

$$\mathbf{r}_k \in R_A := \left\{\mathbf{e}_i - \mathbf{e}_j, \mathbf{e}_i, -\mathbf{e}_j \mid i \in [n],\ j \in A\right\} \quad \text{for } k = 1, \dots, l$$

for some $A \subseteq [n]$. We will refer to R_A as the *elementary set of A*. Note that by Lemma 6 the rays of a tangent cone at a vertex $\mathbf{e}_A$ (corresponding to a set $A \subseteq [n]$) of a matroid polytope or an independence matroid polytope form an elementary set of A. Due to convexity and the assumption that $\mathbf{r}_k$ are extremal, for each $i \in [n]$ and $j \in A$ at most two of the three vectors $\mathbf{e}_i - \mathbf{e}_j, \mathbf{e}_i, -\mathbf{e}_j$ are extremal rays $\mathbf{r}_k$ of $\mathcal{C}$. This implies by construction that, considering all pairs $\mathbf{e}_i - \mathbf{e}_j$ and $\mathbf{e}_i$ or $-\mathbf{e}_j$, the number of generators $\mathbf{r}_k$ of $\mathcal{C}$ is bounded by

$$n|A|+n+|A|. \tag{3}$$

Recall a cone is *simple* if it is generated by linearly independent vectors and it is *unimodular* if its fundamental parallelepiped contains only one lattice point [23]. A triangulation of $\mathcal{C}$ is *unimodular* if it is a polyhedral subdivision such that each subcone is unimodular.

Example 7 (Matroid on K_4) The vertices $\mathbf{e}_{\{2,3,5\}}, \mathbf{e}_{\{2,3,4\}}, \mathbf{e}_{\{1,3,6\}}, \mathbf{e}_{\{1,3,4\}}, \mathbf{e}_{\{1,2,6\}}$ and $\mathbf{e}_{\{1,2,5\}}$ are all adjacent to the vertex $\mathbf{e}_{\{1,2,3\}}$, see Fig. 1. Moreover, the tangent cone $\mathcal{C}_{\mathcal{P}(M(\mathbf{K}_4))}(\mathbf{e}_{\{1,2,3\}})$ is generated by the differences of these vertices with $\mathbf{e}_{\{1,2,3\}}$:

$$\begin{aligned}\mathcal{C}_{\mathcal{P}(M(\mathbf{K}_4))}(\mathbf{e}_{\{1,2,3\}}) = \mathbf{e}_{\{1,2,3\}} + \text{cone}\{&\mathbf{e}_{\{2,3,5\}} - \mathbf{e}_{\{1,2,3\}}, \mathbf{e}_{\{2,3,4\}} - \mathbf{e}_{\{1,2,3\}},\\ &\mathbf{e}_{\{1,3,4\}} - \mathbf{e}_{\{1,2,3\}}, \mathbf{e}_{\{1,2,6\}} - \mathbf{e}_{\{1,2,3\}},\\ &\mathbf{e}_{\{1,2,5\}} - \mathbf{e}_{\{1,2,3\}}\}.\end{aligned}$$

Lemma 8 *Let $\mathcal{C} \subseteq \mathbb{R}^n$ be a cone generated by p extremal rays $\{\mathbf{r}_1, \ldots, \mathbf{r}_p\} \subseteq R_A$ where R_A is an elementary set of some $A \subseteq [n]$. Every triangulation of $\mathcal{C}$ is unimodular.*

Proof Without loss of generality, we can assume $\{\mathbf{r}_1, \ldots, \mathbf{r}_l\}$ are generators of the form $\mathbf{e}_i - \mathbf{e}_j$ and $\{\mathbf{r}_{l+1}, \ldots, \mathbf{r}_p\}$ are generators of the form $\mathbf{e}_i$ or $-\mathbf{e}_j$ for the cone $\mathcal{C}$.

It is easy to see that the matrix $\tilde{T}_{\mathcal{C}} := [\mathbf{r}_1, \ldots, \mathbf{r}_l]$ is totally unimodular. Let $G_{\mathcal{C}}$ be a directed graph with vertex set $[n]$ and an edge from vertex i to j if $\mathbf{r}_k = \mathbf{e}_i - \mathbf{e}_j$ is an extremal ray of $\mathcal{C}$. We can see that $G_{\mathcal{C}}$ is a subgraph of the complete directed graph K_n with two arcs between each pair of vertices; one for each direction. Since $\tilde{T}_{\mathcal{C}} := [\mathbf{r}_1, \ldots, \mathbf{r}_l]$ is the incidence matrix of the graph $G_{\mathcal{C}}$, it is totally unimodular [34, Chap. 19, Example 2], i.e., every subdeterminant is 0, 1 or -1 [34, Chap. 19, Theorem 9]. Therefore $T_{\mathcal{C}} := [\mathbf{r}_1, \ldots, \mathbf{r}_l, \mathbf{r}_{l+1}, \ldots, \mathbf{r}_p]$ is totally unimodular since augmenting $\tilde{T}_{\mathcal{C}}$ by a vector $\mathbf{e}_i$ or $-\mathbf{e}_j$ preserves this subdeterminant property: for any submatrix containing part of a vector $\mathbf{e}_i$ or $-\mathbf{e}_j$ perform the cofactor expansion down the vector $\mathbf{e}_i$ or $-\mathbf{e}_j$ when calculating the determinant.

Since $T_{\mathcal{C}}$ is totally unimodular, each basis of $T_{\mathcal{C}}$ generates the entire integer lattice $\mathbb{Z}^n \cap \text{lin}(\mathcal{C})$ and hence every simplicial cone of a triangulation has normalized volume 1. □

Lemma 9 *Let $\mathcal{C} \subseteq \mathbb{R}^n$ be a cone generated by l extremal rays $\{\mathbf{r}_1, \ldots, \mathbf{r}_l\} \subseteq R_A$ where R_A is an elementary set of some $A \subseteq [n]$, where* $\dim(\mathcal{C}) < n$. *The extremal rays $\{\mathbf{r}_1, \ldots, \mathbf{r}_l\}$ can be augmented by a vector $\tilde{\mathbf{r}}$ such that* $\dim(\text{cone}\{\mathbf{r}_1, \ldots, \mathbf{r}_l, \tilde{\mathbf{r}}\}) = \dim(\mathcal{C}) + 1$, *the vectors $\mathbf{r}_1, \ldots, \mathbf{r}_l, \tilde{\mathbf{r}}$ are all extremal, and $\tilde{\mathbf{r}} \in R_A$.*

Proof It follows from convexity that at most two of $\mathbf{e}_i - \mathbf{e}_j$, $\mathbf{e}_i$ or $-\mathbf{e}_j$ are extremal generators of $\mathcal{C}$ for $i \in [n]$ and $j \in A$. There are at least n possible extremal ray generators, considering two of $\mathbf{e}_i - \mathbf{e}_j$, $\mathbf{e}_i$ or $-\mathbf{e}_j$ for each $i \in [n]$ and $j \in A$. Moreover, all these pairs span $\mathbb{R}^n$. Thus, by the basis augmentation theorem of linear algebra, there exists a vector $\tilde{\mathbf{r}}$ such that $\dim(\text{cone}\{\mathbf{r}_1, \ldots, \mathbf{r}_l, \tilde{\mathbf{r}}\}) = \dim(\mathcal{C}) + 1$ and $\mathbf{r}_1, \ldots, \mathbf{r}_l, \tilde{\mathbf{r}}$ are all extremal. □

Lemma 10 *Let r be a fixed integer, n be an integer, $A \subseteq [n]$ with $|A| \le r$ and let $\mathcal{C} \subseteq \mathbb{R}^n$ be a cone generated by l extremal rays $\{\mathbf{r}_1, \dots, \mathbf{r}_l\} \subseteq R_A$ where R_A is an elementary set of A. Then any triangulation of* $\operatorname{conv}(\{\mathbf{0}, \mathbf{r}_1, \dots, \mathbf{r}_l\})$ *has at most a polynomial in n number of top-dimensional simplices.*

Proof Assume $\dim(\mathcal{C}) = n$. Later, we will show how to remove this restriction. We can see that $\operatorname{conv}\{\mathbf{0}, \mathbf{r}_1, \dots, \mathbf{r}_l\} \subseteq [-1,1]^A \times \tilde{\Delta}_{[n]\setminus A}$ where

$$[-1,1]^A := \left\{ \mathbf{x} \in \mathbb{R}^A \mid |\mathbf{x}_j| \le 1 \quad j \in A \right\} \subseteq \mathbb{R}^A,$$
$$\tilde{\Delta}_{[n]\setminus A} := \operatorname{conv}\left(\left\{ \mathbf{e}_i \mid i \in [n] \setminus A \right\} \cup \{\mathbf{0}\} \right) \subseteq \mathbb{R}^{[n]\setminus A}.$$

The volume of a d-simplex $\operatorname{conv}\{\mathbf{v}_0, \dots, \mathbf{v}_d\}$ is [23]

$$\frac{1}{d!} \left| \det \begin{pmatrix} \mathbf{v}_0 & \cdots & \mathbf{v}_d \\ 1 & \cdots & 1 \end{pmatrix} \right|. \tag{4}$$

Thus the $(n - |A|)$-volume of $\tilde{\Delta}_{[n]\setminus A}$ is $\frac{1}{(n-|A|)!}$ and the $|A|$-volume of $[-1,1]^A$ is $2^{|A|}$. Therefore

$$\operatorname{vol}\left([-1,1]^A \times \tilde{\Delta}_{[n]\setminus A}\right) = 2^{|A|} \frac{1}{(n-|A|)!} = \frac{1}{n!} 2^{|A|} n(n-1) \cdots \left(n - |A| + 1\right).$$

It is also a fact that any integral n-simplex has n-volume bounded below by $\frac{1}{n!}$, using the simplex volume equation (4). Therefore any triangulation of $\operatorname{conv}\{\mathbf{0}, \mathbf{r}_1, \dots, \mathbf{r}_l\}$ has at most

$$2^{|A|} n(n-1) \cdots \left(n - |A| + 1\right) \le 2^r n(n-1) \cdots (n - r + 1)$$

full-dimensional simplices, a polynomial function in n of degree r.

Let $d_{\mathcal{C}} := n - \dim(\mathcal{C})$. If $\dim(\mathcal{C}) < n$, then by Lemma 9, $\{\mathbf{r}_1, \dots, \mathbf{r}_l\}$ can be augmented with vectors $\{\tilde{\mathbf{r}}_1, \dots, \tilde{\mathbf{r}}_{d_{\mathcal{C}}}\}$ where $\tilde{\mathbf{r}}_k \in R_A$ for A above, such that $\dim(\operatorname{cone}\{\mathbf{r}_1, \dots, \mathbf{r}_l, \tilde{\mathbf{r}}_1, \dots, \tilde{\mathbf{r}}_{d_{\mathcal{C}}}\}) = n$ and $\{\mathbf{r}_1, \dots, \mathbf{r}_l, \tilde{\mathbf{r}}_1, \dots, \tilde{\mathbf{r}}_{d_{\mathcal{C}}}\}$ are extremal. Moreover,

$$\begin{aligned} \dim\left(\operatorname{conv}\{\mathbf{0}, \mathbf{r}_1, \dots, \mathbf{r}_l\}\right) &< \dim\left(\operatorname{conv}\{\mathbf{0}, \mathbf{r}_1, \dots, \mathbf{r}_l, \tilde{\mathbf{r}}_1\}\right) < \cdots \\ &< \dim\left(\operatorname{conv}\{\mathbf{0}, \mathbf{r}_1, \dots, \mathbf{r}_l, \tilde{\mathbf{r}}_1, \dots, \tilde{\mathbf{r}}_{d_{\mathcal{C}}-1}\}\right) \\ &< \dim\left(\operatorname{conv}\{\mathbf{0}, \mathbf{r}_1, \dots, \mathbf{r}_l, \tilde{\mathbf{r}}_1, \dots, \tilde{\mathbf{r}}_{d_{\mathcal{C}}}\}\right), \end{aligned}$$

that is, $\tilde{\mathbf{r}}_k \notin \operatorname{aff}\{\mathbf{0}, \mathbf{r}_1, \dots, \mathbf{r}_l, \tilde{\mathbf{r}}_1, \dots, \tilde{\mathbf{r}}_{k-1}\}$ for $1 \le k \le d_{\mathcal{C}}$.

