Discrete Comput Geom 37:443–453 (2007)
DOI: 10.1007/s00454-006-1289-0

Discrete & Computational
Geometry

A Non-Regular Gröbner Fan*

Anders Nedergaard Jensen

Institute for Mathematics and Its Applications, University of Minnesota,
400 Lind Hall, 207 Church Street S.E., Minneapolis, MN 55455-0436, USA
ajensen@imf.au.dk

Abstract. The Gröbner fan of an ideal $I \subset k[x_1, \ldots, x_n]$, defined by Mora and Robbiano, is a complex of polyhedral cones in $\mathbb{R}^n$. The maximal cones of the fan are in bijection with the distinct monomial initial ideals of I as the term order varies. If I is homogeneous the Gröbner fan is complete and is the normal fan of the state polytope of I. In general the Gröbner fan is not complete and therefore not the normal fan of a polytope. We may ask if the *restricted* Gröbner fan, a subdivision of $\mathbb{R}^n_{\geq 0}$, is regular, i.e. the normal fan of a polyhedron. The main result of this paper is an example of an ideal in $n = 4$ variables whose restricted Gröbner fan is not regular.

1. Introduction

Let $R = k[x_1, \ldots, x_n]$ be the polynomial ring in n variables over a field k and let $I \subset R$ be an ideal. The *Gröbner fan* and the *restricted Gröbner fan* of I are n-dimensional polyhedral fans defined in [12]. The main result of this paper is the following.

Theorem 1. *The restricted Gröbner fan of the two-dimensional ideal*

$$I = \langle acd + a^2c - ab, ad^2 - c, ad^4 + ac \rangle \subset \mathbb{Q}[a, b, c, d]$$

is not the normal fan of a polyhedron.

In contrast, when the ideal I is *homogeneous* its Gröbner fan and restricted Gröbner fan are known to be normal fans of polyhedra, see Section 2.

* This research was partially supported by the Faculty of Science, University of Aarhus, Danish Research Training Council (Forskeruddannelsesrådet, FUR), Institute for Operations Research ETH, Grants DMS 0222452 and DMS 0100141 of the U.S. National Science Foundation and the American Institute of Mathematics.

We recall the definition of a *fan* in $\mathbb{R}^n$. A *polyhedron* in $\mathbb{R}^n$ is a set of the form $\{x \in \mathbb{R}^n \colon Ax \le b\}$ where A is a matrix and b is a vector. Bounded polyhedra are called *polytopes*. If $b = 0$ the set is a *polyhedral cone*.

Definition 2. A collection C of polyhedra in $\mathbb{R}^n$ is a polyhedral complex if:

1. all proper faces of a polyhedron $P \in C$ are in C, and
2. the intersection of any two polyhedra $A, B \in C$ is a face of A and a face of B.

A polyhedral complex is a *fan* if it only consists of cones. A simple way to construct a fan is by taking the *normal fan* of a polyhedron.

Definition 3. Let $P \subset \mathbb{R}^n$ be a polyhedron. All non-empty faces of P are of the form

$$face_\omega(P) = \{p \in P \colon \langle \omega, p \rangle = \max_{q \in P} \langle \omega, q \rangle\}$$

for some $\omega \in \mathbb{R}^n$. For a face F of P we define its normal cone

$$N_P(F) := \overline{\{\omega \in \mathbb{R}^n \colon face_\omega(P) = F\}}$$

with the closure being taken in the usual topology. The normal fan of P is the fan consisting of the normal cones $N_P(F)$ as F runs through all non-empty faces of P.

If the union of all cones in a fan is $\mathbb{R}^n$, the fan is said to be *complete*. It is clear that the normal fan of a polytope is complete. Not all fans arise as the normal fan of a polyhedron. Those that do are called *regular*.

If the ideal I is homogeneous, its Gröbner fan is the normal fan of a polytope known as the *state polytope* of I [2], [14, Chapter 2]. In the general case, no similar result exists as the Gröbner fan is not complete. However, we could ask if the *restricted* Gröbner fan of I, a fan in $\mathbb{R}^n_{\ge 0}$, is regular. Theorem 1 gives an example of an ideal in $n = 4$ variables whose Gröbner fan and restricted Gröbner fan are not regular.

