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Improving Rogers' Upper Bound for the Density of Unit Ball Packings via Estimating the Surface Area of Voronoi Cells from Below in Euclidean d-Space for All $d \geq 8$*

Károly Bezdek

Department of Geometry, Eötvös University,
H-1117 Budapest, Hungary
kbezdek@ludens.elte.hu
and
Department of Mathematics, Cornell University,
Ithaca, NY 14853-4201, USA
bezdek@math.cornell.edu

Abstract. The sphere packing problem asks for the densest packing of unit balls in $\mathbf{E}^d$. This problem has its roots in geometry, number theory and information theory and it is part of Hilbert's 18th problem. One of the most attractive results on the sphere packing problem was proved by Rogers in 1958. It can be phrased as follows. Take a regular d-dimensional simplex of edge length 2 in $\mathbf{E}^d$ and then draw a d-dimensional unit ball around each vertex of the simplex. Let σ_d denote the ratio of the volume of the portion of the simplex covered by balls to the volume of the simplex. Then the volume of any Voronoi cell in a packing of unit balls in $\mathbf{E}^d$ is at least ω_d/σ_d, where ω_d denotes the volume of a d-dimensional unit ball. This has the immediate corollary that the density of any unit ball packing in $\mathbf{E}^d$ is at most σ_d. In 1978 Kabatjanskii and Levenštein improved this bound for large d. In fact, Rogers' bound is the presently known best bound for $4 \leq d \leq 42$, and above that the Kabatjanskii–Levenštein bound takes over. In this paper we improve Rogers' upper bound for the density of unit ball packings in Euclidean d-space for all $d \geq 8$ and improve the Kabatjanskii–Levenštein upper bound in small dimensions. Namely, we show that the volume of any Voronoi cell in a packing of unit balls in $\mathbf{E}^d$, $d \geq 8$, is at least $\omega_d/\hat{\sigma}_d$ and so the density of any unit ball packing in $\mathbf{E}^d$, $d \geq 8$, is at most $\hat{\sigma}_d$, where $\hat{\sigma}_d$ is a geometrically well-defined quantity satisfying the inequality $\hat{\sigma}_d < \sigma_d$ for all $d \geq 8$. We prove this by showing that the surface area of any Voronoi cell in a packing of unit balls in $\mathbf{E}^d$, $d \geq 8$, is at least $(d \cdot \omega_d)/\hat{\sigma}_d$.

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Introduction

A family of nonoverlapping d-dimensional balls of radii 1 in the d-dimensional Euclidean space $\mathbf{E}^d$ is called a unit ball packing of $\mathbf{E}^d$. The density of the packing is the proportion of space covered by these unit balls. The sphere packing problem asks for the densest packing of unit balls in $\mathbf{E}^d$. Indubitably, of all problems concerning packing it was the sphere packing problem which attracted the most attention in the past decade. It has its roots in geometry, number theory and information theory and it is part of Hilbert's 18th problem. The reader is referred to [10] (especially the third edition, which has about 800 references covering 1988–1998) for further information, definitions and references. In what follows we report on a few selected developments and then we state the main results of this paper.

The Voronoi cell of a unit ball in a packing of unit balls in $\mathbf{E}^d$ is the set of points that are not farther away from the center of the given ball than from any other ball's center. As is well known (see for example [24]) the Voronoi cells of a unit ball packing in $\mathbf{E}^d$ form a tiling of $\mathbf{E}^d$. One of the most attractive results on the sphere packing problem was proved by Rogers [23] in 1958. It was rediscovered by Baranovskii [2] and extended to spherical and hyperbolic spaces by Böröczky [6]. It can be phrased as follows. Take a regular d-dimensional simplex of edge length 2 in $\mathbf{E}^d$ and then draw a d-dimensional unit ball around each vertex of the simplex. Let σ_d denote the ratio of the volume of the portion of the simplex covered by balls to the volume of the simplex. Then the volume of any Voronoi cell in a packing of unit balls in $\mathbf{E}^d$ is at least ω_d/σ_d, where ω_d denotes the volume of a d-dimensional unit ball. This has the immediate corollary that the (upper) density of any unit ball packing in $\mathbf{E}^d$ is at most σ_d. Daniel's asymptotic formula [24] yields that

$$\sigma_d = \frac{d}{e} 2^{-(0.5+o(1))d} \qquad (\text{as } d \to \infty).$$

Then 20 years later, in 1978, Kabatjanskii and Levenštein [18] improved this bound in the exponential order of magnitude as follows. They showed that the density of any unit ball packing in $\mathbf{E}^d$ is at most

$$2^{-(0.599+o(1))d} \qquad (\text{as } d \to \infty).$$

In fact, Rogers' bound is the presently known best bound for $4 \le d \le 42$ (see also Remark 2 below), and above that the Kabatjanskii–Levenštein bound takes over [10, p. 20].

There has been some important recent progress concerning the existence of economical packings. On the one hand, improving earlier results, Ball [1] proved through a very elegant completely new variational argument that, for each d, there is a lattice packing of unit balls in $\mathbf{E}^d$ with density at least

$$\frac{d-1}{2^{d-1}} \zeta(d),$$

where $\zeta(d) = \sum_{k=1}^{\infty} (1/k^d)$ is the Riemann zeta function. On the other hand, for some small values of d, there are explicit (lattice) packings which give densities (considerably) higher than the bound just stated. In connection with this we briefly mention some of the exciting results of lower dimensions. The reader is referred to [10], [12], [22] and [25] for a comprehensive view of results of this type.

In $\mathbf{E}^3$ the face centered cubic lattice packing of congruent balls with density $\pi/\sqrt{18} = 0.74048\cdots$ was conjectured by Kepler to be the densest packing among all packings of unit balls [10]. It has been proven that the face centered cubic lattice is a locally optimal arrangement (for completely independent approaches to this see [3], [11] and [13]) and that the density of any packing of congruent balls in $\mathbf{E}^3$ cannot be larger than $0.773055\cdots$, which was established by Muder in [21]. Finally, in 1998 Hales [16] announced that the final step in the proof of the Kepler conjecture had been completed: the Kepler conjecture is now a theorem (finishing a several years complex project that started with [14] and [15]). Hales' proof is long and difficult and requires extensive computer calculation. As of October 2001, it has not yet been published but it is widely regarded as being likely to be correct. In $\mathbf{E}^d$, $4 \le d \le 9$, the densest packings of congruent balls known are obtained by the "laminating" or "greedy" construction described in [8]. (In fact, [8] determines all equivalent laminated lattices for $d \le 25$ and produces many of the densest lattices known up to 25.) In $\mathbf{E}^{10}$ we encounter for the first time a nonlattice packing of congruent balls that is denser than all known lattice packings. This packing is obtained from "Construction A" (for more details see [20] and [25]). In $\mathbf{E}^d$, $18 \le d \le 22$, record nonlattice packings of congruent balls have recently been given such as Vardy's construction [26] ("Construction B^*") and Bierbrauer and Edel record packing in $\mathbf{E}^{18}$ (see [25] and also [9]). The packing of congruent balls with proper radii around the points of the Leech lattice is a remarkably dense packing in $\mathbf{E}^{24}$. New packings of congruent balls in $\mathbf{E}^d$, $26 \le d \le 31$, have been recently discovered by Bacher, Borcherds, Conway, Sloane, Vardy and Venkov—see [10] for details. Finally, we mention the Kschischang and Pasupathy lattice packing [19] of balls in $\mathbf{E}^{36}$ and the Mordell–Weil lattice packings of balls in $\mathbf{E}^d$, $80 \le d \le 4096$, discovered by Elkies and Shioda that are the densest lattice packings of balls known in those dimensions—see [10] and [25] for details. For a complete account on record packings we refer to [22]. All these explicit constructions raise the well-known challenging question whether one can find a smaller upper bound than Rogers' bound for the density of unit ball packings, especially in low dimensions. The corollary established in this paper does exactly this by improving Rogers' upper bound for the density of unit ball packings in Euclidean d-space for all $d \ge 8$.

In what follows we state the major results of this paper. As usual, let $\mathrm{lin}(\cdots)$, $\mathrm{aff}(\cdots)$, $\mathrm{conv}(\cdots)$, $\mathrm{Vol}_d(\cdots)$, ω_d, $\mathrm{SVol}_{d-1}(\cdots)$, $\mathrm{dist}(\cdots)$, $\|\cdots\|$ and $\mathbf{o}$ refer to the linear hull, the affine hull, the convex hull in $\mathbf{E}^d$, the d-dimensional Euclidean volume measure, the d-dimensional volume of a d-dimensional unit ball, the $(d-1)$-dimensional spherical volume measure, the distance function in $\mathbf{E}^d$, the standard Euclidean norm and the origin in $\mathbf{E}^d$.

Let $\mathrm{conv}\{\mathbf{o}, \mathbf{w}_1, \ldots, \mathbf{w}_d\}$ be a d-dimensional simplex having the property that the linear hull $\mathrm{lin}\{\mathbf{w}_j - \mathbf{w}_i | i < j \le d\}$ is orthogonal to the vector $\mathbf{w}_i$ in $\mathbf{E}^d$, $d \ge 8$ for all $1 \le i \le d-1$, that is, let

$$\mathrm{conv}\{\mathbf{o}, \mathbf{w}_1, \ldots, \mathbf{w}_d\}$$

be a d-dimensional orthoscheme in $\mathbf{E}^d$, moreover, let

$$\|\mathbf{w}_i\| = \sqrt{\frac{2i}{i+1}} \qquad \text{for all} \quad 1 \le i \le d.$$

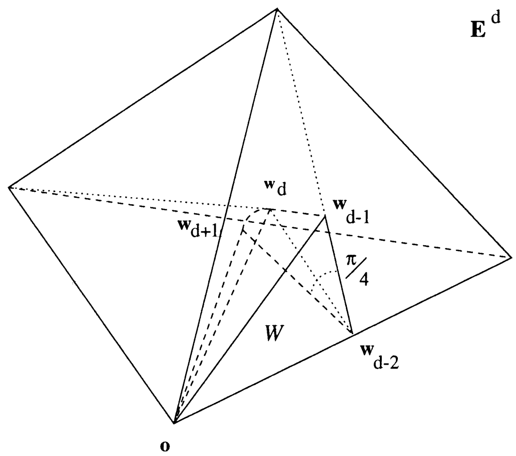

Fig. 1

It is clear that in the right triangle $\triangle\, \mathbf{w}_{d-2}\mathbf{w}_{d-1}\mathbf{w}_d$ with right angle at the vertex $\mathbf{w}_{d-1}$ we have the inequality $\|\mathbf{w}_d - \mathbf{w}_{d-1}\| = \sqrt{2/d(d+1)} < \sqrt{2/(d-1)d} = \|\mathbf{w}_{d-1} - \mathbf{w}_{d-2}\|$ and therefore $\angle\mathbf{w}_{d-1}\mathbf{w}_{d-2}\mathbf{w}_d < \pi/4$ (see Fig. 1). Now, in the plane aff$\{\mathbf{w}_{d-2}, \mathbf{w}_{d-1}, \mathbf{w}_d\}$ of the triangle $\triangle\, \mathbf{w}_{d-2}\mathbf{w}_{d-1}\mathbf{w}_d$ let

$$\lhd\, \mathbf{w}_{d-2}\mathbf{w}_d\mathbf{w}_{d+1}$$

denote the circular sector of central angle $\angle\mathbf{w}_d\mathbf{w}_{d-2}\mathbf{w}_{d+1} = \pi/4 - \angle\mathbf{w}_{d-1}\mathbf{w}_{d-2}\mathbf{w}_d$ and of center $\mathbf{w}_{d-2}$ sitting over the circular arc with endpoints $\mathbf{w}_d, \mathbf{w}_{d+1}$ and radius $\|\mathbf{w}_d - \mathbf{w}_{d-2}\| = \|\mathbf{w}_{d+1} - \mathbf{w}_{d-2}\|$ such that $\lhd\mathbf{w}_{d-2}\mathbf{w}_d\mathbf{w}_{d+1}$ and $\triangle\mathbf{w}_{d-2}\mathbf{w}_{d-1}\mathbf{w}_d$ are adjacent along the line segment $\mathbf{w}_{d-2}\mathbf{w}_d$ and are separated by the line of $\mathbf{w}_{d-2}\mathbf{w}_d$. Then let

$$D(\mathbf{w}_{d-2}, \mathbf{w}_{d-1}, \mathbf{w}_d, \mathbf{w}_{d+1}) = \triangle\, \mathbf{w}_{d-2}\mathbf{w}_{d-1}\mathbf{w}_d \cup\ \lhd\, \mathbf{w}_{d-2}\mathbf{w}_d\mathbf{w}_{d+1}$$

be the convex domain generated by the triangle $\triangle\mathbf{w}_{d-2}\mathbf{w}_{d-1}\mathbf{w}_d$ with constant angle $\angle\mathbf{w}_{d-1}\mathbf{w}_{d-2}\mathbf{w}_{d+1} = \pi/4$.

Now, let

$$W = \operatorname{conv}(\{\mathbf{o}, \mathbf{w}_1, \ldots, \mathbf{w}_{d-3}\} \cup D(\mathbf{w}_{d-2}, \mathbf{w}_{d-1}, \mathbf{w}_d, \mathbf{w}_{d+1}))$$

be the d-dimensional wedge (or cone) with $(d-1)$-dimensional base

$$Q_W = \operatorname{conv}(\{\mathbf{w}_1, \ldots, \mathbf{w}_{d-3}\} \cup D(\mathbf{w}_{d-2}, \mathbf{w}_{d-1}, \mathbf{w}_d, \mathbf{w}_{d+1})) \quad \text{and} \quad \text{apex } \mathbf{o}.$$

Finally, if $B = \{\mathbf{x} \in \mathbf{E}^d \,|\, \operatorname{dist}(\mathbf{o}, \mathbf{x}) = \|\mathbf{x}\| \le 1\}$ denotes the d-dimensional unit ball centered at the origin $\mathbf{o}$ of $\mathbf{E^d}$ and $S = \{\mathbf{x} \in \mathbf{E}^d \,|\, \operatorname{dist}(\mathbf{o}, \mathbf{x}) = \|\mathbf{x}\| = 1\}$ denotes the $(d-1)$-dimensional unit sphere centered at $\mathbf{o}$, then let

$$\hat{\sigma}_d = \frac{\operatorname{SVol}_{d-1}(W \cap S)}{\operatorname{Vol}_{d-1}(Q_W)} = \frac{\operatorname{Vol}_d(W \cap B)}{\operatorname{Vol}_d(W)}$$

be the the surface density (resp., volume density) of the unit sphere S (resp., of the unit ball B) in the wedge W. For the sake of completeness we remark that as the regular d-dimensional simplex of edge length 2 can be dissected into $(d+1)!$ pieces, each being congruent to conv$\{\mathbf{o}, \mathbf{w}_1, \dots, \mathbf{w}_d\}$, therefore

$$\sigma_d = \frac{\mathrm{Vol}_d(\mathrm{conv}\{\mathbf{o}, \mathbf{w}_1, \dots, \mathbf{w}_d\} \cap B)}{\mathrm{Vol}_d(\mathrm{conv}\{\mathbf{o}, \mathbf{w}_1, \dots, \mathbf{w}_d\})}.$$

Now we are ready to state the main result of this paper. Recall that the surface density of any unit sphere in its Voronoi cell in a unit sphere packing of $\mathbf{E}^d$ is defined as the ratio of the surface area of the unit sphere to the surface area of its Voronoi cell. The following theorem improves Theorem 3 of [4] which claims that the surface area of any Voronoi cell in a packing of unit balls in $\mathbf{E}^d$, $d \geq 2$, is at least $(d \cdot \omega_d)/\sigma_d$.

Theorem. *The surface area of any Voronoi cell in a packing of unit balls in the d-dimensional Euclidean space $\mathbf{E}^d$, $d \geq 8$, is at least $(d \cdot \omega_d)/\hat{\sigma}_d$, that is, the surface density of any unit sphere in its Voronoi cell in a unit sphere packing of $\mathbf{E}^d$, $d \geq 8$, is at most $\hat{\sigma}_d$.*

As the volume of a Voronoi cell in a unit ball packing of $\mathbf{E}^d$ is at least as large as $1/d$ times the surface area of the Voronoi cell the following result follows from the theorem.

Corollary. *The volume of any Voronoi cell in a packing of unit balls in $\mathbf{E}^d$, $d \geq 8$, is at least $\omega_d/\hat{\sigma}_d$. Thus, the (upper) density of any unit ball packing in $\mathbf{E}^d$, $d \geq 8$, is at most $\hat{\sigma}_d$.*

Finally, we show that our upper bound $\hat{\sigma}_d$ for the density of unit ball packings in Euclidean d-space is indeed better than Rogers' bound σ_d.

