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Algorithms and Almost Tight Results for 3-Colorability of Small Diameter Graphs

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Abstract The 3-coloring problem is well known to be NP-complete. It is also well known that it remains NP-complete when the input is restricted to graphs with diameter 4. Moreover, assuming the Exponential Time Hypothesis (ETH), 3-coloring cannot be solved in time $2^{o(n)}$ on graphs with n vertices and diameter at most 4. In spite of extensive studies of the 3-coloring problem with respect to several basic parameters, the complexity status of this problem on graphs with small diameter, i.e. with diameter at most 2, or at most 3, has been an open problem. In this paper we investigate graphs with small diameter. For graphs with diameter at most 2, we provide the first subexponential algorithm for 3-coloring, with complexity $2^{O(\sqrt{n \log n})}$. Furthermore we extend the notion of an articulation vertex to that of an *articulation neighborhood*, and we provide a polynomial algorithm for 3-coloring on graphs with diameter 2 that have at least one

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articulation neighborhood. For graphs with diameter at most 3, we establish the complexity of 3-coloring by proving for every $\varepsilon \in [0, 1)$ that 3-coloring is NP-complete on triangle-free graphs of diameter 3 and radius 2 with n vertices and minimum degree $\delta = \Theta(n^{\varepsilon})$. Moreover, assuming ETH, we use three different amplification techniques of our hardness results, in order to obtain for every $\varepsilon \in [0, 1)$ subexponential asymptotic lower bounds for the complexity of 3-coloring on triangle-free graphs with diameter 3 and minimum degree $\delta = \Theta(n^{\varepsilon})$. Finally, we provide a 3-coloring algorithm with running time $2^{O\left(\min\{\delta\Delta,\ \frac{n}{\delta}\log\delta\}\right)}$ for arbitrary graphs with diameter 3, where n is the number of vertices and δ (resp. Δ) is the minimum (resp. maximum) degree of the input graph. To the best of our knowledge, this is the first subexponential algorithm for graphs with $\delta = \omega(1)$ and for graphs with $\delta = O(1)$ and $\Delta = o(n)$. Due to the above lower bounds of the complexity of 3-coloring, the running time of this algorithm is asymptotically almost tight when the minimum degree of the input graph is $\delta = \Theta(n^{\varepsilon})$, where $\varepsilon \in [\frac{1}{2}, 1)$, as its time complexity is $2^{O\left(\frac{n}{\delta}\log\delta\right)} = 2^{O\left(n^{1-\varepsilon}\log n\right)}$ and the corresponding lower bound states that there is no $2^{o\left(n^{1-\varepsilon}\right)}$-time algorithm.

Keywords 3-Coloring · Graph diameter · Graph radius · Subexponential algorithm · NP-complete · Exponential Time Hypothesis

1 Introduction

A *proper k-coloring* (or *k-coloring*) of a graph G is an assignment of k different colors to the vertices of G, such that no two adjacent vertices receive the same color. That is, a k-coloring is a partition of the vertices of G into k independent sets. The corresponding *k-coloring problem* is the problem of deciding whether a given graph G admits a k-coloring of its vertices, and to compute one if it exists. Furthermore, the minimum number k of colors for which there exists a k-coloring is denoted by $\chi(G)$ and is termed the *chromatic number* of G. The *minimum coloring problem* is to compute the chromatic number of a given graph G, and to compute a $\chi(G)$-coloring of G.

One of the most well known complexity results is that the k-coloring problem is NP-complete for every $k \geq 3$, while it can be solved in linear time for $k = 2$ [12]. Therefore, since graph coloring has numerous applications besides its theoretical interest, there has been a considerable interest in studying how several graph parameters affect the tractability of the k-coloring problem, where $k \geq 3$. In view of this, the complexity status of the coloring problem has been established for many graph classes.

It has been proved that 3-coloring remains NP-complete even when the input graph is restricted to be a line graph [15], a triangle-free graph with maximum degree 4 [20], or a planar graph with maximum degree 4 [12]. On the positive side, one of the most famous result in this context has been that the minimum coloring problem can be solved in polynomial time for perfect graphs using the ellipsoid method [13]. Furthermore, polynomial algorithms for 3-coloring have been also presented for classes of non-perfect graphs, such as P_6-free graphs [24] and AT-free graphs [25]. Here, a graph G is a P_6-free graph if G does not contain any path on 6 vertices as an induced subgraph.

Furthermore, a graph G is an AT-free graph if G does not contain any asteroidal triple, i.e. three vertices where every two of them are connected by a path avoiding the neighborhood of the third one. Although the minimum coloring problem is NP-complete on P_5-free graphs, the k-coloring problem is polynomial on these graphs for every fixed k [14]. Courcelle's celebrated theorem states that every problem definable in Monadic Second-Order logic (MSO) can be solved in linear time on graphs with bounded treewidth [9], and thus also the coloring problem can be solved in linear time on such graphs.

For the cases where 3-coloring is NP-complete, considerable attention has been given to devising exact algorithms that are faster than the brute-force algorithm (see e.g. the recent book [11]). In this context, asymptotic lower bounds of the time complexity have been provided for the main NP-complete problems, based on the *Exponential Time Hypothesis (ETH)* [16,17]. ETH states that there exists no deterministic algorithm that solves the 3SAT problem in time $2^{o(n)}$, given a boolean formula with n variables. In particular, assuming ETH, 3-coloring cannot be solved in time $2^{o(n)}$ on graphs with n vertices, even when the input is restricted to graphs with diameter 4 and radius 2 (see [19,22]). Therefore, since it is assumed that no subexponential $2^{o(n)}$ time algorithms exist for 3-coloring, most attention has been given to decreasing the multiplicative factor of n in the exponent of the running time of exact exponential algorithms, see e.g. [5,11,21]. For a more detailed discussion about ETH we refer to Sect. 2.

One of the most central notions in a graph is the distance between two vertices, which is the basis of the definition of other important parameters, such as the diameter, the eccentricity, and the radius of a graph. For these graph parameters, it is known that 3-coloring is NP-complete on graphs with diameter at most 4 (see e.g. the standard proof of [22]). Furthermore, it is straightforward to check that k-coloring is NP-complete for graphs with diameter at most 2, for every $k \geq 4$: we can reduce 3-coloring on arbitrary graphs to 4-coloring on graphs with diameter 2, just by introducing to an arbitrary graph a new vertex that is adjacent to all others.

In contrast, in spite of extensive studies of the 3-coloring problem with respect to several basic parameters, the complexity status of this problem on graphs with small diameter, i.e. with diameter at most 2 or at most 3, has been an open problem, see e.g. [6,8,18]. The complexity status of 3-coloring is open also for triangle-free graphs of diameter 2 and of diameter 3. It is worthwhile mentioning here that a graph is triangle-free and of diameter 2 if and only if it is a maximal triangle free graph. Moreover, it is known that 3-coloring is NP-complete for triangle-free graphs [20], however it is not known whether this reduction can be extended to maximal triangle free graphs. Another interesting result is that almost all graphs have diameter 2 [7]; however, this result cannot be used in order to establish the complexity of 3-coloring for graphs with diameter 2.

Our Contribution In this paper we provide subexponential algorithms and hardness results for the 3-coloring problem on graphs with low diameter, i.e. with diameter 2 and 3. As a preprocessing step, we first present two reduction rules that we apply to an arbitrary graph G such that the resulting graph G' is 3-colorable if and only G is 3-colorable. We use these reduction rules to reduce the size of the given graph and to

Table 1 Current state of the art and our algorithmic and NP-completeness results for k-coloring on graphs with diameter $diam(G)$

$k \backslash diam(G)$	2	3	≥ 4
3	$(*)$ $2^{O(\min\{\delta, \frac{n}{\delta} \log \delta\})}$-time algorithm $(*)$ polynomial algorithm if there is at least one articulation neighborhood	$(*)$ NP-complete for minimum degree $\delta = \Theta(n^{\varepsilon})$, for every $\varepsilon \in [0, 1)$, even if $rad(G) = 2$ and G is triangle-free $(*)$ $2^{O\left(\min\{\delta\Delta,\ \frac{n}{\delta} \log \delta\}\right)}$ time algorithm	NP-complete [22] and no $2^{o(n)}$-time algorithm
≥ 4	NP-complete	NP-complete	NP-complete

Our results are indicated by an asterisk

simplify the algorithms that we present. Whenever these two reduction rules cannot be applied to a graph, we call this graph *irreducible*; for a detailed discussion we refer to Sect. 2.

For graphs with diameter at most 2, we first provide a subexponential algorithm for 3-coloring with running time $2^{O\left(\min\{\delta, \frac{n}{\delta} \log \delta\}\right)}$, where n is the number of vertices and δ is the minimum degree of the input graph. This algorithm is simple and has worst-case running time $2^{O(\sqrt{n \log n})}$. To the best of our knowledge, this is the first subexponential algorithm for graphs with diameter 2. We demonstrate that this is indeed the worst-case of our algorithm by providing, for every $n \geq 1$, a 3-colorable graph $G_n = (V_n, E_n)$ with $\Theta(n)$ vertices, such that G_n has diameter 2 and both its minimum degree and the size of a minimum dominating set is $\Theta(\sqrt{n})$. In addition, this graph is triangle-free and irreducible with respect to the above two reduction rules. Furthermore we define the notion of an *articulation neighborhood* in a graph, which extends the notion of an articulation vertex. For any vertex v of a graph G, the closed neighborhood $N[v] = N(v) \cup \{v\}$ is an articulation neighborhood if $G - N[v]$ is disconnected. We present a polynomial algorithm for 3-coloring irreducible graphs G with diameter 2 having at least one articulation neighborhood.