The definitions of the Gröbner fan and the restricted Gröbner fan appear in Section 2, and the proof of Theorem 1 is given in Section 3. For the reader unfamiliar with Gröbner fans we provide the necessary background in Section 2. It is interesting to consider what happens if we homogenize the example ideal I and project its state polytope back into $\mathbb{R}^4$. In Section 4 we point out why the normal fan of this projection is not the Gröbner fan of I. In particular, we conclude for this example that the third variant of the Gröbner fan, the *extended* Gröbner fan defined in [12], does not agree with the restricted fan in the positive orthant.

An interesting corollary of the restricted Gröbner fan being regular would be an easy proof that the memoryless reverse search algorithm [1] can be used for enumerating the maximal cones in the fan by exploiting the structure of the underlying polyhedron. In light of Theorem 1 the fact that the reverse search method can be used requires a non-trivial proof which is given in [5].

2. The Gröbner Fan of an Ideal

For $\alpha \in \mathbb{N}^n$ we use the notation $x^\alpha := x_1^{\alpha_1} \cdots x_n^{\alpha_n}$. By a term order on R we mean a total ordering on the monomials in R such that:

1. for all $\alpha \in \mathbb{N}^n \backslash \{0\}$, $1 < x^\alpha$ and
2. for $\alpha, \beta, \gamma \in \mathbb{N}^n$, $x^\alpha < x^\beta \Rightarrow x^\alpha x^\gamma < x^\beta x^\gamma$.

Let $\prec$ be a term order. For a non-zero polynomial $f \in R$ we define its *initial term*, $in_\prec(f)$, to be the unique maximal term of f with respect to $\prec$. In the same way for $\omega \in \mathbb{R}^n$ we define the *initial form*, $in_\omega(f)$, to be the sum of all terms of f whose exponents maximize $\langle \cdot, \omega \rangle$. The *initial ideals* of an ideal I with respect to $\prec$ and ω are defined as

$$in_\prec(I) = \langle in_\prec(f)\colon\ f \in I \backslash \{0\} \rangle \quad \text{and} \quad in_\omega(I) = \langle in_\omega(f)\colon\ f \in I \backslash \{0\} \rangle.$$

Note that $in_\prec(I)$ is a monomial ideal while $in_\omega(I)$ might not be. A monomial in $R \backslash in_\prec(I)$ (with coefficient 1) is called a *standard monomial* of $in_\prec(I)$.

Although initial ideals are defined with respect to not necessarily positive vectors, Gröbner bases are only defined with respect to true term orders:

Definition 4. Let $I \subset R$ be an ideal and let $\prec$ be a term order on R. A generating set $\mathcal{G} = \{g_1, \ldots, g_m\}$ for I is called a Gröbner basis for I with respect to $\prec$ if

$$in_\prec(I) = \langle in_\prec(g_1), \ldots, in_\prec(g_m) \rangle.$$

The Gröbner basis $\mathcal{G}$ is minimal if no polynomial can be left out. A minimal Gröbner basis is reduced if the initial term of every $g \in \mathcal{G}$ has coefficient 1 and all other monomials in g are standard monomials of $in_\prec(I)$.

For a term order $\prec$ and an ideal I the reduced Gröbner basis is unique and depends only on I and $in_\prec(I)$. We denote it by $\mathcal{G}_\prec(I)$.

