Discrete Comput Geom 34:1–10 (2005)
DOI: 10.1007/s00454-004-1110-x

Discrete & Computational
Geometry

Forbidden Families of Geometric Permutations in $\mathbb{R}^d$*

Andrei Asinowski and Meir Katchalski

Faculty of Mathematics, Technion – Israel Institute of Technology,
Haifa 32000, Israel
{andrei,meirk}@techunix.technion.ac.il

Abstract. A geometric permutation induced by a transversal line of a finite family $\mathcal{F}$ of disjoint convex sets in $\mathbb{R}^d$ is the order in which the transversal meets the members of the family. We prove that for each natural k, each family of k permutations is realizable (as a family of geometric permutations of some $\mathcal{F}$) in $\mathbb{R}^d$ for $d \geq 2k - 1$, but there is a family of k permutations which is non-realizable in $\mathbb{R}^d$ for $d \leq 2k - 2$.

1. Introduction

Let $\mathcal{F} = \{A_1, A_2, \dots, A_n\}$ be a finite family of n pairwise disjoint convex sets in $\mathbb{R}^d$. A line l is a *transversal* of $\mathcal{F}$ if it intersects all the members of $\mathcal{F}$. Each non-directed transversal intersects the members of $\mathcal{F}$ in an order that can be described by a pair of permutations of $\{1, 2, \dots, n\}$ which are reverses of each other. Such a pair is called a *geometric permutation*. Figure 1 shows an example of a planar family of four sets that admits six geometric permutations.

Earlier results on geometric permutations deal mostly with a maximal number of geometric permutations that a family of n disjoint convex sets can have. For example, in $\mathbb{R}^2$ this number is $2n - 2$ (for $n \geq 4$) [5], [10], and in $\mathbb{R}^d$ it is known to be $O(n^{2d-2})$ [15] and $\Omega(n^{d-1})$ [9]. For n disjoint balls in $\mathbb{R}^d$ the maximal number of geometric permutations is $\Theta(n^{d-1})$ [13]; and if the balls are congruent then this maximal number is bounded by a constant [11] (improved in [3] to 2 for $n \geq 9$). With the restriction that the members of the family are translates of a convex set in $\mathbb{R}^2$, the maximal number of geometric permutations is 3 [8], [9], and a complete characterization of possible families of geometric permutations in this case is known [1]. In contrast, the maximal number of

* These results are part of Andrei Asinowski's Ph.D. Thesis written under the supervision of Meir Katchalski.

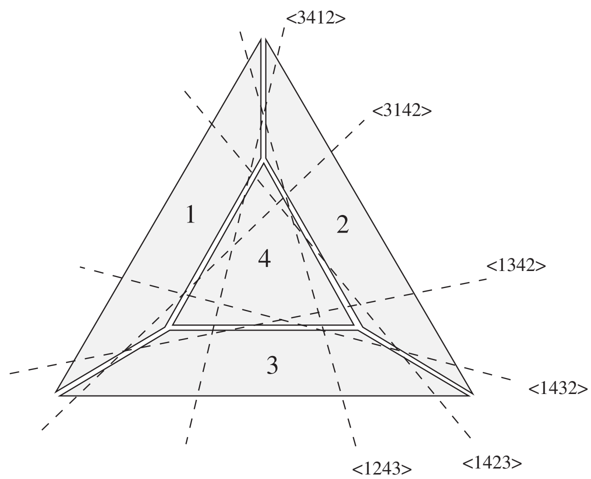

Fig. 1. A family of four sets in $\mathbb{R}^2$ with six geometric permutations.

geometric permutations for all families of n disjoint translates of a convex set in $\mathbb{R}^3$ is $\Omega(n)$ [2].

One of the motivations for studying geometric permutations is Helly-type problems on the existence of common transversals for families of disjoint translates of a convex set: see, for example, Tverberg's solution of Grünbaum's conjecture about transversals of disjoint translates in $\mathbb{R}^2$ [14], and a Helly-type result for disjoint unit balls in $\mathbb{R}^3$ [7]. For other related results see [4], [6] and [16].

This paper deals with the following aspect of geometric permutations. Let $\mathcal{P} = \{p_1, p_2, \ldots, p_k\}$ be some family of permutations on n sets (throughout this paper we assume that all of the permutations are distinct, and, furthermore, no two of them are reverses of each other). It can be asked whether there is a family $\mathcal{F}$ that admits all of them as its geometric permutations. The answer may depend on d (the dimension of the space). If there is such a family in $\mathbb{R}^d$, we say that $\mathcal{P}$ is *realizable* in $\mathbb{R}^d$, otherwise we say that it is *non-realizable* (or *forbidden*) in $\mathbb{R}^d$.