For graphs with diameter at most 3, we establish the complexity of deciding 3-coloring, even for the case of triangle-free graphs. Namely we prove that 3-coloring is NP-complete on irreducible and triangle-free graphs with diameter 3 and radius 2, by providing a reduction from 3SAT. In addition, we provide a 3-coloring algorithm with running time $2^{O\left(\min\{\delta\Delta,\ \frac{n}{\delta} \log \delta\}\right)}$ for arbitrary graphs with diameter 3, where n is the number of vertices and δ (resp. Δ) is the minimum (resp. maximum) degree of the input graph. To the best of our knowledge, this algorithm is the first subexponential algorithm for graphs with $\delta = \omega(1)$ and for graphs with $\delta = O(1)$ and $\Delta = o(n)$. Table 1 summarizes the current state of the art of the complexity of k-coloring, as well as our algorithmic and NP-completeness results.

Furthermore, we provide three different amplification techniques that extend our hardness results for graphs with diameter 3. In particular, we first show that 3-coloring is NP-complete on irreducible and triangle-free graphs G of diameter 3 and radius 2 with n vertices and minimum degree $\delta(G) = \Theta(n^{\varepsilon})$, for every $\varepsilon \in [\frac{1}{2}, 1)$

Table 2 Summary of the results of Theorems 11, 12, and 13

$\delta(G) = \Theta(n^{\varepsilon})$	$0 \le \varepsilon < \frac{1}{3}$	$\frac{1}{3} \le \varepsilon < \frac{1}{2}$	$\frac{1}{2} \le \varepsilon < 1$
Time complexity lower bound	No $2^{o\left(n^{\left(\frac{1-\varepsilon}{2}\right)}\right)}$-time algorithm (cf. Theorem 13)	No $2^{o(n^{\varepsilon})}$-time algorithm (cf. Theorem 12)	No $2^{o\left(n^{1-\varepsilon}\right)}$-time algorithm (cf. Theorem 11)

Time complexity lower bounds for deciding 3-coloring on irreducible and triangle-free graphs G with n vertices, diameter 3, radius 2, and minimum degree $\delta(G) = \Theta(n^{\varepsilon})$, where $\varepsilon \in [0, 1)$, assuming ETH. The lower bound for $\varepsilon \in [\frac{1}{2}, 1)$ is asymptotically almost tight, as there exists an algorithm for arbitrary graphs with diameter 3 with running time $2^{O\left(\frac{n}{\delta}\log\delta\right)} = 2^{O\left(n^{1-\varepsilon}\log n\right)}$ by Theorem 6

and that, for such graphs, there exists no algorithm for 3-coloring with running time $2^{o\left(\frac{n}{\delta}\right)} = 2^{o\left(n^{1-\varepsilon}\right)}$, assuming ETH. This lower bound is asymptotically almost tight, due to our above algorithm with running time $2^{O\left(\frac{n}{\delta}\log\delta\right)}$, which is subexponential when $\delta(G) = \Theta(n^{\varepsilon})$ for some $\varepsilon \in [\frac{1}{2}, 1)$. With our second amplification technique, we show that 3-coloring remains NP-complete also on irreducible and triangle-free graphs G of diameter 3 and radius 2 with n vertices and minimum degree $\delta(G) = \Theta(n^{\varepsilon})$, for every $\varepsilon \in [0, \frac{1}{2})$. Moreover, we prove that for such graphs, when $\varepsilon \in [0, \frac{1}{3})$, there exists no algorithm for 3-coloring with running time $2^{o\left(\sqrt{\frac{n}{\delta}}\right)} = 2^{o\left(n^{\left(\frac{1-\varepsilon}{2}\right)}\right)}$, assuming ETH. Finally, with our third amplification technique, we prove that for such graphs, when $\varepsilon \in [\frac{1}{3}, \frac{1}{2})$, there exists no algorithm for 3-coloring with running time $2^{o(\delta)} = 2^{o(n^{\varepsilon})}$, assuming ETH. Table 2 summarizes our time complexity lower bounds for 3-coloring on irreducible and triangle-free graphs with diameter 3 and radius 2, parameterized by their minimum degree δ.

We note here that the way we use the term "parameterized by" should not be confused with the way this term is being used in the context of parameterized complexity (such as in FPT algorithms). In this paper we use the term "parameterized by" in the sense that our upper and lower bounds are given as a function of several parameters of the graph, such as the minimum degree δ, or the maximum degree Δ.

Organization of the Paper We provide in Sect. 2 details about the ETH, as well as the necessary notation and terminology, our two reduction rules, and the notion of an irreducible graph. In Sects. 3 and 4 we present our results for graphs with diameter 2 and 3, respectively.

2 Preliminaries and Notation

A theorem proving an NP-hardness result for a decision problem does not provide any information about how efficiently (although not polynomially, unless P$\neq$NP) this decision problem can be solved. In the context of providing lower bounds for the time complexity of solving NP-complete problems, Impagliazzo, Paturi, and Zane formulated the *ETH* [17].

Exponential Time Hypothesis (ETH) [17] There exists no algorithm solving 3SAT in time $2^{o(n)}$, where n is the number of variables in the input CNF formula.

ETH is a strong hypothesis which might be true or not. In particular, if ETH is true then it immediately follows that P $\neq$ NP. In addition to formulating ETH, Impagliazzo, Paturi, and Zane also proved the celebrated *Sparsification Lemma* [17], which has the following theorem as a direct consequence. (This result is quite useful for providing lower bounds assuming ETH, as it parameterizes the running time by the size of the input CNF formula, rather than only the number of its variables.)

Theorem 1 ([16]) *3SAT can be solved in time $2^{o(n)}$ if and only if it can be solved in time $2^{o(m)}$, where n is the number of variables and m is the number of clauses in the input CNF formula.*

The following very well known theorem about the 3-coloring problem is based on the fact that there exists a standard polynomial-time reduction from Not-All-Equal-3-SAT to 3-coloring[1] [22], in which the constructed graph has diameter 4 and radius 2, and its number of vertices is linear in the size of the input formula (see also [19]).

Theorem 2 ([19,22]) *Assuming ETH, there exists no $2^{o(n)}$ time algorithm for 3-coloring on graphs G with diameter 4, radius 2, and n vertices.*

Notation We consider in this article simple undirected graphs with no loops or multiple edges. In an undirected graph G, the edge between vertices u and v is denoted by uv, and in this case u and v are said to be *adjacent* in G. Otherwise u and v are called *non-adjacent* or *independent*. Given a graph $G = (V, E)$ and a vertex $u \in V$, denote by $N(u) = \{v \in V : uv \in E\}$ the set of neighbors (or the *open neighborhood*) of u and by $N[u] = N(u) \cup \{u\}$ the *closed neighborhood* of u. Whenever the graph G is not clear from the context, we will write $N_G(u)$ and $N_G[u]$, respectively. Denote by $\deg(u) = |N(u)|$ the *degree* of u in G and by $\delta(G) = \min\{\deg(u) : u \in V\}$ the *minimum degree* of G. Let u and v be two non-adjacent vertices of G. Then, u and v are called (false) *twins* if they have the same set of neighbors, i.e. if $N(u) = N(v)$. Furthermore, we call the vertices u and v *siblings* if $N(u) \subseteq N(v)$ or $N(v) \subseteq N(u)$; note that two twins are always siblings.

Given a graph $G = (V, E)$ and two vertices $u, v \in V$, we denote by $d(u, v)$ the *distance* of u and v, i.e. the length of a shortest path between u and v in G. Furthermore, we denote by $diam(G) = \max\{d(u, v) : u, v \in V\}$ the *diameter* of G and by $rad(G) = \min_{u \in V}\{\max\{d(u, v) : v \in V\}\}$ the *radius* of G. Given a subset $S \subseteq V$, $G[S]$ denotes the subgraph of G *induced* by S. We denote for simplicity by $G - S$ the induced subgraph $G[V \setminus S]$ of G. A subset $S \subseteq V$ is an *independent set* in G if the graph $G[S]$ has no edges. Furthermore, a subset $S \subseteq V$ is a *clique* if the graph $G[S]$ has all $\binom{|S|}{2}$ possible edges among its vertices. A clique with t vertices is denoted by K_t. A graph G that contains no K_t as an induced subgraph is called *K_t-free*. Furthermore, a subset $D \subseteq V$ is a *dominating set* of G if every vertex of $V \setminus D$ has at least one neighbor in D. For simplicity, we refer in the remainder of the article to a

[1] Note that there exists a polynomial-time reduction from 3SAT to Not-All-Equal-3-SAT, such that the size of the output monotone formula is linear in the size of the input formula.

proper k-coloring of a graph G just as a *k-coloring* of G. Throughout the article we perform several times the *merging* operation of two (or more) independent vertices, which is defined as follows: we *merge* the independent vertices $u_1, u_2, \ldots, u_t$ when we replace them by a new vertex u_0 with $N(u_0) = \cup_{i=1}^{t} N(u_i)$. In addition to the well known big-O notation for asymptotic complexity, sometimes we use the O^* notation that suppresses polynomially bounded factors. For instance, for functions f and g, we write $f(n) = O^*(g(n))$ if $f(n) = O(g(n)\ poly(n))$, where $poly(n)$ is a polynomial.

In the following we provide two reduction rules that can be applied to an arbitrary graph G. Throughout the article, we assume that any given graph G of low diameter is *irreducible* with respect to these two reduction rules, i.e. that these reduction rules have been iteratively applied to G until they cannot be applied any more. Note that the iterative application of these reduction rules on a graph with n vertices can be done in time polynomial in n. The correctness of these two reduction rules is almost trivial, therefore they could be considered as folklore. Such reductions are well-known tools in exact exponential and parameterized algorithms.