Proposition. $\hat{\sigma}_d < \sigma_d$ *for all* $d \geq 8$.

Remark 1. It is not hard to see that the proof of the theorem presented in the sections below can be used to prove the following stronger statement. Take a Voronoi cell of a unit ball in a packing of unit balls in the d-dimensional Euclidean space $\mathbf{E}^d$, $d \geq 8$, and then take the intersection of the given Voronoi cell with the closed d-dimensional ball of radius $\sqrt{2d/(d+1)}$ concentric to the unit ball of the Voronoi cell. Then the surface area of the truncated Voronoi cell is at least $(d \cdot \omega_d)/\hat{\sigma}_d$.

Remark 2. Our density bound $\hat{\sigma}_d$ for the density of unit ball packings in $\mathbf{E}^d$, $d \geq 8$, is a geometrically explicit bound for all $d \geq 8$. However, in concrete small dimensions in particular, in dimension 8 our method of proof suggests further improvements via the three-dimensional skeleton of Voronoi polytopes. As a result the best possible numerical value for the density bound generated by our method in dimension 8 is still in progress. In connection with this we mention that Cohn and Elkies just recently announced an analogue for sphere packing of the linear programming bounds for error-correcting codes in [7] and used it to improve Rogers' upper bound for the density of sphere packings for dimensions

4–36. This work of Cohn and Elkies seems to be in progress as well. Finally, Hsiang [17] very recently announced a solution of the eight-dimensional sphere packing problem but details are not yet public. Their methods are apparently quite different from ours.

The organization of the rest of the paper is as follows. The proof of the theorem consists of six major steps which are discussed in consecutive separate sections of the paper. Then a short proof of the proposition is presented in the last section. Finally, we have to emphasize that although most of the methods of this paper work in all dimensions being at least 8 they are designed to act together in an efficient way mostly in low dimensions.

1. Some Metric Properties of Voronoi Cells of Unit Ball Packings in $\mathbf{E}^d$

Let P be a bounded Voronoi cell, i.e. a d-dimensional Voronoi polytope of a packing $\mathcal{P}$ of d-dimensional unit balls in $\mathbf{E}^d$. Without loss of generality we may assume that the unit ball $B = \{\mathbf{x} \in \mathbf{E}^d \mid \text{dist}(\mathbf{o}, \mathbf{x}) = \|\mathbf{x}\| \le 1\}$ centered at the origin $\mathbf{o}$ of $\mathbf{E}^d$ is one of the unit balls of $\mathcal{P}$ with P as its Voronoi cell. Then P is the intersection of finitely many closed halfspaces of $\mathbf{E}^d$, each of which is bounded by a hyperplane which is the perpendicular bisector of a line segment $\mathbf{ox}$, with $\mathbf{x}$ being the center of some unit ball of $\mathcal{P}$. Now, let F_{d-i} be an arbitrary $(d-i)$-dimensional face of P, $1 \le i \le d$. Then clearly there are at least $i+1$ Voronoi cells of $\mathcal{P}$ which meet along the face F_{d-i}, that is, contain F_{d-i} (one of which is, of course, P). Also, it is clear from the construction that the affine hull of centers of the unit balls sitting in all of these Voronoi cells is orthogonal to aff F_{d-i}. Thus, there are unit balls of these Voronoi cells with centers $\{\mathbf{o}, \mathbf{x}_1, \dots, \mathbf{x}_i\}$ such that $X = \text{conv}\{\mathbf{o}, \mathbf{x}_1, \dots, \mathbf{x}_i\}$ is an i-dimensional simplex and, of course, aff X is orthogonal to aff F_{d-i}. Hence, if $R(F_{d-i})$ denotes the radius of the $(i-1)$-dimensional sphere that passes through the vertices of X, then

$$R(F_{d-i}) = \text{dist}(\mathbf{o}, \text{aff}\, F_{d-i}), \qquad \text{where} \quad 1 \le i \le d.$$

Lemma 1. *If $F_{d-i-1} \subset F_{d-i}$ and $R(F_{d-i}) = R < \sqrt{2}$ for some i, $1 \le i \le d-1$, then*

$$\frac{2}{\sqrt{4-R^2}} \le R(F_{d-i-1}).$$

Proof. Recall that if C is the convex hull of the centers of the unit balls of $\mathcal{P}$ whose Voronoi cells contain F_{d-i}, then aff C is an i-dimensional affine subspace totally orthogonal to aff F_{d-i}; moreover, aff $F_{d-i} \cap C = \mathbf{b}$ is the center of the $(i-1)$-dimensional sphere that passes through the vertices of C. As $\mathbf{o}$ is among the vertices of C there exist vertices of C, say, $\mathbf{c}_1, \dots, \mathbf{c}_i$, such that $C_i = \text{conv}\{\mathbf{o}, \mathbf{c}_1, \dots, \mathbf{c}_i\}$ is an i-dimensional simplex containing $\mathbf{b}$ with

$$\|\mathbf{b}-\mathbf{o}\| = \|\mathbf{b}-\mathbf{c}_1\| = \cdots = \|\mathbf{b}-\mathbf{c}_i\| = R(F_{d-i}) = R.$$

From now on we deal only with the case

$$\mathbf{b} \in \text{relint}\, C_i$$

for the reason that the case $\mathbf{b} \in \text{relbd}\, C_i$ follows from this by standard limit procedure.

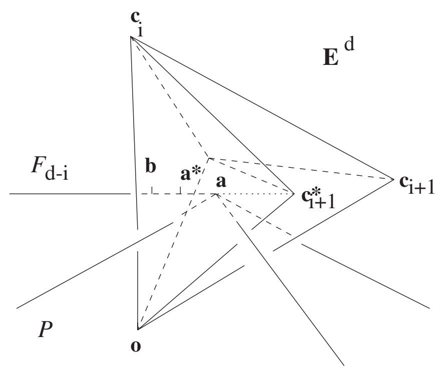

Fig. 2

As $F_{d-i-1} \subset F_{d-i}$ there must be a unit ball of $\mathcal{P}$ with center, say, $\mathbf{c}_{i+1}$ whose Voronoi cell intersects F_{d-i} precisely in F_{d-i-1}. Thus,

$$C_{i+1} = \operatorname{conv}\{\mathbf{o}, \mathbf{c}_1, \ldots, \mathbf{c}_i, \mathbf{c}_{i+1}\}$$

is an $(i+1)$-dimensional simplex; moreover, if the center of the i-dimensional sphere that passes through the vertices of C_{i+1} is $\mathbf{a}$, then

$$\|\mathbf{a} - \mathbf{o}\| = \|\mathbf{a} - \mathbf{c}_1\| = \cdots = \|\mathbf{a} - \mathbf{c}_i\| = \|\mathbf{a} - \mathbf{c}_{i+1}\| = R(F_{d-i-1}).$$

Obviously, all the edges of C_{i+1} have lengths larger than or equal to 2 (see Fig. 2).

Finally, let $\mathbf{c}^*_{i+1} \in \mathbf{E}^d$ be a point such that

$$\|\mathbf{c}^*_{i+1} - \mathbf{o}\| = \|\mathbf{c}^*_{i+1} - \mathbf{c}_1\| = \cdots = \|\mathbf{c}^*_{i+1} - \mathbf{c}_i\| = 2.$$

Then

$$C^*_{i+1} = \operatorname{conv}\{\mathbf{o}, \mathbf{c}_1, \ldots, \mathbf{c}_i, \mathbf{c}^*_{i+1}\}$$

is an $(i+1)$-dimensional simplex; moreover, if $\mathbf{a}^*$ is the center of the i-dimensional sphere that passes through the vertices of C^*_{i+1}, then

$$\mathbf{a}^* \in \operatorname{relint} C^*_{i+1} \quad \text{and}$$

$$\|\mathbf{a}^* - \mathbf{o}\| = \|\mathbf{a}^* - \mathbf{c}_1\| = \cdots = \|\mathbf{a}^* - \mathbf{c}_i\| = \|\mathbf{a}^* - \mathbf{c}^*_{i+1}\| = \frac{2}{\sqrt{4 - R^2}}.$$

Notice that the length of each edge of C_{i+1} is larger than or equal to the length of the corresponding edge of C^*_{i+1}.

Now, recall the following statement which is a special case of Lemma 7 in [5].

Sublemma 1. *Let* $Y^* \subset \mathbf{E}^n, 1 \le n$, *be an* n*-dimensional simplex with vertices* $\{\mathbf{y}^*_0, \mathbf{y}^*_1, \ldots, \mathbf{y}^*_n\} \subset \mathbf{E}^n$ *and let* $\mathbf{p}^* \in \operatorname{int} Y^*$ *be an arbitrary interior point of* Y^*. *If* $Y \subset \mathbf{E}^n$ *is an* n*-dimensional simplex with vertices* $\{\mathbf{y}_0, \mathbf{y}_1, \ldots, \mathbf{y}_n\} \subset \mathbf{E}^n$ *and* $\mathbf{p} \in \mathbf{E}^n$ *is a point with the property that* $\|\mathbf{y}^*_k - \mathbf{y}^*_l\| \le \|\mathbf{y}_k - \mathbf{y}_l\|$ *for all* $0 \le k < l \le n$ *and* $\|\mathbf{p}^* - \mathbf{y}^*_j\| \ge \|\mathbf{p} - \mathbf{y}_j\|$ *for all* $0 \le j \le n$, *then the simplices* Y^* *and* Y *are congruent*; *moreover,* $\|\mathbf{p}^* - \mathbf{y}^*_j\| = \|\mathbf{p} - \mathbf{y}_j\|$ *for all* $0 \le j \le n$.

Thus, if one assumes that $R(F_{d-i-1}) < 2/\sqrt{4-R^2}$, then Sublemma 1 applied to the simplices C^*_{i+1} and C_{i+1} immediately leads to a contradiction. This completes the proof of Lemma 1. □

As an easy corollary of Lemma 1 we get the following well-known inequality [23].

Corollary 1. $\sqrt{2i/(i+1)} \le R(F_{d-i})$ *for all* $1 \le i \le d$.

We will need the following metric property of Voronoi polytopes as well. (For a somewhat weaker version of this see pp. 257–258 of [6].)

Lemma 2. *If* $R(F_{d-i}) < \sqrt{2}$ *for some* i, $1 \le i \le d$, *then the orthogonal projection of* $\mathbf{o}$ *onto* aff F_{d-i} *belongs to* relint F_{d-i} *and so* $R(F_{d-i}) = \operatorname{dist}(\mathbf{o}, F_{d-i})$.

Proof. Take the i-dimensional simplex C_i defined in the beginning of the proof of Lemma 1. The vertices of C_i are $\mathbf{o}, \mathbf{c}_1, \dots, \mathbf{c}_i$ that are centers of unit balls of $\mathcal{P}$ whose Voronoi cells all contain F_{d-i}. Thus, aff C_i is orthogonal to aff F_{d-i}; moreover,

$$\mathbf{b} = C_i \cap \operatorname{aff} F_{d-i} \quad \text{and} \quad \|\mathbf{b}-\mathbf{o}\| = \|\mathbf{b}-\mathbf{c}_1\| = \cdots = \|\mathbf{b}-\mathbf{c}_i\| = R(F_{d-i}).$$

From this it is clear that

$$R(F_{d-i}) = \operatorname{dist}(\mathbf{o}, \operatorname{aff} F_{d-i}) = \operatorname{dist}(\mathbf{o}, \mathbf{b}) \qquad \text{with} \quad \mathbf{b} = C_i \cap \operatorname{aff} F_{d-i}.$$

We are left to show that $\mathbf{b} \in \operatorname{relint} F_{d-i}$.

Assume that $\mathbf{b} \notin \operatorname{relint} F_{d-i}$ (see Fig. 3). Then there is a $(d-i-1)$-dimensional face of F_{d-i}, say, F_{d-i-1}, with the property that aff F_{d-i-1} separates the point $\mathbf{b}$ from F_{d-i} in aff F_{d-i}. This means that there is a unit ball of $\mathcal{P}$ with center, say, $\mathbf{c}_{i+1}$ such that the perpendicular bisector of the line segment $\mathbf{oc}_{i+1}$ intersects aff F_{d-i} in aff F_{d-i-1} and therefore it separates F_{d-i} from the point $\mathbf{b}$ in $\mathbf{E}^d$.

As a result we get that

$$\|\mathbf{b}-\mathbf{c}_{i+1}\| \le \|\mathbf{b}-\mathbf{o}\| = \|\mathbf{b}-\mathbf{c}_1\| = \cdots = \|\mathbf{b}-\mathbf{c}_i\| = R(F_{d-i}).$$

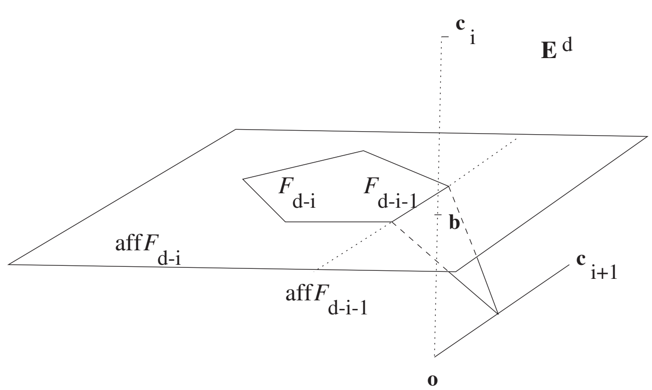

Fig. 3

Now, let H_k^+ (resp., H_0^+) denote the closed halfspace of $\mathbf{E}^d$ that contains the point $\mathbf{c}_k$ (resp., $\mathbf{o}$) and is bounded by the hyperplane of $\mathbf{E}^d$ that passes through the point $\mathbf{b}$ and is perpendicular to the vector $\mathbf{b} - \mathbf{c}_k$ (resp., $\mathbf{b}$), where $1 \le k \le i$. As $\mathbf{b} \in C_i$ it is easy to see that

$$H_0^+ \cup H_1^+ \cup \cdots \cup H_i^+ = \mathbf{E}^d.$$

From this then it is immediate that there exists a k with $0 \le k \le i$ such that $\mathbf{c}_{i+1} \in H_k^+$. In other words, there is a point $\mathbf{c} \in \{\mathbf{o}, \mathbf{c}_1, \ldots, \mathbf{c}_i\}$ such that

$$\angle \mathbf{cbc}_{i+1} \le \frac{\pi}{2}.$$

As $\|\mathbf{b} - \mathbf{c}_{i+1}\| \le \|\mathbf{b} - \mathbf{c}\| = R(F_{d-i}) < \sqrt{2}$ this implies in a straightforward way that $\|\mathbf{c} - \mathbf{c}_{i+1}\| < 2$, a contradiction. This completes the proof of Lemma 2. $\square$

2. Wedges of Type I–III, Truncated Wedges of Type I and II and Some of Their Metric Properties

Let $F_0 \subset F_1 \subset \cdots \subset F_{d-1}$ be an arbitrary flag of the Voronoi polytope P. Then let $\mathbf{r}_i \in F_{d-i}$ be the uniquely determined point of the $(d-i)$-dimensional face F_{d-i} of P that is closest to the center point $\mathbf{o}$ of P, that is, let

$$\mathbf{r}_i \in F_{d-i} \quad \text{such that} \quad \|\mathbf{r}_i\| = \min\{\|\mathbf{x}\| \mid \mathbf{x} \in F_{d-i}\}, \qquad \text{where} \quad 1 \le i \le d.$$

Definition 1. If the vectors $\mathbf{r}_1, \ldots, \mathbf{r}_i$ are linearly independent in $\mathbf{E}^d$, then we call $\operatorname{conv}\{\mathbf{o}, \mathbf{r}_1, \ldots, \mathbf{r}_i\}$ the *i-dimensional Rogers simplex* assigned to the subflag $F_{d-i} \subset \cdots \subset F_{d-1}$ of the Voronoi polytope P, where $1 \le i \le d$. If $\operatorname{conv}\{\mathbf{o}, \mathbf{r}_1, \ldots, \mathbf{r}_d\} \subset \mathbf{E}^d$ is the d-dimensional Rogers simplex assigned to the flag $F_0 \subset \cdots \subset F_{d-1}$ of P, then $\operatorname{conv}\{\mathbf{r}_{d-i}, \ldots, \mathbf{r}_d\}$ is called the *i-dimensional base* of the given d-dimensional Rogers simplex and $\operatorname{dist}(\mathbf{o}, \operatorname{aff}\{\mathbf{r}_{d-i}, \ldots, \mathbf{r}_d\}) = \operatorname{dist}(\mathbf{o}, \operatorname{aff} F_i) = R(F_i)$ is called the *height* assigned to the i-dimensional base, where $1 \le i \le d$.

