Discrete Comput Geom 28:405–409 (2002)
DOI: 10.1007/s00454-002-0744-9

Discrete & Computational
Geometry

Configurations of $2n-2$ Quadrics in $\mathbb{R}^n$ with $3 \cdot 2^{n-1}$ Common Tangent Lines

Gábor Megyesi

Department of Mathematics, UMIST,
P.O. Box 88, Manchester M60 1QD, England
gmegyesi@ma.umist.ac.uk

Abstract. We construct $2n - 2$ smooth quadrics in $\mathbb{R}^n$ whose equations have the same degree 2 homogeneous parts such that these quadrics have $3 \cdot 2^{n-1}$ isolated common real tangent lines. Special cases of the construction give examples of $2n-2$ spheres with affinely dependent centres such that all but one of the radii are equal, and of $2n - 2$ quadrics which are translated images of each other.

Sottile and Theobald proved in Theorem 10 of [3] that given $2n-2$ quadric hypersurfaces in $\mathbb{P}^n_{\mathbb{C}}$ whose intersection with a fixed hyperplane is a given smooth quadric, the number of isolated common tangent lines of these quadrics is at most $3 \cdot 2^{n-1}$. This generalizes the previous result in three dimensions by Theobald [4, Theorem 4], which is motivated by algorithmic problems in computer graphics.

If we consider quadrics in affine space and take the fixed hyperplane to be the hyperplane at infinity, then the condition on the intersection with the hyperplane means that the homogeneous degree 2 parts of the equations must be the same, so, for example, any collection of spheres satisfies this condition.

It was also shown in Theorem 2 of [3] that there exist configurations of $2n - 2$ unit spheres in $\mathbb{R}^n$ which have $3 \cdot 2^{n-1}$ isolated common real tangent lines. In this note we answer two of the questions left open in [3].

Theorem 1.

(a) *Let $n \geq 3$. Given non-zero real numbers $\lambda_3, \ldots, \lambda_n$, there exist $2n - 2$ smooth real quadrics in $\mathbb{R}^n$ such that the homogeneous degree* 2 *part of their equations is $x_1^2 + x_2^2 + \lambda_3 x_3^2 + \cdots + \lambda_n x_n^2$, and such that they have $3 \cdot 2^{n-1}$ isolated common real tangent lines.*

(b) *There exist* $2n - 2$ *spheres with affinely dependent centres such that they have* $3 \cdot 2^{n-1}$ *isolated common real tangent lines. It is possible to make* $2n - 1$ *of the radii equal.*

(c) *If* $\lambda_j > 0$ *for* $3 \le j \le n - 1$ *and* $\lambda_n < 0$, *then it is possible to achieve that the quadrics are translated copies of each other.*

Remarks. A non-degenerate quadratic form can be diagonalized in a suitable orthormal coordinate system. After scaling, it can be brought into the form $\sum_{i=1}^{n} \pm x_i^2$. After a possible change of sign and reordering the variables, the equation will be of the form considered in part (a) of the theorem, so the construction is general enough to cope with all topological types of non-degenerate quadrics.

The question whether it is possible to have $2n-2$ unit spheres with affinely dependent centres which have $3 \cdot 2^{n-1}$ isolated real common tangent lines is still open. With our construction only $2n - 1$ of the radii can be equal. In the three-dimensional case it is known that for unit spheres with affinely dependent centres the bound is 8 [2]. Sottile and Theobald [3] have not been able to construct $2n-2$ unit spheres with affinely dependent centres with more than 2^n isolated common real tangent lines, so one can conjecture that in this case the bound is 2^n, rather than $3 \cdot 2^{n-1}$.

Proof. (a) We shall be working over $\mathbb{R}$, all variables are understood to be real, unless stated otherwise.

For $x = (x_1, x_2, \ldots, x_n)$, $y = (y_1, y_2, \ldots, y_n) \in \mathbb{R}^n$ define $b(x, y) = \sum_{i=1}^{n} \lambda_i x_i y_i$, and $q(x) = b(x, x) = \sum_{i=1}^{n} \lambda_i x_i^2$, where $\lambda_1 = \lambda_2 = 1$.