Given I a natural equivalence relation on $\mathbb{R}^n$ is the one induced by taking initial ideals:

$$u \sim v \quad \Longleftrightarrow \quad in_u(I) = in_v(I).$$

We introduce the following notation for the closures of the equivalence classes:

$$C_\prec(I) = \overline{\{u \in \mathbb{R}^n\colon\ in_u(I) = in_\prec(I)\}} \quad \text{and}$$

$$C_v(I) = \overline{\{u \in \mathbb{R}^n\colon\ in_u(I) = in_v(I)\}}.$$

A well-known fact is that for a fixed ideal I there are only finitely many sets $C_\prec(I)$ and they cover $\mathbb{R}^n_{\geq 0}$, see [12]. Secondly, every initial ideal $in_\prec(I)$ is of the form $in_\omega(I)$ for some $\omega \in \mathbb{R}^n_{>0}$. Consequently, every $C_\prec(I)$ is of the form $C_\omega(I)$. A third observation is that the equivalence classes are not convex in general since we allow the vectors to be anywhere in $\mathbb{R}^n$:

Example 5. Let $I = \langle x_1 - 1, x_2 - 1 \rangle$. The ideal I has five initial ideals: $\langle x_1 - 1, x_2 - 1 \rangle$, $\langle x_1, x_2 \rangle$, $\langle x_1, x_2 - 1 \rangle$, $\langle x_1 - 1, x_2 \rangle$ and $\langle 1 \rangle$. In particular, for $u = (-1, 3)^T$ and $v = (3, -1)^T$ we have $in_u(I) = in_v(I) = \langle 1 \rangle$ but $in_{(1/2)(u+v)}(I) = \langle x_1, x_2 \rangle$.

Theorem 6. *Let $\prec$ be a term order and let $v \in C_\prec(I)$ then for $u \in \mathbb{R}^n$,*

$$in_u(I) = in_v(I) \iff \forall g \in \mathcal{G}_\prec(I), \quad in_u(g) = in_v(g).$$

This theorem is a little more general than Proposition 2.3 in [14] as it allows the vectors to be negative. A proof is given in [5]. Theorem 6 shows that the closures of the equivalence classes are polyhedral cones since, for fixed $\prec$ and fixed v, each $g \in \mathcal{G}_\prec(I)$ introduces the equality $in_u(g) = in_v(g)$, which is equivalent to having u satisfy a set of linear equations and strict linear inequalities. The closure is taken by making the strict inequalities non-strict. Thus, in particular, the set $C_v(I)$ is a convex polyhedral cone if it contains a strictly positive vector.

Definition 7. The *Gröbner fan* of an ideal $I \subset R$ is the set of the closures of all equivalence classes intersecting the positive orthant together with their proper faces.

This is a variation of the definitions appearing in the literature. The advantage of this variant is that it gives well-defined and *nice* fans in the homogeneous and non-homogeneous case simultaneously. By *nice* we mean that all cones in this fan are closures of equivalence classes. It is not clear a priori that the Gröbner fan is a polyhedral complex. A proof that the Gröbner fan is in fact a fan (polyhedral complex) is given in [5]. The support of the Gröbner fan of I is called the *Gröbner region* of I.

For the purpose of this paper it is better to study the *restricted* Gröbner fan as we will see soon. Using the definition we already have, together with the notion of *common refinements* of fans [15], it is straightforward to make a definition equivalent to the original one in [12].

Definition 8. Let F and F' be two polyhedral fans in $\mathbb{R}^n$. Their common refinement is the polyhedral fan $F \wedge F' := \{C \cap C'\}_{(C,C') \in F \times F'}$.

Definition 9 (Definition 2.5 of [12]). The restricted Gröbner fan of an ideal $I \subset R$ is the common refinement of the non-negative orthant $\mathbb{R}^n_{\geq 0}$ with its proper faces and the Gröbner fan of I.

The support of the restricted Gröbner fan is $\mathbb{R}^n_{\geq 0}$.

A fundamental question to ask is the following: *Is the Gröbner fan always the normal fan of a polytope*? The answer to this question is *no* since the Gröbner fan is not always complete. Thus we rephrase the question for restricted Gröbner fans: *Is the restricted Gröbner fan of an ideal always the normal fan of a polyhedron*?

We note that the Gröbner fan being regular is stronger than the restricted Gröbner fan being so. This is because the normal fan of the Minkowski sum of two polyhedra is the common refinement of their normal fans. The claim follows since $\mathbb{R}^n_{\geq 0}$ with its proper faces is the normal fan of $\mathbb{R}^n_{\leq 0}$.