Since $\tilde{\mathbf{r}}_k \notin \operatorname{aff}\{\mathbf{0}, \mathbf{r}_1, \dots, \mathbf{r}_l, \tilde{\mathbf{r}}_1, \dots, \tilde{\mathbf{r}}_{k-1}\}$, any full-dimensional simplex in a triangulation of $\operatorname{conv}\{\mathbf{0}, \mathbf{r}_1, \dots, \mathbf{r}_l, \tilde{\mathbf{r}}_1, \dots, \tilde{\mathbf{r}}_k\}$ must contain $\tilde{\mathbf{r}}_k$, see Fig. 2. If not, then there exists a top-dimensional simplex using the points $\{\mathbf{0}, \mathbf{r}_1, \dots, \mathbf{r}_l, \tilde{\mathbf{r}}_1, \dots, \tilde{\mathbf{r}}_{k-1}\}$, but we know all these points lie in a subspace of one less dimension, a contradiction. Therefore, a bound on the number of simplices in a triangulation of $\operatorname{conv}\{\mathbf{0}, \mathbf{r}_1, \dots, \mathbf{r}_l, \tilde{\mathbf{r}}_1, \dots, \tilde{\mathbf{r}}_k\}$ is a bound on that of $\operatorname{conv}\{\mathbf{0}, \mathbf{r}_1, \dots, \mathbf{r}_l, \tilde{\mathbf{r}}_1, \dots, \tilde{\mathbf{r}}_{k-1}\}$.

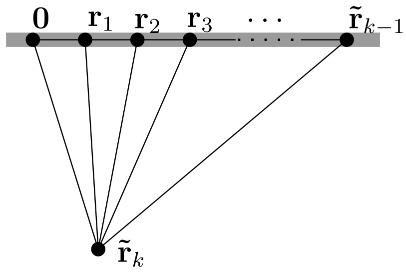

Fig. 2 If $\tilde{\mathbf{r}}_k$ is not contained in the affine span of $\{\mathbf{0}, \mathbf{r}_1, \ldots, \mathbf{r}_p, \tilde{\mathbf{r}}_1, \ldots, \tilde{\mathbf{r}}_{k-1}\}$, then every full-dimensional simplex must contain $\tilde{\mathbf{r}}_k$

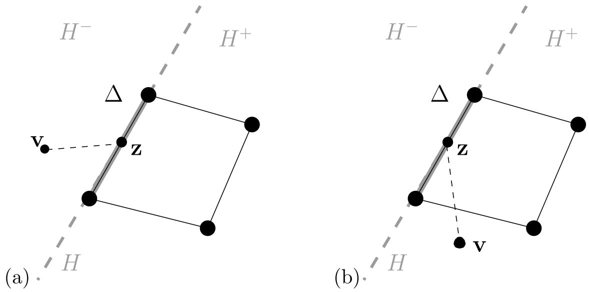

Fig. 3 (**a**) Δ visible to $\mathbf{v}$; (**b**) Δ *not* visible to $\mathbf{v}$

Thus, if $\dim(\mathcal{C}) < n$ we can augment $\mathcal{C}$ by vectors $\tilde{\mathbf{r}}_1, \ldots, \tilde{\mathbf{r}}_{d_\mathcal{C}}$ so that the cone $\tilde{\mathcal{C}} := \text{cone}\{\mathbf{r}_1, \ldots, \mathbf{r}_p, \tilde{\mathbf{r}}_1, \ldots, \tilde{\mathbf{r}}_{d_\mathcal{C}}\}$ is of dimension n and $\mathbf{r}_l, \tilde{\mathbf{r}}_k \in R_A$ for A above. We proved any triangulation of $\text{conv}\{\mathbf{0}, \mathbf{r}_1, \ldots, \mathbf{r}_p, \tilde{\mathbf{r}}_1, \ldots, \tilde{\mathbf{r}}_{d_\mathcal{C}}\}$ has at most polynomially many full-dimensional n-simplices, which implies that any triangulation of $\text{conv}\{\mathbf{0}, \mathbf{r}_1, \ldots, \mathbf{r}_p\}$ has at most polynomially many top-dimensional simplices due to the construction of the generators $\tilde{\mathbf{r}}_k$. □

We have shown that for a cone $\mathcal{C}$ generated by an elementary set of extremal rays $\{\mathbf{r}_1, \ldots, \mathbf{r}_l\} \subseteq R_A$ for some $A \subseteq [n]$, any triangulation of $\text{conv}\{\mathbf{0}, \mathbf{r}_1, \ldots, \mathbf{r}_l\}$ has at most polynomially many simplices. What we need next is an efficient method to compute some triangulation of $\text{conv}\{\mathbf{0}, \mathbf{r}_1, \ldots, \mathbf{r}_l\}$. We will show that the placing triangulation is a suitable candidate.

Let $\mathcal{P} \subseteq \mathbb{R}^n$ be a polytope of dimension n and Δ be a facet of $\mathcal{P}$ and $\mathbf{v} \in \mathbb{R}^n$. There exists a unique hyperplane H containing Δ and $\mathcal{P}$ is contained in one of the closed sides of H, call it H^+. If $\mathbf{v}$ is contained in the interior of H^-, the other closed halfspace defined by H, then Δ is *visible* from $\mathbf{v}$ (see [23], Chap. 14.2); see Fig. 3. The well-known *placing triangulation* is given by an algorithm where a point is iteratively added to an intermediate triangulation by determining which facets are visible to the new point [14, 23]. We recall now how to determine if a facet is visible to a vertex in polynomial time.

Lemma 11 *Let $\mathcal{P} \subseteq \mathbb{R}^n$ be a polytope given by t vertices $\{\mathbf{v}^1, \ldots, \mathbf{v}^t\} \subseteq \mathbb{R}^n$ and $\Delta \subseteq \mathcal{P}$ be a facet of $\mathcal{P}$ given by q vertices $\{\tilde{\mathbf{v}}^1, \ldots, \tilde{\mathbf{v}}^q\} \subseteq \{\mathbf{v}^1, \ldots, \mathbf{v}^t\}$. If $\mathbf{v} \in \mathbb{R}^n$ where $\mathbf{v} \notin \mathcal{P}$, then deciding if Δ is visible to $\mathbf{v}$ can be done in polynomial time in the input $\{\tilde{\mathbf{v}}^1, \ldots, \tilde{\mathbf{v}}^q\}$, $\{\mathbf{v}^1, \ldots, \mathbf{v}^t\}$ and $\mathbf{v}$.*

Proof Let $\mathbf{z} := \frac{1}{q}\sum_{i=1}^{q}\tilde{\mathbf{v}}^i$ so that $\mathbf{z} \in \operatorname{relint}(\Delta)$. We consider the linear program:

$$\left\{ \begin{pmatrix} \mathbf{x} \\ \mathbf{y} \\ \lambda \end{pmatrix} \in \mathbb{R}^{n+t+1} \;\middle|\; \mathbf{x} = \sum_{i=1}^{t} \mathbf{v}^i \mathbf{y}_i,\ \mathbf{y} \geq \mathbf{0},\ \sum_{i=1}^{t} \mathbf{y}_i = 1, \right.$$
$$\left. 0 \leq \lambda < 1,\ \lambda\mathbf{v} + (1-\lambda)\mathbf{z} = \mathbf{x} \right\}. \tag{5}$$

If (5) has a solution, then there exists a point $\bar{\mathbf{x}} \in \mathcal{P}$ between the facet Δ and $\mathbf{v}$, hence Δ is not visible from $\mathbf{v}$. If (5) does not have a solution, then there are no points of $\mathcal{P}$ between $\mathbf{v}$ and Δ, hence Δ is visible from $\mathbf{v}$ (see Lemma 4.2.1 in [14]). It is well known that a strict inequality, such as the one in (5), can be handled by an equivalent linear program which has only one additional variable. Determining if (5) has a solution can be done in polynomial time in the input [34]. □

The placing triangulation is obtained by incrementally adding one point at a time, connecting the new point to the current triangulation. More precisely:

Algorithm 12 (The Placing Triangulation [14, 23])

Input: A set of ordered points $\{\mathbf{v}_1, \ldots, \mathbf{v}_t\} \in \mathbb{R}^n$.
Output: A triangulation $\mathcal{T}$ of $\{\mathbf{v}_1, \ldots, \mathbf{v}_t\}$

$\mathcal{T} := \{\{\mathbf{v}_1\}\}$.
for each $\mathbf{v}_i \in \{\mathbf{v}_2, \ldots, \mathbf{v}_t\}$ **do**
  Let $B \in \mathcal{T}$.
  $P_i := \{\mathbf{v}_1, \ldots, \mathbf{v}_{i-1}\}$
  if $\mathbf{v}_i \notin \operatorname{aff}(P_i)$ **then**
    $\mathcal{T}' := \emptyset$
    for each $D \in \mathcal{T}$ **do**
      $\mathcal{T}' := \mathcal{T}' \cup \{D \cup \{\mathbf{v}_i\}\}$.
  else
    for each $B \in \mathcal{T}$ and each $(|B|-1)$-subset C of B **do**
      Create and solve the linear program (5) with $(P_i, C, \mathbf{v}_i)$ to decide visibility of C to $\mathbf{v}_i$.
      if C is visible to $\mathbf{v}_i$ **then**
        $\mathcal{T}' := \mathcal{T}' \cup \{C \cup \{\mathbf{v}_i\}\}$
  $\mathcal{T} := \mathcal{T}'$
return $\mathcal{T}$

Indeed, Algorithm 12 returns a triangulation [23]. We will show that for certain input (a point set corresponding to a vertex cone of a matroid polytope or independence matroid polytope), it runs in polynomial time. We remark that there are exponentially, in n, many lower dimensional simplices in any given triangulation. But, it is important to note that only the highest dimensional simplices are listed in an intermediate triangulation (and thus the final triangulation) in the placing triangulation algorithm.

Theorem 13 *Let r be a fixed integer, n be an integer, $A \subseteq [n]$ with $|A| \leq r$, and let $\{\mathbf{r}_1, \ldots, \mathbf{r}_l\} \subseteq R_A$. Then the placing triangulation (Algorithm 12) with input $\{\mathbf{0}, \mathbf{r}_1, \ldots, \mathbf{r}_l\}$ runs in polynomial time.*

Proof By (3) there is only a polynomial, in n, number of extremal rays $\{\mathbf{r}_1, \ldots, \mathbf{r}_l\}$. Thus, the for statement on line 2 repeats a polynomial number of times. Step 5 can be done in polynomial time by solving the linear equation $[\mathbf{r}_1, \ldots, \mathbf{r}_{i-1}]\mathbf{x} = \mathbf{v}_i$.

The for statement on line 7 repeats for every simplex D in the triangulation $\mathcal{T}$, and the number of simplices in $\mathcal{T}$ is bounded by the number of simplices in the final triangulation. By Lemma 10 any triangulation of extremal cone generators in R_A with the origin will use at most polynomially many top-dimensional simplices. Hence the number of top-dimensional simplices of any partial triangulation $\mathcal{T}$ will be polynomially bounded since it is a subset of the final triangulation.

The for statement on line 10 repeats for every simplex B and every $(|B| - 1)$-simplex of B. As before, the number of simplices B is polynomially bounded, and there are at most n $(|B| - 1)$-simplices of B. Thus the for statement will repeat a polynomial number of times.

Finally, by Lemma 11, determining if C is visible to $\mathbf{v}_i$ can be done in polynomial time. Therefore Algorithm 12 runs in a polynomial time. □

Corollary 14 *Let r be a fixed integer, n be an integer, $A \subseteq [n]$ with $|A| \leq r$, and let $\mathcal{C} \subseteq \mathbb{R}^n$ be a cone generated by extremal rays $\{\mathbf{r}_1, \ldots, \mathbf{r}_l\} \subseteq R_A$. A triangulation of $\mathcal{C}$ can be computed in polynomial time in the input of the extremal ray generators $\{\mathbf{r}_1, \ldots, \mathbf{r}_l\}$.*

Proof Let $\mathcal{P}_\mathcal{C} := \operatorname{conv}\{\mathbf{0}, \mathbf{r}_1, \ldots, \mathbf{r}_t\}$. We give an algorithm which produces a triangulation of $\mathcal{P}_\mathcal{C} := \operatorname{conv}\{\mathbf{0}, \mathbf{r}_1, \ldots, \mathbf{r}_t\}$ such that each full-dimensional simplex has $\mathbf{0}$ as a vertex; see Fig. 4. Such a triangulation would extend to a triangulation of the cone $\mathcal{C}$. This can be accomplished by applying two placing triangulations: one to triangulate the boundary of $\mathcal{P}_\mathcal{C}$ not incident to $\mathbf{0}$, and another to attach the triangulated boundary faces to $\mathbf{0}$. The algorithm goes as follows:

(1) Triangulate $\mathcal{P}_\mathcal{C}$ using the placing triangulation algorithm. Call it $\mathcal{T}'$.
(2) Triangulate $\mathcal{P}_\mathcal{C}$ using the boundary faces of $\mathcal{T}'$ which do not contain v.