For example, the pair of permutations $\{\langle 1234\rangle, \langle 2143\rangle\}$ is forbidden in $\mathbb{R}^2$ (see Example 1 in Section 3.2). On the other hand, for each natural n, any pair $\{p_1, p_2\}$ of permutations on n sets is realizable in $\mathbb{R}^3$ (this is a special case of Theorem 1).

Our results deal with realizable and forbidden families of permutations in $\mathbb{R}^d$:

Theorem 1. *For each natural k, each family of k permutations is realizable in $\mathbb{R}^{2k-1}$.*

Theorem 2. *For each natural k, there is a family of k permutations which is non-realizable in $\mathbb{R}^{2k-2}$.*

It is clear that if a family of permutations is realizable in $\mathbb{R}^d$, then it is realizable in each $\mathbb{R}^{d'}$ where $d' \geq d$. Thus it follows from our theorems that each family of k permutations is realizable in $\mathbb{R}^d$ for each $d \geq 2k - 1$, but there is a family of k permutations which is non-realizable in $\mathbb{R}^d$ for each $d \leq 2k - 2$.

2. Proof of Theorem 1

Let $\{p_1, p_2, \ldots, p_k\}$ be a family of permutations on n sets. Take k lines $l_1, l_2, \ldots, l_k$ in general position in $\mathbb{R}^{2k-1}$ (by *general position* we mean that their affine hull is $\mathbb{R}^{2k-1}$). For each $j \in \{1, 2, \ldots, k\}$, put n points $P_{j1}, P_{j2}, \ldots, P_{jn}$ on l_j, according to the permutation p_j. For each $i \in \{1, 2, \ldots, n\}$, define $S_i = \operatorname{conv}(\{P_{1i}, P_{2i}, \ldots, P_{ki}\})$. Each S_i is of dimension at most $k-1$.

We prove that the sets $S_1, S_2, \ldots, S_n$ are pairwise disjoint: suppose $S_x \cap S_y \neq \emptyset$ with $x \neq y$. Let τ be the minimal flat containing S_x and S_y. Since S_x and S_y intersect, the dimension of τ is at most $2k-2$. Then for each $j \in \{1, 2, \ldots, k\}$, the points P_{jx} and P_{jy}, and therefore the line l_j, belong to τ. Thus all the lines $l_1, l_2, \ldots, l_k$ lie in τ, contradicting their being in general position.

Thus $\{S_1, S_2, \ldots, S_n\}$ is a family of pairwise disjoint convex sets in $\mathbb{R}^{2k-1}$, and it has $p_1, p_2, \ldots, p_k$ (induced by $l_1, l_2, \ldots, l_k$, respectively) as geometric permutations.

3. Proof of Theorem 2

3.1. *Notations*

Let $\{A_1, A_2, \ldots, A_n\}$ be a family of disjoint convex sets in $\mathbb{R}^d$, and let l be a transversal of this family.

We write $l : (A_{x_1} \prec A_{x_2} \prec \cdots \prec A_{x_n})$, or just $l : (x_1 \prec x_2 \prec \cdots \prec x_n)$, if l is directed, and it meets these sets in this order: A_{x_1} before A_{x_2} before $\cdots$ before A_{x_n}.

We write $l : (A_{x_1} * A_{x_2} * \cdots * A_{x_n})$, or just $l : (x_1 * x_2 * \cdots * x_n)$, either if l is directed and $l : (A_{x_1} \prec A_{x_2} \prec \cdots \prec A_{x_n})$ or $l : (A_{x_n} \prec A_{x_{n-1}} \prec \cdots \prec A_{x_1})$, or if l is undirected but this happens when we choose a direction on it. Of course, $l : (A_{x_1} * A_{x_2} * \cdots * A_{x_n})$ is the same as $l : (A_{x_n} * \cdots * A_{x_2} * A_{x_1})$, and $l : (A_x * A_y * A_z)$ means simply that on l, $l \cap A_y$ is between $l \cap A_x$ and $l \cap A_z$.

For two disjoint convex sets A_x and A_y in $\mathbb{R}^d$, we denote by $H^{(xy)}$ a hyperplane ($(d-1)$-flat) that strictly separates A_x from A_y; by $H_x^{(xy)}$ the open halfspace bounded by $H^{(xy)}$ that contains A_x, and by $H_y^{(xy)}$ the open halfspace bounded by $H^{(xy)}$ that contains A_y.