Observe that, whenever a graph G contains a clique with four vertices as an induced subgraph, then G is not 3-colorable. Furthermore, we can check easily in polynomial time (e.g. with brute-force) whether a given graph G contains a K_4. Therefore we assume in the following that all given graphs are K_4-free. Furthermore, since a graph is 3-colorable if and only if all its connected components are 3-colorable, we assume in the following that all given graphs are connected. In order to present our two reduction rules of an arbitrary K_4-free graph G, recall first that the *diamond* graph is a graph with 4 vertices and 5 edges, i.e. it consists of a K_4 without one edge. The diamond graph is illustrated in Fig. 1a. Suppose that four vertices u_1, u_2, u_3, u_4 of a given graph $G = (V, E)$ induce a diamond graph, and assume without loss of generality that $u_1u_2 \notin E$. Then, it is easy to see that in any 3-coloring of G (if such exists), u_1 and u_2 necessarily obtain the same color. Therefore we can merge u_1 and u_2 into one vertex, as the next reduction rule states, and the resulting graph is 3-colorable if and only if G is 3-colorable.

Reduction Rule 1 (diamond elimination) *Let $G = (V, E)$ be a K_4-free graph. If the quadruple $\{u_1, u_2, u_3, u_4\}$ of vertices in G induces a diamond graph, where $u_1u_2 \notin E$, then merge vertices u_1 and u_2.*

Note that, after performing a diamond elimination in a K_4-free graph G, we may introduce a new K_4 in the resulting graph. An example of such a graph G is illustrated in Fig. 1b. In this example, the graph on the left-hand side has no K_4 but it has two diamonds, namely on the quadruples $\{u_1, u_2, u_3, u_4\}$ and $\{u_4, u_5, u_6, u_7\}$ of vertices. However, after eliminating the first diamond by merging u_1 and u_4, we create a new K_4 on the quadruple $\{u_1, u_5, u_6, u_7\}$ of vertices, cf. the graph of the right-hand side of Fig. 1b.

Suppose now that a graph G has a pair of siblings u and v and assume without loss of generality that $N(u) \subseteq N(v)$. Then, we can extend any proper 3-coloring of $G - \{u\}$ (if such exists) to a proper 3-coloring of G by assigning to u the same color as v. Therefore, we can remove vertex u from G, as the next reduction rule states, and the resulting graph $G - \{u\}$ is 3-colorable if and only if G is 3-colorable.

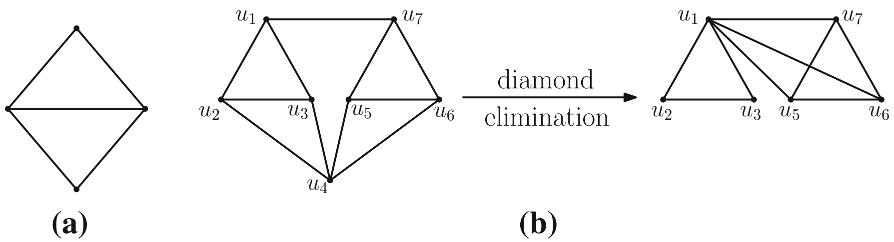

Fig. 1 **a** The diamond graph and **b** an example of a diamond elimination of a K_4-free graph, which creates a new K_4 on the vertices $\{u_1, u_5, u_6, u_7\}$ in the resulting graph

Reduction Rule 2 (siblings elimination) *Let* $G = (V, E)$ *be a* K_4*-free graph and* $u, v \in V$*, such that* $N(u) \subseteq N(v)$*. Then remove* u *from* G*.*

Definition 1 Let $G = (V, E)$ be a K_4-free graph. If neither Reduction Rule 1 nor Reduction Rule 2 can be applied to G, then G is *irreducible*.

Due to Definition 1, a K_4-free graph is irreducible if and only if it is diamond-free and siblings-free. Given a K_4-free graph G with n vertices, clearly we can iteratively execute Reduction Rules 1 and 2 in time polynomial in n, until we either find a K_4 or none of the Reduction Rules 1 and 2 can be further applied. If we find a K_4, then clearly the initial graph G is not 3-colorable. Otherwise, we transform G in polynomial time into an irreducible (K_4-free) graph G' of smaller or equal size, such that G' is 3-colorable if and only if G is 3-colorable.

Observe that during the application of these reduction rules to a graph G, neither the diameter nor the radius of G increase. Moreover, note that in the irreducible graph G', the neighborhood $N_{G'}(u)$ of every vertex u in G' induces a graph with maximum degree at most 1, since otherwise G' would have a K_4 or a diamond as an induced subgraph. That is, the subgraph of G' induced by $N_{G'}(u)$ contains only isolated vertices and isolated edges. Furthermore, if G (and thus also G') is connected and if G' has more than two vertices, then the minimum degree of G' is $\delta(G') \geq 2$, since G' is siblings-free. All these facts are summarized in the next observation. In the remainder of the article, we assume that any given graph G is irreducible.

Observation 1 *Let* $G = (V, E)$ *be a connected* K_4*-free graph and* $G' = (V', E')$ *be an irreducible graph obtained from* G*. If* G' *has more than two vertices, then* $\delta(G') \geq 2$*,* $diam(G') \leq diam(G)$*,* $rad(G') \leq rad(G)$*, and* G' *is* 3*-colorable if and only if* G *is* 3*-colorable. Moreover, for every* $u \in V'$*,* $N_{G'}(u)$ *induces in* G' *a graph with maximum degree* 1*.*

3 Algorithms for 3-Coloring on Graphs with Diameter 2

In this section we present our results on graphs with diameter 2. In particular, we provide in Sect. 3.1 our subexponential algorithm for 3-coloring on such graphs. We then provide, for every n, an example of an irreducible and triangle-free graph G_n with $\Theta(n)$ vertices and diameter 2, which is 3-colorable, has minimum dominating set of size $\Theta(\sqrt{n})$, and its minimum degree is $\delta(G_n) = \Theta(\sqrt{n})$. Furthermore, we define in Sect. 3.2 the notion of an articulation neighborhood, and we provide our polynomial algorithm for irreducible graphs G with diameter 2, which have at least one articulation neighborhood.

3.1 An $2^{O(\sqrt{n}\log n)}$-Time Algorithm for Any Graph with Diameter 2

We first provide in the next lemma a well known algorithm that decides the 3-coloring problem on an arbitrary graph G, using a dominating set (DS) of G [21]. For completeness we provide a short proof of this lemma.

Lemma 1 ([21], the DS-approach) *Let $G = (V, E)$ be a graph and $D \subseteq V$ be a dominating set of G. Then, the 3-coloring problem can be decided in $O^*(3^{|D|})$ time on G.*

Proof We iterate over all possible proper 3-colorings of D. There are at most $3^{|D|}$ such colorings; note that, if there is no proper 3-coloring of $G[D]$, then clearly G is not 3-colorable. Fix now a proper 3-coloring of $G[D]$. If we want to construct a 3-coloring of G that agrees with this precoloring of $G[D]$, then for every vertex of $G \setminus D$ there will be only two possible colors that this vertex can use in such a coloring (if one exists). Therefore, the question of whether this 3-coloring of $G[D]$ can be extended to a 3-coloring of G is a list 2-coloring problem, which can be solved in polynomial time as it can be formulated as a 2SAT instance (a similar approach has been first used in [10], in the context of coloring problems on dense graphs). Therefore, by considering in worst case all possible 3-colorings of the dominating set D, we can decide 3-coloring on G in time $O^*(3^{|D|})$. □

In the next theorem we use Lemma 1 to provide an improved 3-coloring algorithm for the case of graphs with diameter 2. The time complexity of this algorithm is parameterized by the minimum degree δ of the given graph G, as well as by the fraction $\frac{n}{\delta}$.

Theorem 3 *Let $G = (V, E)$ be an irreducible graph with n vertices. Let $diam(G) = 2$ and δ be the minimum degree of G. Then, the 3-coloring problem can be decided in $2^{O\left(\min\{\delta, \frac{n}{\delta}\log\delta\}\right)}$ time on G.*

Proof In an arbitrary graph G with n vertices and minimum degree δ, it is well known how to construct in polynomial time a dominating set D with cardinality $|D| \le n\frac{1+\ln(\delta+1)}{\delta+1}$ [2,3,23] (see also [1]). Therefore we can decide by Lemma 1 the 3-coloring problem on G in time $O^*(3^{n\frac{1+\ln(\delta+1)}{\delta+1}})$. Note by Observation 1 that $\delta \ge 2$, since G is irreducible by assumption. Therefore the latter running time is $2^{O(\frac{n}{\delta}\log\delta)}$.

The DS-approach of Lemma 1 applies to any graph G. However, since G has diameter 2 by assumption, we can design a second algorithm for 3-coloring of G as follows. Consider a vertex $u \in V$ with minimum degree, i.e. $\deg(u) = \delta$. Since $diam(G) = 2$, it follows that for every other vertex $u \in V$, either $d(u, v) = 1$ or $d(u, v) = 2$. Therefore $N(u)$ is a dominating set of G with cardinality δ. Fix a proper

3-coloring of $G[N(u)]$. Similarly to the proof of Lemma 1, note that if we want to construct a 3-coloring of G that agrees with this precoloring of $G[N(u)]$, then for every vertex of $G\backslash N(u)$ there will be only two possible colors that this vertex can use in such a coloring (if one exists). We iterate now for all possible proper 2-colorings of $G[N(u)]$ (instead of all 3-colorings of the dominating set as in the proof of Lemma 1). There are at most 2^{δ} such colorings. Similarly to the algorithm of Lemma 1, for every such 2-coloring of $G[N(u)]$ we solve in polynomial time the corresponding list 2-coloring of $G - N[u]$. Thus, considering at most all possible 2-colorings of $G[N(u)]$, we can decide the 3-coloring problem on G in time $O^*(2^{\delta}) = 2^{O(\delta)}$.

Summarizing, we can combine these two 3-coloring algorithms for G, obtaining an algorithm with time complexity $2^{O\left(\min\{\delta, \frac{n}{\delta} \log \delta\}\right)}$. □

The next corollary provides the first subexponential algorithm for the 3-coloring problem on graphs with diameter 2. Its correctness follows by Theorem 3.

