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# The Maximal Number of Geometric Permutations for *n* Disjoint Translates of a Convex Set in $\mathbb{R}^3$ Is $\Omega(n)$*

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**Abstract.** A geometric permutation induced by a transversal line of a finite family of disjoint convex sets in $\mathbb{R}^d$ is the order in which the transversal meets the members of the family. It is known that the maximal number of geometric permutations in families of $n$ disjoint translates of a convex set in $\mathbb{R}^2$ is 3. We prove that for $d \geq 3$ the maximal number of geometric permutations for such families in $\mathbb{R}^d$ is $\Omega(n)$.

## 1. Introduction

Let $\mathcal{F} = \{A_1, A_2, \dots, A_n\}$ be a finite family of $n$ pairwise disjoint convex sets in $\mathbb{R}^d$. A line $l$ is a *transversal* of $\mathcal{F}$ if it intersects all the members of $\mathcal{F}$. Each non-directed transversal intersects the members of $\mathcal{F}$ in an order which can be described by a pair of permutations of $\{1, 2, \dots, n\}$ which are reverses of each other. Such a pair is called a *geometric permutation*.

There are several results concerning the maximal number of geometric permutations for families of $n$ disjoint convex sets in $\mathbb{R}^d$ (we denote it by $g_d(n)$). Among them: $g_2(n) = 2n - 2$ for $n \geq 4$ ([4] as the upper bound, [10] as the lower bound); $g_d(n) = O(n^{2d-2})$ [14], and $g_d(n) = \Omega(n^{d-1})$ [9]. Some results deal with families with the restriction that the members of the family are disjoint translates of a convex set. Katchalski et al. proved [8], [9] that for such families in $\mathbb{R}^2$, the maximal number of geometric permutations is 3. They also conjectured [9] that for each natural $d$, there is a *constant* upper bound on the number of geometric permutations for families of translates in $\mathbb{R}^d$ (the conjectured upper bound was $\frac{(d+1)!}{2}$). However, the only known upper bound in $\mathbb{R}^d$ is $O(n^{d-1})$ (follows

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from [12]). A constant upper bound is known in a special case: for families of congruent balls in $\mathbb{R}^d$ [11] (improved in [3]; the bound is 2 when $n \geq 9$).

We refute the conjecture mentioned above by proving that there is no constant bound for the maximal number of geometric permutations for families of disjoint translates of a general convex set:

**Theorem 1.** *For each $n \in \mathbb{N}$, $n > 1$, there exists a convex set $X = X(n)$ in $\mathbb{R}^3$ and a family $\mathcal{F} = \mathcal{F}(n)$ of $2n$ disjoint translates of $X$ that admits at least $n + 1$ geometric permutations.*

In Section 3 we show that a stronger results holds: there is a set $Y$ that does not depend on $n$ and satisfies Theorem 1.

A motivation for studying geometric permutations is Helly-type problems on the existence of common transversals for families of disjoint translates of a convex set. For example, geometric permutations were used in Tverberg's solution [13] of Grünbaum's conjecture: if in a family of disjoint translates of a convex set in $\mathbb{R}^2$ each five sets have a transversal line, then the whole family has a transversal. Geometric permutations were also used in the proofs of Helly-type theorems on transversals for disjoint unit balls in $\mathbb{R}^d$. The most recent result, by Cheong et al., is the following: If $\mathcal{F}$ is a family of disjoint unit balls in $\mathbb{R}^d$ such that every $4d - 1$ members of $\mathcal{F}$ have a transversal line, then the whole family $\mathcal{F}$ has a transversal line [2]; weaker results (for $\mathbb{R}^3$) appeared earlier in [1] and [6].