Algorithm 15 (Triangulation joining $\mathbf{0}$ to boundary faces)

Input: A triangulation $\mathcal{T}'$ of $\mathcal{P}_\mathcal{C}$, given by its vertices.
Output: A triangulation $\mathcal{T}$ of $\mathcal{P}_\mathcal{C}$ such that every highest dimension simplex of $\mathcal{T}$ is incident to $\mathbf{0}$.

```
T := ∅
for each C where C is a (|B| − 1)-simplex of B ∈ T′ do
    if C is not a (|A| − 1)-simplex of A ∈ T′ where A ≠ B then
        T := T ∪ { C ∪ {0} }
return T
```

By Theorem 13, triangulating $\mathcal{P}_\mathcal{C}$ using Algorithm 12 can be done in polynomial time. Algorithm 15 indeed produces a triangulation of $\mathcal{P}_\mathcal{C}$. It covers $\mathcal{P}_\mathcal{C}$ since every

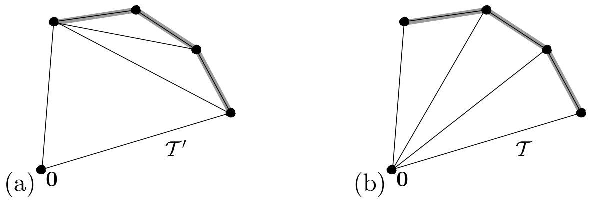

Fig. 4 A triangulation $\mathcal{T}'$ of $\mathcal{P}_{\mathcal{C}}$ can be used to extend to a triangulation $\mathcal{T}$ such that $\mathbf{0}$ is incident to every highest dimensional simplex

extremal ray generator $\mathbf{r}_k$ of $\mathcal{C}$ is on some $(\dim(\mathcal{C}) - 1)$-simplex. Moreover, $\mathcal{T}$ by construction has the property that the intersection of any two simplices of $\mathcal{T}$ is a simplex. Step 3 checks if C is on the boundary, since if C is on the boundary it will not be on the intersection of two higher-dimensional simplices.

Step 2 repeats a polynomial number of times since any triangulation of $\mathcal{P}_{\mathcal{C}}$ has at most a polynomial number of simplices, and each simplex B has at most n $(|B| - 1)$-simplices. Step 3 can be computed in polynomial time since again there are only polynomially many simplices B in the triangulation $\mathcal{T}'$ and at most n $(|B| - 1)$-simplices to check if they are equal to C. Hence, Algorithm 15 runs in polynomial time. □

Example 16 The tangent cone at the vertex $\mathbf{e}_B := \mathbf{e}_{\{1,2,3\}}$ on the polytope $\mathcal{P}(M(K_4))$ can be triangulated as:

$$\big\{\{\mathbf{e}_{\{2,3,5\}} - \mathbf{e}_B, \mathbf{e}_{\{2,3,4\}} - \mathbf{e}_B, \mathbf{e}_{\{1,3,6\}} - \mathbf{e}_B, \mathbf{e}_{\{1,3,4\}} - \mathbf{e}_B, \mathbf{e}_{\{1,2,6\}} - \mathbf{e}_B\},$$
$$\{\mathbf{e}_{\{2,3,5\}} - \mathbf{e}_B, \mathbf{e}_{\{1,3,6\}} - \mathbf{e}_B, \mathbf{e}_{\{1,3,4\}} - \mathbf{e}_B, \mathbf{e}_{\{1,2,6\}} - \mathbf{e}_B, \mathbf{e}_{\{1,2,5\}} - \mathbf{e}_B\},$$
$$\{\mathbf{e}_{\{2,3,5\}} - \mathbf{e}_B, \mathbf{e}_{\{2,3,4\}} - \mathbf{e}_B, \mathbf{e}_{\{1,3,4\}} - \mathbf{e}_B, \mathbf{e}_{\{1,2,6\}} - \mathbf{e}_B, \mathbf{e}_{\{1,2,5\}} - \mathbf{e}_B\}\big\}.$$

2.3 Polymatroids

We will show that certain lemmas from Sect. 2.2 also hold for certain polymatroids. Recall that the rank of the matroid M is the size of any basis of M which equals $\varphi([n])$. Our lemmas from Sect. 2.2 rely on the fact that M has fixed rank, that is, for some $r \in \mathbb{Z}$, $r \geq 0$, $\varphi(A) \leq r$ for all $A \subseteq [n]$. We will show that a similar condition on a polymatroid rank function is sufficient for the lemmas of Sect. 2.2 to hold.

Lemma 17 *Let $\psi : 2^{[n]} \longrightarrow \mathbb{N}$ be an integral polymatroid rank function where $\psi(A) \leq r$ for all $A \subseteq [n]$, where r is a fixed integer. Then the number of vertices of $\mathcal{P}(\psi)$ is bounded by a polynomial in n of degree r.*

Proof It is known that if ψ is integral, then all vertices of $\mathcal{P}(\psi)$ are integral [41]. The number of vertices of $\mathcal{P}(\psi)$ can be bounded by the number of nonnegative integral solutions to $x_1 + \cdots + x_n \leq r$, which has $\binom{n+r}{r}$ solutions, a polynomial in n of degree r [38]. □

Lemma 18 *Let $\psi : 2^{[n]} \longrightarrow \mathbb{N}$ be an integral polymatroid rank function. If $\mathbf{v}$ is a vertex of $\mathcal{P}(\psi)$, then all adjacent vertices of $\mathbf{v}$ can be enumerated in polynomial time.*

Moreover, if $\psi(A) \le r$ for all $A \subseteq [n]$, where r is a fixed integer, then the vertices of $\mathcal{P}(\psi)$ can be enumerated in polynomial time.

Proof If $\mathbf{v}$ is a vertex of $\mathcal{P}(\psi)$, then generating and listing all adjacent vertices to $\mathbf{v}$ can be done in polynomial time by Corollary 5.5 in [39]. If $\psi(A) \le r$ for all $A \subseteq [n]$, where r is a fixed integer, then, by Lemma 17, there is a polynomial number of vertices for $\mathcal{P}(\psi)$. We know that $\mathbf{0} \in \mathbb{R}^n$ is a vertex of any polymatroid. Therefore, beginning with $\mathbf{0}$, we can perform a breadth-first search, which is output-sensitive polynomial time, on the graph of $\mathcal{P}(\psi)$, enumerating all vertices of $\mathcal{P}(\psi)$. □

What remains to be shown is that these polymatroids have cones like the ones in Sect. 2.2. We first recall some needed definitions from [39]. Let $\mathbf{v}, \mathbf{w} \in \mathbb{R}^n$ and define $\Delta(\mathbf{v}, \mathbf{w}) := \{ i \in [n] \mid v_i \neq w_i \}$ and $\mathrm{cl}(\mathbf{v}) := \{ S \mid S \subseteq [n],\ \sum_{i \in S} v_i = \psi(S) \}$. Let $F = \{f_1, \dots, f_{|F|}\}$ be an ordered subset of $[n]$ and $F_i := \{f_1, \dots, f_i\}$. If ψ is a polymatroid rank function, then we construct $\mathbf{v} \in \mathbb{R}^n$ where $v_i = \psi(F_i) - \psi(F_{i-1})$ where $v_j = 0$ when $j \notin F$ and one says F *generates* $\mathbf{v}$. A classical result of Edmonds [18] says that the set of vectors generated by all ordered subsets of $[n]$ is exactly the set of vertices of $\mathcal{P}(\psi)$. Now we can restate an important lemma.

Lemma 19 (See Theorem 4.1 and Sect. 2 in [39]) *Let ψ be a polymatroid rank function. If $\mathbf{v}$ and $\mathbf{w}$ are vertices of the polymatroid $\mathcal{P}(\psi)$, then either*

(i) $|\Delta(\mathbf{v}, \mathbf{w})| = 1$.
(ii) $\mathrm{cl}(\mathbf{v}) = \mathrm{cl}(\mathbf{w})$ *and* $\Delta(\mathbf{v}, \mathbf{w}) = \{c, d\}$ *for some $c, d \in [n]$ where there exists some ordered set $F = \{f_1, \dots, f_{|F|}\}$ which generates $\mathbf{v}$ with $f_{k+1} = d$ and $f_k = c$ for some integer k, $1 \le k \le |F| - 1$; moreover the ordered set $\tilde{F} := \{f_1, \dots, f_{k-1}, f_{k+1}, f_k, f_{k+2}, \dots, f_{|F|}\}$ generates $\mathbf{w}$.*

Lemma 20 *Let ψ be an integral polymatroid rank function and $\mathcal{C}$ the tangent cone of a vertex $\mathbf{v}$ of the polymatroid $\mathcal{P}(\psi)$, translated to the origin. Then $\mathcal{C}$ is generated by extremal ray generators $\{\mathbf{r}_1, \dots, \mathbf{r}_l\} \subseteq R_{\mathrm{supp}(\mathbf{v})}$, where $R_{\mathrm{supp}(\mathbf{v})}$ is an elementary set of* $\mathrm{supp}(\mathbf{v})$.

Proof Let $\psi : 2^{[n]} \longrightarrow \mathbb{Z}$ be a integral polymatroid rank function. Let $\mathbf{v}$ and $\mathbf{w}$ be adjacent vertices of the polymatroid $\mathcal{P}(\psi)$. Using Lemma 19, if $|\Delta(\mathbf{v}, \mathbf{w})| = 1$, then $\mathbf{w} - \mathbf{v} = h\mathbf{e}_i$ where h is some integer and $\mathbf{e}_i$ is the standard ith elementary vector for some $i \in [n]$. If $h < 0$, then certainly $i \in \mathrm{supp}(\mathbf{v})$, else $i \in [n]$. Thus $\mathbf{w} - \mathbf{v}$, a generator of $\mathcal{C}$, is parallel to a vector in $R_{\mathrm{supp}(\mathbf{v})}$.

Let $\mathbf{v}$ and $\mathbf{w}$ be adjacent and satisfy (ii) in Lemma 19, where $\Delta(\mathbf{v}, \mathbf{w}) = \{c, d\}$. Hence there exists an $F = \{f_1, \dots, f_{|F|}\}$ which generates $\mathbf{v}$ with $f_{k+1} = d$ and $f_k = c$ for some integer k, $1 \le k \le |F| - 1$; moreover the ordered set $\tilde{F} := \{f_1, \dots, f_{k-1}, f_{k+1}, f_k, f_{k+2}, \dots, f_{|F|}\}$ generates $\mathbf{w}$. First we note that $\psi(F_{k-1}) = \psi(\tilde{F}_{k-1})$ and $\psi(F_{k+1}) = \psi(\tilde{F}_{k+1})$. By assumption, we know $v_c \neq w_c$, $v_d \neq w_d$ and $v_l = w_l$ for all $l \in [n] \setminus \{c, d\}$. Thus

$$(\mathbf{v} - \mathbf{w})_c = v_c - w_c = \psi(F_{k+1}) - \psi(F_k) - \left(\psi\left(\tilde{F}_k\right) - \psi\left(\tilde{F}_{k-1}\right)\right)$$

$$= \psi(F_{k+1}) - \psi(F_k) - \psi(\tilde{F}_k) + \psi(\tilde{F}_{k-1}),$$

and

$$\begin{aligned}(\mathbf{v}-\mathbf{w})_d = v_d - w_d &= \psi(F_k) - \psi(F_{k-1}) - (\psi(\tilde{F}_{k+1}) - \psi(\tilde{F}_k))\\ &= \psi(F_k) - \psi(\tilde{F}_{k-1}) - (\psi(F_{k+1}) - \psi(\tilde{F}_k))\\ &= -\psi(F_{k+1}) + \psi(F_k) + \psi(\tilde{F}_k) - \psi(\tilde{F}_{k-1}).\end{aligned}$$

Therefore $(\mathbf{v}-\mathbf{w})_c = -(\mathbf{v}-\mathbf{w})_d$ and $\mathbf{w}-\mathbf{v}$ is parallel to $\mathbf{e}_d - \mathbf{e}_c$. Moreover, $c \in \operatorname{supp}(\mathbf{v})$ since $\mathbf{w}, \mathbf{v} \geq \mathbf{0}$ by assumption that $\mathbf{v}, \mathbf{w} \in \mathcal{P}(\psi)$. Thus $\mathbf{w}-\mathbf{v}$, a generator of $\mathcal{C}$, is parallel to a vector in $R_{\operatorname{supp}(\mathbf{v})}$. □

2.4 The Construction of a Short Multivariate Rational Generating Function

From the knowledge of triangulations of tangent cones of matroid polytopes, independence matroid polytopes, and polymatroids we will now recover short multivariate generating functions.