Corollary 1 *Let $G = (V, E)$ be an irreducible graph with n vertices and let $diam(G) = 2$. Then, the 3-coloring problem can be decided in $2^{O(\sqrt{n \log n})}$ time on G.*

Proof Let $\delta = \delta(G)$ be the minimum degree of G. If $\delta \leq \sqrt{n \log n}$, then the 3-coloring problem can be decided in $2^{O(\delta)} = 2^{O(\sqrt{n \log n})}$ time on G by Theorem 3. Suppose now that $\delta > \sqrt{n \log n}$. Note that $\log \delta < \log n$, since $\delta < n$. Therefore $\frac{\log \delta}{\delta} < \frac{\log n}{\sqrt{n \log n}} = \sqrt{\frac{\log n}{n}}$, and thus $\frac{n}{\delta} \log \delta < n\sqrt{\frac{\log n}{n}}$, i.e. $\frac{n}{\delta} \log \delta < \sqrt{n \log n}$. Therefore the 3-coloring problem can be decided in $2^{O\left(\frac{n}{\delta} \log \delta\right)} = 2^{O(\sqrt{n \log n})}$ time on G by Theorem 3. □

Given the statements of Lemma 1 and Theorem 3, a question that arises naturally is whether the worst case complexity of the algorithm of Theorem 3 is indeed $2^{O(\sqrt{n \log n})}$ (as given in Corollary 1). That is, do there exist 3-colorable irreducible graphs G with n vertices and $diam(G) = 2$, such that both $\delta(G)$ and the size of the minimum dominating set of G are $\Theta(\sqrt{n \log n})$, or close to this value? We prove that such graphs exist; therefore our analysis of the DS-approach (cf. Lemma 1 and Theorem 3, see also [1–3,23]) is asymptotically almost tight in the case of 3-coloring of graphs with diameter 2. In particular, we provide in the next theorem for every n an example of an irreducible 3-colorable graph G_n with $\Theta(n)$ vertices and $diam(G_n) = 2$, such that both $\delta(G_n)$ and the size of the minimum dominating set of G are $\Theta(\sqrt{n})$. In addition, each of these graphs G_n is triangle-free, as the next theorem states. The construction of the graphs G_n is based on a suitable and interesting matrix arrangement of the vertices of G_n.

Theorem 4 *Let $n \geq 1$. Then there exists an* irreducible *and* triangle-free 3-colorable *graph $G_n = (V_n, E_n)$ with $\Theta(n)$ vertices, where $diam(G_n) = 2$ and $\delta(G_n) = \Theta(\sqrt{n})$. Furthermore, the size of the minimum dominating set of G_n is $\Theta(\sqrt{n})$.*

Proof We assume without loss of generality that $n = 4k^2 + 1$ for some integer $k > 1$ and we construct a graph $G_n = (V_n, E_n)$ with n vertices (otherwise, if $n \neq 4k^2 + 1$

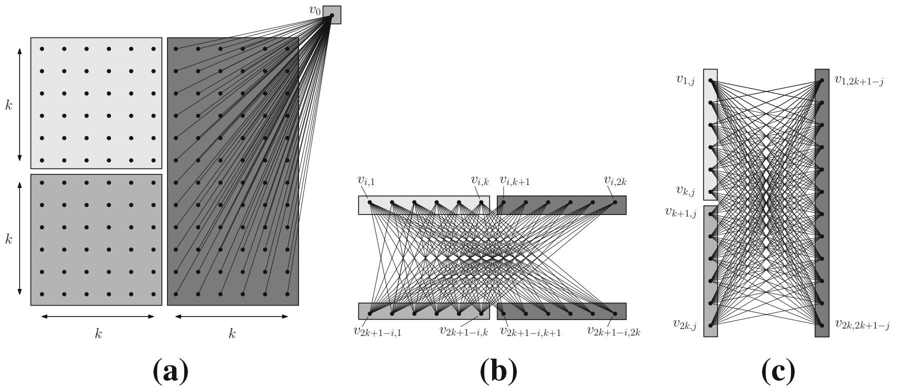

Fig. 2 **a** The $2k \times 2k$ matrix arrangement of the vertices of G_n, where $n = 4k^2 + 1$, **b** the edges of G_n between the ith and the $(2k + 1 - i)$th rows, and **c** the edges of G_n between the jth and the $(2k + 1 - j)$th columns of this matrix arrangement. The three color classes are illustrated by *different shades of gray*

for any $k > 1$, we provide the same construction of G_n with $4\lceil \frac{\sqrt{n}}{2} + 1\rceil^2 + 1 = \Theta(n)$ vertices). We arrange the first $n - 1$ vertices of G_n (i.e. all vertices of G_n except one of them) in a matrix of size $2k \times 2k$. For simplicity of notation, we enumerate these $4k^2$ vertices in the usual fashion, i.e. vertex $v_{i,j}$ is the vertex in the intersection of row i and column j, where $1 \leq i, j \leq 2k$. Furthermore, we denote the $(4k^2 + 1)$th vertex of G_n by v_0. We assign the 3 colors red, blue, green to the vertices of G_n as follows. The vertices $\{v_{i,j} : 1 \leq i \leq k,\ 1 \leq j \leq k\}$ are colored blue, the vertices $\{v_{i,j} : k+1 \leq i \leq 2k,\ 1 \leq j \leq k\}$ are colored green, the vertices $\{v_{i,j} : 1 \leq i \leq 2k,\ k+1 \leq j \leq 2k\}$ are colored red, and vertex v_0 is colored green.

We add edges among vertices of V_n as follows. First, vertex v_0 is adjacent to exactly all vertices that are colored red in the above 3-coloring of G_n. Then, the vertices of jth column $\{v_{1,j}, v_{2,j}, \ldots, v_{2k,j}\}$ and the vertices of the $(2k + 1 - j)$th column $\{v_{1,2k+1-j}, v_{2,2k+1-j}, \ldots, v_{2k,2k+1-j}\}$ form a complete bipartite graph, without the edges $\{v_{i,j}v_{i,2k+1-j} : 1 \leq i \leq 2k\}$, i.e. without the edges between vertices of the same row. Furthermore, the vertices of ith row $\{v_{i,1}, v_{i,2}, \ldots, v_{i,2k}\}$ and the vertices of the $(2k + 1 - i)$th row $\{v_{2k+1-i,1}, v_{2k+1-i,2}, \ldots, v_{2k+1-i,2k}\}$ form a complete bipartite graph, without the edges $\{v_{i,j}v_{2k+1-i,j} : 1 \leq j \leq k\} \cup \{v_{i,j}v_{2k+1-i,\ell} : k+1 \leq j, \ell \leq 2k\}$. That is, there are no edges between vertices of the same column and no edges between vertices colored red in the above 3-coloring of G_n. Note also that there are no edges between vertices colored blue (resp. green, red), and thus this coloring is a proper 3-coloring of G_n. The $2k \times 2k$ matrix arrangement of the vertices of G_n is illustrated in Fig. 2a. In this figure, the three color classes are illustrated by different shades of gray. Furthermore, the edges of G_n between different rows and between different columns in this matrix arrangement are illustrated in Fig. 2b, c, respectively.

It is easy to see by the construction of G_n that $3k \leq \deg(v_{i,j}) \leq 4k - 2$ for every vertex $v_{i,j} \in V_n \setminus \{v_0\}$ and that $\deg(v_0) = 2k^2$. In particular, $\deg(v_{i,j}) = 3k$ (resp. $\deg(v_{i,j}) = 4k - 2$) for every vertex $v_{i,j}$ that has been colored red (resp. blue or green) in the above coloring of G_n. Therefore, since $k = \Theta(\sqrt{n})$, it follows that

$\deg(v_{i,j}) = \Theta(\sqrt{n})$ for every $v_{i,j} \in V_n \setminus \{v_0\}$ and that $\deg(v_0) = \frac{n-1}{2}$. That is, the minimum degree is $\delta(G_n) = \Theta(\sqrt{n})$.

Note now that the set $\{v_0\} \cup \{v_{1,1}, v_{2,1}, \ldots, v_{2k,1}\}$ of vertices is a dominating set of G with $2k+1 = \Theta(\sqrt{n})$ vertices, since for every $i = 1, 2, \ldots, 2k$, vertex $v_{i,1}$ is adjacent to all vertices of the $(2k+1-i)$th row of the matrix, except vertex $v_{2k+1-i,1}$. Suppose that there exists a dominating set $D \subseteq V_n$ with cardinality $o(\sqrt{n})$, i.e. $|D| = \frac{\sqrt{n}}{f(n)}$, where $f(n) = \omega(1)$. Denote $D' = D \cup \{v_0\}$; note that $|D'| \leq \frac{\sqrt{n}}{f(n)} + 1$. Then, since $\deg(v_{i,j}) = \Theta(\sqrt{n})$ for every vertex $v_{i,j} \in V_n \setminus \{v_0\}$, the vertices of D' can be adjacent to at most

$$\frac{n-1}{2} + |D' \setminus \{v_0\}| \cdot \Theta(\sqrt{n}) \leq \frac{n-1}{2} + \frac{\sqrt{n}}{f(n)} \cdot \Theta(\sqrt{n}) = \frac{n-1}{2} + \frac{\Theta(n)}{f(n)}$$

vertices of $V_n \setminus D'$ in total. However $|V_n \setminus D'| \geq n - \frac{\sqrt{n}}{f(n)} - 1 \geq n - \sqrt{n} - 1$ and $\frac{n-1}{2} + \frac{\Theta(n)}{f(n)}$ is asymptotically smaller than $n - \sqrt{n} - 1$, since $f(n) = \omega(1)$. That is, the vertices of D' (and thus also of D) cannot dominate all vertices of $V_n \setminus D' \subseteq V_n \setminus D$. Thus D is not a dominating set, which is a contradiction. Therefore the size of a minimum dominating set of G_n is $\Theta(\sqrt{n})$.