3.2. *Examples of Forbidden Families of Permutations in* $\mathbb{R}^2$, $\mathbb{R}^3$ *and* $\mathbb{R}^4$

In this section we provide three examples of forbidden families of permutations. Examples 1 and 3 illustrate the general idea used in the proof of Theorem 2; Example 2 illustrates a slightly different method, applied to $\mathbb{R}^3$.

Example 1. The pair of permutations

$$\{p_1 = \langle 1\,2 \;\; 3\,4 \rangle,$$
$$p_2 = \langle 2\,1 \;\; 4\,3 \rangle\}$$

is forbidden in $\mathbb{R}^2$.

Remark. This example already appeared in early papers on geometric permutations [8], [10].

Proof. Suppose that this pair is realizable in $\mathbb{R}^2$ with $\mathcal{F} = \{A_1, A_2, A_3, A_4\}$ and transversal lines l_1, l_2 inducing permutations p_1, p_2 respectively. Since parallel transversals clearly induce the same permutation, l_1 and l_2 intersect in a point O. Note that for each possible position of O on the transversals relative to the members of $\mathcal{F}$, there exist A_x and A_y in $\mathcal{F}$ so that $l_1 : (O * x * y)$ and $l_2 : (O * y * x)$, and this contradicts the disjointness of the sets (the bar | denotes the position of O on the transversals; * is dropped):

- l_1: (|1|2|34), l_2: (|2|1|43): take $x = 3$, $y = 4$.
- l_1: (12|3|4|), l_2: (21|4|3|): take $x = 2$, $y = 1$.
- l_1: (|1|234), l_2: (214|3|): take $x = 2$, $y = 4$.
- l_1: (123|4|), l_2: (|2|143): take $x = 3$, $y = 1$. □

Example 2. The triple of permutations

$$\{p_1 = \langle 1\,2\,3\;\;4\,5\,6\rangle,\quad p_2 = \langle 3\,2\,1\;\;6\,5\,4\rangle,\quad p_3 = \langle 2\,4\,6\;\;1\,3\,5\rangle\}$$

is forbidden in $\mathbb{R}^3$.

Proof. Suppose that this triple is realizable in $\mathbb{R}^3$ with $\mathcal{F} = \{A_1, A_2, \ldots, A_6\}$ and transversal lines l_1, l_2, l_3 inducing permutations p_1, p_2, p_3, respectively. Using standard arguments (we mention them later in Section 3.4), it is possible to assume that no two lines among l_1, l_2 and l_3 intersect, and that there is no plane parallel to all of them. Then there exists a line m which is parallel to l_3 and intersects both l_1 and l_2—in points O_1 and O_2, respectively (such a line m exists since the plane τ that contains l_1 and is parallel to l_3, intersects l_2 in a point; denote this point by O_2; m is the line parallel to l_3 that contains O_2). Choose a direction for m, and the same direction for l_3, so that m: $(O_1 \prec O_2)$.

Suppose that there exist $A_x, A_y \in \mathcal{F}$ so that l_1: $(O_1 * A_x * A_y)$ and l_2: $(O_2 * A_y * A_x)$. This implies $O_1 \in H_x^{(xy)}$ and $O_2 \in H_y^{(xy)}$, hence m : $(H_x^{(xy)} \prec H_y^{(xy)})$, and thus also l_3: $(H_x^{(xy)} \prec H_y^{(xy)})$. However, l_3 is a transversal of $\mathcal{F}$, hence l_3: $(A_x \prec A_y)$.

Note that for each possible position of O_1 and O_2 on the transversals relative to the members of $\mathcal{F}$, we can choose two pairs of members of $\mathcal{F}$ so that the previous observation contradicts the actual permutation p_3 (the bar | denotes the position of O_j on the transversal l_j):

- l_1: (|1|23456) and l_2: (|4|5|6123).
  l_1: (|26) and l_2: (|62) imply l_3: $(2 \prec 6)$; l_1: (|36) and l_2: (|63) imply l_3: $(3 \prec 6)$.
  A contradiction to l_3: $(2 * 6 * 3)$.
- l_1: (12|3456) and l_2: (|4|5|6123).
  l_1: (|21) and l_2: (|12) imply l_3: $(2 \prec 1)$; l_1: (|36) and l_2: (|63) imply l_3: $(3 \prec 6)$.
  A contradiction to l_3: $(2 * 6 * 1 * 3)$.