The above question is known to have a positive answer in the following three special cases:

- If the ideal is homogeneous the answer is *yes* since the Gröbner fan is the normal fan of the *state polytope* of I introduced by Bayer and Morrison in [2]. We should mention that in [11] it is shown that the Gröbner fan is not the normal fan of the state polytope as it was defined in [2]. Instead we should use the construction in Chapter 2 of [14]. We take the Minkowski sum of the state polytope with $\mathbb{R}^n_{\leq 0}$ to get a polyhedron having the restricted Gröbner fan as its normal fan.
- The *Newton polytope*, $New(f)$, of a polynomial f is defined to be the convex hull of the exponent vectors of the monomials in f. In the case of a *principal* ideal $I = \langle f \rangle$ the Newton polytope $New(f)$ will almost have the Gröbner fan as its normal fan, since two vectors $u, v \in \mathbb{R}^n$ pick out the same initial ideal of I if and only if they are maximized on the same face of $New(f)$. The only thing that keeps $New(f)$ from having the Gröbner fan of I as its normal fan is that we have not included all equivalence classes in the Gröbner fan. However, the normal fan of the Minkowski sum of $New(f)$ and $\mathbb{R}^n_{\leq 0}$ is the restricted Gröbner fan.
- A third case where we have a similar result is for *zero-dimensional* ideals. The construction of a polytope is similar but simpler than the construction in the homogeneous case as there is only a finite number of standard monomials for each initial ideal. We claim, without proof, that the following construction works: For every term order $\prec$ construct the vector $v_\prec$ equal to the negative of the sum of all exponent vectors of all standard monomials of $in_\prec(I)$. Take the convex hull of all $v_\prec$ as we vary the term order. The Minkowski sum of this polytope with $\mathbb{R}^n_{\geq 0}$ is a polyhedron whose normal fan is the restricted Gröbner fan. Under a certain genericity condition this construction appeared in [13].

In contrast to the above, we have Theorem 1.

3. The Proof

This section contains a proof of Theorem 1. We start by deducing a necessary condition for a fan to be the normal fan of a polyhedron. We then show that the restricted Gröbner fan of the ideal in the theorem violates this condition. Finally we argue that the Gröbner fan has been computed correctly.

3.1. *A Necessary Condition*

Let F be a fan in $\mathbb{R}^n$. Suppose F is the normal fan of a polyhedron $P \subset \mathbb{R}^n$. The non-empty faces of P are in bijection with the cones in F by taking normal cones of the faces. Adjacency is preserved in the sense that two vertices of an edge of P map to cones in F having the normal cone of the edge as a common facet. Furthermore, the edge is perpendicular to the shared facet. If a set of normals of the shared facets in F are specified, then for every bounded edge the difference between its endpoints can be expressed as some scalar times the specified normal of its normal cone. The scalars are considered to be unknowns. Since the adjacency information of the vertices of P is present in F, the bounded edge graph of P can be deduced from F. A necessary condition for F to be the normal fan of P is that every combinatorial cycle in the edge graph is a geometric cycle in space. This condition gives rise to a feasible system of inequalities on the scalars dependent on F alone.

To be more specific about the inequality system, consider the adjacency graph of the n-dimensional cones in F, or equivalently the edge graph of the supposed polyhedron P. Let $V = \{1, \ldots, m\}$ denote the vertices and a subset $E \subset \{(i, j) \in V \times V\colon i < j\}$ denote the edges in the graph. For each shared facet, choose a normal vector $d_{(i,j)} \in \mathbb{R}^n$ such that the ith cone is on the negative side of the hyperplane with normal vector $d_{(i,j)}$ and the jth cone is on the positive side. The graph (V, E) is considered to be undirected when we define its cycles. A vector $f \in \mathbb{R}^E$ is called a flow in (V, E) if

$$\forall j \in V, \qquad \sum_{(i,j)\in E} f_{(i,j)} = \sum_{(j,k)\in E} f_{(j,k)}.$$