Now we will prove that $diam(G_n) = 2$. First note that for every vertex $v_{i,j} \in V_n \setminus \{v_0\}$ we have $d(v_0, v_{i,j}) \leq 2$. Indeed, if $v_{i,j}$ is colored red, then $v_{i,j}$ is adjacent to v_0; otherwise v_0 and $v_{i,j}$ have vertex $v_{2k+1-i,2k}$ as common neighbor. Now consider two arbitrary vertices $v_{i,j}$ and $v_{p,q}$, where $(i, j) \neq (p, q)$. Since any two red vertices have v_0 as common neighbor (and thus they are at distance 2 from each other), assume that at most one of the vertices $\{v_{i,j}, v_{p,q}\}$ is colored red. We will prove that $d(v_{i,j}, v_{p,q}) \leq 2$. If $p = i$, then $v_{i,j}$ and $v_{p,q}$ lie both in the ith row of the matrix. Therefore $v_{i,j}$ and $v_{p,q}$ have all vertices of $\{v_{2k+1-i,1}, v_{2k+1-i,2}, \ldots, v_{2k+1-i,k}\} \setminus \{v_{2k+1-i,j}, v_{2k+1-i,q}\}$ as their common neighbors, and thus $d(v_{i,j}, v_{p,q}) = 2$. If $q = j$, then $v_{i,j}$ and $v_{p,q}$ lie both in the jth column of the matrix. Therefore $v_{i,j}$ and $v_{p,q}$ have all vertices of $\{v_{1,2k+1-j}, v_{2,2k+1-j}, \ldots, v_{2k,2k+1-j}\} \setminus \{v_{i,2k+1-j}, v_{p,2k+1-j}\}$ as their common neighbors, and thus $d(v_{i,j}, v_{p,q}) = 2$. Suppose that $p \neq i$ and $q \neq j$. If $p = 2k+1-i$ or $q = 2k+1-j$, then $v_{i,j}v_{p,q} \in E_n$ by the construction of G_n , and thus $d(v_{i,j}, v_{p,q}) = 1$. Suppose now that also $p \neq 2k+1-i$ and $q \neq 2k+1-j$. Since at most one of the vertices $\{v_{i,j}, v_{p,q}\}$ is colored red, we may assume without loss of generality that $v_{i,j}$ is blue or green, i.e. $j \leq k$. Then there exists the path $(v_{i,j}, v_{2k+1-i,2k+1-q}, v_{p,q})$ in G_n, and thus $d(v_{i,j}, v_{p,q}) = 2$. Summarizing, $d(u, v) \leq 2$ for every pair of vertices u and v in G_n, and thus $diam(G_n) = 2$.

Now we will prove that G_n is a triangle-free graph. First note that v_0 cannot belong to any possible triangle in G_n, since its neighbors are by construction an independent set. Suppose now that the vertices $v_{i,j}, v_{p,q}, v_{r,s}$ induce a triangle in G_n. Since the above coloring of the vertices of V_n is a proper 3-coloring of G_n , we may assume without loss of generality that $v_{i,j}$ is colored blue, $v_{p,q}$ is colored green, and $v_{r,s}$ is colored red in the above coloring of G_n. That is, $i, j \in \{1, 2, \ldots, k\}$, $p \in \{k+1, k+2, \ldots, 2k\}$, $q \in \{1, 2, \ldots, k\}$, $r \in \{1, 2, \ldots, 2k\}$, and $s \in \{k+1, k+2, \ldots, 2k\}$. Thus, since $v_{i,j}v_{p,q} \in E_n$, it follows by the construction of G_n that $p = 2k+1-i$. Furthermore, since $v_{p,q}v_{r,s} \in E_n$, it follows that $p = 2k + 1 - r$ or $q = 2k + 1 - s$. If $p = 2k + 1 - r$, then $r = i$ (since also $p = 2k + 1 - i$). Therefore $v_{i,j}$ and $v_{r,s}$ lie both in the ith row of the matrix, which is a contradiction, since we assumed that $v_{i,j}v_{r,s} \in E_n$. Therefore $q = 2k + 1 - s$. Finally, since $v_{i,j}v_{r,s} \in E_n$, it follows that $r = 2k + 1 - i$ or $s = 2k + 1 - j$. If $r = 2k + 1 - i$, then $r = p$ (since also $p = 2k + 1 - i$). Therefore $v_{p,q}$ and $v_{r,s}$ lie both in the pth row of the matrix, which is a contradiction, since we assumed that $v_{p,q}v_{r,s} \in E_n$. Therefore $s = 2k + 1 - j$. Thus, since also $q = 2k + 1 - s$, it follows that $q = j$, i.e. $v_{i,j}$ and $v_{p,q}$ lie both in the jth column of the matrix. This is again a contradiction, since we assumed that $v_{i,j}v_{p,q} \in E_n$. Therefore, no three vertices of G_n induce a triangle, i.e. G_n is triangle-free.

Note now that G_n is diamond-free, since it is also triangle-free, and thus the Reduction Rule 1 does not apply on G_n. Furthermore, it is easy to check that there exist no pair $v_{i,j}$ and $v_{p,q}$ of vertices such that $N(v_{i,j}) \subseteq N(v_{p,q})$, and that there does not exist any vertex $v_{i,j}$ such that $N(v_{i,j}) \subseteq N(v_0)$ or $N(v_0) \subseteq N(v_{i,j})$. That is, G_n is siblings-free, and thus also the Reduction Rule 2 does not apply on G_n. Therefore G_n is irreducible. This completes the proof of the theorem. □

3.2 A Tractable Subclass of Graphs with Diameter 2

In this section we present a subclass of graphs with diameter 2, which admits an efficient algorithm for 3-coloring. We first introduce the definition of an *articulation neighborhood* in a graph.

Definition 2 Let $G = (V, E)$ be a graph and let $v \in V$. If $G - N[v]$ is disconnected, then $N[v]$ is an *articulation neighborhood* in G.

We prove in Theorem 5 that, given an irreducible graph with $diam(G) = 2$, which has at least one articulation neighborhood, we can decide 3-coloring on G in polynomial time. Note here that there exist instances of K_4-free graphs G with diameter 2, in which G has no articulation neighborhood, but in the irreducible graph G' obtained by G (by iteratively applying the Reduction Rules 1 and 2), $G' - N_{G'}[v]$ becomes disconnected for some vertex v of G'. That is, G' may have an articulation neighborhood, although G has none. Therefore, if we consider as input the irreducible graph G' instead of G, we can decide in polynomial time the 3-coloring problem on G' (and thus also on G).

Theorem 5 *Let $G = (V, E)$ be an irreducible graph with n vertices and $diam(G) = 2$. If G has at least one articulation neighborhood, then we can decide 3-coloring on G in time polynomial in n.*

Proof Let $v_0 \in V$ such that $G - N[v_0]$ is disconnected and let $C_1, C_2, \ldots, C_k$ be the connected components of $G - N[v_0]$. In order to compute a proper 3-coloring of G, the algorithm assigns without loss of generality the color red to vertex v_0. Suppose that at least one component C_i, $1 \leq i \leq k$, is trivial, i.e. C_i is an isolated vertex u. Then, since $u \in V \setminus N[v_0]$ and u has no adjacent vertices in $V \setminus N[v_0]$, it follows that $N(u) \subseteq$

$N(v_0)$, i.e. u and v_0 are siblings in G. This is a contradiction, since G is an irreducible graph. Therefore, every connected component C_i of $G - N[v_0]$ is non-trivial, i.e. it contains at least one edge. Thus, in any proper 3-coloring of G (if such exists), there exists at least one vertex of every connected component C_i of $G - N[v_0]$ that is colored not red, i.e. with a different color than v_0.

Suppose now that G is 3-colorable, and let c be a proper 3 -coloring of G. Assume without loss of generality that c uses the colors red, blue, and green (recall that v_0 is already colored red). Let C_i, C_j be an arbitrary pair of connected components of $G - N[v_0]$; such a pair always exists, as $G - N[v_0]$ is assumed to be disconnected. Let u and v be two arbitrary vertices of C_i and of C_j, respectively, such that both u and v are not colored red in c; such vertices u and v always exist, as we observed above. Then, since $diam(G) = 2$, there exists at least one common neighbor a of u and v, where $a \in N(v_0)$. That is, $a \in N(v_0) \cap N(u) \cap N(v)$. Therefore, since v_0 is colored red in c and u, v are not colored red in c, it follows that both u and v are colored by the same color in c. Assume without loss of generality that both u and v are colored blue in c. Thus, since u and v have been chosen arbitrarily under the single assumption that they are not colored red in c, it follows that for every vertex u of $G - N[v_0]$, u is either colored red or blue in c. Furthermore, since v_0 has been assumed to be red and hence the neighbors of v_0 use colors blue and green, each vertex now has only two possible colors that it can use. This reduces the problem to the list 2-coloring problem that can be solved in polynomial time (by reducing to the 2SAT problem). This completes the proof of the theorem. $\square$

A question that arises now naturally by Theorem 5 is whether there exist any irreducible 3-colorable graph $G = (V, E)$ with $diam(G) = 2$, which has no articulation neighborhood at all. A negative answer to this question would imply that we can decide the 3-coloring problem on *any* graph with diameter 2 in polynomial time using Theorem 5. However, the answer to that question is positive: for every $n \geq 1$, the graph $G_n = (V_n, E_n)$ that has been presented in Theorem 4 is irreducible, 3-colorable, has diameter 2, and $G_n - N[v]$ is connected for every $v \in V_n$, i.e. G_n has no articulation neighborhood. Therefore Theorem 5 cannot be used in a trivial way to decide in polynomial time the 3-coloring problem for an arbitrary graph of diameter 2. We leave the tractability of the 3-coloring problem of arbitrary diameter-2 graphs as an open problem.