Definition 2. The i-dimensional simplex $Y = \operatorname{conv}\{\mathbf{o}, \mathbf{y}_1, \ldots, \mathbf{y}_i\} \subset \mathbf{E}^d$ with vertices $\mathbf{y}_0 = \mathbf{o}, \mathbf{y}_1, \ldots, \mathbf{y}_i$ is called an *i-dimensional orthoscheme* if for each j, $0 \le j \le i-1$, the vector $\mathbf{y}_j$ is orthogonal to the linear hull $\operatorname{lin}\{\mathbf{y}_k - \mathbf{y}_j \mid j+1 \le k \le i\}$, where $1 \le i \le d$.

It is shown in [23] that the union of the d-dimensional Rogers simplices of the Voronoi polytope P is the polytope P itself and their interiors are pairwise disjoint. This fact together with Corollary 1 and Lemma 2 imply the following metric properties of Rogers simplices in a straightforward way.

Lemma 3.

(1) *If* $\operatorname{conv}\{\mathbf{o}, \mathbf{r}_1, \ldots, \mathbf{r}_i\}$ *is an i-dimensional Rogers simplex assigned to the subflag* $F_{d-i} \subset \cdots \subset F_{d-1}$ *of the Voronoi polytope P, then* $\sqrt{2j/(j+1)} \le \|\mathbf{r}_j\|$ *for all* $1 \le j \le i$*, where* $1 \le i \le d$.

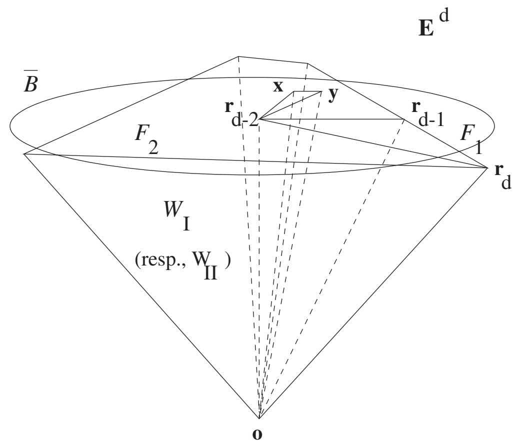

Fig. 4

(2) *If* $F_{d-i} \subset \cdots \subset F_{d-1}$ *is a subflag of the Voronoi polytope* P *with* $R(F_{d-i}) < \sqrt{2}$, *then* $\mathrm{conv}\{\mathbf{o}, \mathbf{r}_1, \ldots, \mathbf{r}_i\}$ *is an* i*-dimensional Rogers simplex which is, in fact, an* i*-dimensional orthoscheme* (*in short, an* i*-dimensional Rogers orthoscheme*) *with the property that each* $\mathbf{r}_j \in \mathrm{relint}\, F_{d-j}$, $1 \le j \le i$, *is the orthogonal projection of* $\mathbf{o}$ *onto* $\mathrm{aff}\, F_{d-j}$, *where* $1 \le i \le d$.

(3) *If* $F_2 \subset \cdots \subset F_{d-1}$ *is a subflag of the Voronoi polytope* $P \subset \mathbf{E}^d$, $3 \le d$, *with* $R(F_2) < \sqrt{2}$, *then the union of the two-dimensional bases of the* d*-dimensional Rogers simplices that contain the orthoscheme* $\mathrm{conv}\{\mathbf{o}, \mathbf{r}_1, \ldots, \mathbf{r}_{d-2}\}$ *is the* (*uniquely determined*) *two-dimensional face* F_2 *of the Voronoi polytope* P *that is totally orthogonal to* $\mathrm{conv}\{\mathbf{o}, \mathbf{r}_1, \ldots, \mathbf{r}_{d-2}\}$ *at the point* $\mathbf{r}_{d-2}$ *and so* $\|\mathbf{r}_{d-2}\| = \mathrm{dist}(\mathbf{o}, \mathrm{aff}\, F_2)$ *with* $\mathbf{r}_{d-2} \in \mathrm{relint}\, F_2$.

Now we are ready for the definitions of wedges and truncated wedges. (As an illustration see Fig. 4.) Recall that for any two-dimensional face F_2 of the Voronoi polytope $P \subset \mathbf{E}^d$, $3 \le d$, we have that $\sqrt{2(d-2)/(d-1)} \le R(F_2)$.

Definition 3.

(1) Let F_2 be a two-dimensional face of the Voronoi polytope $P \subset \mathbf{E}^d$, $3 \le d$, with $\sqrt{2(d-2)/(d-1)} \le R(F_2) < \sqrt{2(d-1)/d}$ and let $\mathrm{conv}\{\mathbf{o}, \mathbf{r}_1, \ldots, \mathbf{r}_{d-2}\}$ be any $(d-2)$-dimensional Rogers simplex with $\mathbf{r}_{d-2} \in \mathrm{relint}\, F_2$. Then the union W_{I} of the d-dimensional Rogers simplices of P that contain the orthoscheme $\mathrm{conv}\{\mathbf{o}, \mathbf{r}_1, \ldots, \mathbf{r}_{d-2}\}$ is called a *wedge of type* I (*generated by the* $(d-2)$*-dimensional Rogers orthoscheme* $\mathrm{conv}\{\mathbf{o}, \mathbf{r}_1, \ldots, \mathbf{r}_{d-2}\}$). F_2 is called the *two-dimensional base* of W_{I} and $\|\mathbf{r}_{d-2}\| = \mathrm{dist}(\mathbf{o}, \mathrm{aff}\, F_2)$ is the *height* of W_{I} assigned to the base F_2.

(2) Let F_2 be a two-dimensional face of the Voronoi polytope $P \subset \mathbf{E}^d$, $3 \le d$, with $\sqrt{2(d-1)/d} \le R(F_2) < \sqrt{2d/(d+1)}$ and let $\mathrm{conv}\{\mathbf{o}, \mathbf{r}_1, \ldots, \mathbf{r}_{d-2}\}$ be any $(d-2)$-dimensional Rogers simplex with $\mathbf{r}_{d-2} \in \mathrm{relint}\, F_2$. Then the union

W_{II} of the d-dimensional Rogers simplices of P that contain the orthoscheme $\text{conv}\{\mathbf{o}, \mathbf{r}_1, \ldots, \mathbf{r}_{d-2}\}$ is called a *wedge of type* II (*generated by the* $(d-2)$*-dimensional Rogers orthoscheme* $\text{conv}\{\mathbf{o}, \mathbf{r}_1, \ldots, \mathbf{r}_{d-2}\}$). F_2 is called the *two-dimensional base* of W_{II} and $\|\mathbf{r}_{d-2}\| = \text{dist}(\mathbf{o}, \text{aff } F_2)$ is the *height* of W_{II} assigned to the base F_2.

(3) Let $\text{conv}\{\mathbf{o}, \mathbf{r}_1, \ldots, \mathbf{r}_d\}$ be the d-dimensional Rogers simplex assigned to the flag $F_0 \subset F_1 \cdots \subset F_{d-1}$ of the Voronoi polytope $P \subset \mathbf{E}^d$, $3 \le d$, with $\sqrt{2d/(d+1)} \le R(F_2)$. Then $W_{\text{III}} = \text{conv}\{\mathbf{o}, \mathbf{r}_1, \ldots, \mathbf{r}_d\}$ is called a *wedge of type* III.

It useful to recall that for any vertex F_0 of the Voronoi polytope $P \subset \mathbf{E}^d$ we have that $\sqrt{2d/(d+1)} \le R(F_0)$.

Definition 4. Let $\bar{B} = \{\mathbf{x} \in \mathbf{E}^d \,|\, \text{dist}(\mathbf{o}, \mathbf{x}) = \|\mathbf{x}\| \le \sqrt{2d/(d+1)}\}$.

(1) If W_{I} is a wedge of type I with the two-dimensional base F_2 which is generated by the $(d-2)$-dimensional Rogers orthoscheme $\text{conv}\{\mathbf{o}, \mathbf{r}_1, \ldots, \mathbf{r}_{d-2}\}$ of the Voronoi polytope $P \subset \mathbf{E}^d$, $3 \le d$, then

$$\bar{W}_{\text{I}} = \text{conv}((\bar{B} \cap F_2) \cup \{\mathbf{o} = \mathbf{r}_0, \ldots, \mathbf{r}_{d-3}\})$$

is called the *truncated wedge of type* I with the *two-dimensional base* $\bar{B} \cap F_2$ generated by the $(d-2)$-dimensional Rogers orthoscheme

$$\text{conv}\{\mathbf{o}, \mathbf{r}_1, \ldots, \mathbf{r}_{d-2}\}.$$

(2) If W_{II} is a wedge of type II with the two-dimensional base F_2 which is generated by the $(d-2)$-dimensional Rogers orthoscheme $\text{conv}\{\mathbf{o}, \mathbf{r}_1, \ldots, \mathbf{r}_{d-2}\}$ of the Voronoi polytope $P \subset \mathbf{E}^d$, $3 \le d$, then

$$\bar{W}_{\text{II}} = \text{conv}((\bar{B} \cap F_2) \cup \{\mathbf{o} = \mathbf{r}_0, \ldots, \mathbf{r}_{d-3}\})$$

is called the *truncated wedge of type* II with the *two-dimensional base* $\bar{B} \cap F_2$ generated by the $(d-2)$-dimensional Rogers orthoscheme

$$\text{conv}\{\mathbf{o}, \mathbf{r}_1, \ldots, \mathbf{r}_{d-2}\}.$$

As the following claim can be proved by Lemma 3 in a straightforward way, we omit its simple proof (see also Fig. 4).

Sublemma 2.

(1) *Let* W_{I} (*resp*., W_{II}) *denote the wedge of type* I (*resp*., *of type* II) *with the two-dimensional base* F_2 *which is generated by the* $(d-2)$*-dimensional Rogers orthoscheme* $\text{conv}\{\mathbf{o}, \mathbf{r}_1, \ldots, \mathbf{r}_{d-2}\}$ *of the Voronoi polytope* $P \subset \mathbf{E}^d$, $3 \le d$. *If the points* $\mathbf{x}, \mathbf{y} \in \text{aff } F_2$ *are chosen so that the triangle* $\triangle \mathbf{r}_{d-2}\mathbf{x}\mathbf{y}$ *has a right angle at the vertex* $\mathbf{x}$, *then* $\text{conv}\{\mathbf{o}, \mathbf{r}_1, \ldots, \mathbf{r}_{d-2}, \mathbf{x}, \mathbf{y}\}$ *is a* d*-dimensional orthoscheme*. *Moreover*, *if* $\mathbf{z} \in \text{aff } F_2$ *is an arbitrary point*, *then* $\text{conv}\{\mathbf{o} = \mathbf{r}_0, \ldots, \mathbf{r}_{d-3}, \mathbf{z}\}$ *is a* $(d-2)$*-dimensional orthoscheme*.

(2) *Let* W_{I} *denote the wedge of type* I *with the two-dimensional base* F_2 *which is generated by the* $(d-2)$*-dimensional Rogers orthoscheme* $\mathrm{conv}\{\mathbf{o} = \mathbf{r}_0, \mathbf{r}_1, \dots, \mathbf{r}_{d-2}\}$ *of the Voronoi polytope* $P \subset \mathbf{E}^d$, $3 \le d$. *Let* $Q_2 \subset \mathrm{aff} F_2$ *and* $Q_2^* \subset \mathrm{aff} F_2$ *be compact convex sets with* $\mathrm{relint}\, Q_2 \cap \mathrm{relint}\, Q_2^* = \emptyset$. *If* $K_2 = Q_2$ (*resp.*, $K_2^* = Q_2^*$) *and* $K_j = \mathrm{conv}(K_{j-1} \cup \{\mathbf{r}_{d-j}\})$ (*resp.*, $K_j^* = \mathrm{conv}(K_{j-1}^* \cup \{\mathbf{r}_{d-j}\})$) *for* $j = 3, \dots, d$, *then* $K_d = \mathrm{conv}(Q_2 \cup \{\mathbf{o} = \mathbf{r}_0, \dots, \mathbf{r}_{d-3}\})$ (*resp.*, $K_d^* = \mathrm{conv}(Q_2^* \cup \{\mathbf{o} = \mathbf{r}_0, \dots, \mathbf{r}_{d-3}\})$), *moreover*, $\mathrm{relint}\, K_d \cap \mathrm{relint}\, K_d^* = \emptyset$. *A similar statement holds for* W_{II}.

(3) *Let* W_{I} (*resp.*, $\bar{W}_{\mathrm{I}}$) *denote the wedge of type* I (*resp.*, *truncated wedge of type* I) *with the two-dimensional base* F_2 (*resp.*, $\bar{B} \cap F_2$) *which is generated by the* $(d-2)$*-dimensional Rogers orthoscheme* $\mathrm{conv}\{\mathbf{o} = \mathbf{r}_0, \mathbf{r}_1, \dots, \mathbf{r}_{d-2}\}$ *of the Voronoi polytope* $P \subset \mathbf{E}^d$, $3 \le d$. *If* $K_2 = F_2$ (*resp.*, $K_2 = \bar{B} \cap F_2$) *and* $K_j = \mathrm{conv}(K_{j-1} \cup \{\mathbf{r}_{d-j}\})$ *for* $j = 3, \dots, d$, *then* $K_d = W_{\mathrm{I}}$ (*resp.*, $K_d = \bar{W}_{\mathrm{I}}$). *Similar statements hold for* W_{II} *and* $\bar{W}_{\mathrm{II}}$.

The core part of this section is Corollary 2 that follows from Lemma 4 in a trivial way.

Lemma 4. *Let* F_2 *be an arbitrary two-dimensional face of the Voronoi polytope* $P \subset \mathbf{E}^d$ *of dimension* $d \ge 8$. *Then the number of sides* F_1 *of the face* F_2 *with* $R(F_1) < \sqrt{2d/(d+1)}$ *is at most four.*

Proof. Assume that there are five sides (i.e. boundary line segments) say, E_1, E_2, E_3, E_4 and E_5 of the face F_2 of the Voronoi polytope $P \subset \mathbf{E}^d$, $d \ge 8$, with the property that

$$R(E_m) = \mathrm{dist}(\mathbf{o}, \mathrm{aff}\, E_m) < \sqrt{\frac{2d}{d+1}}, \qquad \text{for all} \quad 1 \le m \le 5.$$

First (as we have seen above), we can pick $(d-2)$ unit balls of $\mathcal{P}$ with centers, say, $\mathbf{c}_1, \dots, \mathbf{c}_{d-2}$ such that each of their Voronoi cells contains F_2 (and so $\mathrm{aff}\{\mathbf{o}, \mathbf{c}_1, \dots, \mathbf{c}_{d-2}\} = \mathrm{lin}\{\mathbf{c}_1, \dots, \mathbf{c}_{d-2}\}$ is totally orthogonal to $\mathrm{aff}\, F_2$); moreover, the point $\mathbf{b} = \mathrm{lin}\{\mathbf{c}_1, \dots, \mathbf{c}_{d-2}\} \cap \mathrm{aff}\, F_2$ belongs to the $(d-2)$-dimensional simplex $C_{d-2} = \mathrm{conv}\{\mathbf{o}, \mathbf{c}_1, \dots, \mathbf{c}_{d-2}\}$. (Also, notice that $\mathbf{b}$ is the center of the $(d-3)$-dimensional sphere that passes through the vertices of C_{d-2}.) The existence of the sides E_m, $1 \le m \le 5$, implies that $R(F_2) < \sqrt{2d/(d+1)} < \sqrt{2}$ and so, via Lemma 2, we get that $\mathbf{b} \in \mathrm{relint}\, F_2$.

Second, for each side E_m, $1 \le m \le 5$, there exists a unit ball of $\mathcal{P}$ with center, say, $\mathbf{c}_{f(m)}$ such that its Voronoi cell intersects F_2 in E_m. Let the hyperplane of $\mathbf{E}^d$ spanned by the $(d-2)$-dimensional simplex C_{d-2} and the center point $\mathbf{c}_{f(m)}$ be denoted by $H_{f(m)}$. As the hyperplanes $H_{f(m)}$, $1 \le m \le 5$, all contain the $(d-2)$-dimensional linear subspace $\mathrm{lin}\{\mathbf{c}_1, \dots, \mathbf{c}_{d-2}\}$ there must be i and j such that the angle between H_i and H_j is $2\pi/5$ or less. We complete the proof by showing that this forces $\mathrm{dist}(\mathbf{c}_i, \mathbf{c}_j) < 2$, an impossibility.