Remark 21 Notice that formula (2) is of exponential size, even when the triangulation $\mathcal{T}$ only has polynomially many simplicial cones of maximal dimension. The reason is that, when the dimension n is allowed to vary, there are exponentially many intersecting proper faces in the set $\hat{\mathcal{T}}$. Therefore, we cannot use (2) to compute the multivariate rational generating function of $\mathcal{C}$ in polynomial time for varying dimension.

To obtain a shorter formula, we use the technique of *half-open exact decompositions* [29], which is a refinement of the method of "irrational" perturbations [6, 27]. We use the following result; see also Figs. 5 and 6.

Lemma 22

(a) *Let*

$$\sum_{i\in I_1} \varepsilon_i[\mathcal{C}_i] + \sum_{i\in I_2} \varepsilon_i[\mathcal{C}_i] = 0 \tag{6}$$

be a linear identity (with rational coefficients ε_i) of indicator functions of cones $\mathcal{C}_i \subseteq \mathbb{R}^n$, where the cones $\mathcal{C}_i$ are full-dimensional for $i \in I_1$ and lower-dimensional for $i \in I_2$. Let each cone be given as

$$\mathcal{C}_i = \{\mathbf{x} \in \mathbb{R}^n : \langle \mathbf{b}^*_{i,j}, \mathbf{x}\rangle \leq 0 \text{ for } j \in J_i\}. \tag{7}$$

Let $\mathbf{y} \in \mathbb{R}^n$ be a vector such that $\langle \mathbf{b}^*_{i,j}, \mathbf{y}\rangle \neq 0$ for all $i \in I_1 \cup I_2$, $j \in J_i$. For $i \in I_1$, we define the "half-open cone"

$$\begin{aligned}\tilde{\mathcal{C}}_i = \{\mathbf{x} \in \mathbb{R}^d : \langle \mathbf{b}^*_{i,j}, \mathbf{x}\rangle \leq 0 \text{ for } j \in J_i \text{ with } \langle \mathbf{b}^*_{i,j}, \mathbf{y}\rangle < 0,\\ \langle \mathbf{b}^*_{i,j}, \mathbf{x}\rangle < 0 \text{ for } j \in J_i \text{ with } \langle \mathbf{b}^*_{i,j}, \mathbf{y}\rangle > 0\}.\end{aligned} \tag{8}$$

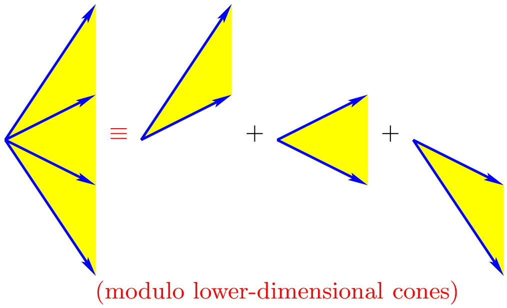

Fig. 5 An identity, valid modulo lower-dimensional cones, corresponding to a polyhedral subdivision of a cone

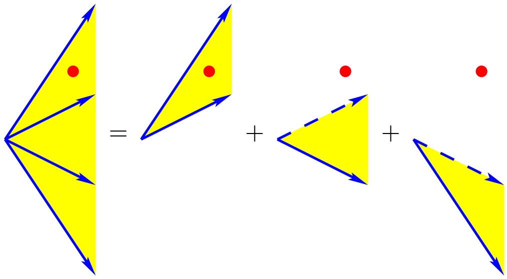

Fig. 6 The technique of half-open exact decomposition. The relative location of the vector **y** (represented by a *dot*) determines which defining inequalities are strict (*broken lines*) and which are weak (*solid lines*)

Then

$$\sum_{i\in I_1} \varepsilon_i \bigl[\tilde{\mathcal{C}}_i\bigr] = 0. \tag{9}$$

(b) *In particular, let*

$$[\mathcal{C}] = \sum_{\mathcal{C}_i\in\mathcal{T}} [\mathcal{C}_i] + \sum_{\mathcal{C}_i\in\hat{\mathcal{T}}} \varepsilon_i\,[\mathcal{C}_i] \tag{10}$$

be the identity corresponding to a triangulation of the cone $\mathcal{C}$, where $\mathcal{T}$ is the set of simplicial cones of maximal dimension and $\hat{\mathcal{T}}$ is the set of intersecting proper faces. Then there exists a polynomial-time algorithm to construct a vector $\mathbf{y}\in\mathbb{Q}^n$ such that the above construction yields the identity

$$[\mathcal{C}] = \sum_{\mathcal{C}_i\in\mathcal{T}} \bigl[\tilde{\mathcal{C}}_i\bigr], \tag{11}$$

which describes a partition of $\mathcal{C}$ into half-open cones of maximal dimension.

Proof Part (a) is a slightly less general form of Theorem 3 in [29]. Part (b) follows from the discussion in [29, Sect. 2]. □

Since the cones in a triangulation $\mathcal{T}$ of all tangent cones $\mathcal{C}_{\mathcal{P}}(\mathbf{v})$ of our polytopes are unimodular by Lemma 8, we can efficiently write the multivariate generating functions of their half-open counterparts.

Lemma 23 (Lemma 9 in [29]) *Let $\tilde{\mathcal{C}} \subseteq \mathbb{R}^n$ be an N-dimensional half-open pointed simplicial affine cone with an integral apex $\mathbf{v} \in \mathbb{Z}^n$ and the ray description*

$$\tilde{\mathcal{C}} = \left\{ \mathbf{v} + \sum_{j=1}^{N} \lambda_j \mathbf{b}_j : \lambda_j \geq 0 \text{ for } j \in J_{\leq} \text{ and } \lambda_j > 0 \text{ for } j \in J_{<} \right\}, \tag{12}$$

where $J_{\leq} \cup J_{<} = \{1, \ldots, N\}$ and $\mathbf{b}_j \in \mathbb{Z}^n \setminus \{\mathbf{0}\}$. We further assume that $\tilde{\mathcal{C}}$ is unimodular, i.e., the vectors $\mathbf{b}_j$ form a basis of the lattice $(\mathbb{R}\mathbf{b}_1 + \cdots + \mathbb{R}\mathbf{b}_N) \cap \mathbb{Z}^n$. Then the unique point in the fundamental parallelepiped of the half-open cone $\tilde{\mathcal{C}}$ is

$$\mathbf{a} = \mathbf{v} + \sum_{j \in J_{<}} \mathbf{b}_j, \tag{13}$$

and the generating function of $\mathcal{C}$ is given by

$$g_{\mathcal{C}}(\mathbf{z}) = \frac{\mathbf{z}^{\mathbf{a}}}{\prod_{j=1}^{N}(1 - \mathbf{z}^{\mathbf{b}_j})}. \tag{14}$$

Taking all results together, we obtain:

Corollary 24 *Let r be a fixed integer. There exist algorithms that, given*

(a) *a matroid M on n elements, presented by an evaluation oracle for its rank function φ, which is bounded above by r, or*
(b) *an evaluation oracle for an integral polymatroid rank function $\psi : 2^{[n]} \longrightarrow \mathbb{N}$, which is bounded above by r*

compute in time polynomial in n vectors $\mathbf{a}_i \in \mathbb{Z}^n$, $\mathbf{b}_{i,j} \in \mathbb{Z}^n \setminus \{\mathbf{0}\}$, and $\mathbf{v}_i \in \mathbb{Z}^n$ for $i \in I$ (a polynomial-size index set) and $j = 1, \ldots, N$, where $N \leq n$, such that the multivariate generating function of $\mathcal{P}(M)$, $\mathcal{P}^{\mathcal{I}}(M)$ and $\mathcal{P}(\psi)$, respectively, is the sum of rational functions

$$g_{\mathcal{P}}(\mathbf{z}) = \sum_{i \in I} \frac{\mathbf{z}^{\mathbf{a}_i}}{\prod_{j=1}^{N}(1 - \mathbf{z}^{\mathbf{b}_{i,j}})}, \tag{15}$$

and the kth dilation of the polytope has the multivariate rational generating function

$$g_{k\mathcal{P}}(\mathbf{z}) = \sum_{i \in I} \frac{\mathbf{z}^{\mathbf{a}_i + (k-1)\mathbf{v}_i}}{\prod_{j=1}^{N}(1 - \mathbf{z}^{\mathbf{b}_{i,j}})}. \tag{16}$$

Proof Lemma 4 implies that finding the multivariate generating function of $\mathcal{P}(M)$, $\mathcal{P}^{\mathcal{I}}(M)$ or $\mathcal{P}(\psi)$ can be reduced to finding the multivariate generating functions of their tangent cones. Moreover, $\mathcal{P}(M)$, $\mathcal{P}^{\mathcal{I}}(M)$ and $\mathcal{P}(\psi)$ have only polynomially in n many vertices as described in Sect. 2.2 or Lemma 17. Enumerating their vertices can be done in polynomial time by Lemmas 6 or 18.

Given a vertex $\mathbf{v}$ of $\mathcal{P}(M)$, $\mathcal{P}^{\mathcal{I}}(M)$ or $\mathcal{P}(\psi)$, its neighbors can be computed in polynomial time by Lemmas 6 or 18. The tangent cone $\mathcal{C}(\mathbf{v})$ at $\mathbf{v}$ is generated by elements in R_A, where R_A is an elementary set for some $A \subseteq [n]$. See Sect. 2.2 or Lemma 20. We also proved in Lemma 8 that every triangulation of $\mathcal{C}(\mathbf{v})$ generated by elements in R_A is unimodular and Lemma 10 states that any triangulation of $\mathcal{C}(\mathbf{v})$ has at most a polynomial in n number of top-dimensional simplices. Moreover, a triangulation of the cone $\mathcal{C}(\mathbf{v})$ can be computed in polynomial time by Lemma 13. Finally, using Lemmas 22 and 23 we can write the polynomial sized multivariate generating function of $\mathcal{C}(\mathbf{v})$ in polynomial time. Therefore we can write the multivariate generating function of $\mathcal{P}(M)$, $\mathcal{P}^{\mathcal{I}}(M)$ or $\mathcal{P}(\psi)$ in polynomial time. □

2.5 Polynomial-Time Specialization of Rational Generating Functions in Varying Dimension

We now compute the Ehrhart polynomial $i(\mathcal{P}, k) = \#(k\mathcal{P} \cap \mathbb{Z}^n)$ from the multivariate rational generating function $g_{k\mathcal{P}}(\mathbf{z})$ of Corollary 24. This amounts to the problem of evaluating or *specializing* a rational generating function $g_{k\mathcal{P}}(\mathbf{z})$, depending on a parameter k, at the point $\mathbf{z} = \mathbf{1}$. This is a pole of each of its summands, but a regular point (removable singularity) of the function itself. From now on we call this the *specialization problem*. We explain a very general procedure to solve it which we hope will allow future applications.

To this end, let the generating function of a polytope $\mathcal{P} \subseteq \mathbb{R}^n$ be given in the form

$$g_{\mathcal{P}}(\mathbf{z}) = \sum_{i \in I} \varepsilon_i \frac{\mathbf{z}^{\mathbf{a}_i}}{\prod_{j=1}^{s_i}(1 - \mathbf{z}^{\mathbf{b}_{ij}})}, \tag{17}$$

where $\varepsilon_i \in \{\pm 1\}$, $\mathbf{a}_i \in \mathbb{Z}^n$, and $\mathbf{b}_{ij} \in \mathbb{Z}^n \setminus \{\mathbf{0}\}$. Let $s = \max_{i \in I} s_i$ be the maximum number of binomials in the denominators. In general, if s is allowed to grow, more poles need to be considered for each summand, so the evaluation will need more computational effort.