In other words the flow entering j is the same as the flow leaving j. The set of flows is a subspace of $\mathbb{R}^E$. We introduce a vector $s \in \mathbb{R}^E_{>0}$ of unknown scalars such that the true vector from vertex i to vertex j is $s_{(i,j)} d_{(i,j)}$. Each cycle in the graph can be represented by a flow $f \in \mathbb{R}^E$ being 0 on the edges not appearing in the cycle and ± 1 elsewhere depending on the relative orientation of the cycle and the edge. For such an f the condition that the cycle forms a loop in space can be expressed as

$$\sum_{(i,j)\in E} f_{(i,j)} s_{(i,j)} d_{(i,j)} = 0. \tag{1}$$

Note that (1) is a system of n equations—one for each coordinate of $d_{(i,j)}$. If F is the normal fan of a polyhedron P, there exist positive scalars $s_{(i,j)}$ satisfying (1) for every flow f since the cycle flows span the vector space of flows. By linearity this is equivalent to having the scalars satisfy (1) for a basis of the vector space of flows rather than the entire space. In matrix form we may express the necessary condition as the system

$$As = 0 \quad \text{and} \quad s_{(i,j)} > 0 \qquad \text{for all} \quad (i, j) \in E \tag{2}$$

having a solution where A is a suitable $nl \times |E|$ matrix with l being the dimension of the vector space of flows.

3.2. *The Certificate*

Proof of Theorem 1. The restricted Gröbner fan of the ideal

$$I = \langle acd + a^2c - ab, ad^2 - c, ad^4 + ac\rangle \subset \mathbb{Q}[a, b, c, d]$$

has 81 full-dimensional cones each corresponding to a monomial initial ideal. Their adjacency graph (V, E) has 163 edges, with each edge direction normal to the shared facet. The list of full-dimensional cones, reduced Gröbner bases and monomial initial

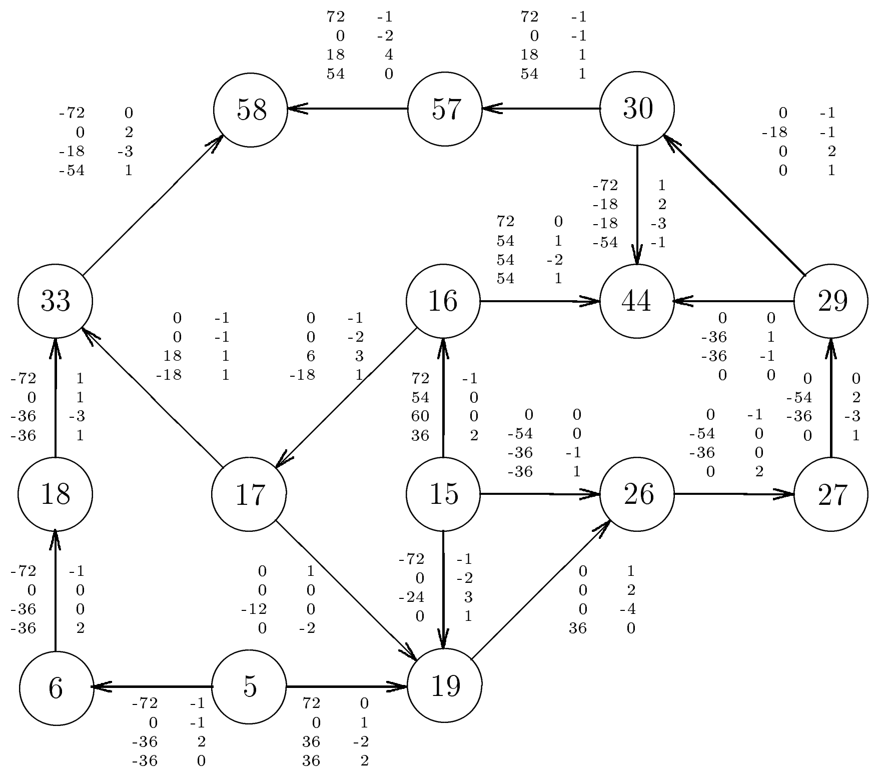

Fig. 1. The certificate subgraph.

ideals can be found on the webpage [10]. We present a certificate that the fan is not the normal fan of a polyhedron. Only the subgraph in Fig. 1 is needed to describe it. Two vectors are written for each edge in the subgraph. The vector to the right is the edge direction $d_{(i,j)}$ and the vectors to the left describe four flows in the subgraph.