Holmsen and Matoušek showed [7] that in $\mathbb{R}^3$ there is no Helly-type theorem analogous to Tverberg's result mentioned above (the proof of Grünbaum's conjecture). For each $n \in \mathbb{N}$ they construct a family of disjoint translates of a convex set such that each $n$ members of the family have a transversal line, while the entire family does not. In general, the idea of their construction is to take first a family of disjoint sets that have the desired transversal properties, but are not translates of each other, and then to append them one to another in order to obtain translates, preserving their disjointness and transversal properties. Our construction uses a similar idea. Both constructions involve the hyperbolic paraboloid $\Sigma = \{(x, y, z) \in \mathbb{R}^3\colon\ z = xy\}$. This surface has been used earlier for the construction of several examples with transversal lines (see, for example, Theorem 2.9 by Aronov et al. on p. 171 of [5], and the construction described there).

## 2. The Construction

Let $n \in \mathbb{N}$, $n > 1$.

*Planes and Lines*

Denote by $\Sigma$ the hyperbolic paraboloid $\Sigma = \{(x, y, z) \in \mathbb{R}^3\colon\ z = xy\}$. For each $i \in \{0, 1, \ldots, n\}$, let $\lambda_i$ be the plane $y = i$, and let $l_i$ be the line $\lambda_i \cap \Sigma = \{(x, y, z)\colon\ y = i, z = xi\}$. These $n + 1$ lines will be transversal lines for $\mathcal{F}$, inducing different geometric permutations.

For each $m \in \{1, 2, \ldots, n\}$, let $u_m$ denote the plane $x = 2mn^2$, let $u'_m$ denote the plane $x = 2mn^2 + 1$, and let $w_m$ denote the plane $x = 2mn^2 + n^2 + \frac{1}{2}$. These planes will be used in the construction of $\mathcal{F}$, and in the proof of the disjointness of its members.

*The Set X*

For each $m \in \{1, 2, \ldots, n\}$, define four points on $\Sigma$ as follows:

$$\begin{aligned} P_{m,1} &= l_{m-1} \cap u_m = (2mn^2, m-1, 2mn^2 \cdot (m-1)), \\ P_{m,2} &= l_m \cap u'_m = (2mn^2+1, m, (2mn^2+1) \cdot m); \\ Q_{m,1} &= l_m \cap u_m = (2mn^2, m, 2mn^2 \cdot m), \\ Q_{m,2} &= l_{m-1} \cap u'_m = (2mn^2+1, m-1, (2mn^2+1) \cdot (m-1)). \end{aligned}$$

Note that $P_{m,1}, P_{m,2} \in \pi_m$, and $Q_{m,1}, Q_{m,2} \in \sigma_m$, where $\pi_m$ is the plane $y = x - 2mn^2 + m - 1$ and $\sigma_m$ is the plane $y = -x + 2mn^2 + m$. The planes $\pi_m$ and $\sigma_m$ are parallel to the planes $y = x$ and $y = -x$, respectively.

Let $a_m$ be the segment that contains $P_{m,1}$ and $P_{m,2}$ with endpoints in the planes $\lambda_0$ and $\lambda_n$, and let $b_m$ be the segment that contains $Q_{m,1}$ and $Q_{m,2}$ with endpoints in the planes $\lambda_0$ and $\lambda_n$. Figures 1 and 2 show $a_i$'s and $b_i$'s for $n = 3$. (In these figures the solid parts of the segments are above $\Sigma$, and the dashed are below it. Note that the figures are not drawn to scale: in fact, the segments are much further apart.)

In what follows, "the highest (lowest) point" (of a set) means "the point with the maximal (minimal) $z$-coordinate." (This is used only when such points are unique.)