- l_1: $(|1|2|3|456)$ and l_2: $(456|1|2|3|)$.
  l_1: $(|46)$ and l_2: $(|64)$ imply l_3: $(4 \prec 6)$; l_1: $(|56)$ and l_2: $(|65)$ imply l_3: $(5 \prec 6)$.
  A contradiction to l_3: $(4 * 6 * 5)$.
- l_1: $(123|4|5|6|)$ and l_2: $(|4|5|6|123)$.
  l_1: $(|21)$ and l_2: $(|12)$ imply l_3: $(2 \prec 1)$; l_1: $(|31)$ and l_2: $(|13)$ imply l_3: $(3 \prec 1)$.
  A contradiction to l_3: $(2 * 1 * 3)$.
- l_1: $(1234|56)$ and l_2: $(4561|2|3|)$.
  l_1: $(|41)$ and l_2: $(|14)$ imply l_3: $(4 \prec 1)$; l_1: $(|56)$ and l_2: $(|65)$ imply l_3: $(5 \prec 6)$.
  A contradiction to l_3: $(4 * 6 * 1 * 5)$.
- l_1: $(12345|6|)$ and l_2: $(4561|2|3|)$.
  l_1: $(|41)$ and l_2: $(|14)$ imply l_3: $(4 \prec 1)$; l_1: $(|51)$ and l_2: $(|15)$ imply l_3: $(5 \prec 1)$.
  A contradiction to l_3: $(4 * 1 * 5)$. □

Remark. In this forbidden triple, replacing p_3 by one of the seven permutations obtained from it by rearranging some of the pairs $\{2, 4\}, \{1, 6\}, \{3, 5\}$ (for example $\langle 421635 \rangle$) also gives a forbidden triple. This can be proved using the same method.

Example 3. The triple of permutations

$$\{p_1 = \langle 123\ \ 456\ \ 789 \rangle,\ p_2 = \langle 312\ \ 564\ \ 978 \rangle,\ p_3 = \langle 231\ \ 645\ \ 897 \rangle\}$$

is forbidden in $\mathbb{R}^4$.

Proof. Suppose that this triple is realizable in $\mathbb{R}^4$ with $\mathcal{F} = \{A_1, A_2, \ldots, A_9\}$ and transversal lines l_1, l_2, l_3 inducing permutations p_1, p_2, p_3, respectively. It is possible to assume that there is a line s that intersects each of l_1, l_2, l_3 (this follows from Lemma 4 to be proved later). For $i \in \{1, 2, 3\}$, choose a point $O_i \in l_i \cap s$. It is also possible to assume that $O_3 \in \text{conv}(\{O_1, O_2\})$: it is easy to see, using the symmetry of the permutations, that the two other possibilities can be obtained from this by relabeling the sets. Note that for each possible position of O_1, O_2, O_3 on the transversals relative to the members of $\mathcal{F}$, there exist A_x and A_y in $\mathcal{F}$ so that l_1: $(O_1 * x * y)$ and l_2: $(O_2 * x * y)$, but l_3: $(O_3 * y * x)$:

- l_1: $(|1|2|3|4|5|6|789)$, l_2: $(|3|1|2|5|6|4|9|78)$, l_3: $(|2|3|1|6|4|5|897)$: take $x = 7$, $y = 8$.
- l_1: $(12|3|4|5|6|7|8|9|)$, l_2: $(312|5|6|4|9|7|8|)$, l_3: $(231|6|4|5|8|9|7|)$: take $x = 2$, $y = 1$.
- l_1: $(|1|2|3|4|5|6789)$, l_2: $(|3|1|2|5|64978)$, l_3: $(2316458|9|7|)$: take $x = 6$, $y = 8$.
- l_1: $(12345|6|7|8|9|)$, l_2: $(3125|6|4|9|7|8|)$, l_3: $(|2|3|1645897)$: take $x = 5$, $y = 1$.
- l_1: $(|1|2|3|456789)$, l_2: $(|3|1|2|5|6|4978)$, l_3: $(2316458|9|7|)$: take $x = 4$, $y = 8$.
- l_1: $(1234|5|6|7|8|9|)$, l_2: $(312564|9|7|8|)$, l_3: $(|2|3|1645897)$: take $x = 4$, $y = 1$.
- l_1: $(|1|2|3|456789)$, l_2: $(312564|9|7|8|)$, l_3: $(231645|8|9|7|)$: take $x = 4$, $y = 5$.
- l_1: $(|1|2|3|456789)$, l_2: $(312564|9|7|8|)$, l_3: $(|2|3|1|645897)$: take $x = 4$, $y = 6$.
- l_1: $(|1|2|3|4|5|6|789)$, l_2: $(31256497|8|)$, l_3: $(|2|3|1|6|4|5|8|97)$: take $x = 7$, $y = 9$.