Let $c_1 = (0, 1, 0, \ldots, 0)$, $c_2 = (\sqrt{3}/2, -\frac{1}{2}, 0, \ldots, 0)$, $c_3 = (-\sqrt{3}/2, -\frac{1}{2}, 0, \ldots, 0)$, $c_4 = (0, 0, \ldots, 0)$, and $c_{2j-1} = (0, \ldots, 0, a_j, 0, \ldots, 0)$, $c_{2j} = (0, \ldots, 0, -a_j, 0, \ldots, 0)$ with $\pm a_j$ in the jth position for $3 \le j \le n - 1$. The value of the $a_j \in \mathbb{R}\backslash\{0\}$, $3 \le j \le n - 1$, will be determined later.

The $2n - 2$ quadrics will be $Q_j = \{x \in \mathbb{R}^n \mid q(x - c_j) = R_j\}$, $1 \le j \le 2n - 2$, where the $R_j \in \mathbb{R}$, $1 \le j \le 2n - 2$, are also to be determined later.

Let $G \subset O(n)$ be the subgroup generated by the symmetry group of the regular triangle $c_1c_2c_3$ and the reflections $x_i \to -x_i$, for $3 \le i \le n$. G fixes $c_1, c_2, \ldots, c_{2n-2}$ as a set and $|G| = 3 \cdot 2^{n-1}$.

The key idea is that if $R_1 = R_2 = R_3$ and $R_{2j-1} = R_{2j}$ for $3 \le j \le n - 1$, then the collection of quadrics is invariant under G and if we find a common tangent line ℓ which is not fixed by any element of G, then its images under G give exactly $3 \cdot 2^{n-1}$ common tangent lines to the $2n - 2$ quadrics.

Instead of choosing the quadrics first and then trying to find a common tangent line, we follow the easier route of choosing ℓ first and then choosing the quadrics.

As in [3], we represent ℓ by a direction vector $v = (v_1, v_2, \ldots, v_n)$ and a point $p = (p_1, p_2, \ldots, p_n) \in \ell$. If $q(v) \neq 0$, then we can make p unique by requiring that $b(p, v) = 0$.

The line ℓ is tangent to the quadric given by the equation $q(x - c) = R$ if and only if R is the critical value of the quadratic function $q(x - c)$ restricted to ℓ. The points of ℓ can be written in the form $p + tv$ for some $t \in \mathbb{R}$, and the quadratic function restricted to ℓ is

$$q(p+tv-c) = q(p-c) + 2tb(v, p-c) + t^2 q(v).$$

If $q(v) \neq 0$, the critical value is

$$M(c, \ell) = q(p-c) - \frac{(b(v, p-c))^2}{q(v)} = q(p-c) - \frac{(b(v, c))^2}{q(v)}, \tag{1}$$

since $b(p, v) = 0$. ℓ is tangent to the quadric $q(x-c) = R$ if and only if $M(c, \ell) = R$. If $q(v) = 0$, then the quadratic function restricted to ℓ has degree at most 1, and it has a critical value if and only if $b(v, p-c) = 0$, when it is a constant function.

From now on we assume $q(v) \neq 0$. We now look at the conditions that ℓ is tangent to both Q_{2j-1} and Q_{2j}, and $R_{2j-1} = R_{2j}$ for $3 \leq j \leq n-1$. We have $b(c_{2j-1}, v) = a_j v_j$, while $b(c_{2j}, v) = -a_j v_j$, so $M(c_{2j-1}, \ell) = M(c_{2j}, \ell)$ implies $q(p-c_{2j-1}) = q(p-c_{2j})$. Hence $\lambda_j a_j p_j = 0$, and since $\lambda_j \neq 0$ and $a_j \neq 0$, we must have $p_j = 0$.

Similarly the conditions that ℓ is tangent to Q_1, Q_2 and Q_3, $R_1 = R_2 = R_3$ and $b(p, v) = 0$ determine p_1, p_2 and p_n uniquely as $p_1 = v_1 v_2 / 2q(v)$, $p_2 = (v_1^2 - v_2^2)/4q(v)$ and $p_n = v_2(v_2^2 - 3v_1^2)/4\lambda_n v_n q(v)$, assuming that $v_n \neq 0$.