4 Almost Tight Results for Graphs with Diameter 3

In this section we present our results on graphs with diameter 3. In particular, we first provide in Sect. 4.1 our algorithm for 3-coloring arbitrary graphs with diameter 3 that has running time $2^{O(\min\{\delta\Delta,\ \frac{n}{\delta}\log\delta\})}$, where n is the number of vertices and δ (resp. Δ) is the minimum (resp. maximum) degree of the input graph. Then we prove in Sect. 4.2 that 3-coloring is NP-complete on irreducible and triangle-free graphs with diameter 3 and radius 2 by providing a reduction from 3SAT. Finally, we provide in Sect. 4.3 our three different amplification techniques that extend our hardness results of Sect. 4.2. In particular, we provide in Theorems 9 and 10 our NP-completeness amplifications, and in Theorems 11, 12, and 13 our lower bounds for the time complexity of 3-coloring, assuming ETH.

4.1 An $2^{O\left(\min\{\delta\Delta,\ \frac{n}{\delta}\log\delta\}\right)}$-Time Algorithm for Any Graph with Diameter 3

In the next theorem we use the DS-approach of Lemma 1 to provide a 3-coloring algorithm for the case of graphs with diameter 3. The time complexity of this algorithm is parameterized by the minimum degree δ of the given graph G, as well as by the fraction $\frac{n}{\delta}$.

Theorem 6 *Let $G = (V, E)$ be an irreducible graph with n vertices and $diam(G) = 3$. Let δ and Δ be the minimum and the maximum degree of G, respectively. Then, the 3-coloring problem can be decided in $2^{O\left(\min\{\delta\Delta,\ \frac{n}{\delta}\log\delta\}\right)}$ time on G.*

Proof First recall that, in an arbitrary graph G with n vertices and minimum degree δ, we can construct in polynomial time a dominating set D with cardinality $|D| \leq n\frac{1+\ln(\delta+1)}{\delta+1}$ [2]. Therefore, similarly to the proof of Theorem 3, we can use the DS-approach of Lemma 1 to obtain an algorithm that decides 3-coloring on G in $2^{O\left(\frac{n}{\delta}\log\delta\right)}$ time.

The DS-approach of Lemma 1 applies to any graph G. However, since G has diameter 3 by assumption, we can design a second algorithm for 3-coloring of G as follows. Consider a vertex $u \in V$ with minimum degree, i.e. $\deg(u) = \delta$. Since $diam(G) = 3$, we can partition the vertices of $V \backslash N[u]$ into two sets A and B, such that $A = \{v \in V \backslash N[u] : d(u, v) = 2\}$ and $B = \{v \in V \backslash N[u] : d(u, v) = 3\}$. Note that every vertex $v \in A$ is adjacent to at least one vertex of $N(u)$. Therefore, since $|N(u)| = \delta$ and the maximum degree in G is Δ, it follows that $|A| \leq \delta \cdot \Delta$. Furthermore, the set $A \cup \{u\}$ is a dominating set of G with cardinality at most $\delta\Delta + 1$. Thus we can decide 3-coloring on G by considering in the worst case all possible 3-colorings of $A \cup \{u\}$ in $O^*(3^{\delta\Delta+1}) = 2^{O(\delta\Delta)}$ time by using the DS-approach of Lemma 1.

Summarizing, we can combine these two 3-coloring algorithms for G, obtaining an algorithm with time complexity $2^{O\left(\min\{\delta\Delta,\ \frac{n}{\delta}\log\delta\}\right)}$. $\square$

To the best of our knowledge, the algorithm of Theorem 6 is the first subexponential algorithm for graphs with diameter 3, whenever $\delta = \omega(1)$, as well as whenever $\delta = O(1)$ and $\Delta = o(n)$. As we will later prove in Sect. 4.3, the running time provided in Theorem 6 is asymptotically almost tight whenever $\delta = \Theta(n^{\varepsilon})$, for any $\varepsilon \in [\frac{1}{2}, 1)$.

Note now that for any graph G with n vertices and $diam(G) = 3$, the maximum degree Δ of G is $\Delta = \Omega(n^{\frac{1}{3}})$. Indeed, suppose otherwise that $\Delta = o(n^{\frac{1}{3}})$, and let u be any vertex of G. Then, there are at most Δ vertices in G at distance 1 from u, at most Δ^2 vertices at distance 2 from u, and at most Δ^3 vertices at distance 3 from u. That is, G contains at most $1 + \Delta + \Delta^2 + \Delta^3 = o(n)$ vertices, since we assumed that $\Delta = o(n^{\frac{1}{3}})$, which is a contradiction. Therefore $\Delta = \Omega(n^{\frac{1}{3}})$. Furthermore note that, whenever $\delta = \Omega(n^{\frac{1}{3}}\sqrt{\log n})$ in a graph with n vertices and diameter 3, we have $\delta\Delta = \Omega(\frac{n}{\delta}\log\delta)$. Indeed, since $\Delta = \Omega(n^{\frac{1}{3}})$ as we proved above, it follows that in this case $\delta^2\Delta = \Omega(n \log n) = \Omega(n \log \delta)$, since $\delta < n$, and thus $\delta\Delta = \Omega(\frac{n}{\delta} \log \delta)$. Therefore, if the minimum degree of G is $\delta = \Omega(n^{\frac{1}{3}}\sqrt{\log n})$, the running time of the algorithm of Theorem 6 becomes $2^{O(\frac{n}{\delta} \log \delta)} = 2^{O\left(n^{\frac{2}{3}}\sqrt{\log n}\right)}$.

4.2 The 3-Coloring Problem is NP-Complete on Graphs with Diameter 3 and Radius 2

In this section we provide a reduction from the 3SAT problem to the 3-coloring problem of triangle-free graphs with diameter 3 and radius 2. Given a boolean formula ϕ in conjunctive normal form with three literals in each clause (3-CNF), ϕ is *satisfiable* if there is a truth assignment of ϕ, such that every clause contains at least one true literal. The problem of deciding whether a given 3-CNF formula is satisfiable, i.e. the *3SAT* problem, is one of the most known NP-complete problems [12]. Let ϕ be a 3-CNF formula with n variables $x_1, x_2, \ldots, x_n$ and m clauses $\alpha_1, \alpha_2, \ldots, \alpha_m$. We can assume in the following without loss of generality that each clause has three distinct literals. We now construct an irreducible and triangle-free graph $H_\phi = (V_\phi, E_\phi)$ with diameter 3 and radius 2, such that ϕ is satisfiable if and only if H_ϕ is 3-colorable. Before we construct H_ϕ, we first construct an auxiliary graph $G_{n,m}$ that depends only on the number n of the variables and the number m of the clauses in ϕ, rather than on ϕ itself.

We construct the graph $G_{n,m} = (V_{n,m}, E_{n,m})$ as follows. Let v_0 be a vertex with $8m$ neighbors $v_1, v_2, \ldots, v_{8m}$, which induce an independent set. Consider also the sets $U = \{u_{i,j} : 1 \leq i \leq n + 5m, 1 \leq j \leq 8m\}$ and $W = \{w_{i,j} : 1 \leq i \leq n + 5m, 1 \leq j \leq 8m\}$ of vertices. Each of these sets has $(n + 5m)8m$ vertices. The set $V_{n,m}$ of vertices of $G_{n,m}$ is defined as $V_{n,m} = U \cup W \cup \{v_0, v_1, v_2, \ldots, v_{8m}\}$. That is, $|V_{n,m}| = 2 \cdot (n + 5m)8m + 8m + 1$, and thus $|V_{n,m}| = \Theta(m^2)$, since $m = \Omega(n)$.

The set $E_{n,m}$ of the edges of $G_{n,m}$ is defined as follows. Define first for every $j \in \{1, 2, \ldots, 8m\}$ the subsets $U_j = \{u_{1,j}, u_{2,j}, \ldots, u_{n+5m,j}\}$ and $W_j = \{w_{1,j}, w_{2,j}, \ldots, w_{n+5m,j}\}$ of U and W, respectively. Then define $N(v_j) = \{v_0\} \cup U_j \cup W_j$ for every $j \in \{1, 2, \ldots, 8m\}$, where $N(v_j)$ denotes the set of neighbors of vertex v_j in $G_{n,m}$. For simplicity of the presentation, we arrange the vertices of $U \cup W$ on a rectangle matrix of size $2(n + 5m) \times 8m$, cf. Fig. 3a. In this matrix arrangement, the (i, j)th element is vertex $u_{i,j}$ if $i \leq n + 5m$, and vertex $w_{i-n-5m,j}$ if $i \geq n + 5m + 1$. In particular, for every $j \in \{1, 2, \ldots, 8m\}$, the jth column of this matrix contains exactly the vertices of $U_j \cup W_j$, cf. Fig. 3a. Note that, for every $j \in \{1, 2, \ldots, 8m\}$, vertex v_j is adjacent to all vertices of the jth column of this matrix. Denote now by $\ell_i = \{u_{i,1}, u_{i,2}, \ldots, u_{i,8m}\}$ (resp. $\ell'_i = \{w_{i,1}, w_{i,2}, \ldots, w_{i,8m}\}$) the ith (resp. the $(n + 5m + i)$th) row of this matrix, cf. Fig. 3a. For every $i \in \{1, 2, \ldots, n + 5m\}$, the vertices of ℓ_i and of ℓ'_i induce two independent sets in $G_{n,m}$. We then add between the vertices of ℓ_i and the vertices of ℓ'_i all possible edges, except those of $\{u_{i,j}w_{i,j} : 1 \leq j \leq 8m\}$. That is, we add all possible $(8m)^2 - 8m$ edges between the vertices of ℓ_i and of ℓ'_i, such that they induce a complete bipartite graph without a perfect matching between the vertices of ℓ_i and of ℓ'_i, cf. Fig. 3b. Note by the construction of $G_{n,m}$ that both U and W are independent sets in $G_{n,m}$. Further-