Let $\mathbf{a}_i$ (resp., $\mathbf{a}_j$) be the center of the $(d-2)$-dimensional sphere that passes through the points $\mathbf{o}, \mathbf{c}_1, \dots, \mathbf{c}_{d-2}, \mathbf{c}_i$ (resp., $\mathbf{o}, \mathbf{c}_1, \dots, \mathbf{c}_{d-2}, \mathbf{c}_j$) (see Fig. 5). By assumption, $\|\mathbf{a}_i\| = \mathrm{dist}(\mathbf{o}, \mathbf{a}_i) = R(E_{f^{-1}(i)}) < \sqrt{2d/(d+1)}$ (resp., $\|\mathbf{a}_j\| = \mathrm{dist}(\mathbf{o}, \mathbf{a}_j) = R(E_{f^{-1}(j)}) < \sqrt{2d/(d+1)}$) and so Lemma 2 implies that $\mathbf{a}_i \in \mathrm{relint}\, E_{f^{-1}(i)}$ (resp., $\mathbf{a}_j \in \mathrm{relint}\, E_{f^{-1}(j)}$). Now, let $\mathbf{a}_i^*$ (resp., $\mathbf{a}_j^*$) be the uniquely determined point on the halfline that emanates from

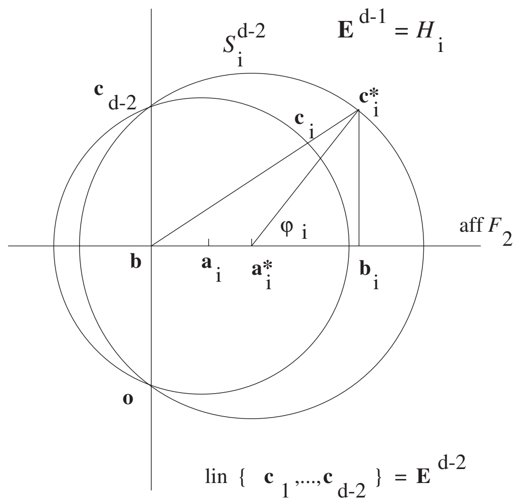

Fig. 5

b and passes through $\mathbf{a}_i$ (resp., $\mathbf{a}_j$) with the property that $\|\mathbf{a}_i^*\| = \sqrt{2d/(d+1)}$ (resp., $\|\mathbf{a}_j^*\| = \sqrt{2d/(d+1)}$). Moreover, if $S_i^{d-2} = \{\mathbf{x} \in H_i \mid \|\mathbf{x} - \mathbf{a}_i^*\| = \sqrt{2d/(d+1)}\}$ (resp., $S_j^{d-2} = \{\mathbf{x} \in H_j \mid \|\mathbf{x} - \mathbf{a}_j^*\| = \sqrt{2d/(d+1)}\}$), then let $\mathbf{c}_i^*$ (resp., $\mathbf{c}_j^*$) be the intersection of S_i^{d-2} (resp., S_j^{d-2}) with the halfline emanating from **b** and passing through the point $\mathbf{c}_i$ (resp., $\mathbf{c}_j$). Finally, let $\mathbf{b}_i$ (resp., $\mathbf{b}_j$) be the orthogonal projection of the point $\mathbf{c}_i^*$ (resp., $\mathbf{c}_j^*$) onto the plane aff F_2. As $\text{lin}\{\mathbf{c}_1, \ldots, \mathbf{c}_{d-2}\}$ is orthogonal to aff F_2 and $\text{lin}\{\mathbf{c}_1, \ldots, \mathbf{c}_{d-2}\} \subset H_i$ (resp., $\text{lin}\{\mathbf{c}_1, \ldots, \mathbf{c}_{d-2}\} \subset H_j$) the points $\mathbf{b}, \mathbf{a}_i, \mathbf{a}_i^*, \mathbf{b}_i$ (resp., $\mathbf{b}, \mathbf{a}_j, \mathbf{a}_j^*, \mathbf{b}_j$) lie on a line of aff F_2.

Sublemma 3. $\text{dist}(\mathbf{c}_i, \mathbf{c}_j) < \text{dist}(\mathbf{c}_i^*, \mathbf{c}_j^*)$.

Proof. Let $s_i = \|\mathbf{c}_i - \mathbf{b}\|$ and $s_i^* = \|\mathbf{c}_i^* - \mathbf{b}\|$ (resp., $s_j = \|\mathbf{c}_j - \mathbf{b}\|$ and $s_j^* = \|\mathbf{c}_j^* - \mathbf{b}\|$). If $\psi = \angle \mathbf{c}_i \mathbf{b} \mathbf{c}_j = \angle \mathbf{c}_i^* \mathbf{b} \mathbf{c}_j^*$, then

$$\|\mathbf{c}_i - \mathbf{c}_j\|^2 = s_i^2 + s_j^2 - 2 s_i s_j \cos\psi \quad \text{and}$$
$$\|\mathbf{c}_i^* - \mathbf{c}_j^*\|^2 = (s_i^*)^2 + (s_j^*)^2 - 2(s_i^*)(s_j^*) \cos\psi.$$

As $s_i < s_i^*$ and $s_j < s_j^*$ it is sufficient to show that

$$\frac{\partial}{\partial s_i}\|\mathbf{c}_i - \mathbf{c}_j\|^2 = 2(s_i - s_j\cos\psi) > 0 \quad \text{and}$$
$$\frac{\partial}{\partial s_j}\|\mathbf{c}_i - \mathbf{c}_j\|^2 = 2(s_j - s_i\cos\psi) > 0.$$

By symmetry it is sufficient to show that $s_i - s_j \cos\psi > 0$. Assume that $s_i \le s_j \cos\psi$. Then, on the one hand, $\pi/2 \le \angle \mathbf{b}\mathbf{c}_i\mathbf{c}_j$ and so $4 < s_i^2 + \|\mathbf{c}_i - \mathbf{c}_j\|^2 \le s_j^2$. On the other hand,

$s_j \le \|\mathbf{a}_j - \mathbf{b}\| + \|\mathbf{c}_j - \mathbf{a}_j\| = \sqrt{\|\mathbf{a}_j\|^2 - \|\mathbf{b}\|^2} + \|\mathbf{a}_j\|$. By assumption and Corollary 1 we know that $\|\mathbf{a}_j\| < \sqrt{2d/(d+1)}$ and $\sqrt{2(d-2)/(d-1)} \le \|\mathbf{b}\|$ and so

$$s_j < \sqrt{\frac{2d}{d+1} - \frac{2(d-2)}{d-1}} + \sqrt{\frac{2d}{d+1}} \le 2 \qquad \text{for all} \quad d \ge 3,$$

a contradiction. This completes the proof of Sublemma 3. □

Thus, it will suffice to prove that $\operatorname{dist}(\mathbf{c}_i^*, \mathbf{c}_j^*) \le 2$.

Let $\varphi_i = \pi - \angle \mathbf{b}\mathbf{a}_i^*\mathbf{c}_i^*$ (resp., $\varphi_j = \pi - \angle \mathbf{b}\mathbf{a}_j^*\mathbf{c}_j^*$). Sublemma 4, which we prove at the end of this section, shows that $0 \le \varphi_i \le \pi/2$ (resp., $0 \le \varphi_j \le \pi/2$) as indicated in Fig. 5.

If $\phi = \angle \mathbf{a}_i\mathbf{b}\mathbf{a}_j$ and $l_i = \|\mathbf{b}_i - \mathbf{b}\|$ (resp., $l_j = \|\mathbf{b}_j - \mathbf{b}\|$), then we get that

$$\begin{aligned}\|\mathbf{c}_i^* - \mathbf{c}_j^*\|^2 &\le \|\mathbf{b}_i - \mathbf{b}_j\|^2 + (\|\mathbf{c}_i^* - \mathbf{b}_i\| + \|\mathbf{c}_j^* - \mathbf{b}_j\|)^2 \\ &= (l_i^2 + l_j^2 - 2l_il_j\cos\phi) + \frac{2d}{d+1}(\sin\varphi_i + \sin\varphi_j)^2.\end{aligned}$$

By assumption $0 \le \phi \le 2\pi/5$ and so

$$\|\mathbf{c}_i^* - \mathbf{c}_j^*\|^2 \le \left(l_i^2 + l_j^2 - 2l_il_j\cos\frac{2\pi}{5}\right) + \frac{2d}{d+1}(\sin\varphi_i + \sin\varphi_j)^2.$$

Substituting $l_i = \sqrt{\frac{2d}{d+1} - \|\mathbf{b}\|^2} + \sqrt{\frac{2d}{d+1}}\cos\varphi_i$ $\left(\text{resp., } l_j = \sqrt{\frac{2d}{d+1} - \|\mathbf{b}\|^2} + \sqrt{\frac{2d}{d+1}}\cos\varphi_j\right)$ yields

$$\begin{aligned}\|\mathbf{c}_i^* - \mathbf{c}_j^*\|^2 \le{} & \left(2 - \cos\frac{2\pi}{5}\right)\frac{4d}{d+1} - 2\left(1 - \cos\frac{2\pi}{5}\right)\|\mathbf{b}\|^2 \\ & + \frac{4d}{d+1}\sin\varphi_i\sin\varphi_j - \left(\frac{4d}{d+1}\cos\frac{2\pi}{5}\right)\cos\varphi_i\cos\varphi_j \\ & + 2\left(1 - \cos\frac{2\pi}{5}\right)\sqrt{\frac{2d}{d+1}}(\cos\varphi_i + \cos\varphi_j)\sqrt{\frac{2d}{d+1} - \|\mathbf{b}\|^2}.\end{aligned}$$

Corollary 1 implies that $\sqrt{2(d-2)/(d-1)} \le \|\mathbf{b}\|$ and so

$$\begin{aligned}\|\mathbf{c}_i^* - \mathbf{c}_j^*\|^2 \le{} & \left(2 - \cos\frac{2\pi}{5}\right)\frac{4d}{d+1} - \left(1 - \cos\frac{2\pi}{5}\right)\frac{4(d-2)}{d-1} \\ & + \frac{4d}{d+1}\sin\varphi_i\sin\varphi_j - \left(\frac{4d}{d+1}\cos\frac{2\pi}{5}\right)\cos\varphi_i\cos\varphi_j \\ & + 2\left(1 - \cos\frac{2\pi}{5}\right)\sqrt{\frac{2d}{d+1}}(\cos\varphi_i + \cos\varphi_j)\sqrt{\frac{2d}{d+1} - \frac{2(d-2)}{d-1}} \\ ={} & G(\varphi_i, \varphi_j).\end{aligned}$$

Then straightforward computation yields

$$\frac{\partial}{\partial \varphi_i} G(\varphi_i, \varphi_j)$$
$$= \frac{4d}{d+1} \sin\varphi_j \cos\varphi_i + \sin\varphi_i \left(\frac{4d}{d+1} \cos\frac{2\pi}{5} \cos\varphi_j - 4\left(1 - \cos\frac{2\pi}{5}\right)\sqrt{\frac{2d}{(d+1)^2(d-1)}} \right)$$

and

$$\frac{\partial}{\partial \varphi_j} G(\varphi_i, \varphi_j)$$
$$= \frac{4d}{d+1} \sin\varphi_i \cos\varphi_j + \sin\varphi_j \left(\frac{4d}{d+1} \cos\frac{2\pi}{5} \cos\varphi_i - 4\left(1 - \cos\frac{2\pi}{5}\right)\sqrt{\frac{2d}{(d+1)^2(d-1)}} \right).$$

Now we complete the proof of Lemma 4 by Sublemma 4, which we are left to prove at the end of this section. The details are as follows.

According to Sublemma 4 the angle φ_i (resp., φ_j) is maximized when $\|\mathbf{b}\| = \sqrt{2(d-2)/(d-1)}$ (i.e. when the norm of $\mathbf{b}$ is minimized, i.e. when the $(d-2)$-dimensional simplex $C_{d-2} = \operatorname{conv}\{\mathbf{o}, \mathbf{c}_1, \ldots, \mathbf{c}_{d-2}\}$ is a regular simplex of edge length 2) and the unit ball centered at $\mathbf{c}_i^*$ (resp., $\mathbf{c}_j^*$) is tangent to exactly $(d-2)$ unit balls centered at $(d-2)$ points out of $\mathbf{o}, \mathbf{c}_1, \ldots, \mathbf{c}_{d-2}$. In other words, as is claimed below in Sublemma 4,

$$\frac{\sqrt{2}}{3}\frac{2d-1}{\sqrt{d(d-1)}} \le \cos\varphi_i \quad \text{and} \quad \frac{\sqrt{2}}{3}\frac{2d-1}{\sqrt{d(d-1)}} \le \cos\varphi_j \qquad \text{for all} \quad d \ge 2.$$

From this it follows in a straightforward way that

$$\frac{\partial}{\partial \varphi_i} G(\varphi_i, \varphi_j) \ge 0 \quad \text{and} \quad \frac{\partial}{\partial \varphi_j} G(\varphi_i, \varphi_j) \ge 0 \qquad \text{for all} \quad d \ge 4.$$

Therefore $G(\varphi_i, \varphi_j)$ is maximized when φ_i and φ_j are as large as possible, that is, using Sublemma 4 again when $\cos\varphi_i = \cos\varphi_j = (\sqrt{2}/3)((2d-1)/\sqrt{d(d-1)})$. This yields

$$\|\mathbf{c}_i^* - \mathbf{c}_j^*\|^2$$
$$\le \frac{(40 - 32\cos(2\pi/5))d^2 + (56 - 64\cos(2\pi/5))d + (16 - 32\cos(2\pi/5))}{9(d^2-1)}$$
$$\le 4 \qquad \text{for all} \quad d \ge 8$$

as desired.

We are left to prove the following statement.

Sublemma 4.

$$\frac{\sqrt{2}}{3}\frac{2d-1}{\sqrt{d(d-1)}} \le \cos\varphi_i \quad \textit{and} \quad \frac{\sqrt{2}}{3}\frac{2d-1}{\sqrt{d(d-1)}} \le \cos\varphi_j \qquad \textit{for all} \quad d \ge 4.$$

Proof. The extremal problem whose solution leads to the stated inequalities is the following one. Let the $(d-1)$-dimensional simplex $X = \operatorname{conv}\{\mathbf{x}_0, \mathbf{x}_1, \dots, \mathbf{x}_{d-2}, \mathbf{x}_{d-1}\}$ of all edge lengths being at least 2 be inscribed in the $(d-2)$-dimensional sphere $S^{d-2} = \{\mathbf{y} \in \mathbf{E}^{d-1} \mid \|\mathbf{y} - \mathbf{a}\| = \sqrt{2d/(d+1)}\}$ centered at the point $\mathbf{a}$ in $\mathbf{E}^{d-1}$, $d \ge 4$. Assume that the center $\mathbf{b}$ of the $(d-3)$-dimensional sphere that passes through the vertices $\mathbf{x}_0, \mathbf{x}_1, \dots, \mathbf{x}_{d-2}$ of X belongs to $\operatorname{conv}\{\mathbf{x}_0, \mathbf{x}_1, \dots, \mathbf{x}_{d-2}\}$. Now, if $\varphi = \pi - \angle \mathbf{b}\mathbf{a}\mathbf{x}_{d-1}$, then find the maximum of φ for the simplices X. (Notice that the given metric conditions force the points $\mathbf{a}$ and $\mathbf{x}_{d-1}$ to lie in the same open halfspace bounded by the hyperplane $\operatorname{aff}\{\mathbf{x}_0, \mathbf{x}_1, \dots, \mathbf{x}_{d-2}\}$ in $\mathbf{E}^{d-1}$.)

For the sake of simplicity let X itself denote an extremal simplex, i.e. a $(d-1)$-dimensional simplex of the above type for which φ is maximal. Now it is easy to show—by contradiction and by moving the vertex $\mathbf{x}_{d-1}$ in the proper direction on S^{d-2}—that there exists a $(d-3)$-dimensional face of the $(d-2)$-dimensional simplex $\operatorname{conv}\{\mathbf{x}_0, \mathbf{x}_1, \dots, \mathbf{x}_{d-2}\}$, say, $\operatorname{conv}\{\mathbf{x}_1, \dots, \mathbf{x}_{d-2}\}$, such that all of its vertices lie at distance 2 from the point $\mathbf{x}_{d-1}$, i.e.

$$\|\mathbf{x}_1 - \mathbf{x}_{d-1}\| = \|\mathbf{x}_2 - \mathbf{x}_{d-1}\| = \cdots = \|\mathbf{x}_{d-2} - \mathbf{x}_{d-1}\| = 2.$$

However, then again it is easy to prove—by contradiction and by moving the vertex $\mathbf{x}_0$ in the proper direction on S^{d-2}—that the vertices of the $(d-3)$-dimensional simplex $\operatorname{conv}\{\mathbf{x}_1, \dots, \mathbf{x}_{d-2}\}$ must lie at distance 2 from the vertex $\mathbf{x}_0$ as well, i.e.

$$\|\mathbf{x}_1 - \mathbf{x}_0\| = \|\mathbf{x}_2 - \mathbf{x}_0\| = \cdots = \|\mathbf{x}_{d-2} - \mathbf{x}_0\| = 2.$$

Now, let $\mathbf{c}$ be the center of the $(d-4)$-dimensional sphere S^{d-4} that passes through the vertices of the $(d-3)$-dimensional simplex $\operatorname{conv}\{\mathbf{x}_1, \dots, \mathbf{x}_{d-2}\}$ in $\operatorname{aff}\{\mathbf{x}_1, \dots, \mathbf{x}_{d-2}\}$ (see Fig. 6).