The conclusion is that given any direction vector v with $q(v) \neq 0$ and $v_n \neq 0$, there exists a unique line ℓ with this direction vector such that $M(c_1, \ell) = M(c_2, \ell) = M(c_3, \ell)$ and $M(c_{2j-1}, \ell) = M(c_{2j}, \ell)$ for $3 \leq j \leq n-1$. Let $R_i = M(c_i, \ell)$ for $1 \leq i \leq 2n-2$, then ℓ is tangent to each quadric Q_i, $1 \leq i \leq 2n-2$. This set of quadrics is invariant under G, therefore if ℓ is not fixed by any element of G, which is true for general v, then its images under G give exactly $3 \cdot 2^{n-1}$ common tangent lines. (The precise condition for v not to be invariant under any element of G is $v_j \neq 0$ for $1 \leq j \leq n$, and $(v_1^2 - 3v_2^2)(3v_1^2 - v_2^2) \neq 0$.)

We now prove that there are no other common tangent lines to these quadrics, not even complex ones, which will show that they are not part of a higher-dimensional family. We assume that there is a line ℓ' with direction vector $v' = (v'_1, v'_2, \ldots, v'_n) \in \mathbb{C}^n$ which is tangent to the same $2n-2$ quadrics.

We first consider the possibility that $q(v') = 0$. Let $p' \in \ell'$. We must have $b(v', p' - c_i) = 0$ for all i, $1 \leq i \leq 2n-2$. It is easy to see that this implies $v'_1 = v'_2 = \cdots = v'_{n-1} = 0$, and then $q(v') = 0$ forces $v'_n = 0$, a contradiction.

Hence $q(v') \neq 0$, and without loss of generality we may assume that $q(v) = q(v')$.

Let H be the group generated by G and the map $v \to -v$. G and H act naturally on $\mathbb{C}[x_1, x_2, \ldots, x_n]$. For $x = (x_1, x_2, \ldots, x_n)$, let $I_1(x) = x_1^2 + x_2^2$, $I_2(x) = x_2^2(x_2^2 - 3x_1^2)^2$ and $I_j(x) = x_j^2$ for $3 \leq j \leq n$. These elements are clearly invariant under H and we claim that they generate the ring of invariants of $\mathbb{C}[x_1, x_2, \ldots, x_n]$ under the action of H.

H is generated by reflections, so by Chevalley [1, Theorems A, B], the ring of invariants can be generated by n elements, and the product of the degrees of any minimal set of generators is the order of the group H, which is $3 \cdot 2^n$. There is no invariant in degree 1, and we have $n-1$ linearly independent invariants in degree 2, $I_j(x)$, $1 \leq j \leq n$, $j \neq 2$. Therefore any minimal set of generators for the ring of invariants must consist of $n-1$ generators in degree 2 and one generator in degree 6. This generator in degree 6 can be any invariant element not in the subring generated by the degree 2 invariants. $I_2(x)$ is clearly not in the subring generated by $I_j(x)$, $1 \le j \le n$, $j \ne 2$, therefore $I_1(x)$, $I_2(x), \ldots, I_n(x)$ generate the ring of invariants as claimed.

We shall prove that $I_k(v) = I_k(v')$ for all k, $1 \le k \le n$, which will imply that v and v' are in the same H-orbit, therefore v' or $-v'$ is in the same G-orbit as v. As the direction vector determines the common tangent line uniquely, ℓ' must be the image of ℓ under an element of G.

We have the identities

$$R_1 - R_4 = 1 - \frac{v_1^2 + v_2^2}{2q(v)} \tag{2}$$

and

$$R_1 - R_{2j} = 1 - \lambda_j a_j^2 - \frac{v_1^2 + v_2^2 - 2\lambda_j^2 v_j^2 a_j^2}{2q(v)} \tag{3}$$

for $3 \le j \le n-1$, which can be verified by direct calculation. The same identities hold if we replace v by v', therefore we can deduce that $I_1(v) = I_1(v')$ and $I_j(v) = I_j(v')$ for $3 \le j \le n-1$. These together with $q(v) = q(v')$ imply that $I_n(v) = I_n(v')$, too.