Let V' be the set of vertices appearing in the subgraph and let E' be the edges. Let f^1, f^2, f^3 and f^4 denote the flows above. Suppose the restricted Gröbner fan was the normal fan of a polyhedron P. Equality system (1) implies

$$\forall (r,t) \in \{1,2,3,4\} \times \{1,2,3,4\}, \qquad \sum_{(i,j)\in E'} f^r_{(i,j)} s_{(i,j)} d_{(i,j)_t} = 0. \tag{3}$$

In particular, the sum of the equations in (3) for $(r,t) = (1,1), (2,2), (3,3), (4,4)$ is zero. Therefore,

$$0 = \sum_{r=1}^{4} \sum_{(i,j)\in E'} s_{(i,j)} d_{(i,j)_r} f^r_{(i,j)} = \sum_{(i,j)\in E'} s_{(i,j)} \sum_{r=1}^{4} d_{(i,j)_r} f^r_{(i,j)}.$$

The local contribution at each edge except the edge (29, 30) is zero because $d_{(i,j)} \cdot (f^1_{(i,j)}, f^2_{(i,j)}, f^3_{(i,j)}, f^4_{(i,j)})^T = 0$ (check this in the picture). Consequently,

$$0 = s_{(29,30)} d_{(29,30)} \cdot f_{(29,30)} = 18 s_{(29,30)}$$

implying $s_{(29,30)} = 0$. Hence vertices 29 and 30 have the same coordinates which contradicts that P is a polyhedron with the required edge graph. □

Remark 10. Another way to argue is by observing that we have applied the trivial direction of Farkas' lemma to (3). With A' being the 16×20 matrix representing the equalities in (3) a variant of Farkas' lemma says

$$\exists y\colon\ y^T A' \geq 0 \quad \text{and} \quad y^T A' \neq 0 \quad \Longleftrightarrow \quad \nexists s > 0\colon\ A's = 0.$$

In our case $y = (1, 0, 0, 0, 0, 1, 0, 0, 0, 0, 1, 0, 0, 0, 0, 1)^T$ where the four non-zero components correspond to the equations (1, 1), (2, 2), (3, 3) and (4, 4).

3.3. *Correctness of the Subgraph*

For completeness, a positive interior point in each of the 15 maximal cones of the restricted Gröbner fan leading to the inconsistency is given in the top part of Fig. 2. Further, a positive vector in the relative interior of every shared facet is given in the bottom part. To verify the correctness of the certificate the following procedure is suggested: It is straightforward to check that the flows are flows and that the dot products of flows and listed directions are 0 except for the edge (29, 30). The question is how to check the correctness of the edge subgraph and the listed directions. For each of the listed edges (i, j) with $i < j$ compute the corresponding reduced Gröbner bases $\mathcal{G}_i$ and $\mathcal{G}_j$ and use Theorem 6 to compute their cones C_i and C_j. Check that the listed facet vector for the edge (i, j) is in the closure of both cones C_i and C_j and that the listed direction vector non-strictly separates C_i and C_j with C_j being on the non-negative side. Checking that the listed facet vector is in the relative interior of a facet of C_i completes the verification. The non-straightforward part of this test was implemented as a 230 line script in Singular [7]. The script itself is available on the internet, see [10].