As the vertices $\mathbf{x}_1, \dots, \mathbf{x}_{d-2}$ lie at distance 2 from the vertices $\mathbf{x}_0$ and $\mathbf{x}_{d-1}$ it is clear that $\operatorname{aff}\{\mathbf{x}_0, \mathbf{x}_{d-1}, \mathbf{c}\}$ is orthogonal to $\operatorname{aff}\{\mathbf{x}_1, \dots, \mathbf{x}_{d-2}\}$ (and, of course, $\mathbf{c} = \operatorname{aff}\{\mathbf{x}_0, \mathbf{x}_{d-1}, \mathbf{c}\} \cap$

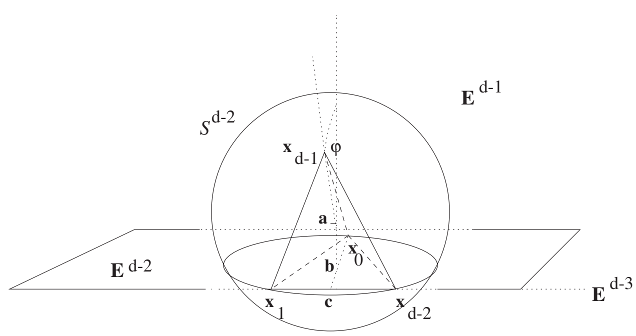

Fig. 6

aff$\{\mathbf{x}_1, \ldots, \mathbf{x}_{d-2}\}$). Moreover, as $\mathbf{b} \in \text{conv}\{\mathbf{x}_0, \mathbf{x}_1, \ldots, \mathbf{x}_{d-2}\}$ and $\mathbf{x}_0$ (resp., $\mathbf{b}$) lie equidistant from the vertices $\mathbf{x}_1, \ldots, \mathbf{x}_{d-2}$, therefore $\mathbf{b}$ belongs to the line segment $\mathbf{x}_0\mathbf{c}$. Finally, as $\mathbf{a} - \mathbf{b}$ is orthogonal to aff$\{\mathbf{x}_0, \mathbf{x}_1, \ldots, \mathbf{x}_{d-2}\}$ we get that $\mathbf{a} \in \text{aff}\{\mathbf{x}_0, \mathbf{x}_{d-1}, \mathbf{c}\}$ and so $\angle \mathbf{bax}_{d-1}$ and φ are both represented in the plane aff$\{\mathbf{x}_0, \mathbf{x}_{d-1}, \mathbf{c}\}$. Now, let $\mathbf{u}, \mathbf{v}$ be any pair of diametrically opposite points of S^{d-4}. Then by construction aff$\{\mathbf{u}, \mathbf{v}\}$ is orthogonal to aff$\{\mathbf{x}_0, \mathbf{x}_{d-1}, \mathbf{c}\}$ and

$$\|\mathbf{u} - \mathbf{x}_0\| = \|\mathbf{u} - \mathbf{x}_{d-1}\| = \|\mathbf{v} - \mathbf{x}_0\| = \|\mathbf{v} - \mathbf{x}_{d-1}\| = 2,$$

$$\|\mathbf{u} - \mathbf{a}\| = \|\mathbf{v} - \mathbf{a}\| = \|\mathbf{x}_0 - \mathbf{a}\| = \|\mathbf{x}_{d-1} - \mathbf{a}\| = \sqrt{\frac{2d}{d+1}},$$

$$\|\mathbf{u} - \mathbf{b}\| = \|\mathbf{v} - \mathbf{b}\| = \|\mathbf{x}_0 - \mathbf{b}\|,$$

moreover, Corollary 1 applied to the $(d-3)$-dimensional simplex conv$\{\mathbf{x}_1, \ldots, \mathbf{x}_{d-2}\}$ yields

$$\sqrt{\frac{2(d-3)}{d-2}} \le \|\mathbf{u} - \mathbf{c}\| = \|\mathbf{v} - \mathbf{c}\| < \sqrt{\frac{2d}{d+1}}.$$

From now on we work in the three-dimensional Euclidean space aff$\{\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{x}_0, \mathbf{x}_{d-1}, \mathbf{u}, \mathbf{v}\}$ (see Fig. 7). Let

$$x = \|\mathbf{u} - \mathbf{c}\| = \|\mathbf{v} - \mathbf{c}\|,$$

$$l = \|\mathbf{u} - \mathbf{a}\| = \|\mathbf{v} - \mathbf{a}\| = \|\mathbf{x}_0 - \mathbf{a}\| = \|\mathbf{x}_{d-1} - \mathbf{a}\| \quad \text{and}$$

$$\rho = \angle \mathbf{bac}, \qquad \tau = \angle \mathbf{cax}_{d-1}.$$

Easy elementary geometry yields

$$\cos\rho = \sqrt{\frac{l^2(4-x^2)-4}{(4-x^2)(l^2-x^2)}} \quad \text{and} \quad \cos\tau = \frac{l^2-2}{l\sqrt{l^2-x^2}}.$$

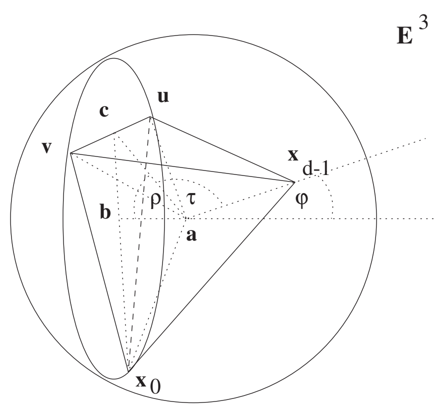

Fig. 7

Therefore differentiating ρ and τ as functions of x we get that

$$\frac{d\rho}{dx} = \frac{-x[l^2x^4 - 8(l^2-1)x^2 + 12l^2 - 16]}{(2-x^2)(4-x^2)(l^2-x^2)\sqrt{4l^2-4-l^2x^2}} \quad \text{and}$$

$$\frac{d\tau}{dx} = \frac{(2-l^2)x}{(l^2-x^2)\sqrt{4l^2-4-l^2x^2}}.$$

From these, using $1 \le x < l < \sqrt{2}$, we easily obtain that $d(\rho+\tau)/dx > 0$ if and only if

$$x^4 + \frac{10-7l^2}{l^2-1}x^2 + \frac{10l^2-16}{l^2-1} < 0.$$

Therefore $l = \sqrt{2d/(d+1)}$ yields that $d(\rho+\tau)/dx > 0$ if and only if

$$f(x) = x^4 - \frac{4d-10}{d-1}x^2 + \frac{4d-16}{d-1} < 0.$$

Notice that $f(\sqrt{2(d-4)/(d-1)}) = f(\sqrt{2}) = 0$ and $f(z) < 0$ for all $\sqrt{2(d-4)/(d-1)} < z < \sqrt{2}$. Finally, as $2 \le \|\mathbf{x}_{d-1} - \mathbf{x}_0\|$ an easy computation shows that, in fact, $x \le \sqrt{2(d-2)/(d-1)}$ and so

$$\sqrt{\frac{2(d-4)}{d-1}} < \sqrt{\frac{2(d-3)}{d-2}} \le x \le \sqrt{\frac{2(d-2)}{d-1}} < \sqrt{\frac{2d}{d+1}} < \sqrt{2}.$$

Thus, $\rho + \tau$ as a function of x is strictly increasing on the interval

$$\left[\sqrt{\frac{2(d-3)}{d-2}}, \sqrt{\frac{2(d-2)}{d-1}}\right].$$

Hence, $\pi - (\rho+\tau)$ as a function of x is strictly decreasing on the same interval. Consequently, φ as the largest possible value of $\pi - (\rho+\tau)$ is attained at $x = \sqrt{2(d-3)/(d-2)}$. This yields $\cos\varphi = (\sqrt{2}/3)((2d-1)/\sqrt{d(d-1)})$ finishing the proof of Sublemma 4. □

This completes the proof of Lemma 4. □

As an immediate corollary of Lemma 4 we get the following statement.

Corollary 2. *Let $\bar{B} \cap F_2$ be the two-dimensional base of the type* I *truncated wedge* $\bar{W}_{\mathrm{I}}$ (*resp*., *type* II *truncated wedge* $\bar{W}_{\mathrm{II}}$) *in the Voronoi polytope $P \subset \mathbf{E}^d$ of dimension $d \ge 8$. Then the number of line segments of positive length in* $\operatorname{relbd}(\bar{B} \cap F_2)$ *is at most four.*

3. The Lemma of Comparison and the Integral Representation of the Surface Density in (Truncated) Wedges of Type I and II

Recall that $B = \{\mathbf{x} \in \mathbf{E}^d \mid \operatorname{dist}(\mathbf{o}, \mathbf{x}) = \|\mathbf{x}\| \le 1\}$ and let

$$S = \{\mathbf{x} \in \mathbf{E}^d \mid \operatorname{dist}(\mathbf{o}, \mathbf{x}) = \|\mathbf{x}\| = 1\}.$$

Then let $H \subset \mathbf{E}^d$ be a hyperplane disjoint from the interior of the unit ball B and let $Q \subset H$ be an arbitrary $(d-1)$-dimensional compact convex set. If $[\mathbf{o}, Q]$ denotes the convex cone $\mathrm{conv}(\{\mathbf{o}\} \cup Q)$ with apex $\mathbf{o}$ and base Q, then *the* (*volume*) *density* $\delta([\mathbf{o}, Q], B)$ of the unit ball B in the cone $[\mathbf{o}, Q]$ is defined as

$$\delta([\mathbf{o}, Q], B) = \frac{\mathrm{Vol}_d([\mathbf{o}, Q] \cap B)}{\mathrm{Vol}_d([\mathbf{o}, Q])},$$

where $\mathrm{Vol}_d(\cdots)$ refers to the corresponding d-dimensional Euclidean volume measure [4]. It is natural to introduce the following very similar notion.

Definition 5. *The surface density* $\hat{\delta}([\mathbf{o}, Q], S)$ of the unit sphere S in the convex cone $[\mathbf{o}, Q]$ with apex $\mathbf{o}$ and base Q is defined by

$$\hat{\delta}([\mathbf{o}, Q], S) = \frac{\mathrm{SVol}_{d-1}([\mathbf{o}, Q] \cap S)}{\mathrm{Vol}_{d-1}(Q)},$$

where $\mathrm{SVol}_{d-1}(\cdots)$ refers to the corresponding $(d-1)$-dimensional spherical volume measure.

If $h = \mathrm{dist}(\mathbf{o}, H)$, then clearly $h \cdot \delta([\mathbf{o}, Q], B) = \hat{\delta}([\mathbf{o}, Q], S)$. We will need the following statement, the first part of which is due to Rogers [23] and the second part of which has been recently proved in Theorem 1 of [4].

Lemma 5 (Lemma of Comparison). *Let* $U = \mathrm{conv}\{\mathbf{o}, \mathbf{u}_1, \ldots, \mathbf{u}_d\}$ *be a d-dimensional orthoscheme in* $\mathbf{E}^d$ *and let* $V = \mathrm{conv}\{\mathbf{o}, \mathbf{v}_1, \ldots, \mathbf{v}_d\}$ *be a d-dimensional simplex of* $\mathbf{E}^d$ *such that* $\|\mathbf{v}_i\| = \mathrm{dist}(\mathbf{o}, \mathrm{conv}\{\mathbf{v}_i, \mathbf{v}_{i+1}, \ldots, \mathbf{v}_d\})$ *for all* $1 \le i \le d-1$. *If* $1 \le \|\mathbf{u}_i\| \le \|\mathbf{v}_i\|$ *holds for all* $1 \le i \le d$, *then*

(1) $\delta(U, B) \ge \delta(V, B)$ *and*
(2) $\hat{\delta}(U, S) \ge \hat{\delta}(V, S)$.

At this point it is useful to introduce the following notations. (Notice that Sublemma 2 provides the necessary geometry for Definition 7.)

Definition 6. Let $\mathbf{x}_1, \ldots, \mathbf{x}_n$, $n \ge 1$, be points in $\mathbf{E}^d$, $d \ge 1$, and let $X \subset \mathbf{E}^d$ be an arbitrary convex set. If $X_0 = X$ and $X_m = \mathrm{conv}(\{\mathbf{x}_{n-(m-1)}\} \cup X_{m-1})$ for $m = 1, \ldots, n$, then we denote the final convex set X_n by

$$[\mathbf{x}_1, \ldots, \mathbf{x}_n, X].$$

Definition 7. Let W_{I} (resp., $\bar{W}_{\mathrm{I}}$) denote the wedge (resp., truncated wedge) of type I with the two-dimensional base F_2 (resp., $\bar{B} \cap F_2$) which is generated by the $(d-2)$-dimensional Rogers orthoscheme $\mathrm{conv}\{\mathbf{o}, \mathbf{r}_1, \ldots, \mathbf{r}_{d-2}\}$ of the Voronoi polytope $P \subset \mathbf{E}^d$, $d \ge 4$. Then let

$$Q_{\mathrm{I}} = [\mathbf{r}_1, \ldots, \mathbf{r}_{d-3}, F_2] \qquad (\text{resp., } \bar{Q}_{\mathrm{I}} = [\mathbf{r}_1, \ldots, \mathbf{r}_{d-3}, \bar{B} \cap F_2])$$

be called the $(d-1)$-*dimensional base* of the type I wedge $W_{\mathrm{I}} = [\mathbf{o}, Q_{\mathrm{I}}]$ (resp., type I truncated wedge $\bar{W}_{\mathrm{I}} = [\mathbf{o}, \bar{Q}_{\mathrm{I}}]$). Similarly, we define the $(d-1)$-dimensional bases Q_{II} and $\bar{Q}_{\mathrm{II}}$ of W_{II} and $\bar{W}_{\mathrm{II}}$. Finally, let

$$h_1 = \|\mathbf{r}_1\|, \quad h_2 = \|\mathbf{r}_2 - \mathbf{r}_1\|, \quad \ldots, \quad h_{d-2} = \|\mathbf{r}_{d-2} - \mathbf{r}_{d-3}\|.$$

Sublemma 5. *Let* W_{I} (*resp.*, W_{II}) *denote the wedge of type* I (*resp., of type* II) *with the two-dimensional base* F_2 *which is generated by the* $(d-2)$*-dimensional Rogers orthoscheme* $\mathrm{conv}\{\mathbf{o}, \mathbf{r}_1, \ldots, \mathbf{r}_{d-2}\}$ *of the Voronoi polytope* $P \subset \mathbf{E}^d$, $d \geq 4$. *Then we have the following volume formulas*:

(1)

$$\mathrm{Vol}_{d-1}(Q_{\mathrm{I}}) = \frac{2}{(d-1)!}\left(\prod_{i=2}^{d-2} h_i\right)\mathrm{Vol}_2(F_2) \quad \textit{and}$$

$$\mathrm{Vol}_d(W_{\mathrm{I}}) = \frac{h_1}{d}\mathrm{Vol}_{d-1}(Q_{\mathrm{I}}) = \frac{2}{d!}\left(\prod_{i=1}^{d-2} h_i\right)\mathrm{Vol}_2(F_2).$$

Similar formulas hold for the corresponding dimensional volumes of $\bar{Q}_{\mathrm{I}}$, $\bar{W}_{\mathrm{I}}$, Q_{II}, W_{II}, $\bar{Q}_{\mathrm{II}}$ *and* $\bar{W}_{\mathrm{II}}$.

(2) *In general, if* $K \subset \mathrm{aff}\, F_2$ *is a convex domain, then*

$$\mathrm{Vol}_{d-1}([\mathbf{r}_1, \ldots, \mathbf{r}_{d-3}, K]) = \frac{2}{(d-1)!}\left(\prod_{i=2}^{d-2} h_i\right)\mathrm{Vol}_2(K) \quad \textit{and}$$

$$\begin{aligned}\mathrm{Vol}_d([\mathbf{o}, \mathbf{r}_1, \ldots, \mathbf{r}_{d-3}, K) &= \frac{h_1}{d}\mathrm{Vol}_{d-1}([\mathbf{r}_1, \ldots, \mathbf{r}_{d-3}, K]) \\ &= \frac{2}{d!}\left(\prod_{i=1}^{d-2} h_i\right)\mathrm{Vol}_2(K).\end{aligned}$$

Proof. The proof follows from Sublemma 2 and Lemma 3 (part (3)) in a straightforward way. □

The central notion of this section is the limiting surface density introduced as follows (see also Fig. 8).