In previous literature, the specialization problem has been considered, but not in sufficient generality for our purpose. In the original paper by [1, Lemma 4.3], the dimension n is fixed, and each summand has exactly $s_i = n$ binomials in the denominator. The same restriction can be found in the survey by [3]. In the more general algorithmic theory of monomial substitutions developed by [4] and [42], there is no assumption on the dimension n, but the number s of binomials in the denominators is fixed. The same restriction appears in the paper by [40, Lemma 2.15]. In a recent paper, [2, Sect. 5] gives a polynomial-time algorithm for the specialization problem for rational functions of the form

$$g(\mathbf{z}) = \sum_{i \in I} \varepsilon_i \frac{\mathbf{z}^{\mathbf{a}_i}}{\prod_{j=1}^{s}(1 - \mathbf{z}^{\mathbf{b}_{ij}})^{\gamma_{ij}}}, \tag{18}$$

where the dimension n is fixed, the number s of different binomials in each denominator equals n, but the multiplicity γ_{ij} is varying.

We will show that the technique from [2, Sect. 5] can be implemented in a way such that we obtain a polynomial-time algorithm even for the case of a general formula (17), when the dimension and the number of binomials are allowed to grow.

Theorem 25 (Polynomial-time specialization)

(a) *There exists an algorithm for computing the specialization of a rational function of the form*

$$g_{\mathcal{P}}(\mathbf{z}) = \sum_{i\in I} \varepsilon_i \frac{\mathbf{z}^{\mathbf{a}_i}}{\prod_{j=1}^{s_i}(1-\mathbf{z}^{\mathbf{b}_{ij}})} \tag{19}$$

at its removable singularity $\mathbf{z} = \mathbf{1}$, *which runs in time polynomial in the encoding size of its data* $\epsilon_i \in \mathbb{Q}$, $\mathbf{a}_i \in \mathbb{Z}^n$ *for* $i \in I$ *and* $\mathbf{b}_{ij} \in \mathbb{Z}^n$ *for* $i \in I$, $j = 1, \ldots, s_i$, *even when the dimension* n *and the numbers* s_i *of terms in the denominators are not fixed.*

(b) *In particular, there exists a polynomial-time algorithm that, given data* $\epsilon_i \in \mathbb{Q}$, $\mathbf{a}_i \in \mathbb{Z}^n$ *for* $i \in I$ *and* $\mathbf{b}_{ij} \in \mathbb{Z}^n$ *for* $i \in I$, $j = 1, \ldots, s_i$ *describing a rational function in the form* (19), *computes a vector* $\boldsymbol{\lambda} \in \mathbb{Q}^n$ *with* $\langle \boldsymbol{\lambda}, \mathbf{b}_{ij}\rangle \neq 0$ *for all* i, j *and rational weights* $w_{i,l}$ *for* $i \in I$ *and* $l = 0, \ldots, s_i$. *Then the number of integer points is given by*

$$\#\big(\mathcal{P} \cap \mathbb{Z}^n\big) = \sum_{i\in I} \varepsilon_i \sum_{l=0}^{s_i} w_{i,l} \langle \boldsymbol{\lambda}, \mathbf{a}_i\rangle^l. \tag{20}$$

(c) *Likewise, given a parametric rational function for the dilations of an integral polytope* $\mathcal{P}$,

$$g_{k\mathcal{P}}(\mathbf{z}) = \sum_{i\in I} \varepsilon_i \frac{\mathbf{z}^{\mathbf{a}_i+(k-1)\mathbf{v}_i}}{\prod_{j=1}^{d}(1-\mathbf{z}^{\mathbf{b}_{i,j}})}, \tag{21}$$

the Ehrhart polynomial $i(\mathcal{P}, k) = \#(k\mathcal{P} \cap \mathbb{Z}^n)$ *is given by the explicit formula*

$$i(\mathcal{P}, k) = \sum_{m=0}^{M} \left(\sum_{i\in I} \varepsilon_i \langle \boldsymbol{\lambda}, \mathbf{v}_i\rangle^m \sum_{l=m}^{s_i} \binom{l}{m} w_{i,l} \langle \boldsymbol{\lambda}, \mathbf{a}_i - \mathbf{v}_i\rangle^{l-m} \right) k^m, \tag{22}$$

where $M = \min\{s, \dim \mathcal{P}\}$.

Proof of Theorem 1 Corollary 24 and Theorem 25 imply Theorem (Theorem 1) directly. □

The remainder of this section contains the proof of Theorem 25. We follow [3] and recall the definition of Todd polynomials. We will prove that they can be efficiently evaluated in rational arithmetic.

Definition 26 We consider the function

$$H(x, \xi_1, \ldots, \xi_s) = \prod_{i=1}^{s} \frac{x\xi_i}{1 - \exp\{-x\xi_i\}},$$

a function that is analytic in a neighborhood of $\mathbf{0}$. The mth (s-variate) *Todd polynomial* is the coefficient of x^m in the Taylor expansion

$$H(x,\xi_1,\ldots,\xi_s)=\sum_{m=0}^{\infty}\mathrm{td}_m(\xi_1,\ldots,\xi_s)x^m.$$

We remark that, when the numbers s and m are allowed to vary, the Todd polynomials have an exponential number of monomials.

Theorem 27 *The Todd polynomial* $\mathrm{td}_m(\xi_1,\ldots,\xi_s)$ *can be evaluated for given rational data* $\xi_1,\ldots,\xi_s$ *in time polynomial in* s, m, *and the encoding length of* $\xi_1,\ldots,\xi_s$.

The proof makes use of the following lemma.

Lemma 28 *The function* $h(x)=x/(1-\mathrm{e}^{-x})$ *is a function that is analytic in a neighborhood of* 0. *Its Taylor series about* $x=0$ *is of the form*

$$h(x)=\sum_{n=0}^{\infty}b_nx^n \quad \textit{where } b_n=\frac{1}{n!\,(n+1)!}c_n \tag{23}$$

with integer coefficients c_n *that have a binary encoding length of* $\mathrm{O}(n^2\log n)$. *The coefficients* c_n *can be computed from the recursion*

$$\begin{aligned} c_0&=1,\\ c_n&=\sum_{j=1}^{n}(-1)^{j+1}\binom{n+1}{j+1}\frac{n!}{(n-j+1)!}c_{n-j} \quad \textit{for } n=1,2,\ldots. \end{aligned} \tag{24}$$

Proof The reciprocal function $h^{-1}(x)=(1-\mathrm{e}^{-x})/x$ has the Taylor series

$$h^{-1}(x)=\sum_{i=0}^{\infty}a_nx^n \quad \text{with } a_n=\frac{(-1)^n}{(n+1)!}.$$

Using the identity $h^{-1}(x)h(x)=\left(\sum_{n=0}^{\infty}a_nx^n\right)\left(\sum_{n=0}^{\infty}b_nx^n\right)=1$, we obtain the recursion

$$\begin{aligned} b_0&=\tfrac{1}{a_0}=1,\\ b_n&=-(a_1b_{n-1}+a_2b_{n-2}+\cdots+a_nb_0) \quad \text{for } n=1,2,\ldots. \end{aligned} \tag{25}$$

We prove (23) inductively. Clearly $b_0=c_0=1$. For $n=1,2,\ldots$, we have

$$\begin{aligned} c_n&=n!\,(n+1)!\,b_n\\ &=-n!\,(n+1)!\,(a_1b_{n-1}+a_2b_{n-2}+\cdots+a_nb_0)\\ &=n!\,(n+1)!\sum_{j=1}^{n}\frac{(-1)^{j+1}}{(j+1)!}\cdot\frac{1}{(n-j)!\,(n-j+1)!}c_{n-j} \end{aligned}$$

$$= \sum_{j=1}^{n} (-1)^{j+1} \frac{(n+1)!}{(j+1)!\,(n-j)!} \cdot \frac{n!}{(n-j+1)!} c_{n-j}.$$

Thus we obtain the recursion formula (24), which also shows that all c_n are integers. A rough estimate shows that

$$|c_n| \le n(n+1)!\,n!\,|c_{n-1}| \le \big((n+1)!\big)^2 |c_{n-1}|,$$

thus $|c_n| \le ((n+1)!)^{2n}$, so c_n has a binary encoding length of $\mathrm{O}(n^2 \log n)$. □

Proof of Theorem 27 By definition, we have

$$H(x, \xi_1, \dots, \xi_s) = \sum_{m=0}^{\infty} \mathrm{td}_m(\xi_1, \dots, \xi_s) x^m = \prod_{j=1}^{s} h(x\xi_j).$$

From Lemma 28 we have

$$h(x\xi_j) = \sum_{n=0}^{m} \beta_{j,n} x^n + o\big(x^m\big) \quad \text{where } \beta_{j,n} = \frac{\xi_j^n}{n!\,(n+1)!} c_n \tag{26}$$

with integers c_n given by the recursion (24). Thus we can evaluate $\mathrm{td}_m(\xi_1, \dots, \xi_s)$ by summing over all the possible compositions $n_1 + \dots + n_s = m$ of the order m from the orders n_j of the factors:

$$\mathrm{td}_m(\xi_1, \dots, \xi_s) = \sum_{\substack{(n_1,\dots,n_s) \in \mathbb{Z}_+^s \\ n_1+\dots+n_s=m}} \beta_{1,n_1} \cdots \beta_{s,n_s}. \tag{27}$$

We remark that the length of the above sum is equal to the number of compositions of m into s nonnegative parts,

$$\begin{aligned} C'_s(m) &= \binom{m+s-1}{s-1} \\ &= \frac{(m+s-1)(m+s-2)\cdots(m+s-(s-1))}{(s-1)(s-2)\cdots 2 \cdot 1} \\ &= \Omega\left(\left(1 + \frac{m}{s-1}\right)^s\right), \end{aligned}$$

which is *exponential* in s (whenever $m \ge s$). Thus we cannot evaluate the formula (27) efficiently when s is allowed to grow.

However, we show that we can evaluate $\mathrm{td}_m(\xi_1, \dots, \xi_s)$ more efficiently. To this end, we multiply up the s truncated Taylor series (26), one factor at a time, truncating after order m. Let us denote

$$\begin{aligned} H_1(x) &= h(x\xi_1), \\ H_2(x) &= H_1(x) \cdot h(x\xi_2), \end{aligned}$$

$$\vdots$$

$$H_s(x) = H_{s-1}(x) \cdot h(x\xi_s) = H(x, \xi_1, \ldots, \xi_s).$$

Each multiplication can be implemented in $\mathrm{O}(m^2)$ elementary rational operations. We finally show that all numbers appearing in the calculations have polynomial encoding size. Let Ξ be the largest binary encoding size of any of the rational numbers $\xi_1, \ldots, \xi_s$. Then every $\beta_{j,n}$ given by (26) has a binary encoding size $\mathrm{O}(\Xi n^5 \log^3 n)$. Let $H_j(x)$ have the truncated Taylor series $\sum_{n=0}^{m} \alpha_{j,n} x^n + o(x^m)$ and let A_j denote the largest binary encoding length of any $\alpha_{j,n}$ for $n \le m$. Then

$$H_{j+1}(x) = \sum_{n=0}^{m} \alpha_{j+1,n} x^n + o\big(x^m\big) \quad \text{with } \alpha_{j+1,n} = \sum_{l=0}^{n} \alpha_{j,l} \beta_{j,n-l}.$$

Thus the binary encoding size of $\alpha_{j+1,n}$ (for $n \le m$) is bounded by $A_j + \mathrm{O}(\Xi m^5 \log^3 m)$. Thus, after s multiplication steps, the encoding size of the coefficients is bounded by $\mathrm{O}(s \Xi m^5 \log^3 m)$, a polynomial quantity. □

Proof of Theorem 25 Parts (a) *and* (b). We recall the technique of [1, Lemma 4.3], refined by [2, Sect. 5].