5	(10, 2, 5, 3)	18	(4, 3, 5, 4)	30	(15, 1, 3, 11)
6	(14, 4, 11, 5)	19	(5, 1, 2, 2)	33	(3, 1, 2, 3)
15	(7, 6, 5, 3)	26	(7, 1, 2, 3)	44	(7, 5, 4, 4)
16	(7, 11, 8, 4)	27	(17, 1, 4, 9)	57	(7, 1, 2, 7)
17	(5, 2, 3, 3)	29	(10, 1, 2, 6)	58	(7, 1, 3, 8)

5	6	(3, 1, 2, 1)	16	44	(5, 7, 5, 3)	29	30	(8, 1, 2, 5)
5	19	(8, 4, 5, 3)	17	19	(4, 1, 2, 2)	29	44	(9, 3, 3, 5)
6	18	(2, 1, 2, 1)	17	33	(6, 1, 3, 4)	30	44	(6, 5, 4, 4)
15	16	(6, 8, 6, 3)	18	33	(4, 1, 3, 4)	30	57	(13, 1, 3, 11)
15	19	(5, 3, 3, 2)	19	26	(10, 1, 3, 4)	33	58	(6, 1, 3, 7)
15	26	(9, 2, 3, 3)	26	27	(18, 2, 5, 9)	57	58	(10, 1, 3, 11)
16	17	(8, 15, 11, 5)	27	29	(13, 1, 3, 7)			

Fig. 2. Representative weight vectors for cones in the certificate.

4. Further Remarks

4.1. *Homogenizing the Ideal*

In [12] a complete fan in $\mathbb{R}^n$ called the *extended* Gröbner fan is defined for any (not necessarily homogeneous) ideal $I \subset R$. This is done by homogenizing the ideal with a new variable. The *extended* Gröbner fan is defined as the Gröbner fan of the homogenized ideal intersected with $\mathbb{R}^n$. Every cone in the Gröbner fan is a union of cones in the extended Gröbner fan. It is clear that the extended Gröbner fan is regular as the Gröbner fan of the homogenized ideal is regular and the normal fan of the projection of its polytope to $\mathbb{R}^n \times \{0\}$ is the intersection of the Gröbner fan of the homogenized ideal with $\mathbb{R}^n \times \{0\}$. Therefore our example shows that the restricted Gröbner fan of an ideal and its extended Gröbner fan need not agree in $\mathbb{R}^n_{>0}$.

In our example the procedure works as follows. We homogenize the ideal I using the variable "e" to get

$$\begin{aligned} {}^hI = \langle & cd^2+ace, -c^2e+c^2d+abd, c^2e+c^3-bce-bcd-abd+abc, -ce^2+ad^2, \\ & -c^2e+acd-abe, c^2e-bce+ac^2-abd, c^2e+a^2c, bce+a^2b\rangle. \end{aligned}$$

The Gröbner fan of the new ideal is a complete fan in $\mathbb{R}^5$. Intersecting this fan with $\mathbb{R}^4 \times \{0\}$ we get the extended Gröbner fan, a regular fan that refines the Gröbner fan of I in the positive orthant. The extended Gröbner fan has 479 full-dimensional cones. The 81 full-dimensional Gröbner cones of I are covered by 353 such cones. Only 156 (restricted) cones are needed to cover the 81 cones in the restricted Gröbner fan of I. Considering only the non-negative orthant, 62 restricted Gröbner cones are preserved, 11 cones are subdivided into two while the remaining 8 cones are subdivided into 3, 3, 5, 5, 6, 8, 10 and 32 cones, respectively, when we pass to the extended fan. Exactly how the cones are subdivided is shown on the webpage [10]. The subgraph listed in Fig. 1 is valid for the extended fan except that the cone at vertex 57 is divided into two.

4.2. *A Program for Finding the Example*

A C++ program was written for finding non-regular Gröbner fans. The input for the program is a set of generators for an ideal I and the output is either a coordinatization of a polyhedron with the restricted Gröbner fan as its normal fan or a certificate for its non-existence. The program works in two steps:

- In step 1 it calls the software package [9] being developed by the author for computing Gröbner fans of polynomial ideals. This work is presented in [5]. The package computes the maximal cones (n-dimensional) of the Gröbner fan of I storing all facets (($n-1$)-dimensional). This is done using exhaustive search on the graph whose vertices are the maximal cones of the fan, with two maximal cones being connected if they share a facet. At each maximal cone the reduced Gröbner basis is known, its facets are computed using linear programming and the Gröbner bases of its neighbors are computed using the local basis change procedure in [3]. A specialized implementation for toric ideals was worked out in [8].