Now define two sets $X^{\mathrm{L}}$ and $X^{\mathrm{U}}$. Each of them is a polygonal line:

$$X^{\mathrm{L}} = \tilde{a}_1 \cup \tilde{a}_2 \cup \cdots \cup \tilde{a}_n, \qquad X^{\mathrm{U}} = \tilde{b}_1 \cup \tilde{b}_2 \cup \cdots \cup \tilde{b}_n,$$

where each $\tilde{a}_m$ is a translate of $a_m$, and each $\tilde{b}_i$ is a translate of $b_i$, so that:

- the lowest point of $\tilde{a}_1$ is $(0, 0, 0)$, and for each $m \in \{2, 3, \ldots n\}$ the lowest point of $\tilde{a}_m$ coincides with the highest point of $\tilde{a}_{m-1}$;
- the highest point of $\tilde{b}_1$ is $(0, n^2, H)$ (where $H$ is a positive number that ensures that $X_{\mathrm{L}}$ is situated much higher than $X_{\mathrm{U}}$), and for each $m \in \{2, 3, \ldots, n\}$ the highest point of $\tilde{b}_m$ coincides with the lowest point of $\tilde{b}_{m-1}$.

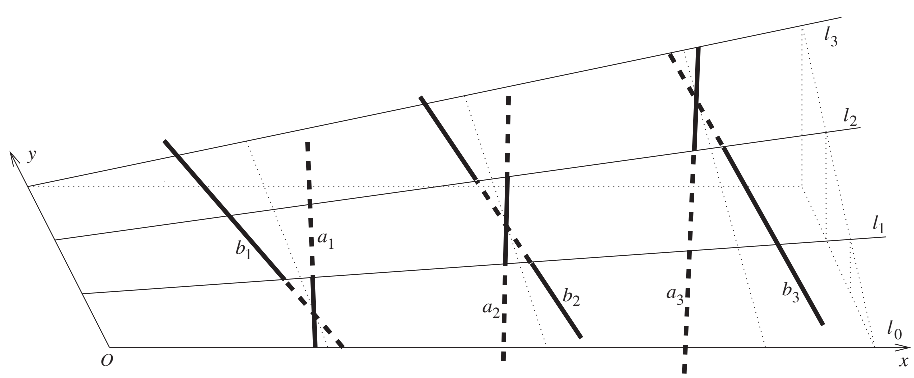


**Fig. 1.** The segments $a_i$ and $b_i$, for $n = 3$. The solid parts are above $\Sigma$, and the dashed are below it.

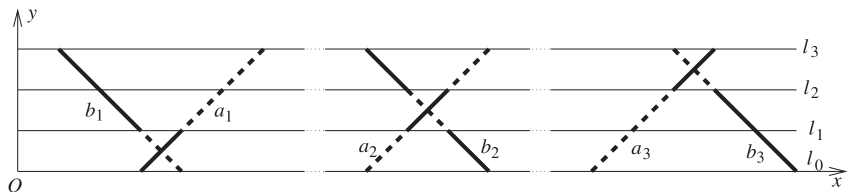


**Fig. 2.** The segments $a_i$ and $b_i$, for $n = 3$, viewed from above.

We make some observations on $X^{\mathrm{L}}$ and $X^{\mathrm{U}}$:

1. The polygonal line $X^{\mathrm{L}}$ lies in the plane $y = x$ (this is true since each $a_m$ lies in $\pi_m$ which is a vertical plane parallel to $y = x$, see Fig. 2; and $\tilde{a}_1$ contains the point $(0, 0, 0)$). Similarly, $X^{\mathrm{U}}$ lies in the plane $y = -x + n^2$.
2. Each $a_m$ contributes $n$ to the lengths of the $x$- and $y$-projections of $X^{\mathrm{L}}$, and each $b_m$ contributes $n$ to the lengths of the $x$- and $y$-projections of $X^{\mathrm{U}}$. Thus the $x$- and $y$-projections of $X^{\mathrm{L}}$ and $X^{\mathrm{U}}$ are $[0, n^2]$.
3. The slopes of $a_m$ and $b_m$, relative to the plane $z = 0$, are, respectively,

$$\frac{1}{\sqrt{2}}((2mn^2 + 1)m - 2mn^2(m - 1)) = \frac{1}{\sqrt{2}}(2n^2 + 1)m \quad \text{and}$$
$$\frac{1}{\sqrt{2}}((2mn^2 + 1)(m - 1) - 2m^2n^2) = -\frac{1}{\sqrt{2}}((2n^2 - 1)m + 1).$$

   This means that if $m < m'$ then the slope of $a_m$ is smaller than that of $a_{m'}$, and the slope of $b_m$ is smaller than that of $b_{m'}$. It follows that $X^{\mathrm{L}}$ is a downward convex polygonal line, and $X^{\mathrm{U}}$ is an upward convex polygonal line.