- l_1 : $(|1|23456789)$, l_2 : $(312|5|6|4|9|7|8|)$, l_3 : $(23|1|6|4|5|8|9|7|)$: take $x = 2$, $y = 3$.
- l_1 : $(123456|7|8|9|)$, l_2 : $(|3|1|2|5|64978)$, l_3 : $(23164|5|8|9|7|)$: take $x = 6$, $y = 4$.
- l_1 : $(12345|6|7|8|9|)$, l_2 : $(|3|1|2|564978)$, l_3 : $(|2|3|1|6|45897)$: take $x = 5$, $y = 4$.

This contradicts the disjointness of the members of $\mathcal{F}$: $O_1, O_2 \in H_x^{(xy)}$, and $O_3 \in H_y^{(xy)}$. However, also $O_3 \in H_x^{(xy)}$ since $O_3 \in \text{conv}(\{O_1, O_2\})$. Thus, O_3 belongs both to $H_x^{(xy)}$ and to $H_y^{(xy)}$, a contradiction. □

3.3. *A $(k-2)$-Flat that Intersects All the Transversals*

We prove two lemmas that imply the existence of a $(k-2)$-flat that intersects a transversal line for each of the k geometric permutations.

Definition. A family $\mathcal{L}$ of s-flats in $\mathbb{R}^d$ is an *open family* if for any $L \in \mathcal{L}$, there are $s+1$ open balls $B_1, B_2, \ldots, B_{s+1}$ so that L intersects each of them, and any s-flat that intersects all these balls belongs to $\mathcal{L}$.

Remark. This definition implies that for each member of an open family, a small perturbation results in another member of the family. For $s = 0$, an open family (of points) is just an open set in the usual sense.

Lemma 3. *Let $k \in \mathbb{N}$. Let $\mathcal{L}_1, \mathcal{L}_2, \ldots, \mathcal{L}_k$ be open families of lines in $\mathbb{R}^{2k-1}$, and let P be a point in $\mathbb{R}^{2k-1}$. Then there exist lines $l_i \in \mathcal{L}_i$ and a $(k-1)$-flat S so that $P \in S$ and for each $i \in \{1, 2, \ldots, k\}$, l_i intersects S.*

Lemma 4. *Let $k \in \mathbb{N}$, $k > 1$. Let $\mathcal{L}_1, \mathcal{L}_2, \ldots, \mathcal{L}_k$ be open families of lines in $\mathbb{R}^{2k-2}$. Then there exist lines $l_i \in \mathcal{L}_i$, and a $(k-2)$-flat S so that for each $i \in \{1, 2, \ldots, k\}$, l_i intersects S.*

Proof of Lemma 3. For $k = 1$ the statement is obvious.

Suppose the lemma holds for $k-1$.

For $1 \le i \le k$, it is possible to choose $l_i \in \mathcal{L}_i$ so that: $A = \text{aff}\{l_1, l_2, \ldots, l_{k-1}\}$ is a $(2k-3)$-flat; $B = \text{aff}\{l_k, P\}$ is a 2-flat; and $A \cap B$ is a point Q different from P, so that the line PQ intersects l_k.

For $1 \le i \le k-1$, let $\mathcal{L}'_i = \mathcal{L}_i \cap A$. Each $\mathcal{L}'_i$ is an open family of lines relative to A. By the induction assumption applied to $\mathcal{L}'_1, \mathcal{L}'_2, \ldots, \mathcal{L}'_{k-1}$ and Q in the $(2k-3)$-flat A, there exist $l'_i \in \mathcal{L}'_i \subseteq \mathcal{L}_i$ $(1 \le i \le k-1)$, and a $(k-2)$-flat T that contains Q and intersects each l'_i.

Let $S = \text{aff}(T, P)$. Clearly, $P \in S$. The flat S intersects each l'_i since $T \subseteq S$, and it intersects l_k since the line PQ lies in S. Since Q is the only point in $A \cap B$, S is a $(k-1)$-flat.

Thus, S and the lines $l'_1, l'_2, \ldots, l'_{k-1}, l_k$ satisfy the conclusion of the lemma. □

Proof of Lemma 4. For $1 \le i \le k$, it is possible to choose $l_i \in \mathcal{L}_i$ so that $C = \text{aff}\{l_1, l_2, \ldots, l_{k-1}\}$ is a $(2k-3)$-flat, and $l_k \cap C$ is a point P.

For $1 \le i \le k-1$, let $\mathcal{L}'_i = \mathcal{L}_i \cap C$. Each $\mathcal{L}'_i$ is an open family of lines in C. By Lemma 3 applied to $\mathcal{L}'_1, \mathcal{L}'_2, \ldots, \mathcal{L}'_{k-1}$ and P in the $(2k-3)$-flat C, there exist $l'_i \in \mathcal{L}'_i$ $(1 \le i \le k-1)$, and a $(k-2)$-flat S that intersects each l'_i and contains P (that belongs to l_k).