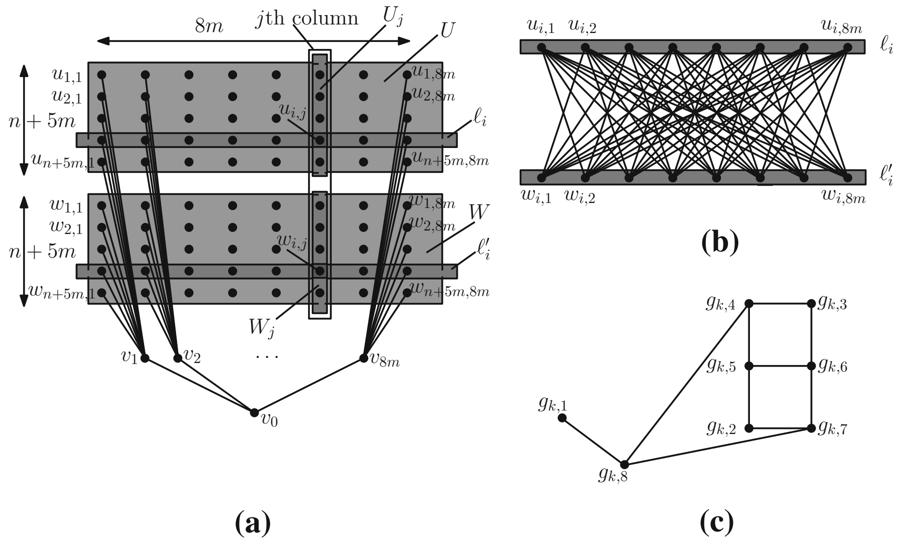

Fig. 3 **a** The $2(n+5m)\times 8m$-matrix arrangement of the vertices $U \cup W$ of $G_{n,m}$ and their connections with the vertices $\{v_0, v_1, v_2, \ldots, v_{8m}\}$, **b** the edges between the vertices of the ith row ℓ_i and the $(n+5m+i)$th row ℓ'_i in this matrix, and **c** the gadget with 8 vertices and 10 edges that we associate in H_ϕ to the clause α_k of ϕ, where $1 \le k \le m$

more note that the minimum degree in $G_{n,m}$ is $\delta(G_{n,m}) = \Theta(m)$ and the maximum degree is $\Delta(G_{n,m}) = \Theta(n+m)$. Thus, since $m = \Omega(n)$, we have that $\delta(G_{n,m}) = \Delta(G_{n,m}) = \Theta(m)$. The construction of the graph $G_{n,m}$ is illustrated in Fig. 3.

Lemma 2 *For every $n, m \ge 1$, the graph $G_{n,m}$ has diameter* 3 *and radius* 2.

Proof First note that for any $j \in \{1, 2, \ldots, 8m\}$, every vertex $u_{i,j}$ (resp. $w_{i,j}$) is adjacent to v_j. Therefore, since $v_j \in N(v_0)$ for every $j \in \{1, 2, \ldots, 8m\}$, it follows that $d(v_0, u) \le 2$ for every $u \in V_{n,m} \setminus \{v_0\}$, and thus $G_{n,m}$ has radius 2. Furthermore note that $d(v_j, v_k) \le 2$ for every $j, k \in \{1, 2, \ldots, 8m\}$, since $v_j, v_k \in N(v_0)$. Consider now an arbitrary vertex $u_{i,j}$ (resp. $w_{i,j}$) and an arbitrary vertex v_k. If $j = k$, then $d(v_k, u_{i,j}) = 1$ (resp. $d(v_k, w_{i,j}) = 1$). Otherwise, if $j \ne k$, then there exists the path $(v_k, v_0, v_j, u_{i,j})$ (resp. $(v_k, v_0, v_j, w_{i,j})$) of length 3 between v_k and $u_{i,j}$ (resp. $w_{i,j}$). Therefore also $d(v_k, u) \le 3$ for every $k = 1, 2, \ldots, 8m$ and every $u \in V_{n,m} \setminus \{v_k\}$.

We will now prove that $d(u_{i,j}, u_{p,q}) \le 3$ for every $(i, j) \ne (p, q)$. If $q \ne j$, then there exists the path $(u_{i,j}, v_j, w_{p,j}, u_{p,q})$ of length 3 between $u_{i,j}$ and $u_{p,q}$. If $q = j$, then $d(u_{i,j}, u_{p,q}) = 2$, as $u_{i,j}, u_{p,q} \in N(v_j)$. Therefore $d(u_{i,j}, u_{p,q}) \le 3$ for every $(i, j) \ne (p, q)$. Similarly we can prove that $d(w_{i,j}, w_{p,q}) \le 3$ for every $(i, j) \ne (p, q)$. It remains to prove that $d(u_{i,j}, w_{p,q}) \le 3$ for every (i, j) and every (p, q). If $p \ne i$ and $q \ne j$, then there exists the path $(u_{i,j}, v_j, u_{p,j}, w_{p,q})$ of length 3 between $u_{i,j}$ and $w_{p,q}$. If $q = j$, then $d(u_{i,j}, w_{p,q}) = 2$, since $u_{i,j}, w_{p,q} \in N(v_j)$. If $p = i$ and $q \ne j$, then $w_{p,q} = w_{i,q}$, and thus $u_{i,j} w_{p,q} \in E_{n,m}$. Therefore $d(u_{ij}, w_{pq}) \le 3$ for every (i, j) and every (p, q), and thus $G_{n,m}$ has diameter 3. $\square$

Lemma 3 *For every $n, m \geq 1$, the graph $G_{n,m}$ is irreducible and triangle-free.*

Proof First observe that, by construction, $G_{n,m}$ has no pair of sibling vertices, and thus the Reduction Rule 2 does not apply to $G_{n,m}$. Thus, in order to prove that $G_{n,m}$ is irreducible, it suffices to prove that $G_{n,m}$ is triangle-free, and thus also diamond-free, i.e. also the Reduction Rule 1 does not apply to $G_{n,m}$. Suppose otherwise that $G_{n,m}$ has a triangle with vertices a, b, c. Note that vertex v_0 does not belong to any triangle in $G_{n,m}$, since its neighbors $N(v_0) = \{v_1, v_2, \ldots, v_{8m}\}$ induce an independent set in $G_{n,m}$. Furthermore, note that also vertex v_j, where $1 \leq j \leq 8m$, does not belong to any triangle in $G_{n,m}$. Indeed, by construction of $G_{n,m}$, its neighbors $N(v_j) = \{v_0\} \cup U_j \cup W_j$ induce an independent set in $G_{n,m}$. Therefore all the vertices a, b, c of the assumed triangle of $G_{n,m}$ belong to $U \cup W$. Therefore, at least two vertices among a, b, c belong to U, or at least two of them belong to W. This is a contradiction, since by the construction of $G_{n,m}$ the sets U and W induce two independent sets. Therefore $G_{n,m}$ is triangle-free. Thus $G_{n,m}$ is also diamond-free, i.e. the Reduction Rule 1 does not apply to $G_{n,m}$. Summarizing, $G_{n,m}$ is irreducible and triangle-free. □

We now construct the graph $H_\phi = (V_\phi, E_\phi)$ from ϕ by adding $10m$ edges to $G_{n,m}$ as follows. Let $k \in \{1, 2, \ldots, m\}$ and consider the clause $\alpha_k = (l_{k,1} \vee l_{k,2} \vee l_{k,3})$, where $l_{k,p} \in \{x_{i_{k,p}}, \overline{x_{i_{k,p}}}\}$ for $p \in \{1, 2, 3\}$ and $i_{k,1}, i_{k,2}, i_{k,3} \in \{1, 2, \ldots, n\}$. For this clause α_k , we add on the vertices of $G_{n,m}$ an isomorphic copy of the gadget in Fig. 3c, which has 8 vertices and 10 edges, as follows. Let $p \in \{1, 2, 3\}$. The literal $l_{k,p}$ corresponds to vertex $g_{k,p}$ of this gadget. If $l_{k,p} = x_{i_{k,p}}$, we set $g_{k,p} = u_{i_{k,p},8k+1-p}$. Otherwise, if $l_{k,p} = \overline{x_{i_{k,p}}}$, we set $g_{k,p} = w_{i_{k,p},8k+1-p}$. Furthermore, for $p \in \{4, \ldots, 8\}$, we set $g_{k,p} = u_{n+5k+4-p,8k+1-p}$.

Note that, by construction, the graphs H_ϕ and $G_{n,m}$ have the same vertex set, i.e. $V_\phi = V_{n,m}$, and that $E_{n,m} \subset E_\phi$. Therefore $diam(H_\phi) \leq 3$ and $rad(H_\phi) \leq 2$, since $diam(G_{n,m}) = 3$ and $rad(G_{n,m}) = 2$ by Lemma 2. Observe now that every positive literal of ϕ is associated to a vertex of U, while every negative literal of ϕ is associated to a vertex of W. In particular, each of the $3m$ literals of ϕ corresponds by this construction to a different column in the matrix arrangement of the vertices of $U \cup W$. If a literal of ϕ is the variable x_i (resp. the negated variable $\overline{x_i}$), where $1 \leq i \leq n$, then the vertex of U (resp. W) that is associated to this literal lies in the ith row ℓ_i (resp. in the $(n + 5m + i)$th row ℓ'_i) of the matrix.

Moreover, note by the above construction that each of the $8m$ vertices $\{g_{k,1}, g_{k,2}, \ldots, g_{k,8}\}_{k=1}^m$ corresponds to a different column in the matrix of the vertices of $U \cup W$. Finally, each of the $5m$ vertices $\{g_{k,4}, g_{k,5}, g_{k,6}, g_{k,7}, g_{k,8}\}_{k=1}^m$ corresponds to a different row in the matrix of the vertices of U. We emphasize here that, by construction, each copy of the gadget of Fig. 3c is an induced subgraph of H_ϕ.