Let $X = \mathrm{conv}(X^{\mathrm{L}} \cup X^{\mathrm{U}})$ (see Fig. 3; note that in fact $X^{\mathrm{U}}$ is situated high above $X^{\mathrm{L}}$).

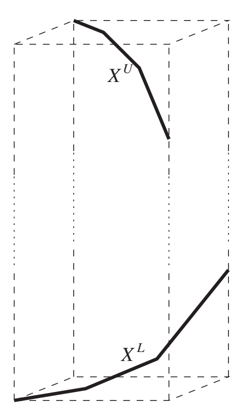


**Fig. 3.** The set $X$ for $n = 3$: the bold polygonal lines are $X^{\mathrm{L}}$ and $X^{\mathrm{U}}$; $X$ is their convex hull.

*The Family $\mathcal{F}$ of Disjoint Translates of X*

For each $m \in \{1, 2, \ldots, n\}$, define $A_m$ to be a translate of $X$ with $\tilde{a}_m$ translated to $a_m$, and define $B_m$ to be a translate of $X$ with $\tilde{b}_m$ translated to $b_m$. Denote by $A_m^{\mathrm{U}}$ ($A_m^{\mathrm{L}}$) the polygonal line on $A_m$ that corresponds to $X^{\mathrm{U}}$ ($X^{\mathrm{L}}$) on $X$, and by $B_m^{\mathrm{U}}$ ($B_m^{\mathrm{L}}$) the polygonal line on $B_m$ that corresponds to $X^{\mathrm{U}}$ ($X^{\mathrm{L}}$) on $X$.

The family $\mathcal{F} = \{A_1, A_2, \ldots, A_n, B_1, B_2, \ldots, B_n\}$ is a family of $2n$ translates of $X$. We prove that they are pairwise disjoint, and that $\mathcal{F}$ has at least $n+1$ geometric permutations.

*Disjointness of the Members of $\mathcal{F}$*

First, note that for each $m \in \{1, 2, \ldots, n-1\}$, the sets $A_m$ and $B_m$ have points ($P_{m,1}$ and $Q_{m,1}$, respectively) with the $x$-coordinate $2mn^2$, and the sets $A_{m+1}$ and $B_{m+1}$ have points ($P_{m+1,2}$ and $Q_{m+1,2}$, respectively) with the $x$-coordinate $2(m+1)n^2+1$. The $x$-lengths of the sets are $n^2$. It follows that the plane $w_m$ (recall that this plane is $x = 2mn^2+n^2+\frac{1}{2}$) separates the sets $A_m$, $B_m$ from the sets $A_{m+1}$, $B_{m+1}$. Hence for $m \neq m'$, $A_m \cap A_{m'}$, $A_m \cap B_{m'}$, and $B_m \cap B_{m'}$ are $\emptyset$.