Thus, S and the lines $l'_1, l'_2, \ldots, l'_{k-1}, l_k$ satisfy the conclusion of the lemma. □

3.4. *Idea of the Proof of Theorem* 2

Let $\mathcal{F} = \{A_1, A_2, \ldots, A_n\}$ be a family of disjoint convex sets in $\mathbb{R}^{2k-2}$ that has permutations $p_1, p_2, \ldots, p_k$. After a slight expansion of the members of $\mathcal{F}$, for each geometric permutation there is a transversal line that intersects all the members of $\mathcal{F}$ in interior points. Then, for each i, the family of all the transversal lines that induce p_i contains an open family of lines. Hence, by Lemma 4, it is possible to choose transversals $l_1, l_2, \ldots, l_k$ (inducing $p_1, p_2, \ldots, p_k$, respectively) so that there is a $(k-2)$-dimensional flat S that intersects each of these transversals.

For each $j \in \{1, 2, \ldots, k\}$, let $O_i \in l_i \cap S$. These are k points in a $(k-2)$-flat, thus **by Radon's theorem** [12] they can be partitioned into two non-empty sets whose convex hulls intersect: $\{1, 2, \ldots, k\} = K \cup L$, $K \cap L = \emptyset$, $K, L \neq \emptyset$, and $\text{conv}(\{O_j : j \in K\}) \cap \text{conv}(\{O_j : j \in L\}) \neq \emptyset$.

Suppose that there are two sets A_x and A_y in $\mathcal{F}$ so that for each $j \in K$, $l_j : (O_j * A_x * A_y)$, and for each $j \in L$, $l_j : (O_j * A_y * A_x)$. Then for each $j \in K$, $O_j \in H_x^{(xy)}$, and for each $j \in L$, $O_j \in H_y^{(xy)}$. Since the open halfspaces $H_x^{(xy)}$ and $H_y^{(xy)}$ are convex sets, it follows that $\text{conv}(\{O_j : j \in K\}) \subseteq H_x^{(xy)}$ and $\text{conv}(\{O_j : j \in L\}) \subseteq H_y^{(xy)}$. However, then each point common to $\text{conv}(\{O_j : j \in K\})$ and $\text{conv}(\{O_j : j \in L\})$ belongs to both $H_x^{(xy)}$ and $H_y^{(xy)}$, which is clearly impossible.

Thus, we have proved the following:

Observation 5. *If a family of permutations* $\{p_1, p_2, \ldots, p_k\}$ *for* $\mathcal{F}$ *is such that for each partition of* $\{1, 2, \ldots, k\}$ *into two disjoint non-empty sets* K *and* L, *and for each possible position of the* O_j*'s in the* p_j*'s relative to the members of* $\mathcal{F}$, *there are two sets* A_x *and* A_y *in* $\mathcal{F}$ (*that depend on the partition and on the position of the* O_j*'s*) *so that for each* $j \in K$, $l_j : (O_j * A_x * A_y)$, *and for each* $j \in L$, $l_j : (O_j * A_y * A_x)$—*such a family of permutations is forbidden in* $\mathbb{R}^{2k-2}$.

3.5. *Construction of a Forbidden Family of Permutations*

We construct a family of k permutations $\mathcal{P} = \{p_1, p_2, \ldots, p_k\}$ that has the property mentioned in Observation 5. The permutations involve $(k+1) \cdot (2^{k-1}+1)$ sets. In the first step we construct their subpermutations $\pi_1, \pi_2, \ldots, \pi_k$ which are permutations of $\{0, 1, \ldots, 2^{k-1}\}$.

Let $S_1, S_2, \ldots, S_{2^{k-1}}$ be the 2^{k-1} subsets of $\{1, 2, \ldots, k\}$ that contain 1 (numbered in some way). Define $\pi_1, \pi_2, \ldots, \pi_k$ to be permutations of $\{0, 1, \ldots, 2^{k-1}\}$ that satisfy:

In π_j : 0 is before $i \Leftrightarrow j \in S_i$.

Note that the permutations $\pi_1, \pi_2, \ldots, \pi_k$ are not defined uniquely.