We first construct a rational vector $\boldsymbol{\lambda} \in \mathbb{Z}^n$ such that $\langle \boldsymbol{\lambda}, \mathbf{b}_{ij} \rangle \neq 0$ for all i, j. One such construction is to consider the *moment curve* $\boldsymbol{\lambda}(\xi) = (1, \xi, \xi^2, \ldots, \xi^{n-1}) \in \mathbb{R}^n$. For each exponent vector $\mathbf{b}_{ij}$ occurring in a denominator of (17), the function $f_{ij} : \xi \mapsto \langle \boldsymbol{\lambda}(\xi), \mathbf{b}_{ij} \rangle$ is a polynomial function of degree at most $n-1$. Since $\mathbf{b}_{ij} \neq \mathbf{0}$, the function f_{ij} is not identically zero. Hence f_{ij} has at most $n-1$ zeros. By evaluating all functions f_{ij} for $i \in I$ and $j = 1, \ldots, s_i$ at $M = (n-1)s|I| + 1$ different values for ξ, for instance at the integers $\xi = 0, \ldots, M$, we can find one $\xi = \bar{\xi}$ that is not a zero of any f_{ij}. Clearly this search can be implemented in polynomial time, even when the dimension n and the number s of terms in the denominators are not fixed. We set $\boldsymbol{\lambda} = \boldsymbol{\lambda}(\bar{\xi})$.

For $\tau > 0$, let us consider the points $\mathbf{z}_\tau = \mathbf{e}^{\tau\boldsymbol{\lambda}} = (\exp\{\tau\lambda_1\}, \ldots, \exp\{\tau\lambda_n\})$. We have

$$\mathbf{z}_\tau^{\mathbf{b}_{ij}} = \prod_{l=1}^{n} \exp\{\tau \lambda_l b_{ijl}\} = \exp\big\{\tau \langle \boldsymbol{\lambda}, \mathbf{b}_{ij} \rangle\big\};$$

since $\langle \boldsymbol{\lambda}, \mathbf{b}_{ij} \rangle \neq 0$ for all i, j, all the denominators $1 - \mathbf{z}_\tau^{\mathbf{b}_{ij}}$ are nonzero. Hence for every $\tau > 0$, the point $\mathbf{z}_\tau$ is a regular point not only of $g(\mathbf{z})$ but also of the individual summands of (17). We have

$$\begin{aligned} g(\mathbf{1}) &= \lim_{\tau \to 0^+} \sum_{i \in I} \varepsilon_i \frac{\mathbf{z}_\tau^{\mathbf{a}_i}}{\prod_{j=1}^{s_i} (1 - \mathbf{z}_\tau^{\mathbf{b}_{ij}})} \\ &= \lim_{\tau \to 0^+} \sum_{i \in I} \varepsilon_i \frac{\exp\{\tau \langle \boldsymbol{\lambda}, \mathbf{a}_i \rangle\}}{\prod_{j=1}^{s_i} (1 - \exp\{\tau \langle \boldsymbol{\lambda}, \mathbf{b}_{ij} \rangle\})} \end{aligned}$$

$$= \lim_{\tau\to 0^+} \sum_{i\in I} \varepsilon_i \, \tau^{-s_i} \exp\bigl\{\tau\langle\boldsymbol{\lambda},\mathbf{a}_i\rangle\bigr\} \prod_{j=1}^{s_i} \frac{\tau}{1-\exp\{\tau\langle\boldsymbol{\lambda},\mathbf{b}_{ij}\rangle\}}$$

$$= \lim_{\tau\to 0^+} \sum_{i\in I} \varepsilon_i \, \tau^{-s_i} \exp\bigl\{\tau\langle\boldsymbol{\lambda},\mathbf{a}_i\rangle\bigr\} \prod_{j=1}^{s_i} \frac{-1}{\langle\boldsymbol{\lambda},\mathbf{b}_{ij}\rangle} h\bigl(-\tau\langle\boldsymbol{\lambda},\mathbf{b}_{ij}\rangle\bigr)$$

$$= \lim_{\tau\to 0^+} \sum_{i\in I} \varepsilon_i \frac{(-1)^{s_i}}{\prod_{j=1}^{s_i}\langle\boldsymbol{\lambda},\mathbf{b}_{ij}\rangle} \tau^{-s_i} \exp\bigl\{\tau\langle\boldsymbol{\lambda},\mathbf{a}_i\rangle\bigr\} H\bigl(\tau,-\langle\boldsymbol{\lambda},\mathbf{b}_{i1}\rangle,\dots,-\langle\boldsymbol{\lambda},\mathbf{b}_{is_i}\rangle\bigr),$$

where $H(x,\xi_1,\dots,\xi_{s_i})$ is the function from Definition 26. We will compute the limit by finding the constant term of the Laurent expansion of each summand about $\tau=0$. Now the function $\tau\mapsto\exp\{\tau\langle\boldsymbol{\lambda},\mathbf{a}_i\rangle\}$ is holomorphic and has the Taylor series

$$\exp\bigl\{\tau\langle\boldsymbol{\lambda},\mathbf{a}_i\rangle\bigr\} = \sum_{l=0}^{s_i} \alpha_{i,l}\tau^l + o\bigl(\tau^{s_i}\bigr) \quad \text{where } \alpha_{i,l} = \frac{\langle\boldsymbol{\lambda},\mathbf{a}_i\rangle^l}{l!}, \tag{28}$$

and $H(\tau,\xi_1,\dots,\xi_{s_i})$ has the Taylor series

$$H(\tau,\xi_1,\dots,\xi_s) = \sum_{m=0}^{s_i} \mathrm{td}_m(\xi_1,\dots,\xi_s)\tau^m + o\bigl(\tau^{s_i}\bigr).$$

Because of the factor τ^{-s_i}, which gives rise to a pole of order s_i in the summand, we can compute the constant term of the Laurent expansion by summing over all the possible compositions $s_i = l + (s_i - l)$ of the order s_i:

$$g(\mathbf{1}) = \sum_{i\in I} \varepsilon_i \frac{(-1)^{s_i}}{\prod_{j=1}^{s_i}\langle\boldsymbol{\lambda},\mathbf{b}_{ij}\rangle} \sum_{l=0}^{s_i} \frac{\langle\boldsymbol{\lambda},\mathbf{a}_i\rangle^l}{l!} \mathrm{td}_{s_i-l}\bigl(-\langle\boldsymbol{\lambda},\mathbf{b}_{i1}\rangle,\dots,-\langle\boldsymbol{\lambda},\mathbf{b}_{is_i}\rangle\bigr). \tag{29}$$

We use the notation

$$w_{i,l} = (-1)^{s_i} \frac{\mathrm{td}_{s_i-l}(-\langle\boldsymbol{\lambda},\mathbf{b}_{i,1}\rangle,\dots,-\langle\boldsymbol{\lambda},\mathbf{b}_{i,s_i}\rangle)}{l!\cdot\langle\boldsymbol{\lambda},\mathbf{b}_{i,1}\rangle\cdots\langle\boldsymbol{\lambda},\mathbf{b}_{i,s_i}\rangle} \quad \text{for } i\in I \quad \text{and} \quad l=0,\dots,s_i;$$

these rational numbers can be computed in polynomial time using Theorem 27. We now obtain the formula of the claim,

$$g(\mathbf{1}) = \sum_{i\in I} \varepsilon_i \sum_{l=0}^{s_i} w_{i,l}\langle\boldsymbol{\lambda},\mathbf{a}_i\rangle^l.$$

Part (c) Applying the same technique to the parametric rational function (21), we obtain

$$\begin{aligned}\#\bigl(k\mathcal{P}\cap\mathbb{Z}^n\bigr) &= g_{k\mathcal{P}}(\mathbf{1}) \\ &= \sum_{i\in I} \varepsilon_i \sum_{l=0}^{s_i} w_{i,l}\bigl\langle\boldsymbol{\lambda},\mathbf{a}_i+(k-1)\mathbf{v}_i\bigr\rangle^l\end{aligned}$$

Table 1 Coefficients of the Ehrhart polynomials of selected matroids in [8, 36]

	Ehrhart Polynomial
Speyer1	$1, \frac{21}{5}, \frac{343}{45}, \frac{63}{8}, \frac{91}{18}, \frac{77}{40}, \frac{29}{90}$
Speyer2	$1, \frac{135}{28}, \frac{3691}{360}, \frac{1511}{120}, \frac{88}{9}, \frac{39}{8}, \frac{529}{360}, \frac{89}{420}$
BJR1	$1, \frac{109}{30}, \frac{23}{4}, \frac{59}{12}, \frac{9}{4}, \frac{9}{20}$
BJR2	$1, \frac{211}{60}, \frac{125}{24}, \frac{33}{8}, \frac{43}{24}, \frac{43}{120}$
BJR3	$1, \frac{83}{20}, \frac{2783}{360}, \frac{199}{24}, \frac{391}{72}, \frac{247}{120}, \frac{61}{180}$
BJR4	$1, \frac{25}{6}, \frac{353}{45}, \frac{101}{12}, \frac{193}{36}, \frac{23}{12}, \frac{53}{180}$

$$= \sum_{i\in I} \varepsilon_i \sum_{l=0}^{s_i} w_{i,l} \sum_{m=0}^{l} \binom{l}{m} \langle \boldsymbol{\lambda}, \mathbf{a}_i - \mathbf{v}_i\rangle^{l-m} k^m \langle \boldsymbol{\lambda}, \mathbf{v}_i\rangle^m$$

$$= \sum_{m=0}^{s} \left(\sum_{i\in I} \varepsilon_i \langle \boldsymbol{\lambda}, \mathbf{v}_i\rangle^m \sum_{l=m}^{s_i} \binom{l}{m} w_{i,l} \langle \boldsymbol{\lambda}, \mathbf{a}_i - \mathbf{v}_i\rangle^{l-m} \right) k^m,$$

an explicit formula for the Ehrhart polynomial. We remark that, since the Ehrhart polynomial is of degree equal to the dimension of $\mathcal{P}$, all coefficients of k^m for $m > \dim \mathcal{P}$ must vanish. Thus we obtain the formula of the claim, where we sum only up to $\min\{s, \dim \mathcal{P}\}$ instead of s. □

3 Algebraic Properties of h^*-vectors and Ehrhart Polynomials of Matroid Polytopes

Using the programs cdd+ [21], LattE [13] and LattE macchiato [28] we explored patterns for the Ehrhart polynomials of matroid polytopes. Since previous authors proposed other invariants of a matroid (e.g., the Tutte polynomials and the invariants of [8, 36]) we wished to know how well the Ehrhart polynomial distinguishes nonisomorphic matroids. It is natural to compare it with other known invariants. Some straightforward properties are immediately evident. For example, the Ehrhart polynomial of a matroid and that of its dual are equal. Also the Ehrhart polynomial of a direct sum of matroids is the product of their Ehrhart polynomials.

We call the last two matroids in Fig. 2 in [36] *Speyer1* and *Speyer2* and the matroids of Fig. 2 in [8] *BJR1*, *BJR2*, *BJR3*, and *BJR4*, and list their Ehrhart polynomials in Table 1. We note that *BJR3* and *BJR4* have the same Tutte polynomial, yet their Ehrhart polynomials are different. This proves that the Ehrhart polynomial cannot be computed using deletion and contractions, as is the case for the Tutte polynomial. Examples *BJR1* and *BJR2* show that the Ehrhart polynomial of a matroid may help to distinguish nonisomorphic matroids: These two matroids are not isomorphic yet they have the same Tutte polynomials and the same quasi-symmetric function studied in [8]. Although they share some properties, there does not seem to be an obvious relation to Speyer's univariate polynomials introduced in [36]; examples *Speyer1* and *Speyer2* show they are relatively prime with their corresponding Ehrhart polynomials.