It remains to prove that $A_m \cap B_m = \emptyset$ for each $m \in \{1, 2, \ldots, n\}$. Let $\tau_m$ be the plane that contains the point $(2mn^2 + \frac{1}{2}, m - \frac{1}{2}, (2mn^2 + \frac{1}{2})(m - \frac{1}{2}))$, and is parallel to $a_m$ and $b_m$. We claim that this plane separates $A_m$ from $B_m$. Let $t_m = \pi_m \cap \tau_m$ and $s_m = \sigma_m \cap \tau_m$ (the planes $\pi_m$ and $\sigma_m$ were defined at the beginning of this section). The segment $a_m$ is parallel to $\tau_m$ and lies in $\pi_m$, hence $a_m$ is parallel to $t_m$, and it is easy to check that $a_m$ is above $t_m$ in the vertical plane $\pi_m$. We have observed that $X^{\mathrm{L}}$ is downward convex. Hence $A_m^{\mathrm{L}}$, being a translate of $X^{\mathrm{L}}$, is also above $t_m$, and thus above $\tau_m$. Similarly, the segment $b_m$ is parallel to $s_m$ and is below it in the vertical plane $\sigma_m$. We have observed that $X^{\mathrm{U}}$ is upward convex. Hence $B_m^{\mathrm{U}}$, being a translate of $X^{\mathrm{U}}$, is also below $s_m$, and thus above $\tau_m$ (see Fig. 4).

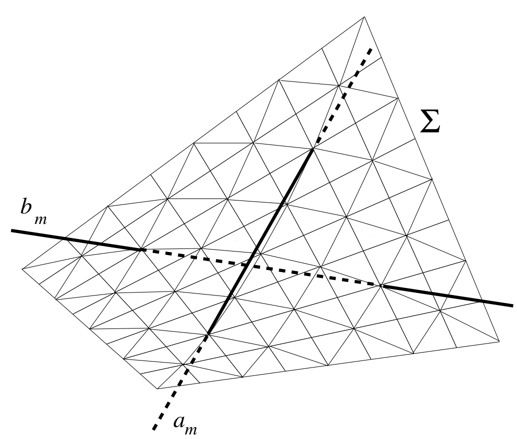


**Fig. 4.** Illustration of the proof of the disjointness of $A_m$ and $B_m$: the segment $a_m$ is above $b_m$.

If $H$ in the definition of $X_U$ is large enough, then $A_m^U$ is also above $\tau_m$, and $B_m^L$ is below $\tau_m$. Since $A_m = \mathrm{conv}(A_m^L \cup A_m^U)$ and $B_m = \mathrm{conv}(B_m^L \cup B_m^U)$, it follows that $A_m$ is above $\tau_m$, and $B_m$ is below $\tau_m$. Hence $A_m \cap B_m = \emptyset$.

*Transversal Properties*

We prove that each $A_m$ and each $B_m$ meets each $l_i$.

- For $A_m$:
  For $i = m - 1$, $P_{m,1} = l_{m-1} \cap a_m$; for $i = m$, $P_{m,2} = l_m \cap a_m$.
  For each $i \neq m - 1, m$: since the curve $\Sigma \cap \pi_m$ is downward convex, the point $\lambda_i \cap a_m$, which belongs to $A_m^L$, lies below $\Sigma$ and hence below $l_i$. Since the projection of $A_m^U$ on the $y$-axis is equal to the projection of $A_m^L$, there is a point of $A_m^U$ that belongs to $\lambda_i$. Since $A_m^U$ is high above $A_m^L$, this point is above $l_i$, and it follows that $A_m$ meets $l_i$.
- For $B_m$:
  For $i = m - 1$, $Q_{m,2} = l_{m-1} \cap b_m$; for $i = m$, $Q_{m,1} = l_m \cap b_m$.
  For each $i \neq m - 1, m$: since the curve $\Sigma \cap \sigma_m$ is upward convex, the point $\lambda_i \cap b_m$, which belongs to $B_m^U$, lies above $\Sigma$ and hence above $l_i$. Since the projection of $B_m^L$ on the $y$-axis is equal to the projection of $B_m^U$, there is a point of $B_m^L$ that belongs to $\lambda_i$. Since $B_m^U$ is high above $B_m^L$, this point is below $l_i$, and it follows that $B_m$ meets $l_i$.