After that, for each $j \in \{1, 2, \ldots, k\}$, construct a permutation p_j by duplication of π_j $k+1$ times as follows: for $\pi_j = (\alpha_0, \alpha_1, \ldots, \alpha_{2^{k-1}})$, define $p_j = ((\alpha_0, 1), \ldots, (\alpha_{2^{k-1}}, 1), (\alpha_0, 2), \ldots, (\alpha_{2^{k-1}}, 2), \ldots\ldots, (\alpha_0, k+1), \ldots, (\alpha_{2^{k-1}}, k+1))$. The permutations $p_1, p_2, \ldots, p_k$ are permutations of the members of the set $\{0, 1, \ldots, 2^{k-1}\} \times \{1, 2, \ldots, k+1\}$. For each $m \in \{1, 2, \ldots, k+1\}$, we call the subpermutation $((\alpha_0, m), \ldots, (\alpha_{2^{k-1}}, m))$ *the* m*th interval of* p_i.

Example of the Construction for $k = 3$. Let $S_1 = \{1, 2, 3\}$, $S_2 = \{1, 2\}$, $S_3 = \{1, 3\}$, $S_4 = \{1\}$. The permutations π_1, π_2, π_3 should be defined so that:

In π_1: $0 \prec 1, 0 \prec 2, 0 \prec 3, 0 \prec 4$.
In π_2: $0 \prec 1, 0 \prec 2, 3 \prec 0, 4 \prec 0$.
In π_3: $0 \prec 1, 2 \prec 0, 0 \prec 3, 4 \prec 0$.

For example, take $\pi_1 = (01234)$, $\pi_2 = (34012)$, $\pi_3 = (24013)$.
Then, for these choices of π_1, π_2 and π_3,

$$p_1 = (\underbrace{(0,1),(1,1),(2,1),(3,1),(4,1)}_{\text{1st interval}}, \underbrace{(0,2),(1,2),(2,2),(3,2),(4,2)}_{\text{2nd interval}}, \ldots,$$
$$\ldots, \underbrace{(0,4),(1,4),(2,4),(3,4),(4,4)}_{\text{4th interval}}),$$
$$p_2 = ((3,1),(4,1),(0,1),(1,1),(2,1),(3,2),(4,2),(0,2),(1,2),(2,2),\ldots,$$
$$\ldots,(3,4),(4,4),(0,4),(1,4),(2,4)),$$
$$p_3 = ((2,1),(4,1),(0,1),(1,1),(3,1),(2,2),(4,2),(0,2),(1,2),(3,2),\ldots,$$
$$\ldots,(2,4),(4,4),(0,4),(1,4),(3,4)).$$

The family of permutations $\{p_1, p_2, p_3\}$ is forbidden in $\mathbb{R}^4$.

3.6. *Why the Construction Gives a Forbidden Family*

We prove that the family of permutations $\{p_1, p_2, \ldots, p_k\}$ defined in Section 3.5 is forbidden in $\mathbb{R}^{2k-2}$. Suppose that there exists a family $\mathcal{F}$ of convex disjoint sets in $\mathbb{R}^{2k-2}$ that admits $p_1, p_2, \ldots, p_k$ as geometric permutations. Let $l_1, l_2, \ldots, l_k$ be transversals giving these geometric permutations, and let S be a $(k-2)$-flat that intersects each l_j, and let $O_j \in l_j \cap S$. Since each p_j consists of $(k+1)$ "intervals", there is $m \in \{1, 2, \ldots, k+1\}$ so that for each $j \in \{1, 2, \ldots, k\}$, O_j does not belong to the mth interval of p_j. After dropping the "second component" (m), the mth interval of p_j is identical to π_j, and O_j is either before or after all of its sets.

Let $K \cup L$ be a partition of $\{1, 2, \ldots, k\}$ into two disjoint non-empty sets. Define $M = \{j \in \{1, 2, \ldots, k\} : O_j \text{ is before } \pi_j\}$ and $N = \{j \in \{1, 2, \ldots, k\} : O_j \text{ is after } \pi_j\}$. Define $K' = (K \cap M) \cup (L \cap N)$. Note that K' is a subset of $\{1, 2, \ldots, k\}$, and it is possible to assume that $1 \in K'$ (otherwise we interchange K and L). Hence $K' = S_a$ for some $a \in \{1, 2, \ldots, 2^{k-1}\}$.

Consider four cases:

- If $j \in K \cap M$, then $j \in K' = S_a$, hence $l_j : (O_j \prec A_0 \prec A_a)$.
- If $j \in K \cap N$, then $j \notin K' = S_a$, hence $l_j : (A_a \prec A_0 \prec O_j)$.
- If $j \in L \cap M$, then $j \notin K' = S_a$, hence $l_j : (O_j \prec A_a \prec A_0)$.
- If $j \in L \cap N$, then $j \in K' = S_a$, hence $l_j : (A_0 \prec A_a \prec O_j)$.