Our experiments included, among others, many examples coming from small graphical matroids, random realizable matroids over fields of small positive characteristic, and the classical examples listed in the Appendix of [32] for which we list

Table 2 Coefficients of the Ehrhart polynomials and h^*-vectors of selected matroids in [32]

	n	r	h^*-vector	Ehrhart Polynomial
K_4	6	3	1, 10, 20, 10, 1	$1, \frac{107}{30}, \frac{21}{4}, \frac{49}{12}, \frac{7}{4}, \frac{7}{20}$
W^3	6	3	1, 11, 24, 11, 10	$1, \frac{18}{5}, \frac{11}{2}, \frac{9}{2}, 2, \frac{2}{5}$
Q_6	6	3	1, 12, 28, 12, 1	$1, \frac{109}{30}, \frac{23}{4}, \frac{59}{12}, \frac{9}{4}, \frac{9}{20}$
P_6	6	3	1, 13, 32, 13, 1	$1, \frac{11}{3}, 6, \frac{16}{3}, \frac{5}{2}, \frac{1}{2}$
R_6	6	3	1, 12, 28, 12, 1	$1, \frac{109}{30}, \frac{23}{4}, \frac{59}{12}, \frac{9}{4}, \frac{9}{20}$
F_7	7	3	21, 98, 91, 21, 1	$1, \frac{21}{5}, \frac{343}{45}, \frac{63}{8}, \frac{91}{18}, \frac{77}{40}, \frac{29}{90}$
F_7^-	7	3	21, 101, 97, 22, 1	$1, \frac{253}{60}, \frac{2809}{360}, \frac{33}{4}, \frac{193}{36}, \frac{61}{30}, \frac{121}{360}$
P^7	7	3	21, 104, 103, 23, 1	$1, \frac{127}{30}, \frac{479}{60}, \frac{69}{8}, \frac{17}{3}, \frac{257}{120}, \frac{7}{20}$
$AG(3,2)$	8	4	1, 62, 561, 1014, 449, 48, 1	$1, \frac{209}{42}, \frac{1981}{180}, \frac{881}{60}, \frac{119}{9}, \frac{95}{12}, \frac{499}{180}, \frac{89}{210}$
$AG'(3,2)$	8	4	1, 62, 562, 1023, 458, 49, 1	$1, \frac{299}{60}, \frac{4007}{360}, \frac{5401}{360}, \frac{122}{9}, \frac{2911}{360}, \frac{1013}{360}, \frac{77}{180}$
R_8	8	4	1, 62, 563, 1032, 467, 50, 1	$1, \frac{524}{105}, \frac{1013}{90}, \frac{1379}{90}, \frac{125}{9}, \frac{743}{90}, \frac{257}{90}, \frac{136}{315}$
F_8	8	4	1, 62, 563, 1032, 467, 50, 1	$1, \frac{524}{105}, \frac{1013}{90}, \frac{1379}{90}, \frac{125}{9}, \frac{743}{90}, \frac{257}{90}, \frac{136}{315}$
Q_8	8	4	1, 62, 564, 1041, 476, 51, 1	$1, \frac{2099}{420}, \frac{4097}{360}, \frac{1877}{120}, \frac{128}{9}, \frac{337}{40}, \frac{1043}{360}, \frac{61}{140}$
S_8	8	4	1, 44, 337, 612, 305, 40, 1	$1, \frac{1021}{210}, \frac{377}{36}, \frac{475}{36}, \frac{193}{18}, \frac{511}{90}, \frac{65}{36}, \frac{67}{252}$
V_8	8	4	1, 62, 570, 1095, 530, 57, 1	$1, \frac{2117}{420}, \frac{4367}{360}, \frac{2107}{120}, \frac{146}{9}, \frac{1133}{120}, \frac{1133}{360}, \frac{193}{420}$
T_8	8	4	1, 62, 564, 1041, 476, 51, 1	$1, \frac{2099}{420}, \frac{4097}{360}, \frac{1877}{120}, \frac{128}{9}, \frac{337}{40}, \frac{1043}{360}, \frac{61}{140}$
V_8^+	8	4	1, 62, 569, 1086, 521, 56, 1	$1, \frac{151}{30}, \frac{2161}{180}, \frac{3103}{180}, \frac{143}{9}, \frac{1669}{180}, \frac{559}{180}, \frac{41}{90}$
L_8	8	4	1, 62, 567, 1068, 503, 54, 1	$1, \frac{527}{105}, \frac{529}{45}, \frac{83}{5}, \frac{137}{9}, \frac{134}{15}, \frac{136}{45}, \frac{47}{105}$
J	8	4	1, 44, 339, 630, 323, 42, 1	$1, \frac{512}{105}, \frac{193}{18}, \frac{83}{6}, \frac{205}{18}, \frac{361}{60}, \frac{17}{9}, \frac{23}{84}$
P_8	8	4	1, 62, 565, 1050, 485, 52, 1	$1, \frac{1051}{210}, \frac{2071}{180}, \frac{2873}{180}, \frac{131}{9}, \frac{1547}{180}, \frac{529}{180}, \frac{277}{630}$
W_4	8	4	1, 38, 262, 475, 254, 37, 1	$1, \frac{135}{28}, \frac{3691}{360}, \frac{1511}{120}, \frac{88}{9}, \frac{39}{8}, \frac{529}{360}, \frac{89}{420}$
W^4	8	4	1, 38, 263, 484, 263, 38, 1	$1, \frac{169}{35}, \frac{467}{45}, \frac{581}{45}, \frac{91}{9}, \frac{227}{45}, \frac{68}{45}, \frac{68}{315}$
$K_{3,3}$	9	5	78, 1116, 3492, 3237, 927, 72, 1	$1, \frac{307}{56}, \frac{137141}{10080}, \frac{3223}{160}, \frac{37807}{1920}, \frac{211}{16}, \frac{5743}{960}, \frac{1889}{1120}, \frac{8923}{40320}$
$AG(2,3)$	9	3	1, 147, 1230, 1885, 714, 63, 1	$1, \frac{1453}{280}, \frac{41749}{3360}, \frac{581}{32}, \frac{34069}{1920}, \frac{927}{80}, \frac{4541}{960}, \frac{239}{224}, \frac{449}{4480}$
Pappus	9	3	1, 147, 1230, 1915, 744, 66, 1	$1, \frac{729}{140}, \frac{3573}{280}, \frac{381}{20}, \frac{1499}{80}, \frac{243}{20}, \frac{49}{10}, \frac{153}{140}, \frac{57}{560}$
Non-Pappus	9	3	1, 147, 1230, 1925, 754, 67, 1	$1, \frac{4379}{840}, \frac{25951}{2016}, \frac{9287}{480}, \frac{21967}{1152}, \frac{987}{80}, \frac{2855}{576}, \frac{3701}{3360}, \frac{275}{2688}$
$Q_3(GF(3)^*)$	9	3	1, 147, 1098, 1638, 632, 59, 1	$1, \frac{433}{84}, \frac{3079}{252}, \frac{4193}{240}, \frac{5947}{360}, \frac{167}{16}, \frac{601}{144}, \frac{787}{840}, \frac{149}{1680}$
R_9	9	3	1, 147, 1142, 1717, 656, 60, 1	$1, \frac{723}{140}, \frac{49}{4}, \frac{88}{5}, \frac{24217}{1440}, \frac{1291}{120}, \frac{625}{144}, \frac{821}{840}, \frac{133}{1440}$

the results in Table 2. For a comprehensive list of all our calculations visit [15]. Soon it became evident that all instances verified both parts of our Conjecture 2.

By far the most comprehensive study we made was for the family of uniform matroids. In this case we based our computations on the theory of Veronese algebras as developed by M. Katzman in [25]. There, Katzman gave an explicit equation for the h^*-vector of uniform matroid polytopes (again, using the language of Veronese algebras). For this family we were able to verify computationally the conjecture is true for all uniform matroids up to 75 elements and to prove partial unimodality as explained in the introduction.

Lemma 29 *The coefficients of the Ehrhart polynomial of the matroid polytope of the uniform matroid $U^{2,n}$ are positive.*

Proof We begin with the expression in Corollary 2.2 in [25] which explicitly gives the Ehrhart polynomial of $\mathcal{P}(U^{r,n})$ as

$$i\big(\mathcal{P}\big(U^{r,n}\big),k\big)=\sum_{s=0}^{r-1}(-1)^s\binom{n}{s}\binom{k(r-s)-s+n-1}{n-1}. \tag{30}$$

Letting $r=2$, (30) becomes

$$\frac{(2k+n-1)(2k+n-2)\cdots(2k+1)}{(n-1)!}-n\frac{(k+n-2)(k+n-3)\cdots(k)}{(n-1)!}.$$

We next consider the coefficient of k^{n-p-1} for $0\le p\le n-1$, which can be written as

$$\begin{aligned}
&2^{n-p-1}\sum_{\substack{1\le j_1<\cdots<j_p\le n-1\\ j_q\in\mathbb{Z}}}\frac{1}{(n-j_1)\cdots(n-j_p)}\\
&\quad-\frac{n}{n-1}\sum_{\substack{1\le j_1<\cdots<j_{p-1}\le n-2\\ j_q\in\mathbb{Z}}}\frac{1}{(n-j_1)\cdots(n-j_{p-1})}\\
&\quad=2^{n-p-1}\sum_{\substack{1\le j_1<\cdots<j_{p-1}\le n-2\\ j_q\in\mathbb{Z}}}\left(\frac{1}{(n-j_1)\cdots(n-j_{p-1})}\sum_{j_p=1+j_{p-1}}^{n-1}\frac{1}{n-j_p}\right)\\
&\quad-\frac{n}{n-1}\sum_{\substack{1\le j_1<\cdots<j_{p-1}\le n-2\\ j_q\in\mathbb{Z}}}\frac{1}{(n-j_1)\cdots(n-j_{p-1})}\\
&\quad=\sum_{\substack{1\le j_1<\cdots<j_{p-1}\le n-2\\ j_q\in\mathbb{Z}}}\left(\frac{1}{(n-j_1)\cdots(n-j_{p-1})}\right.\\
&\qquad\left.\times\left[2^{n-p-1}\sum_{j_p=1+j_{p-1}}^{n-1}\frac{1}{n-j_p}-\frac{n}{n-1}\right]\right).
\end{aligned}\tag{31}$$

It is known that the constant in any Ehrhart polynomial is 1 [37], thus we only need to show that (31) is positive for $0\le p\le n-2$. It is sufficient to show that the square-bracketed term of (31),

$$2^{n-p-1}\sum_{j_p=1+j_{p-1}}^{n-1}\frac{1}{n-j_p}-\left(1+\frac{1}{n-1}\right), \tag{32}$$

is positive for $0\le p\le n-2$ and all $1\le j_1\le\cdots\le j_{p-1}\le n-2$. We can see that $\sum_{j_p=1+j_{p-1}}^{n-1}\frac{1}{n-j_p}\ge 1$. Moreover, since $0\le p\le n-2$, then $2^{n-p-1}\ge 2$ and $1+\frac{1}{n-1}\le 2$ since $n\ge 2$, proving the result. □

To present our results about the h^*-vector we begin explaining the details with the following numbers introduced in [25], which we refer to as the *Katzman coefficients*.

Definition 30 For any positive integers n and r define the coefficients $A_i^{n,r}$ by

$$\sum_{i=0}^{n(r-1)} A_i^{n,r} T^i = \left(1 + T + \cdots + T^{r-1}\right)^n.$$

We also define the vector $\mathbf{A}^{n,r}$ as $(A_0^{n,r}, A_1^{n,r}, \ldots, A_{n(r-1)}^{n,r})$.

Looking at the definition of the Katzman coefficient, we see that $A_j^{n,2} = \binom{n}{j}$ and $A_j^{n,1} = 0$ unless $j = 0$, in which case we have $A_0^{n,1} = 1$.

In the following we derive some new and useful equalities and prove symmetry and unimodality for the Katzman coefficients. Katzman [25] gave an explicit equation for the h^*-vector of uniform matroid polytopes and the coefficients of their Ehrhart polynomials, although he did not use the same language. We restate it here for our purposes.

Lemma 31 (See Corollary 2.9 in [25]) *Let $\mathcal{P}(U^{r,n})$ be the matroid polytope of the uniform matroid of rank r on n elements. Then the h^*-polynomial of $\mathcal{P}(U^{r,n})$ is given by*

$$\sum_{s=0}^{r-1} \sum_{j=0}^{s} \sum_{k=0}^{j} \sum_{l \geq k} (-1)^{s+j+k} \left[\binom{n}{s} \binom{s}{j} \binom{j}{k} A_{(l-k)(r-s)}^{n-j,r-s} \right] T^l. \tag{33}$$

That is, for $\mathbf{h}^*(\mathcal{P}(U^{n,r})) = (h_0^*, \ldots, h_r^*)$,

$$h_l^* = \sum_{s=0}^{r-1} \sum_{j=0}^{s} \sum_{k=0}^{j} (-1)^{s+j+k} \binom{n}{s} \binom{s}{j} \binom{j}{k} A_{(l-k)(r-s)}^{n-j,r-s}.$$

For $r = 2$ the h^*-polynomial of $\mathcal{P}(U)^{n,2}$ is

$$\left(\sum_{l \geq 0} \binom{n}{2l} T^l \right) - nT. \tag{34}$$

The following lemma is a direct consequence of Corollary 2.9 in [25].