*Geometric Permutations*

Let $T_m^A$ and $T_m^B$ be the open halfspaces bounded by $\tau_m$ that contain $A_m$ and $B_m$, respectively. Let $O_i = (0, i, 0) \in l_i$. For each $i$, order the points of $l_i$ by the values of their $x$-coordinates, i.e., $(x_1, i, x_1 i) \prec (x_2, i, x_2 i)$ if and only if $x_1 < x_2$. It follows that for each $m$, on each $l_i$, $O_i \prec A_m$ and $O_i \prec B_m$. Note also that for $m < m'$, on each $l_i$ we have $A_m \prec A_{m'}$, $A_m \prec B_{m'}$, $B_m \prec A_{m'}$, and $B_m \prec B_{m'}$, since the planes $w_m$ separate such pairs of sets. However, the order of $A_m$ and $B_m$ on $l_i$ depends on $i$:

- For $i = m - 1$: on $l_{m-1}$, $O_{m-1} \prec P_{m,1} \prec Q_{m,2}$, and thus $A_m \prec B_m$. Note that $O_{m-1} \in T_m^A$.
- For $i = m$: on $l_m$, $O_m \prec Q_{m,1} \prec P_{m,2}$, and thus $B_m \prec A_m$. Note that $O_m \in T_m^B$.
- For any $i$: we have observed that $O_{m-1} \in T_m^A$ and $O_m \in T_m^B$. It follows that $O_i \in T_m^A$ for $i \leq m - 1$, and $O_i \in T_m^B$ for $i \geq m$. Since both $A_m$ and $B_m$ meet each $l_i$ after $O_i$, we have $A_m \prec B_m$ on $l_i$ for $i \leq m - 1$ and $B_m \prec A_m$ on $l_i$ for $i \geq m$.

We obtain the following geometric permutations for $\mathcal{F}$:

$$
\begin{aligned}
l_0&\colon (A_1, B_1, A_2, B_2, A_3, B_3, \ldots, A_n, B_n),\\
l_1&\colon (B_1, A_1, A_2, B_2, A_3, B_3, \ldots, A_n, B_n),\\
l_2&\colon (B_1, A_1, B_2, A_2, A_3, B_3, \ldots, A_n, B_n),\\
l_3&\colon (B_1, A_1, B_2, A_2, B_3, A_3, \ldots, A_n, B_n),\\
&\cdots\\
l_n&\colon (B_1, A_1, B_2, A_2, B_3, A_3, \ldots, B_n, A_n).
\end{aligned}
$$

Thus $\mathcal{F}$ is a family of $2n$ disjoint translates of the convex set $X$ that has the $n + 1$ geometric permutations listed above (they are distinct if $n > 1$).

## 3. Concluding Remarks

**1.** The set $X = X(n)$ that has been constructed in Section 2, depends on $n$. Here we explain, leaving some details to the reader, how to construct a set that satisfies Theorem 1 for all values of $n \in \mathbb{N}$.

Recall the constuction of $X(n)$. It is easy to see that it is possible to modify the construction so that $\lambda_i$ is the plane $y = i\varepsilon$ for some positive constant $\varepsilon \leq 1$; $l_i = \lambda_i \cap \Sigma$; all the segments $a_m$ and $b_m$ are still parallel to the planes $y = x$ and $y = -x$, respectively, but have $x$- and $y$-lengths $n\varepsilon$ (then the $x$- and $y$-lengths of $X$ are $n^2\varepsilon$). Choosing $\varepsilon \leq 1/n^2$, it is possible to modify the construction so that the planes $x = 5(2m - 1)$ and $x = 5(2m + 1) + \varepsilon$ play the roles of $u_m$ and $u'_m$ (respectively) in the definition of the points $P_{m,j}$ and $Q_{m,j}$, and the planes $x = 10m$ play the role of $w_m$ in the proof of the disjointness of the translates (note that in this case the $x$- and $y$-lengths of $X$ are less than 1). Once this is done, it is possible to "squeeze" the construction (applying the transformation $(x, y, z) \mapsto (x, y, \delta z)$ for a constant $0 < \delta \leq 1$) so that the slopes of all $a_m$'s and $b_m$'s will be positive but less than a prescribed constant $\alpha$.