- From the Gröbner fan computed above, inequality system (2) is deduced. Linear programming methods are used for checking its feasibility. The result is either positive scalars leading to a coordinatization of the vertices of the polyhedron or a certificate for its non-existence.

The software libraries [6] and [4] were used for doing the arithmetic and solving linear programming problems, respectively.

Knowing that we should avoid homogeneous, zero-dimensional and principal ideals, it was not hard to find the example when the C++ program had been written. A practical issue is that we are restricted to ideals with not too complex Gröbner fans as the entire edge graph must be handled by the LP-code. In looking for a three-variable example this seems to be an unfortunate restriction as nothing interesting happens in the small manageable examples we have tried. Thus it remains an open problem if a smaller example exists or if the ideal can be replaced by a prime ideal.

Acknowledgments

The author is thankful to the following people and institutions for supporting this research: Komei Fukuda and Hans-Jakob Lüthi (Institute for Operations Research, ETH Zürich), Douglas Lind and Rekha Thomas (University of Washington, Seattle) and the American Institute of Mathematics. In the writing process of this paper Niels Lauritzen, Komei Fukuda and, especially, Rekha Thomas have been very helpful. Thanks also to the many people who proofread this paper.

References

1. David Avis and Komei Fukuda. A basis enumeration algorithm for convex hulls and vertex enumeration of arrangements and polyhedra. *Discrete Comput. Geom.*, 8:295–313, 1992.
2. David Bayer and Ian Morrison. Standard bases and geometric invariant theory I. Initial ideals and state polytopes. *J. Symbolic Comput.*, 6(2/3):209–217, 1988.
3. Stéphane Collart, Michael Kalkbrener, and Daniel Mall. Converting bases with the Gröbner walk. *J. Symbolic Comput.*, 24(3/4):465–469, 1997.
4. Komei Fukuda. *cddlib reference manual*, *cddlib Version* 093*b*. Swiss Federal Institute of Technology, Lausanne and Zürich, Switzerland, 2003. http://www.ifor.math.ethz.ch/~fukuda/cdd_home/cdd.html.
5. Komei Fukuda, Anders Jensen, and Rekha Thomas. Computing Gröbner fans. *Math. Comput.*, to appear. Available as a preprint at arxiv.org.
6. Torbjörn Granlund et al. GNU multiple precision arithmetic library 4.1.2. December 2002. http://swox.com/gmp/.
7. Gert-Martin Greuel, Gerhard Pfister, and Hans Schönemann. SINGULAR 2.0.5. A computer algebra system for polynomial computations. Centre for Computer Algebra, University of Kaiserslautern, 2004. http://www.singular.uni-kl.de.
8. Birkett Huber and Rekha R. Thomas. Computing Gröbner fans of toric ideals. *Experiment. Math.*, 9(3/4):321–331, 2000.
9. Anders Jensen. Gfan, a software system for Gröbner fans. Available at http://home.imf.au.dk/ajensen/software/gfan/gfan.html, 2005.
10. Anders Jensen. A Singular script for verifying a Gröbner fan computation. Available at http://home.imf.au.dk/ajensen/papers/singularscript.html, 2005.

11. Daniel Mall. *Gröbner Fans and Projective Schemes*. Volume 15 of Progress in Computer Science and Applied Logic. Birkhaüser Bosten, Boston, MA, 1998.
12. Teo Mora and Lorenzo Robbiano. The Gröbner fan of an ideal. *J. Symbol. Comput.*, 6(2/3):183–208, 1988.
13. Shmuel Onn and Bernd Sturmfels. Cutting corners. *Adv. in Appl. Math.*, 23:29–48, 1999.
14. Bernd Sturmfels. *Gröbner Bases and Convex Polytopes*. Volume 8 of University Lecture Series. American Mathematical Society, Providence, RI, 1996.
15. Günter Ziegler. *Lectures on Polytopes*. Volume 152 of Graduate Texts in Mathematics. Springer-Verlag, New York, 1994.

Received March 2, 2006, *and in revised form June* 9, 2006. *Online publication March* 2, 2007.