We can also express R_1 in terms of the invariants. By (1),

$$R_1 = q(p - c_1) - \frac{(b(v, c_1))^2}{q(v)}.$$

By using the formulae for p_1, p_2, p_n ($p_j = 0$ if $3 \le j \le n-1$), and expressing $q(v)$ in terms of the invariants as $q(v) = I_1(v) + \lambda_3 I_3(v) + \cdots + \lambda_n I_n(v)$, after some simple algebraic manipulations we can write R_1 in terms of the invariants as

$$R_1 = \frac{I_2(v) + \lambda_n I_n(v)(3I_1(v) + (4\lambda_3 I_3(v) + \cdots + \lambda_n I_n(v)))^2}{16\lambda_n I_n(v)(I_1(v) + \lambda_3 I_3(v) + \cdots + \lambda_n I_n(v))^2}. \tag{4}$$

The same identity holds with v' instead of v, and since $I_j(v) = I_j(v')$ for $j \ne 2$, we can deduce that $I_2(v) = I_2(v')$, too.

As we noted before, the equality of the invariants implies that ℓ' is the image of ℓ under an element of G, so there are no other common tangent lines.

The identities (2), (3) and (4) also show that none of the R_i is identically 0, so for general choice of v and a_j, $3 \le j \le n$, we will obtain smooth quadrics.

(b) To obtain spheres, we just need to set $\lambda_j = 1$ for $3 \le j \le n$. The centres are affinely dependent as they are in the hyperplane $x_n = 0$.

We now try to make the radii equal. In the case of spheres, R_i is the square of the radius of the ith sphere. By (3), the solution of the equation $R_1 = R_{2j}$ ($3 \le j \le n-1$) in general is

$$a_j = \pm\sqrt{\frac{2q(v) - (v_1^2 + v_2^2)}{2\lambda_j (q(v) - \lambda_j v_j^2)}}.$$

In the case of spheres the denominator does not vanish and the solutions for the a_j are clearly real. Therefore given any sufficiently general direction vector v, we can choose the a_j such that all the spheres, except for the one centred at the origin, have the same radius. The identity (2) shows that we cannot make all the radii equal.

(c) We now consider the possibility of making all the R_i equal, so that the quadrics are translated images of each other. By combining (2) and (3), we obtain $\lambda_j a_j^2(q(v) - \lambda_j v_j^2) = 0$, so $\lambda_j v_j^2 = q(v)$ for $3 \leq j \leq n$. By (2), $2q(v) = v_1^2 + v_2^2 > 0$, so we must have $\lambda_j > 0$ for $3 \leq j \leq n$. We have

$$q(v) = \sum_{i=1}^{n} \lambda_i v_i^2 = (n-1)q(v) + \lambda_n v_n^2,$$

hence $\lambda_n v_n^2 = (2-n)q(v) < 0$, so $\lambda_n < 0$.

Given $\lambda_3, \ldots, \lambda_{n-1} > 0$ and $\lambda_n < 0$, the construction is the following. Choose real numbers v_1 and v_2 such that $v_1 v_2 (v_1^2 - 3v_2^2)(3v_1^2 - v_2^2) \neq 0$. Set $v_j = \sqrt{(v_1^2 + v_2^2)/2\lambda_j}$ for $3 \leq j \leq n-1$, and $v_n = \sqrt{((2-n)(v_1^2 + v_2^2))/2\lambda_n}$. $v_3, \ldots, v_n$ are all positive real numbers. a_j for $3 \leq j \leq n$ can be an arbitrary non-zero real number. With these parameters, the R_i, $1 \leq i \leq 2n-2$, are equal, so the quadrics constructed are translated copies of each other.

By substituting the above values for v_j, $3 \leq j \leq n$, into (4), R_1 can be expressed in terms of v_1 and v_2 as

$$R_1 = \frac{v_2^2(v_2^2 - 3v_1^2)^2}{2(2-n)(v_1^2 + v_2^2)^3} + \frac{1}{4}.$$

This expression is not identically 0, so for general choice of v_1 and v_2 the quadrics will be smooth. The maximum of R_1 is $\frac{1}{4}$, and the minimum is $(n-4)/(4(n-2))$. If $n = 3$, then R_1 can take both positive and negative values, so this construction can give hyperboloids of both one sheet and two sheets. If $n \geq 4$, then the minimum is non-negative, and we obtain quadrics of one topological type only. $\square$

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Received August 22, 2001, *and in revised form March* 5, 2002. *Online publication July* 10, 2002.