Using these observations, we construct a set $Y$ that satisfies Theorem 1 for each $n \in \mathbb{N}$.

For each $n \in \mathbb{N}$, construct *modified* $X^{\mathrm{L}}(n)$ and $X^{\mathrm{U}}(n)$, and *modified* $X(n) = \mathrm{conv}(X^{\mathrm{L}}(n) \cup X^{\mathrm{U}}(n))$, so that:

1. The $x$- and $y$-lengths of $X^{\mathrm{L}}(n)$ and $X^{\mathrm{U}}(n)$ are $1/2^n$.
2. The slopes of all $a_m(n)$'s and $b_m(n)$'s are positive but less than the slopes of all $a_m(n - 1)$'s and $b_m(n - 1)$'s, and less than $1/2^n$.

Append $X^{\mathrm{L}}(n)$'s ($X^{\mathrm{U}}(n)$'s) in order to obtain a polygonal line $Y^{\mathrm{L}}$ ($Y^{\mathrm{U}}$) in the way similar to the joining of the segments $a_m$ ($b_m$) in the construction of $X^{\mathrm{L}}$ ($X^{\mathrm{U}}$). That is, let $Y^{\mathrm{L}} = \bigcup_{n=1}^{\infty} \tilde{X}^{\mathrm{L}}(n)$ and $Y^{\mathrm{U}} = \bigcup_{n=1}^{\infty} \tilde{X}^{\mathrm{U}}(n)$ where $\tilde{X}^{\mathrm{L}}(n)$ ($\tilde{X}^{\mathrm{U}}(n)$) is a translate of *modified* $X^{\mathrm{L}}(n)$ ($X^{\mathrm{U}}(n)$), and the lowest point of $\tilde{X}^{\mathrm{L}}(n)$ coincides with the highest point of $\tilde{X}^{\mathrm{L}}(n+1)$ (the highest point of $\tilde{X}^{\mathrm{U}}(n)$ coincides with the lowest point of $\tilde{X}^{\mathrm{U}}(n+1)$). Because of conditions 1 and 2 above, the sequences of the lowest points of $\tilde{X}^{\mathrm{L}}(n)$ and of the highest points of $\tilde{X}^{\mathrm{U}}(n)$ converge, and the polygonal lines $Y^{\mathrm{L}}$ and $Y^{\mathrm{U}}$ have $x$- and $y$-lengths 1 ($= \sum_{n \in \mathbb{N}} 1/2^n$), and finite $z$-lengths. It remains to put $Y^{\mathrm{U}}$ high above $Y^{\mathrm{L}}$ (so that they have the same $x$- and $y$-projections, say, [0, 1]), and to define $Y = \mathrm{conv}(Y^{\mathrm{L}} \cup Y^{\mathrm{U}})$. The set $Y$ looks similar to the set from Fig. 2, but the polygonal lines $Y^{\mathrm{L}}$ and $Y^{\mathrm{U}}$ consist of an infinite number of segments.

For each natural $n$, it is possible to place $2n$ translates of $Y$ so that the segments of $Y^{\mathrm{L}}$ ($Y^{\mathrm{U}}$) that correspond to $a_m(n)$'s ($b_m(n)$'s) coincide with these segments in $2n$ translates of the *modified* $X(n)$. These translates are disjoint, and they have the $n + 1$ geometric permutations mentioned above. This proves our statement.

**2.** Theorem 1 shows that for $d \geq 3$, for families of disjoint translates of a convex set, $g_d(n) = \Omega(n)$. We think that it is possible, using similar constructions, to improve this bound for $d > 3$.

**3.** To summarize, for families of disjoint translates of a convex set, $g_3(n) = O(n^2)$ and $\Omega(n)$. The problem of narrowing this gap remains open.

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