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Note on Integral Distances*

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Abstract. A planar point set S is called an *integral set* if all the distances between the elements of S are integers. We prove that any integral set contains many collinear points or the minimum distance should be relatively large if $|S|$ is large.

1. Introduction

A point set S is called an *integral set* if all the distances between the elements of S are integers. There are several results and conjectures about integral sets. There is an overview of the topic by Harborth [4] and further interesting results can be found in [5], [9], and in [10]. One of the oldest unsolved problems is due to Ulam: is there an everywhere dense set in the plane such that all distances between the points are rational? Erdős conjectured that the answer is negative [3]. On the other hand, it is not known whether there exists a planar integral set of seven points so that no three of them are collinear and no four are on a circle.

There are only few examples known for integral point sets. Using elliptic curves, Huff [8] and Peeples [11] found integral point sets for every $k > 4$ such that exactly $k - 4$ points are on one line. It is not known whether one can find for every pair n and m of natural numbers a planar integral set of $m + n$ points such that a line contains exactly m points of them.

The present note deals with minimum and maximum distances in integral sets.

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2. Bounds of Integral Sets

The first result related to integral sets of points dates from 1945. It is the following theorem of Anning and Erdős [1]:

Theorem. *If H is an infinite integral set in the plane, then the points of H lie on one line.*

A few months later Erdős gave a simpler proof in [2] (in the proof PQ stands for the distance between two points P and Q):

"If A, B, C are three points not in a straight line and $k = \max(AB, BC)$, then there are at most $4(k+1)2$ points P such that $PA-PB$ and $PB-PC$ are integral. For $|PA-PB|$ is at most AB and therefore assumes one of the values $0, 1, \dots, k$, that is, P lies on one of $k+1$ hyperbolas. Similarly P lies on one of the $k+1$ hyperbolas determined by B and C. These (distinct) hyperbolas intersect in at most $4(k+1)2$ points."

When AB is an integer then two hyperbolas $|PA-PB| = 0$ and $|PA-PB| = AB$ can be handled as one hyperbola. This compound object intersects with any other hyperbola in at most four points. This argument gives an upper bound on the size of a planar integral set H in terms of the distances between two point-pairs A, B and C, D of H if these pairs of points are in *general position*. By this we mean that the lines through A, B and C, D and the two bisector lines are distinct:

$$|H| \leq 4(AB)(CD). \tag{1}$$

With a modification of the argument, we show that the distance between two points A and B alone yields an upper bound on the number of points outside of the line through A and B with integral distance to A and B in an integral set H.

Theorem 1. *If A and B are two points in a planar integral set H at a distance $d > 1$ from each other, and the size of H is at least d^3, then, with the possible exception of at most $2d-1$ points, all points of H are situated on the line through A and B.*

To prove Theorem 1 we need an observation.

Observation 1. *If a triangle T has integer side-lengths $a \leq b \leq c$, then the minimal height m of it is at least $(a - \frac{1}{4})^{1/2}$.*

Proof. As the side-lengths are integers, the triangle inequality in this case is

$$a + b \geq c + 1.$$

The minimal height belongs to the side with length c, and we have, by an easy consequence of Heron's formula,

$$m^2 = a^2 - \left(\frac{c^2 + a^2 - b^2}{2c}\right)^2.$$

For fixed a and b larger c gives smaller height, so we can suppose that $c = a + b - 1$. Then

$$\frac{c^2 + a^2 - b^2}{c} = c + \frac{a+b}{c}(a-b) \geq c + a - b = 2a - 1$$

and equality is possible only when $a = b$. In this latter case we have $m = (a - \frac{1}{4})^{1/2}$. □

Proof of Theorem 1. Let H be a planar integral point set, $A, B \in H$, $AB = d$, and suppose that $|H| > d^3$. Let $C, D \in H$ be a pair of points in general position to A and B. Then, in view of (1), their distance CD is larger than $d^2/4$. As d and DC are integers, it follows that $CD \geq (d^2 + 3)/4$. By Observation 2, this means that each of the triangles ACD and BCD has minimal height larger than $d/2$. It follows that the line determined by C and D cannot separate the points A and B. Otherwise either ACD or BCD would be a triangle with minimal height smaller than $d/2$. Observe that if we have three points in a hyperbola, then two of them are always in general position to the foci and determine a line which separates the foci. Therefore, each of the hyperbolas $|PA - PB| = m$, $0 < m < d$, contains at most two points of H outside of the line through A and B.

The only remaining part of the proof is to bound the number of points of H on the bisector line l of A and B. The line through A and B divides l into two parts, we denote them by l_1 and l_2. Suppose that P_1 and P_2 are two points of H not on the line through A and B such that $P_1 \in l_1$ and $P_2 \in l_2$. As the pairs of points A, P_1 and A, P_2 are in general position to A and B, we have $AP_i \geq (d^2 + 3)/4$ $(i = 1, 2)$. It easily follows that then also $P_1P_2 \geq (d^2 + 3)/4$. Thus, AP_1P_2 is a triangle whose sides are at least $(d^2 + 3)/4$ and whose minimal height is $d/2$. This is impossible by Observation 1. Therefore we can suppose that all the points of $l \cap H$ are points of l_1, say.

Let $P_1, \dots, P_s$ be the points of H which are on l_1 but not on the line through A and B. Let t be the minimum distance between these points. Then

$$(s - 1)t < \frac{d^2 + 3}{4}. \tag{2}$$

Otherwise, A and the two furthest points of H on l_1 form a triangle with sides at least $(d^2 + 1)/4$ and minimal height less than $d/2$.