Lemma 32 *Let $\mathcal{P}(U^{2,n})$ be the matroid polytope of the uniform matroid of rank 2 on n elements. The h^*-vector of $\mathcal{P}(U^{2,n})$ is unimodal.*

The rank 2 case is an interesting example. Although the h^*-vector is unimodal, it is not always symmetric. Next we present some useful lemmas, the first a combinatorial description of the Katzman coefficients.

Lemma 33 *For $i = 0, \dots, n(r-1)$ we have*

$$A_i^{n,r} = \sum_{\substack{0a_0+1a_1+\cdots+(r-1)a_{r-1}=i \\ a_0+a_1+\cdots+a_{r-1}=n}} \binom{n}{a_0, a_1, \dots, a_{r-1}}, \tag{35}$$

where $a_0, \dots, a_{r-1}$ run through nonnegative integers.

Proof Using the multinomial formula [38] we have

$$\begin{aligned}
\sum_{i=0}^{n(r-1)} A_i^{n,r} T^i &= \left(1 + T + \cdots + T^{r-1}\right)^n \\
&= \sum_{a_0+a_1+\cdots+a_{r-1}=n} \binom{n}{a_0, a_1, \dots, a_{r-1}} 1^{a_0} T^{a_1} T^{2a_2} \cdots T^{(r-1)a_{r-1}} \\
&= \sum_{a_0+a_1+\cdots+a_{r-1}=n} \binom{n}{a_0, a_1, \dots, a_{r-1}} T^{0a_0+1a_1+\cdots+(r-1)a_{r-1}}.
\end{aligned}$$

By grouping the powers of T we get (35). □

Next we present a generalization of a property of the binomial coefficients. The following lemma relates the Katzman coefficients to Katzman coefficients with one less element.

Lemma 34

$$A_i^{n,r} = \sum_{k=i-r+1}^{i} A_k^{n-1,r}, \tag{36}$$

where we define $A_p^{n-1,r} := 0$ when $p < 0$ or $p > (n-1)(r-1)$.

Proof

$$\begin{aligned}
&\sum_{i=0}^{n(r-1)} A_i^{n,r} T^i \\
&\quad = \left(1 + T + \cdots + T^{r-1}\right)^n = \left(1 + T + \cdots + T^{r-1}\right)^{n-1}\left(1 + T + \cdots + T^{r-1}\right) \\
&\quad = \left(\sum_{i=0}^{(n-1)(r-1)} A_i^{n-1,r} T^i\right)\left(1 + T + \cdots + T^{r-1}\right) \\
&\quad = \sum_{i=0}^{(n-1)(r-1)} A_i^{n-1,r} T^i + \sum_{i=0}^{(n-1)(r-1)} A_i^{n-1,r} T^{i+1} + \cdots + \sum_{i=0}^{(n-1)(r-1)} A_i^{n-1,r} T^{i+r-1}
\end{aligned}$$

$$= \sum_{i=0}^{(n-1)(r-1)} A_i^{n-1,r} T^i + \sum_{i=1}^{(n-1)(r-1)+1} A_{i-1}^{n-1,r} T^i + \cdots + \sum_{i=r-1}^{(n-1)(r-1)+r-1} A_{i-r+1}^{n-1,r} T^i$$

$$= \sum_{i=0}^{(n-1)(r-1)+r-1} \big(A_i^{n-1,r} + A_{i-1}^{n-1,r} + \cdots + A_{i-r+1}^{n-1,r}\big) T^i.$$

Thus we get (36). □

The following lemma relates the Katzman coefficients of rank r with those of rank $r-1$.

Lemma 35

$$\sum A_i^{n,r} T^i = \sum_{k=0}^{n} \binom{n}{k} T^k \Bigg(\sum_{l=0}^{k(r-2)} A_l^{k,r-1} T^l \Bigg)$$

or in other words

$$A_i^{n,r} = \sum_{\substack{k+l=i \\ 0\le k\le n \\ 0\le l\le k(r-2)}} \binom{n}{k} A_l^{k,r-1}.$$

Proof From Definition 30

$$\sum_{i=0}^{n(r-1)} A_i^{n,r} T^i = \big(1 + T + \cdots + T^{r-1}\big)^n = \big(1 + \big[T + \cdots + T^{r-1}\big]\big)^n$$

$$= \sum_{k=0}^{n} \binom{n}{k} \big[T + \cdots + T^{r-1}\big]^k = \sum_{k=0}^{n} \binom{n}{k} T^k \big[1 + \cdots + T^{r-2}\big]^k$$

$$= \sum_{k=0}^{n} \binom{n}{k} T^k \Bigg(\sum_{l=0}^{k(r-2)} A_l^{k,r-1} T^l \Bigg).$$

□

Lemma 36 *The Katzman coefficients are unimodal and symmetric in the index i. That is, the vector $(A_0^{n,r}, A_1^{n,r}, \dots, A_{n(r-1)}^{n,r})$ is unimodal and symmetric.*

Proof We first prove symmetry. Considering (35) we assume that

$$0a_0 + 1a_1 + \cdots + (r-1)a_{r-1} = i \quad \text{and} \quad a_0 + a_1 + \cdots + a_{r-1} = n.$$

These two assumptions imply that

$$\big((r-1) - 0\big)a_0 + \big((r-1) - 1\big)a_1 + \cdots + \big((r-1) - (r-1)\big)a_{r-1}$$
$$= (r-1)a_0 + (r-1)a_1 + \cdots + (r-1)a_n - 0a_0 - 1a_1 - \cdots - (r-1)a_{r-1}$$
$$= (r-1)n - i.$$

Therefore

$$A_i^{n,r} = \sum_{\substack{0a_0+1a_1+\cdots+(r-1)a_{r-1}=i\\ a_0+a_1+\cdots+a_{r-1}=n}} \binom{n}{a_0, a_1, \ldots, a_{r-1}}$$

$$= \sum_{\substack{(r-1)a_0+(r-2)a_1+\cdots+0a_{r-1}=i\\ a_0+a_1+\cdots+a_{r-1}=n}} \binom{n}{a_0, a_1, \ldots, a_{r-1}} = A^{n,r}_{n(r-1)-i}.$$

To prove unimodality we proceed by induction on n, where r is fixed. First, $\mathbf{A}^{1,r}$ is unimodal. Assume for $n-1$ that $\mathbf{A}^{n-1,r} = (A_0^{n-1,r}, A_1^{n-1,r}, \ldots, A^{n-1,r}_{(n-1)(r-1)})$ is unimodal. Using (36) and the fact that $\mathbf{A}^{n-1,r}$ is symmetric we get that $\mathbf{A}^{n,r}$ is unimodal. To help see this, one can view (36) as a sliding window over r elements of the vector $\mathbf{A}^{n-1,r}$, that is, $A_i^{n,r}$ is equal to the sum of the r elements in a window over the vector $\mathbf{A}^{n-1,r}$. As the window slides up the vector $\mathbf{A}^{n-1,r}$, the sum will increase. When the window is on the center of $\mathbf{A}^{n-1,r}$ symmetry and unimodality of $\mathbf{A}^{n-1,r}$ will imply unimodality of $\mathbf{A}^{n,r}$. □

Now we use the explicit equation for the h^*-vector of uniform matroid polytopes to prove partial unimodality of rank 3 uniform matroids. First we note that the coefficient of T^l, h_l^*, in (33) is

$$h_l^* = \sum_{s=0}^{r-1}\sum_{j=0}^{s}\sum_{k=0}^{j}(-1)^{s+j+k}\binom{n}{s}\binom{s}{j}\binom{j}{k}A^{n-j,r-s}_{(p-k)(r-s)}.$$

Letting the rank $r=3$, and using (33), we get the h^*-polynomial (which is grouped by values of s from (33)),

$$\sum_{l\geq 0}\Bigg[\binom{n}{0}A^{n,3}_{3l} + \binom{n}{1}\big(-A^{n,2}_{2l} + A^{n-1,2}_{2l} - A^{n-1,2}_{2(l-1)}\big)$$
$$+ \binom{n}{2}\big(A^{n,1}_{l} - 2A^{n-1,1}_{l} + 2A^{n-1,1}_{l-1} + A^{n-2,1}_{l} - 2A^{n-2,1}_{l-1} + A^{n-2,1}_{l-2}\big)\Bigg]T^l.$$

Now using that $A_i^{n,2} = \binom{n}{i}$ and $A_i^{n,1} = \delta_0(i)$, where

$$\delta_j(p) = \begin{cases} 1 & \text{if } p=j,\\ 0 & \text{else}\end{cases}$$

we get

$$\sum_{l\geq 0}\Bigg[A^{n,3}_{3l} + n\left(-\binom{n}{2l} + \binom{n-1}{2l} - \binom{n-1}{2l-2}\right)$$

$$+\binom{n}{2}\big(\delta_0(l)-2\delta_0(l)+2\delta_1(l)+\delta_0(l)-2\delta_1(l)+\delta_2(l)\big)\Bigg]T^l$$

$$=\sum_{l\geq 0}\left[A_{3l}^{n,3}+n\left(-\binom{n}{2l}+\binom{n-1}{2l}-\binom{n-1}{2l-2}\right)+\delta_2(l)\binom{n}{2}\right]T^l.$$

Using properties of the binomial coefficients, we see that

$$\begin{aligned}\left[-\binom{n}{2l}\right]+\binom{n-1}{2l}-\binom{n-1}{2l-2}&=\left[-\binom{n-1}{2l}-\binom{n-1}{2l-1}\right]\\&\quad+\binom{n-1}{2l}-\binom{n-1}{2l-2}\\&=-\binom{n-1}{2l-1}-\binom{n-1}{2l-2}\\&=-\binom{n}{2l-1}.\end{aligned}$$

So the h^*-polynomial of rank three uniform matroid polytopes is

$$\sum_{l\geq 0}\left[A_{3l}^{n,3}-n\binom{n}{2l-1}+\delta_2(l)\binom{n}{2}\right]T^l. \tag{37}$$

Using Lemma 36, the coefficient of T^l, if $3l\leq n$, is

$$\begin{aligned}h_l^* &= \sum_{\substack{k+p=3l\\0\leq p\leq k\leq n}}\binom{n}{k}\binom{k}{p}-n\binom{n}{2l-1}+\delta_2(l)\binom{n}{2}\\&=\left[\binom{n}{3l}\binom{3l}{0}+\binom{n}{3l-1}\binom{3l-1}{1}+\cdots+\binom{n}{3l-\lfloor 3l/2\rfloor}\binom{3l-\lfloor 3l/2\rfloor}{\lfloor 3l/2\rfloor}\right]\\&\quad-n\binom{n}{2l-1}+\delta_2(l)\binom{n}{2}.\end{aligned}\tag{38}$$

Next we show that when g is fixed $A_g^{n,3}$ is a polynomial of degree g in the indeterminate n, with positive leading coefficient. Assume $g\leq n$. Considering Lemma 35 and when $g\leq n$,

$$A_g^{n,3}=\sum_{\substack{k+p=g\\0\leq p\leq k\leq n}}\binom{n}{k}\binom{k}{p}, \tag{39}$$

where $\binom{n}{q}$ is a polynomial of degree q with positive leading coefficient. The highest degree polynomial in the sum is $\binom{n}{g}$, a degree g polynomial. Hence $A_g^{n,3}$ is a polynomial of degree g in the indeterminate n, with positive leading coefficient. If $g\geq n$, then $A_g^{n,3}=A_{n-g}^{n,3}$ since the Katzman coefficients are symmetric by Lemma 36.

Proof of Theorem 3 Part (2) Let I be a nonnegative integer. From above we see that $A_g^{n,3}$ is a degree g polynomial in the indeterminate n, with positive leading coefficient. Equation (37) is the h^*-polynomial of $U^{3,n}$, which is a sum of polynomials in n, the highest degree polynomial being $A_{3l}^{n,3}$, a polynomial of degree $3l$. So, $h_l^* - h_{l-1}^*$ is the difference of a degree $3l$ and $3(l-1)$ polynomial, hence $h_l^* - h_{l-1}^*$ is a degree $3l$ polynomial with positive leading coefficient. For sufficiently large n, call it $n(I)$, $h_l^* - h_{l-1}^*$ is positive for $0 \le l \le I$. Hence, the h^*-vector of $U^{3,n}$ is nondecreasing up to the index I for $n \ge n(I)$. □

One might ask if (39) has a simpler form. We ran the *WZ* algorithm on our expression, which proved that (39) cannot be written as a linear combination of a fixed number of hypergeometric terms (*closed form*) [33]. There is still the possibility that this expression has a simpler form, though not a closed form as described above.

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