Definition 8. Let W_{I} (resp., W_{II}) denote the wedge of type I (resp., of type II) with the two-dimensional base F_2 which is generated by the $(d-2)$-dimensional Rogers orthoscheme $\mathrm{conv}\{\mathbf{o}, \mathbf{r}_1, \ldots, \mathbf{r}_{d-2}\}$ of the Voronoi polytope $P \subset \mathbf{E}^d$, $d \geq 4$. Then choose a coordinate system with two perpendicular axes in the plane $\mathrm{aff}\, F_2$ meeting at the point $\mathbf{r}_{d-2}$. Now, if $\mathbf{x}$ is an arbitrary point of the plane $\mathrm{aff}\, F_2$, then for a positive integer n let $T_n(\mathbf{x}) \subset \mathrm{aff}\, F_2$ denote the square centered at $\mathbf{x}$ having sides of length $1/n$ parallel to the fixed coordinate axes. Then *the limiting surface density* $\hat{\delta}_{\mathrm{lim}}([\mathbf{o}, \mathbf{r}_1, \ldots, \mathbf{r}_{d-3}, \mathbf{x}], S)$ of the unit sphere S in the $(d-2)$-dimensional orthoscheme $[\mathbf{o}, \mathbf{r}_1, \ldots, \mathbf{r}_{d-3}, \mathbf{x}]$ is

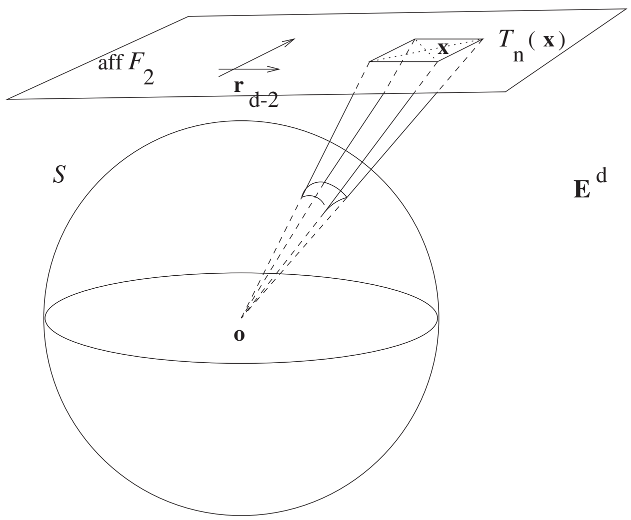

Fig. 8

defined by

$$\hat{\delta}_{\lim}([\mathbf{o}, \mathbf{r}_1, \ldots, \mathbf{r}_{d-3}, \mathbf{x}], S) = \lim_{n\to\infty} \hat{\delta}([\mathbf{o}, \mathbf{r}_1, \ldots, \mathbf{r}_{d-3}, T_n(\mathbf{x})], S).$$

Based on this we are able to give an integral representation of the surface density in a (truncated) wedge.

Lemma 6. *Let W_{I} (resp., W_{II}) denote the wedge of type* I *(resp., of type* II*) with the two-dimensional base F_2 which is generated by the $(d-2)$-dimensional Rogers orthoscheme* $\mathrm{conv}\{\mathbf{o}, \mathbf{r}_1, \ldots, \mathbf{r}_{d-2}\}$ *of the Voronoi polytope $P \subset \mathbf{E}^d$, $d \geq 4$.*

(1) *If $\mathbf{x} \in \mathrm{aff}\, F_2$ and $\mathbf{y} \in \mathrm{aff}\, F_2$ are points such that $\|\mathbf{x}\| \leq \|\mathbf{y}\|$, then*

$$\hat{\delta}_{\lim}([\mathbf{o}, \mathbf{r}_1, \ldots, \mathbf{r}_{d-3}, \mathbf{x}], S) \geq \hat{\delta}_{\lim}([\mathbf{o}, \mathbf{r}_1, \ldots, \mathbf{r}_{d-3}, \mathbf{y}], S).$$

(2) *For the surface densities of the unit sphere S in the wedge W_{I} and in the truncated wedge $\bar{W}_{\mathrm{I}}$ we have the following formulas*:

$$\begin{aligned}\hat{\delta}(W_{\mathrm{I}}, S) &= \frac{\mathrm{SVol}_{d-1}([\mathbf{o}, Q_{\mathrm{I}}] \cap S)}{\mathrm{Vol}_{d-1}(Q_{\mathrm{I}})} \\ &= \frac{1}{\mathrm{Vol}_2(F_2)} \int_{F_2} \hat{\delta}_{\lim}([\mathbf{o}, \mathbf{r}_1, \ldots, \mathbf{r}_{d-3}, \mathbf{x}], S)\, dx\end{aligned}$$

and

$$\begin{aligned}\hat{\delta}(\bar{W}_{\mathrm{I}}, S) &= \frac{\mathrm{SVol}_{d-1}([\mathbf{o}, \bar{Q}_{\mathrm{I}}] \cap S)}{\mathrm{Vol}_{d-1}(\bar{Q}_{\mathrm{I}})} \\ &= \frac{1}{\mathrm{Vol}_2(\bar{B} \cap F_2)} \int_{\bar{B}\cap F_2} \hat{\delta}_{\lim}([\mathbf{o}, \mathbf{r}_1, \ldots, \mathbf{r}_{d-3}, \mathbf{x}], S)\, dx,\end{aligned}$$

where dx stands for the Euclidean area element in the plane aff F_2. *Similar formulas hold for* W_{II} *and* $\bar{W}_{\mathrm{II}}$.

(3) *In general, if* $K \subset \operatorname{aff} F_2$ *is a convex domain, then the surface density of the unit sphere* S *in the* d*-dimensional convex cone* $[\mathbf{o}, \mathbf{r}_1, \ldots, \mathbf{r}_{d-3}, K]$ *with apex* $\mathbf{o}$ *and* $(d-1)$*-dimensional base* $[\mathbf{r}_1, \ldots, \mathbf{r}_{d-3}, K]$ *can be computed as follows*:

$$\hat{\delta}([\mathbf{o}, \mathbf{r}_1, \ldots, \mathbf{r}_{d-3}, K], S) = \frac{1}{\mathrm{Vol}_2(K)} \int_K \hat{\delta}_{\lim}([\mathbf{o}, \mathbf{r}_1, \ldots, \mathbf{r}_{d-3}, \mathbf{x}], S)\, dx.$$

Proof. (1) It is sufficient to look at the case $\|\mathbf{x}\| < \|\mathbf{y}\|$. (The case $\|\mathbf{x}\| = \|\mathbf{y}\|$ follows from this by standard limit procedure.) Then recall that

$$\hat{\delta}([\mathbf{o}, \mathbf{r}_1, \ldots, \mathbf{r}_{d-3}, T_n(\mathbf{x})], S) = h_1 \delta([\mathbf{o}, \mathbf{r}_1, \ldots, \mathbf{r}_{d-3}, T_n(\mathbf{x})], S) \quad \text{and}$$

$$\hat{\delta}([\mathbf{o}, \mathbf{r}_1, \ldots, \mathbf{r}_{d-3}, T_n(\mathbf{y})], S) = h_1 \delta([\mathbf{o}, \mathbf{r}_1, \ldots, \mathbf{r}_{d-3}, T_n(\mathbf{y})], S).$$

Thus, it is sufficient to show that if n is sufficiently large, then

$$\delta([\mathbf{o}, \mathbf{r}_1, \ldots, \mathbf{r}_{d-3}, T_n(\mathbf{x})], S) \geq \delta([\mathbf{o}, \mathbf{r}_1, \ldots, \mathbf{r}_{d-3}, T_n(\mathbf{y})], S).$$

This we can get as follows. We can approximate the d-dimensional convex cone $[\mathbf{o}, \mathbf{r}_1, \ldots, \mathbf{r}_{d-3}, T_n(\mathbf{x})]$ (resp., $[\mathbf{o}, \mathbf{r}_1, \ldots, \mathbf{r}_{d-3}, T_n(\mathbf{y})]$) arbitrarily close with a finite (but possible large) number of nonoverlapping d-dimensional orthoschemes, each containing the $(d-3)$-dimensional orthoscheme $[\mathbf{o}, \mathbf{r}_1, \ldots, \mathbf{r}_{d-3}]$ as a face and each having all the edge lengths of the three edges going out from the vertex $\mathbf{o}$ and not lying on the face $[\mathbf{o}, \mathbf{r}_1, \ldots, \mathbf{r}_{d-3}]$ close to $\|\mathbf{x}\|$ (resp., $\|\mathbf{y}\|$) for n sufficiently large (see also Sublemma 2). Thus, the claim follows from part (1) of Lemma 5 (Lemma of Comparison) rather easily.

(2), (3) It is sufficient to prove the corresponding formula for K.

A typical term of the Riemann–Lebesgue sum of

$$\frac{1}{\mathrm{Vol}_2(K)} \int_K \hat{\delta}_{\lim}([\mathbf{o}, \mathbf{r}_1, \ldots, \mathbf{r}_{d-3}, \mathbf{x}], S)\, dx$$

is equal to

$$\frac{1}{\mathrm{Vol}_2(K)} \hat{\delta}([\mathbf{o}, \mathbf{r}_1, \ldots, \mathbf{r}_{d-3}, T_n(\mathbf{x}_m)], S)\, \mathrm{Vol}_2(T_n(\mathbf{x}_m)), \qquad m \in M.$$

Using Sublemma 5 this turns out to be equal to

$$\frac{\mathrm{Vol}_{d-1}([\mathbf{r}_1, \ldots, \mathbf{r}_{d-3}, T_n(\mathbf{x}_m)])}{\mathrm{Vol}_{d-1}([\mathbf{r}_1, \ldots, \mathbf{r}_{d-3}, K])} \hat{\delta}([\mathbf{o}, \mathbf{r}_1, \ldots, \mathbf{r}_{d-3}, T_n(\mathbf{x}_m)], S) = \frac{\mathrm{SVol}_{d-1}([\mathbf{o}, \mathbf{r}_1, \ldots, \mathbf{r}_{d-3}, T_n(\mathbf{x}_m)] \cap S)}{\mathrm{Vol}_{d-1}([\mathbf{r}_1, \ldots, \mathbf{r}_{d-3}, K])}.$$

Finally, as the union of the nonoverlapping squares $T_n(\mathbf{x}_m)$, $m \in M$, is a good approximation of the convex domain K in the plane aff F_2 we get that

$$\sum_{m \in M} \frac{\mathrm{SVol}_{d-1}([\mathbf{o}, \mathbf{r}_1, \ldots, \mathbf{r}_{d-3}, T_n(\mathbf{x}_m)] \cap S)}{\mathrm{Vol}_{d-1}([\mathbf{r}_1, \ldots, \mathbf{r}_{d-3}, K])} = \frac{\sum_{m \in M} \mathrm{SVol}_{d-1}([\mathbf{o}, \mathbf{r}_1, \ldots, \mathbf{r}_{d-3}, T_n(\mathbf{x}_m)] \cap S)}{\mathrm{Vol}_{d-1}([\mathbf{r}_1, \ldots, \mathbf{r}_{d-3}, K])}$$

is a good approximation of

$$\frac{\mathrm{SVol}_{d-1}([\mathbf{o}, \mathbf{r}_1, \ldots, \mathbf{r}_{d-3}, K] \cap S)}{\mathrm{Vol}_{d-1}([\mathbf{r}_1, \ldots, \mathbf{r}_{d-3}, K])} = \hat{\delta}([\mathbf{o}, \mathbf{r}_1, \ldots, \mathbf{r}_{d-3}, K], S).$$

This completes the proof of Lemma 6. □

4. Truncation of Wedges Increases the Surface Density

Lemma 7. *Let* W_{I} (*resp*., W_{II}) *denote the wedge of type* I (*resp*., *of type* II) *with the two-dimensional base* F_2 *which is generated by the* $(d-2)$*-dimensional Rogers orthoscheme* $\mathrm{conv}\{\mathbf{o}, \mathbf{r}_1, \ldots, \mathbf{r}_{d-2}\}$ *of the Voronoi polytope* $P \subset \mathbf{E}^d$, $d \geq 4$. *Then*

$$\hat{\delta}(W_{\mathrm{I}}, S) \leq \hat{\delta}(\bar{W}_{\mathrm{I}}, S) \qquad (\textit{resp}., \hat{\delta}(W_{\mathrm{II}}, S) \leq \hat{\delta}(\bar{W}_{\mathrm{II}}, S)).$$

Proof. Notice that part (1) of Lemma 6 easily implies that if $0 < \mathrm{Vol}_2(F_2 \backslash \bar{B})$, then for any $\mathbf{x}^* \in F_2$ with $\|\mathbf{x}^*\| = \sqrt{2d/(d+1)}$ we have that

$$\begin{aligned}
&\frac{1}{\mathrm{Vol}_2(F_2 \backslash \bar{B})} \int_{F_2 \backslash \bar{B}} \hat{\delta}_{\mathrm{lim}}([\mathbf{o}, \mathbf{r}_1, \ldots, \mathbf{r}_{d-3}, \mathbf{x}], S)\, dx \\
&\quad \leq \hat{\delta}_{\mathrm{lim}}([\mathbf{o}, \mathbf{r}_1, \ldots, \mathbf{r}_{d-3}, \mathbf{x}^*], S) \\
&\quad \leq \frac{1}{\mathrm{Vol}_2(\bar{B} \cap F_2)} \int_{\bar{B} \cap F_2} \hat{\delta}_{\mathrm{lim}}([\mathbf{o}, \mathbf{r}_1, \ldots, \mathbf{r}_{d-3}, \mathbf{x}], S)\, dx.
\end{aligned}$$

Thus, if $0 < \mathrm{Vol}_2(F_2 \backslash \bar{B})$, then part (2) of Lemma 6 yields that

$$\begin{aligned}
\hat{\delta}(W_{\mathrm{I}}, S) &= \frac{1}{\mathrm{Vol}_2(F_2)} \int_{F_2} \hat{\delta}_{\mathrm{lim}}([\mathbf{o}, \mathbf{r}_1, \ldots, \mathbf{r}_{d-3}, \mathbf{x}], S)\, dx \\
&= \frac{\mathrm{Vol}_2(\bar{B} \cap F_2)}{\mathrm{Vol}_2(F_2)} \cdot \frac{1}{\mathrm{Vol}_2(\bar{B} \cap F_2)} \int_{\bar{B} \cap F_2} \hat{\delta}_{\mathrm{lim}}([\mathbf{o}, \mathbf{r}_1, \ldots, \mathbf{r}_{d-3}, \mathbf{x}], S)\, dx \\
&\quad + \frac{\mathrm{Vol}_2(F_2 \backslash \bar{B})}{\mathrm{Vol}_2(F_2)} \cdot \frac{1}{\mathrm{Vol}_2(F_2 \backslash \bar{B})} \int_{F_2 \backslash \bar{B}} \hat{\delta}_{\mathrm{lim}}([\mathbf{o}, \mathbf{r}_1, \ldots, \mathbf{r}_{d-3}, \mathbf{x}], S)\, dx \\
&\leq \frac{\mathrm{Vol}_2(\bar{B} \cap F_2)}{\mathrm{Vol}_2(F_2)} \cdot \frac{1}{\mathrm{Vol}_2(\bar{B} \cap F_2)} \int_{\bar{B} \cap F_2} \hat{\delta}_{\mathrm{lim}}([\mathbf{o}, \mathbf{r}_1, \ldots, \mathbf{r}_{d-3}, \mathbf{x}], S)\, dx \\
&\quad + \frac{\mathrm{Vol}_2(F_2 \backslash \bar{B})}{\mathrm{Vol}_2(F_2)} \cdot \frac{1}{\mathrm{Vol}_2(\bar{B} \cap F_2)} \int_{\bar{B} \cap F_2} \hat{\delta}_{\mathrm{lim}}([\mathbf{o}, \mathbf{r}_1, \ldots, \mathbf{r}_{d-3}, \mathbf{x}], S)\, dx \\
&= \frac{1}{\mathrm{Vol}_2(\bar{B} \cap F_2)} \int_{\bar{B} \cap F_2} \hat{\delta}_{\mathrm{lim}}([\mathbf{o}, \mathbf{r}_1, \ldots, \mathbf{r}_{d-3}, \mathbf{x}], S)\, dx = \hat{\delta}(\bar{W}_{\mathrm{I}}, S).
\end{aligned}$$