Now, H has at most $2(d - 1)$ points on the hyperbolas $|PA - PB| = m$, $0 < m < d$, and s points on l_1. The number of points on the line through A and B is at most $2t$, because this line intersects each of the t hyperbolas determined by the closest points on l_1 in at most two points. Thus

$$|H| \leq 2(d - 1) + s + 2t.$$

If $s > 1$, then (2) implies that $t \leq \lfloor (d^2 + 1)/4 \rfloor$ and $s \leq \lfloor (d^2 + 5)/4 \rfloor$, therefore

$$|H| \leq 2(d - 1) + \left\lfloor \frac{d^2 + 5}{4} \right\rfloor + 2 \left\lfloor \frac{d^2 + 1}{4} \right\rfloor.$$

The right side is less than d^3 for $d \geq 2$. Hence, there is at most one point on l_1 and the total number of H not on the line through A and B is at most $2d - 1$, as claimed. □

Erdős asked in [3] whether inequality (1) can be improved. When AB is much smaller than CD and the point-pairs are in general position, then the bound

$$|H| \leq \max\{AB^3, 2(AB + CD - 1)\}$$

is better than (1). This inequality follows immediately from Theorem 1. If $|H| > AB^3$, then at most $2(AB - 1)$ points are not on the line through A and B and the integer hyperbolas with foci C and D determine $2CD$ crossings with this line.

When the points of the planar integral set H are in general position, namely no three of them are collinear, then, as a consequence of Theorem 1, we have

Corollary 1. *If an integral set H contains no collinear triples, then the minimum distance in $|H|$ is at least $|H|^{1/3}$.*

The similar statement is not true in general. In what follows we give a construction for arbitrarily large integral sets, such that not all of the points are in a line and there are two points so that their distance is 2.

Construction. In the construction we represent a planar point by its x- and y-coordinates. Let us choose $x_0 = 0$, $y_0 = (m^2 - 1)^{1/2}$ such that m is a natural number and $m^2 - 1$ has many, at least k, divisors ($k \gg 1$). For example let us choose

$$m = 3^{2^{k-2}}.$$

Then $m^2 - 1$ is divisible by 2^i for any $0 \leq i \leq k$. This follows from the Euler–Fermat Theorem stating that $a^{\varphi(b)} \equiv 1 (\bmod\ b)$, if $(a, b) = 1$. (For details and notations see [7]). Thus $m^2 - 1$ can be written in the form $n2^k$ where n is a natural number. We define $x_i = n2^{i-1} - 2^{k-i-1}$, $y_i = 0$ and $x'_i = -n2^{i-1} + 2^{k-i-1}$, $y'_i = 0$, for $2 \leq i \leq k - 1$. Set $x_1 = 1$, $y_1 = 0$, and $x'_1 = -1$, $y'_1 = 0$. Now the point set $H = \{(x_0, y_0), (x_1, y_1), \ldots, (x_k, y_k), (x'_1, y'_1), \ldots, (x'_k, y'_k)\}$ is an integral set, because the distance between (x_0, y_0) and (x_i, y_i) is $n2^{i-1} + 2^{k-i-1}$ for $1 < i \leq k - 1$, which follows easily from the equation

$$(n2^{i-1} - 2^{k-i-1})^2 + n2^k = (n2^{i-1} + 2^{k-i-1})^2.$$

On the other hand, the distance between (x_1, y_1) and (x'_1, y'_1) is 2.

For the minimum diameter of integral sets in general position, Harborth et al. gave an upper bound [6]. They found integral point sets H of cardinality $|H| = n$ on a circle whose diameter satisfies the inequality

$$\text{Diam}(H) < n^{c_1 \log\log(n)}$$

for a constant $c_1 > 0$. When the points of an integral set are not necessarily in general position, the diameter can be as small as $n - 1$, if the points are on one line. In [6] Harborth et al. determined the minimum diameter of non-collinear integral point sets of cardinality n for small values of n. The minimum diameters for $n = 3, 4, 5, 6, 7, 8, 9$ are 1, 4, 7, 8, 17, 21, 29, respectively. Now we show that the diameter is always at least linear in n.

Theorem 2. *There is a positive constant c such that every n-element planar integral set has diameter at least cn.*

First we prove the following:

Observation 2. *In every planar n-element point set H $(n > 3)$ with diameter Δ and having at most $n/2$ collinear points, one can find two point-pairs A, B and C, D in general position such that both distances AB and CD are less than $12\Delta/n^{1/2}$.*

Proof. There is a circle with radius Δ and center o which contains all points of H. Let S be a circle with radius 2Δ and center o. A point $P \in H$ is called *single* if there is at most one other point of H with distance smaller than $12\Delta/n^{1/2}$ from P. Draw a circle of radius $6\Delta/n^{1/2}$ around each single point in H. The intersection of any three of these circles is empty and all of them are contained in S. Therefore, using a simple volume counting, the number of single points in H is less than $n/2$.

Remove the single points. In the remaining set H' any point P has two points $Q_1, Q_2 \in H'$ such that the distances PQ_1 and PQ_2 are less than $12\Delta/n^{1/2}$. H' is not collinear, so there are two point-pairs $A, B \in H'$ and $C, D \in H'$ in general position such that both AB and CD are less than $12\Delta/n^{1/2}$. □

Proof of Theorem 2. If there are more than $n/2$ points on a line, then the diameter is at least $n/2$, so we can suppose that this is not the case. By inequality (1) and Observation 1 we get

$$n \leq 4\left(\frac{12\Delta}{n^{1/2}}\right)^2,$$

which means that there is a positive constant c such that $\Delta > cn$. □

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