In each case, for each $j \in K, l_j : (O_j * A_0 * A_a)$, and for each $j \in L, l_j : (O_j * A_a * A_0)$. Then Observation 5 implies that the family of permutations is forbidden.

4. Bounds on the Minimal Number of Sets in a Forbidden Family

By Theorems 1 and 2, for each natural d, each family of $\lceil d/2 \rceil$ permutations is realizable in $\mathbb{R}^d$, but there is a forbidden family of $\lceil d/2 \rceil + 1$ permutations. What is the minimal number of sets that must be involved in such a forbidden family? Denote this minimal number by φ_d. By our proof, $\varphi_d \leq (\lceil d/2 \rceil + 2) \cdot (2^{\lceil d/2 \rceil} + 1)$. This gives $\varphi_2 \leq 9$ and $\varphi_3, \varphi_4 \leq 20$, whereas, by the examples from Section 3.2, $\varphi_2 \leq 4$, $\varphi_3 \leq 6$ and $\varphi_4 \leq 9$.

On the other hand, for each natural d, there is a family of $d + 1$ disjoint convex sets in $\mathbb{R}^d$ that have all possible $(d + 1)!/2$ geometric permutations [9]. It follows that $\varphi_d \geq d + 2$. Thus, $\varphi_2 = 4$, $5 \leq \varphi_3 \leq 6$, $6 \leq \varphi_4 \leq 9$. It seems that the exponential upper bound for φ_d can be improved substantially. It is clear that $d \leq d'$ implies $\varphi_d \leq \varphi_{d'}$. However, we do not even know whether φ_d is strictly monotone as a function of d.

References

1. A. Asinowski, A. Holmsen, and M. Katchalski. The triples of geometric permutations for families of disjoint translations. *Discrete Math.* **241** (2001), 23–32.
2. A. Asinowski and M. Katchalski. The maximal number of geometric permutations for n disjoint translates of a convex set in $\mathbb{R}^3$ is $\Omega(n)$. Submitted to *Discrete Comput. Geom.*
3. O. Cheong, X. Goaoc, and H.-S. Na. Geometric permutations of disjoint unit spheres. Accepted for publication in *Comput. Geom. Theory Applic.*
4. J. Eckhoff. Helly, Radon and Carathéodory type theorems. In *Handbook of Convex Geometry*, North-Holland, Amsterdam, 1993, pp. 389–448.
5. H. Edelsbrunner and M. Sharir. The maximum number of ways to stab n convex nonintersecting sets in the plane is $2n - 2$. *Discrete Comput. Geom.* **5** (1990), 35–42.
6. J.E. Goodman, R. Pollack, and R. Wenger. Geometric transversal theory. In *New Trends in Discrete and Computational Geometry*, J. Pach, ed., vol. 10 of Algorithms and Combinatorics, Springer-Verlag, Heidelberg, 1993, pp. 163–198.
7. A. Holmsen, M. Katchalski, and T. Lewis. A Helly-type theorem for line transversals to disjoint unit balls. *Discrete Comput. Geom.* **29** (2003), 595–602.
8. M. Katchalski, T. Lewis, and A. Liu. Geometric permutations of disjoint translates of convex sets. *Discrete Math.* **65** (1987), 249–260.

9. M. Katchalski, T. Lewis, and A. Liu. The different ways of stabbing disjoint sets. *Discrete Comput. Geom.* **7** (1992), 197–206.
10. M. Katchalski, T. Lewis, and J. Zaks. Geometric permutations for convex sets. *Discrete Math.* **54** (1985), 271–284.
11. M. Katchalski, S. Suri, and Y. Zhou. A constant bound for geometric permutations of disjoint unit balls. *Discrete Comput. Geom.* **29** (2003), 161–173.
12. J. Radon. Mengen konvexer Körper, die einen gemeinsamen Punkt enthalten. *Math. Ann.* **83** (1921), 113–115.
13. S. Smorodinsky, J.S.B. Mitchell, and M. Sharir. Sharp bounds on geometric permutations of pairwise disjoint balls in $\mathbb{R}^d$. *Discrete Comput. Geom.* **23** (2000), 247–259.
14. H. Tverberg. Proof of Grünbaum's conjecture on common transversals for translates. *Discrete Comput. Geom.* **4** (1989), 191–203.
15. R. Wenger. Upper bounds on geometric permutations for convex sets. *Discrete Comput. Geom.* **5** (1990), 27–33.
16. R. Wenger. Helly-type theorems and geometric transversals. In *Handbook of Discrete and Computational Geometry*, CRC Press, Boca Raton, FL, 1997, pp. 63–82.

Received November 11, 2003, *and in revised form March* 3, 2004. *Online publication July* 26, 2004.