As the same method works for W_{II} and $\bar{W}_{\mathrm{II}}$ this completes the proof of Lemma 7. □

5. Maximum Surface Density in Truncated Wedges of Type I and II

The Case of Truncated Wedges of Type I. Let $\bar{W}_{\mathrm{I}}$ denote the truncated wedge of type I with the two-dimensional base $\bar{B} \cap F_2$ which is generated by the $(d-2)$-dimensional Rogers orthoscheme $\mathrm{conv}\{\mathbf{o}, \mathbf{r}_1, \ldots, \mathbf{r}_{d-2}\}$ of the Voronoi polytope $P \subset \mathbf{E}^d$, $d \geq 8$. By assumption F_2 is a two-dimensional face of the Voronoi polytope P with

$$\sqrt{\frac{2(d-2)}{d-1}} \leq h = R(F_2) < \sqrt{\frac{2(d-1)}{d}}.$$

Let $G_0 \subset \mathrm{aff}\, F_2$ (resp., $G \subset \mathrm{aff}\, F_2$) denote the closed circular disk of radius $g_0(h) = \sqrt{2d/(d+1) - h^2}$ (resp., $g(h) = (2-h^2)/\sqrt{4-h^2}$) centered at the point $\mathbf{r}_{d-2}$. It is easy to see that $G \subset \mathrm{relint}\, G_0$ for all $\sqrt{2(d-2)/(d-1)} \leq h < \sqrt{2(d-1)/d}$. (Moreover, $G = G_0$ for $h = \sqrt{2(d-1)/d}$.) Notice that $G_0 = \bar{B} \cap \mathrm{aff}\, F_2$, thus Corollary 1 implies that there is no vertex of the face F_2 belonging to the relative interior of G_0 (Fig. 9). Moreover, as $h = R(F_2) < \sqrt{2}$ Lemma 1 yields that $2/\sqrt{4-h^2} \leq R(F_1)$ holds for any side F_1 of the face F_2, hence, $G \subset F_2$ and, of course, $G \subset \bar{B} \cap F_2 = G_0 \cap F_2$. Now, let $M \subset \mathrm{aff}\, F_2$ be a square circumscribed about G. A straightforward computation yields that $g_0(h)/g(h)$ is a strictly decreasing function on the interval $[\sqrt{2(d-2)/(d-1)}, \sqrt{2(d-1)/d})$ (i.e. $(d(g_0(h)/g(h)))/dh < 0$ on the interval $(\sqrt{2(d-2)/(d-1)}, \sqrt{2(d-1)/d})$) and

$$\frac{g_0(\sqrt{2(d-2)/(d-1)})}{g(\sqrt{2(d-2)/(d-1)})} = \sqrt{\frac{2d}{d+1}} < \sqrt{2}.$$

Thus, the vertices of the square M do not belong to G_0. Finally, as $d \geq 8$ Corollary 2 implies that there are at most four sides of the face F_2 that intersect the relative interior of G_0.

The following statement is the core part of this section.

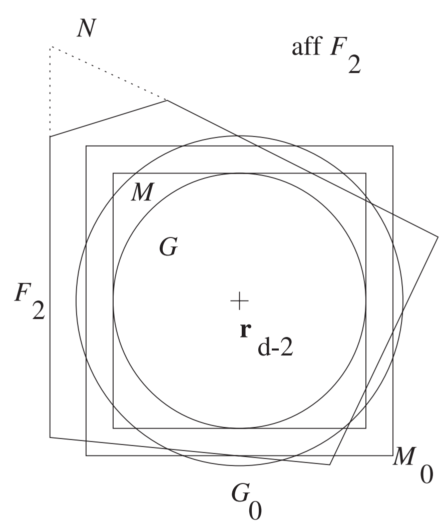

Fig. 9

Lemma 8. *Let $\bar{W}_{\rm I}$ denote the truncated wedge of type* I *with the two-dimensional base $\bar{B}\cap F_2$ which is generated by the $(d-2)$-dimensional Rogers orthoscheme* $\mathrm{conv}\{\mathbf{o}, \mathbf{r}_1, \ldots, \mathbf{r}_{d-2}\}$ *of the Voronoi polytope $P \subset \mathbf{E}^d$, $d \geq 8$. Then*

$$\hat{\delta}(\bar{W}_{\rm I}, S) \leq \hat{\delta}([\mathbf{o}, \mathbf{r}_1, \ldots, \mathbf{r}_{d-3}, G_0 \cap M], S).$$

Proof. Recall that according to part (2) of Lemma 6

$$\hat{\delta}(\bar{W}_{\rm I}, S) = \frac{1}{\mathrm{Vol}_2(\bar{B} \cap F_2)} \int_{\bar{B}\cap F_2} \hat{\delta}_{\lim}([\mathbf{o}, \mathbf{r}_1, \ldots, \mathbf{r}_{d-3}, \mathbf{x}], S)\, dx.$$

Moreover, Corollary 2 guarantees that the number of sides (i.e. of line segments of positive lengths) of $\bar{B} \cap F_2 = G_0 \cap F_2$ is at most four. Thus, all these facts and part (1) of Lemma 6 imply that without loss of generality we may assume that there exists a convex quadrangle $N \subset \mathrm{aff}\, F_2$ with

$$G_0 \cap F_2 = G_0 \cap N.$$

Now, if $G_0 \cap N \neq G_0$ (resp., $G_0 \cap N = G_0$), then let $M_0 \subset \mathrm{aff}\, F_2$ be a square (resp., a smallest square) centered at $\mathbf{r}_{d-2}$ with the property that

$$\mathrm{Vol}_2(G_0 \cap N) = \mathrm{Vol}_2(G_0 \cap M_0).$$

Obviously, no vertex of M_0 belongs to G_0.

Sublemma 6.

$$\hat{\delta}(\bar{W}_{\rm I}, S) = \hat{\delta}([\mathbf{o}, \mathbf{r}_1, \ldots, \mathbf{r}_{d-3}, G_0 \cap N], S) \leq \hat{\delta}([\mathbf{o}, \mathbf{r}_1, \ldots, \mathbf{r}_{d-3}, G_0 \cap M_0], S).$$

Proof. Take all possible convex quadrilaterals $N' \subset \mathrm{aff}\, F_2$ with the property that no vertex of N' belongs to the relative interior of G_0 and $G \subset N'$, moreover, $\mathrm{Vol}_2(G_0 \cap N') = \mathrm{Vol}_2(G_0 \cap N)$.

Obviously, there is convex quadrilateral of this family, say, N', for which

$$\hat{\delta}([\mathbf{o}, \mathbf{r}_1, \ldots, \mathbf{r}_{d-3}, G_0 \cap N'], S) = \frac{1}{\mathrm{Vol}_2(G_0 \cap N')} \int_{G_0\cap N'} \hat{\delta}_{\lim}([\mathbf{o}, \mathbf{r}_1, \ldots, \mathbf{r}_{d-3}, \mathbf{x}], S)\, dx$$

is maximal (see also part (3) of Lemma 6). We claim that all sides of N' lie at the same distance from $\mathbf{r}_{d-2}$. We prove this by contradiction. Assume that there are two sides s_1' and s_2' of $G_0 \cap N'$ such that the length of s_1' is larger than the length of s_2' (Fig. 10). Without loss of generality we may assume that the endpoints of s_1' and s_2' are not vertices of N'. Then we move s_1' (resp., s_2') farther away from $\mathbf{r}_{d-2}$ (resp., closer to $\mathbf{r}_{d-2}$) by a small amount to get the new side $\tilde{s}_1$ (resp., $\tilde{s}_2$) such that the parallel strips Δs_1 and Δs_2 in G_0 determined by $s_1', \tilde{s}_1$ and $s_2', \tilde{s}_2$ have the same area. Then let $\tilde{N}$ be the new convex quadrilateral obtained from N' in the above manner. Now it is easy to show that we can partition Δs_1 (resp., Δs_2) into an arbitrary large number, say, n, of equal area convex subregions, picking in each a point $\mathbf{x}_i$, $1 \leq i \leq n$ (resp., $\mathbf{y}_i$, $1 \leq i \leq n$), such

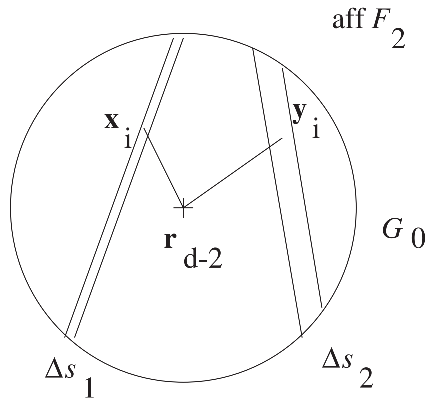

Fig. 10

that $\|\mathbf{x}_i - \mathbf{r}_{d-2}\| < \|\mathbf{y}_i - \mathbf{r}_{d-2}\|$, i.e. $\|\mathbf{x}_i\| < \|\mathbf{y}_i\|$ for all $1 \le i \le n$. As a result the proof of part (1) of Lemma 6 yields the following inequality (see also the Lemma of Strict Comparison in Section 7):

$$\hat{\delta}_{\lim}([\mathbf{o}, \mathbf{r}_1, \ldots, \mathbf{r}_{d-3}, \mathbf{x}_i], S) > \hat{\delta}_{\lim}([\mathbf{o}, \mathbf{r}_1, \ldots, \mathbf{r}_{d-3}, \mathbf{y}_i], S) \qquad \text{for all} \quad 1 \le i \le n.$$

Hence, part (3) of Lemma 6 implies in a straightforward way that

$$\hat{\delta}([\mathbf{o}, \mathbf{r}_1, \ldots, \mathbf{r}_{d-3}, G_0 \cap \tilde{N}], S) > \hat{\delta}([\mathbf{o}, \mathbf{r}_1, \ldots, \mathbf{r}_{d-3}, G_0 \cap N'], S),$$

a contradiction. Thus, indeed all sides of N' must lie at the same distance from $\mathbf{r}_{d-2}$ and as a result we get via Lemma 6 that

$$\hat{\delta}([\mathbf{o}, \mathbf{r}_1, \ldots, \mathbf{r}_{d-3}, G_0 \cap N], S) \le \hat{\delta}([\mathbf{o}, \mathbf{r}_1, \ldots, \mathbf{r}_{d-3}, G_0 \cap M_0], S),$$

finishing the proof of Sublemma 6. □

Sublemma 7.

$$\hat{\delta}([\mathbf{o}, \mathbf{r}_1, \ldots, \mathbf{r}_{d-3}, G_0 \cap M_0], S) \le \hat{\delta}([\mathbf{o}, \mathbf{r}_1, \ldots, \mathbf{r}_{d-3}, G_0 \cap M], S).$$

Proof. Without loss of generality we may assume that M and M_0 are homothetic with respect to $\mathbf{r}_{d-2}$ and $M \subset \operatorname{relint} M_0$. Let $\mathbf{u} \in G_0 \cap M$, $\mathbf{v} \in G_0 \cap M_0$ be midpoints of two corresponding and parallel sides of $G_0 \cap M$ and $G_0 \cap M_0$ (Fig. 11). Then let $\mathbf{x} \in G_0 \cap M$ and $\mathbf{y} \in G_0 \cap M_0$ be the vertices of the above two corresponding and parallel sides of $G_0 \cap M$ and $G_0 \cap M_0$ lying on the same side of the line $\mathbf{uv}$ in aff F_2. Let $\mathbf{z} \in \operatorname{relbd}(G_0 \cap M)$ be the point lying on the same side of the line $\mathbf{uv}$ in aff F_2 as the points $\mathbf{x}, \mathbf{y}$ such that $\angle \mathbf{u}\mathbf{r}_{d-2}\mathbf{z} = \mathbf{v}\mathbf{r}_{d-2}\mathbf{z} = \pi/4$. Let $U = \operatorname{conv}\{\mathbf{r}_{d-2}, \mathbf{u}, \mathbf{x}\}$, $V = \operatorname{conv}\{\mathbf{r}_{d-2}, \mathbf{v}, \mathbf{y}\}$. Moreover, let X (resp., Y) be the circular sector of G_0 spanned by the center $\mathbf{r}_{d-2}$ and the shorter circular arc of relbd G_0 between the points $\mathbf{x}, \mathbf{z}$ (resp., $\mathbf{y}, \mathbf{z}$). Now, part (1) of Sublemma 2, part (2) of Lemma 5 (Lemma of Comparison) and

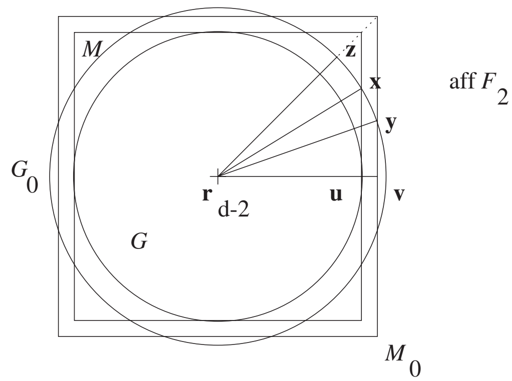

Fig. 11

Lemma 6 easily yield

$$\begin{aligned}&\hat{\delta}([\mathbf{o}, \mathbf{r}_1, \dots, \mathbf{r}_{d-3}, \mathbf{r}_{d-2}, \mathbf{u}, \mathbf{x}], S)\\ &\qquad = \hat{\delta}([\mathbf{o}, \mathbf{r}_1, \dots, \mathbf{r}_{d-3}, U], S) \ge \hat{\delta}([\mathbf{o}, \mathbf{r}_1, \dots, \mathbf{r}_{d-3}, \mathbf{r}_{d-2}, \mathbf{v}, \mathbf{y}], S)\\ &\qquad = \hat{\delta}([\mathbf{o}, \mathbf{r}_1, \dots, \mathbf{r}_{d-3}, V], S) \ge \hat{\delta}([\mathbf{o}, \mathbf{r}_1, \dots, \mathbf{r}_{d-3}, X], S)\\ &\qquad = \hat{\delta}([\mathbf{o}, \mathbf{r}_1, \dots, \mathbf{r}_{d-3}, Y], S).\end{aligned}$$

(Of course, in the case when M_0 is circumscribed about G_0 the points $\mathbf{y}, \mathbf{v}$ coincide and so, in the above inequalities, one has to replace $\hat{\delta}([\mathbf{o}, \mathbf{r}_1, \dots, \mathbf{r}_{d-3}, V], S)$ by $\hat{\delta}_{\lim}([\mathbf{o}, \mathbf{r}_1, \dots, \mathbf{r}_{d-3}, V], S)$.) Hence, these inequalities and the following obvious inequality (partly based on Sublemma 5)

$$\frac{\mathrm{Vol}_{d-1}([\mathbf{r}_1, \dots, \mathbf{r}_{d-3}, X])}{\mathrm{Vol}_{d-1}([\mathbf{r}_1, \dots, \mathbf{r}_{d-3}, U])} = \frac{\mathrm{Vol}_2(X)}{\mathrm{Vol}_2(U)} \le \frac{\mathrm{Vol}_2(Y)}{\mathrm{Vol}_2(V)} = \frac{\mathrm{Vol}_{d-1}([\mathbf{r}_1, \dots, \mathbf{r}_{d-3}, Y])}{\mathrm{Vol}_{d-1}([\mathbf{r}_1, \dots, \mathbf{r}_{d-3}, V])}$$

imply (via part (2) of Sublemma 2) in a straightforward way that

$$\begin{aligned}&\hat{\delta}([\mathbf{o}, \mathbf{r}_1, \dots, \mathbf{r}_{d-3}, U \cup X], S)\\ &\qquad = \frac{\mathrm{Vol}_{d-1}([\mathbf{r}_1, \dots, \mathbf{r}_{d-3}, U])}{\mathrm{Vol}_{d-1}([\mathbf{r}_1, \dots, \mathbf{r}_{d-3}, U \cup X])}\hat{\delta}([\mathbf{o}, \mathbf{r}_1, \dots, \mathbf{r}_{d-3}, U], S)\\ &\qquad\quad + \frac{\mathrm{Vol}_{d-1}([\mathbf{r}_1, \dots, \mathbf{r}_{d-3}, X])}{\mathrm{Vol}_{d-1}([\mathbf{r}_1, \dots, \mathbf{r}_{d-3}, U \cup X])}\hat{\delta}([\mathbf{o}, \mathbf{r}_1, \dots, \mathbf{r}_{d-3}, X], S)\\ &\qquad \ge \frac{\mathrm{Vol}_{d-1}([\mathbf{r}_1, \dots, \mathbf{r}_{d-3}, V])}{\mathrm{Vol}_{d-1}([\mathbf{r}_1, \dots, \mathbf{r}_{d-3}, V \cup Y])}\hat{\delta}([\mathbf{o}, \mathbf{r}_1, \dots, \mathbf{r}_{d-3}, V], S)\\ &\qquad\quad + \frac{\mathrm{Vol}_{d-1}([\mathbf{r}_1, \dots, \mathbf{r}_{d-3}, Y])}{\mathrm{Vol}_{d-1}([\mathbf{r}_1, \dots, \mathbf{r}_{d-3}, V \cup Y])}\hat{\delta}([\mathbf{o}, \mathbf{r}_1, \dots, \mathbf{r}_{d-3}, Y], S)\\ &\qquad = \hat{\delta}([\mathbf{o}, \mathbf{r}_1, \dots, \mathbf{r}_{d-3}, V \cup Y], S).\end{aligned}$$

Hence, by symmetry the inequality

$$\hat{\delta}([\mathbf{o}, \mathbf{r}_1, \dots, \mathbf{r}_{d-3}, G_0 \cap M], S) \geq \hat{\delta}([\mathbf{o}, \mathbf{r}_1, \dots, \mathbf{r}_{d-3}, G_0 \cap M_0], S)$$

follows, finishing the proof of Sublemma 7. □

Thus, Sublemmas 6 and 7 imply that

$$\begin{aligned}\hat{\delta}(\bar{W}_{\mathrm{I}}, S) &= \hat{\delta}([\mathbf{o}, \mathbf{r}_1, \dots, \mathbf{r}_{d-3}, G_0 \cap N], S) \leq \hat{\delta}([\mathbf{o}, \mathbf{r}_1, \dots, \mathbf{r}_{d-3}, G_0 \cap M_0], S)\\ &\leq \hat{\delta}([\mathbf{o}, \mathbf{r}_1, \dots, \mathbf{r}_{d-3}, G_0 \cap M], S).\end{aligned}$$

This completes the proof of Lemma 8. □

It is clear from the construction that we can write $\hat{\delta}([\mathbf{o}, \mathbf{r}_1, \dots, \mathbf{r}_{d-3}, G_0 \cap M], S)$ as a function of $d - 2$ variables, namely,

$$\hat{\Delta}(\xi_1, \dots, \xi_{d-3}, \xi_{d-2}) = \hat{\delta}([\mathbf{o}, \mathbf{r}_1, \dots, \mathbf{r}_{d-3}, G_0 \cap M], S),$$

where $\xi_1 = \|\mathbf{r}_1\|, \dots, \xi_{d-3} = \|\mathbf{r}_{d-3}\|, \xi_{d-2} = \|\mathbf{r}_{d-2}\| = h$. Corollary 1 and the assumption on h imply that

$$m_1 = 1 \leq \xi_1, \quad \dots, \quad m_i = \sqrt{\frac{2i}{i+1}} \leq \xi_i, \quad \dots, \quad m_{d-3} = \sqrt{\frac{2(d-3)}{d-2}} \leq \xi_{d-3},$$

$$m_{d-2} = \sqrt{\frac{2(d-2)}{d-1}} \leq \xi_{d-2} = h < \sqrt{\frac{2(d-1)}{d}}.$$

Notice that if $\|\mathbf{r}_i\| = m_i$ for all $1 \leq i \leq d-2$, then $[\mathbf{o}, \mathbf{r}_1, \dots, \mathbf{r}_{d-3}, G_0 \cap M]$ can be dissected into four pieces, each being congruent to W and therefore $\hat{\delta}([\mathbf{o}, \mathbf{r}_1, \dots, \mathbf{r}_{d-3}, G_0 \cap M], S) = \hat{\sigma}_d$.

Lemma 9. $\hat{\Delta}(\xi_1, \dots, \xi_{d-3}, \xi_{d-2}) \leq \hat{\Delta}(m_1, \dots, m_{d-3}, m_{d-2}) = \hat{\sigma}_d$.

Proof. For any fixed $\xi_{d-2} = h$, part (2) of Lemma 5 (Lemma of Comparison) easily implies that

$$\hat{\Delta}(\xi_1, \dots, \xi_{d-3}, h) \leq \hat{\Delta}(m_1, \dots, m_{d-3}, h).$$

Finally, following the monotonicity idea of the proof of Sublemma 7 it is easy to show that the function $\hat{\Delta}(m_1, \dots, m_{d-3}, h)$ as a function of h is decreasing on the interval $[\sqrt{2(d-2)/(d-1)}, \sqrt{2(d-1)/d})$. (The proof is essentially based on the fact that $\|\mathbf{u}\|, \|\mathbf{x}\|, \|\mathbf{z}\|, \mathrm{Vol}_2(X)/\mathrm{Vol}_2(U)$ are all increasing functions of h.) From this it follows that

$$\hat{\Delta}(m_1, \dots, m_{d-3}, h) \leq \hat{\Delta}(m_1, \dots, m_{d-3}, m_{d-2}) = \hat{\sigma}_d,$$

finishing the proof of Lemma 9. □

We conclude this section with the following immediate corollary of Lemmas 8 and 9.

Corollary 3. *Let $\bar{W}_{\mathrm{I}}$ denote the truncated wedge of type* I *with the two-dimensional base $\bar{B} \cap F_2$ which is generated by the $(d-2)$-dimensional Rogers orthoscheme* $\mathrm{conv}\{\mathbf{o}, \mathbf{r}_1, \ldots, \mathbf{r}_{d-2}\}$ *of the Voronoi polytope $P \subset \mathbf{E}^d$, $d \geq 8$. Then*

$$\hat{\delta}(\bar{W}_{\mathrm{I}}, S) \leq \hat{\sigma}_d.$$

The Case of Truncated Wedges of Type II. It is sufficient to prove the following statement.

Lemma 10. *Let $\bar{W}_{\mathrm{II}}$ denote the truncated wedge of type* II *with the two-dimensional base $\bar{B} \cap F_2$ which is generated by the $(d-2)$-dimensional Rogers orthoscheme* $\mathrm{conv}\{\mathbf{o}, \mathbf{r}_1, \ldots, \mathbf{r}_{d-2}\}$ *of the Voronoi polytope $P \subset \mathbf{E}^d$, $d \geq 4$. Then*

$$\hat{\delta}(\bar{W}_{\mathrm{II}}, S) \leq \hat{\sigma}_d.$$

Proof. By assumption F_2 is a two-dimensional face of the Voronoi polytope P with

$$\sqrt{\frac{2(d-1)}{d}} \leq h = R(F_2) < \sqrt{\frac{2d}{d+1}}.$$

Let $G_0 \subset \mathrm{aff}\, F_2$ denote the closed circular disk of radius $g_0(h) = \sqrt{2d/(d+1) - h^2}$ centered at the point $\mathbf{r}_{d-2}$. As $h = R(F_2) < \sqrt{2}$ Lemma 1 yields that

$$\sqrt{\frac{2d}{d+1}} \leq \frac{2}{\sqrt{4-h^2}} \leq R(F_1)$$

holds for any side F_1 of the face F_2. Thus,

$$\bar{B} \cap F_2 = G_0$$

and so

$$\hat{\delta}(\bar{W}_{\mathrm{II}}, S) = \hat{\delta}([\mathbf{o}, \mathbf{r}_1, \ldots, \mathbf{r}_{d-3}, G_0], S).$$

It is clear from the construction that we can write $\hat{\delta}([\mathbf{o}, \mathbf{r}_1, \ldots, \mathbf{r}_{d-3}, G_0], S)$ as a function of $d-2$ variables, namely,

$$\hat{\Delta}^*(\xi_1, \ldots, \xi_{d-3}, \xi_{d-2}) = \hat{\delta}([\mathbf{o}, \mathbf{r}_1, \ldots, \mathbf{r}_{d-3}, G_0], S),$$

where $\xi_1 = \|\mathbf{r}_1\|, \ldots, \xi_{d-3} = \|\mathbf{r}_{d-3}\|, \xi_{d-2} = \|\mathbf{r}_{d-2}\| = h$. Corollary 1 and the assumption on h imply that

$$m_1 = 1 \leq \xi_1, \quad \ldots, \quad m_i = \sqrt{\frac{2i}{i+1}} \leq \xi_i, \quad \ldots, \quad m_{d-3} = \sqrt{\frac{2(d-3)}{d-2}} \leq \xi_{d-3},$$

$$m^*_{d-2} = \sqrt{\frac{2(d-1)}{d}} \leq \xi_{d-2} = h < \sqrt{\frac{2d}{d+1}}.$$

For any fixed $\xi_{d-2} = h$, part (2) of Lemma 5 (Lemma of Comparison) easily implies that

$$\hat{\Delta}^*(\xi_1, \ldots, \xi_{d-3}, h) \leq \hat{\Delta}^*(m_1, \ldots, m_{d-3}, h).$$

Finally, applying again part (2) of Lemma 5 we immediately get that

$$\hat{\Delta}^*(m_1, \dots, m_{d-3}, h) \le \hat{\Delta}^*(m_1, \dots, m_{d-3}, m^*_{d-2}) \le \hat{\sigma}_d.$$

This completes the proof of Lemma 10. □

6. Proof of the Theorem

Let P be a d-dimensional Voronoi polytope of a packing $\mathcal{P}$ of d-dimensional unit balls in $\mathbf{E}^d$, $d \ge 8$. Without loss of generality we may assume that the unit ball $B = \{\mathbf{x} \in \mathbf{E}^d \mid \mathrm{dist}(\mathbf{o}, \mathbf{x}) = \|\mathbf{x}\| \le 1\}$ centered at the origin $\mathbf{o}$ of $\mathbf{E}^d$ is one of the unit balls of $\mathcal{P}$ with P as its Voronoi cell. As before, let S denote the boundary of B.

First, we dissect P into d-dimensional Rogers simplices. Then let $\mathrm{conv}\{\mathbf{o}, \mathbf{r}_1, \dots, \mathbf{r}_d\}$ be one of these d-dimensional Rogers simplices assigned to the flag, say, $F_0 \subset \cdots \subset F_{d-1}$ of P. As $\mathbf{r}_i \in F_{d-i}$, $1 \le i \le d$, it is clear that $\mathrm{aff}\{\mathbf{r}_{d-2}, \mathbf{r}_{d-1}, \mathbf{r}_d\} = \mathrm{aff}\, F_2$ and so

$$\mathrm{dist}(\mathbf{o}, \mathrm{aff}\{\mathbf{r}_{d-2}, \mathbf{r}_{d-1}, \mathbf{r}_d\}) = \mathrm{dist}(\mathbf{o}, \mathrm{aff}\, F_2) = R(F_2).$$

Notice that Corollary 1 implies that $\sqrt{2(d-2)/(d-1)} \le R(F_2)$.

Second, we group the d-dimensional Rogers simplices of P as follows:

(1) If $\sqrt{2(d-2)/(d-1)} \le R(F_2) < \sqrt{2(d-1)/d}$, then we assign the Rogers simplex $\mathrm{conv}\{\mathbf{o}, \mathbf{r}_1, \dots, \mathbf{r}_d\}$ to the type I wedge W_{I} with the two-dimensional base F_2 generated by the $(d-2)$-dimensional Rogers orthoscheme $\mathrm{conv}\{\mathbf{o}, \mathbf{r}_1, \dots, \mathbf{r}_{d-2}\}$ of the Voronoi polytope $P \subset \mathbf{E}^d$, $d \ge 8$.
(2) If $\sqrt{2(d-1)/d} \le R(F_2) < \sqrt{2d/(d+1)}$, then we assign the Rogers simplex $\mathrm{conv}\{\mathbf{o}, \mathbf{r}_1, \dots, \mathbf{r}_d\}$ to the type II wedge W_{II} with the two-dimensional base F_2 generated by the $(d-2)$-dimensional Rogers orthoscheme $\mathrm{conv}\{\mathbf{o}, \mathbf{r}_1, \dots, \mathbf{r}_{d-2}\}$ of the Voronoi polytope $P \subset \mathbf{E}^d$, $d \ge 8$.
(3) If $\sqrt{2d/(d+1)} \le R(F_2)$, then we assign the Rogers simplex $\mathrm{conv}\{\mathbf{o}, \mathbf{r}_1, \dots, \mathbf{r}_d\}$ to itself as the type III wedge W_{III}.

As the wedges of type I–III of the given Voronoi polytope P sit over the 2-skeleton of P and form a tiling of P it is clear that each d-dimensional Rogers simplex of P belongs to exactly one of them. As a result, in order to show that the surface density $\hat{\delta}(P, S) = \mathrm{SVol}_{d-1}(S)/\mathrm{Vol}_{d-1}(\mathrm{bd}\, P) = d\omega_d/\mathrm{Vol}_{d-1}(\mathrm{bd}\, P)$ of the unit sphere S in the Voronoi polytope P is bounded from above by $\hat{\sigma}_d$, it is sufficient to prove the following inequalities:

($\hat{1}$) $\hat{\delta}(W_{\mathrm{I}}, S) \le \hat{\sigma}_d$;
($\hat{2}$) $\hat{\delta}(W_{\mathrm{II}}, S) \le \hat{\sigma}_d$;
($\hat{3}$) $\hat{\delta}(W_{\mathrm{III}}, S) \le \hat{\sigma}_d$.

This final task left is now easy. Namely, Lemma 7, Corollary 3 and Lemma 10 yield ($\hat{1}$) and ($\hat{2}$) in a straightforward way. Finally, ($\hat{3}$) follows with the help of part (2) of Lemma 5 rather easily.

This completes the proof of the theorem.

7. Proof of the Proposition

The proof of Lemma 5 published in [4] (part (2)) and in [23] (part (1)) can be modified in a straightforward way such that it leads to the following somewhat stronger version of the Lemma of Comparison.

Lemma 11 (Lemma of Strict Comparison). *Let* $U = \operatorname{conv}\{\mathbf{o}, \mathbf{u}_1, \ldots, \mathbf{u}_d\}$ *be a* d*-dimensional orthoscheme in* $\mathbf{E}^d$ *and let* $V = \operatorname{conv}\{\mathbf{o}, \mathbf{v}_1, \ldots, \mathbf{v}_d\}$ *be a* d*-dimensional simplex of* $\mathbf{E}^d$ *such that* $\|\mathbf{v}_i\| = \operatorname{dist}(\mathbf{o}, \operatorname{conv}\{\mathbf{v}_i, \mathbf{v}_{i+1}, \ldots, \mathbf{v}_d\})$ *for all* $1 \le i \le d-1$. *If* $1 \le \|\mathbf{u}_i\| \le \|\mathbf{v}_i\|$ *holds for all* $1 \le i \le d$ *and there is an* i_0, $1 \le i_0 \le d$, *such that* $1 \le \|\mathbf{u}_{i_0}\| < \|\mathbf{v}_{i_0}\|$, *then*

(1) $\delta(U, B) > \delta(V, B)$ *and*
(2) $\hat{\delta}(U, S) > \hat{\delta}(V, S)$.

First, recall that

$$\hat{\sigma}_d = \frac{\operatorname{Vol}_d(W \cap B)}{\operatorname{Vol}_d(W)}.$$

Second, an easy application of part (1) of Lemma 11 implies that

$$\begin{aligned}\lambda_d &= \frac{\operatorname{Vol}_d([\mathbf{o}, \mathbf{w}_1, \ldots, \mathbf{w}_{d-3}, \lhd \mathbf{w}_{d-2}\mathbf{w}_d\mathbf{w}_{d+1}] \cap B)}{\operatorname{Vol}_d([\mathbf{o}, \mathbf{w}_1, \ldots, \mathbf{w}_{d-3}, \lhd \mathbf{w}_{d-2}\mathbf{w}_d\mathbf{w}_{d+1}])} \\ &< \frac{\operatorname{Vol}_d([\mathbf{o}, \mathbf{w}_1, \ldots, \mathbf{w}_d] \cap B)}{\operatorname{Vol}_d([\mathbf{o}, \mathbf{w}_1, \ldots, \mathbf{w}_d])} = \sigma_d.\end{aligned}$$

Thus,

$$\begin{aligned}\hat{\sigma}_d &= \frac{\operatorname{Vol}_d(W \cap B)}{\operatorname{Vol}_d(W)} \\ &= \frac{\lambda_d \operatorname{Vol}_d([\mathbf{o}, \mathbf{w}_1, \ldots, \mathbf{w}_{d-3}, \lhd \mathbf{w}_{d-2}\mathbf{w}_d\mathbf{w}_{d+1}]) + \sigma_d \operatorname{Vol}_d([\mathbf{o}, \mathbf{w}_1, \ldots, \mathbf{w}_d])}{\operatorname{Vol}_d(W)} < \sigma_d.\end{aligned}$$

This completes the proof of the proposition.

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