Observation 2 *The gadget of Fig. 3c has no proper 2-coloring, as it contains an induced cycle of length 5.*

Observation 3 *Consider the gadget of Fig. 3c. If we assign to vertices $g_{k,1}, g_{k,2}, g_{k,3}$ the same color, we cannot extend this coloring to a proper 3-coloring of the gadget. Furthermore, if we assign to vertices $g_{k,1}, g_{k,2}, g_{k,3}$ in total two or three colors, then we can extend this coloring to a proper 3-coloring of the gadget.*

The next observation follows by the construction of H_ϕ and by our initial assumption that each clause of ϕ has three distinct literals.

Observation 4 *For every $i \in \{1, 2, \ldots, n + 5m\}$, there exists no pair of adjacent vertices in the same row ℓ_i or ℓ'_i in H_ϕ.*

Lemma 4 *For every formula ϕ, the graph H_ϕ is irreducible and triangle-free.*

Proof First observe that, similarly to $G_{n,m}$, the graph H_ϕ has no pair of sibling vertices, and thus the Reduction Rule 2 does not apply to H_ϕ. We will now prove that H_ϕ is triangle-free. Suppose otherwise that H_ϕ has a triangle with vertices a, b, c. Similarly to $G_{n,m}$, note that the neighbors of v_0 in H_ϕ induce an independent set, and thus vertex v_0 does not belong to any triangle in H_ϕ. Furthermore, note by the construction of H_ϕ that we do not add any edge between vertices $u_{i,j}$ and $w_{i,j}$, where $1 \le i \le n+5m$ and $1 \le j \le 8m$. Therefore, similarly to $G_{n,m}$, the neighbors of v_j induce an independent set in H_ϕ, where $j \in \{1, 2, \ldots, 8m\}$, and thus v_j does not belong to any triangle in H_ϕ. Therefore all the vertices a, b, c of the assumed triangle of H_ϕ belong to $U \cup W$. Since $G_{n,m}$ is triangle-free by Lemma 3, it follows that at least one edge in this triangle belongs to $E_\phi \setminus E_{n,m}$, i.e. to at least one of the copies of the gadget in Fig. 3c. Note that not all of the three vertices a, b, c belong to the same copy of this gadget, since the gadget is triangle-free, cf. Fig. 3c. Assume without loss of generality that vertices a and b (and thus also the edge ab) of the assumed triangle of H_ϕ belong to the copy of the gadget that corresponds to clause α_k, where $1 \le k \le m$. Then, since vertex c does not belong to this gadget, the edges ac and bc of the assumed triangle belong also to the graph $G_{n,m}$. Therefore, since $a, b, c \in U \cup W$, it follows by the construction of $G_{n,m}$ that a and b belong to some row ℓ_i and c belongs to the row ℓ'_i, or a and b belong to some row ℓ'_i and c belongs to the row ℓ_i. This is a contradiction, since no pair of adjacent vertices (such as a and b) belong to the same row ℓ_i or ℓ'_i in H_ϕ by Observation 4. Therefore H_ϕ is triangle-free, and thus also diamond-free, i.e. the Reduction Rule 2 does not apply to H_ϕ. Summarizing, H_ϕ is irreducible and triangle-free. □

We are now ready to state the main theorem of this section.

Theorem 7 *The formula ϕ is satisfiable if and only if H_ϕ is 3-colorable.*

Proof We will first prove that $G_{n,m}$ is always 3-colorable. Recall that both U and W are independent sets in $G_{n,m}$, and that the only edges among the vertices of $U \cup W$ in $G_{n,m}$ are all possible edges between the rows ℓ_i (that contains only vertices of U) and ℓ'_i (that contains only vertices of W), except for a perfect matching between the vertices of ℓ_i and of ℓ'_i. Consider three colors, say red, green, and blue. We assign to vertex v_0 the color red. Furthermore we assign arbitrarily the color blue or green to each of its neighbors v_j, $1 \le j \le 8m$. For each of these 2^{8m} different colorings of vertex v_0 and its neighbors, we can construct 2^{n+5m} different proper 3-colorings of $G_{n,m}$ as follows. For every $i \in \{1, 2, \ldots, n + 5m\}$, we have at least two possibilities for coloring the vertices of ℓ_i and of ℓ'_i: (a) color all vertices of ℓ_i red, and for every vertex $w_{i,j}$ of ℓ'_i, color $w_{i,j}$ blue (resp. green) if v_j is colored green (resp. blue), and (b) color all vertices of ℓ'_i red, and for every vertex $u_{i,j}$ of ℓ_i, color $u_{i,j}$ blue (resp. green) if v_j is colored green (resp. blue). Therefore, there are at least $2^{8m} \cdot 2^{n+5m}$ different proper 3-colorings of $G_{n,m}$, in which vertex v_0 obtains color red.

($\Rightarrow$) Suppose first that ϕ is satisfiable, and let τ be a satisfying truth assignment of ϕ. Given this truth assignment τ, we construct a proper 3-coloring χ_ϕ of H_ϕ as follows. First assign to v_0 the color red in χ_ϕ. By construction, this coloring χ_ϕ will be one of the above $2^{8m} \cdot 2^{n+5m}$ proper 3-colorings of $G_{n,m}$. That is, for every $i \in \{1, 2, \ldots, n + 5m\}$, either all vertices of row ℓ_i are red and all vertices of row ℓ'_i are green or blue in χ_ϕ, or vice versa. For every $i \in \{1, 2, \ldots, n + 5m\}$, if the vertices of row ℓ_i (resp. of row ℓ'_i) are red in χ_ϕ, then we call row ℓ_i (resp. row ℓ'_i) a *red line*. Otherwise, if the vertices of row ℓ_i (resp. of row ℓ'_i) are not red in χ_ϕ, then we call row ℓ_i (resp. row ℓ'_i) a *white line*. Furthermore, for every vertex $u_{i,j}$ (resp. $w_{i,j}$) of a white line ℓ_i (resp. of a white line ℓ'_i), the color of $u_{i,j}$ (resp. of $w_{i,j}$) in χ_ϕ is uniquely determined by the color of its neighbor v_j in χ_ϕ. That is, if v_j is blue then $u_{i,j}$ (resp. $w_{i,j}$) is green in χ_ϕ. Otherwise, if v_j is green then $u_{i,j}$ (resp. $w_{i,j}$) is blue in χ_ϕ.

Let x_i be an arbitrary variable in ϕ, where $1 \leq i \leq n$. If $x_i = 0$ in τ, we define row ℓ_i to be a red line and row ℓ'_i to be a white line in χ_ϕ, respectively. Otherwise, if $x_i = 1$ in τ, we define row ℓ'_i to be a red line and row ℓ_i to be a white line in χ_ϕ, respectively. Consider an arbitrary clause $\alpha_k = (l_{k,1} \vee l_{k,2} \vee l_{k,3})$ of ϕ, where $l_{k,p} \in \{x_{i_{k,p}}, \overline{x_{i_{k,p}}}\}$ for $p \in \{1, 2, 3\}$. Furthermore consider the copy of the gadget of Fig. 3c that is associated to clause α_k in H_ϕ. Recall by the construction of H_ϕ that the literals $l_{k,1}$, $l_{k,2}$, and $l_{k,3}$ correspond to the vertices $g_{k,1}$, $g_{k,2}$, and $g_{k,3}$ of this gadget, respectively. Since τ is a satisfying assignment of ϕ, at least one of the literals $l_{k,1}$, $l_{k,2}$, and $l_{k,3}$ is true in τ. Therefore at least one of the vertices $g_{k,1}$, $g_{k,2}$, and $g_{k,3}$ belongs to a white line in χ_ϕ, i.e. at least one of them is green or blue in χ_ϕ. If one (resp. two) of the vertices $g_{k,1}, g_{k,2}, g_{k,3}$ belongs (resp. belong) to a red line of χ_ϕ, then we color the other two (resp. the other one) green in χ_ϕ. Otherwise, if all three of the vertices $g_{k,1}$, $g_{k,2}$, and $g_{k,3}$ belong to a white line in χ_ϕ, then we color $g_{k,1}$, $g_{k,2}$ green and vertex $g_{k,3}$ blue in χ_ϕ. For every $p \in \{1, 2, 3\}$, if we color vertex $g_{k,p}$ green (resp. blue) in χ_ϕ, then we color its neighbor v_{8k+1-p} blue (resp. green) in χ_ϕ, cf. the construction of the graph H_ϕ. Otherwise, if we color vertex $g_{k,p}$ red in χ_ϕ, then we color its neighbor v_{8k+1-p} either blue or green in χ_ϕ (both choices lead to a proper 3-coloring of H_ϕ).

Once we have colored the vertices $g_{k,1}, g_{k,2}, g_{k,3}$ with two colors in total, we extend the coloring of these three vertices to a proper 3-coloring χ_k of the gadget associated to clause α_k (cf. Observation 3). Let $p \in \{4, 5, 6, 7, 8\}$. If $g_{k,p}$ is colored green (resp. blue) in χ_k, then we color its neighbor v_{8k+1-p} blue (resp. green) in χ_ϕ, and we define row $\ell_{n+5k+4-p}$ to be a white line and row $\ell'_{n+5k+4-p}$ to be a red line in χ_ϕ, respectively. Otherwise, if $g_{k,p}$ is colored red in χ_k, then we define row $\ell_{n+5k+4-p}$ to be a red line and row $\ell'_{n+5k+4-p}$ to be a white line in χ_ϕ, respectively. Furthermore, in this case we color the neighbor v_{8k+1-p} of $g_{k,p}$ either blue or green χ_ϕ (both choices lead to a proper 3 -coloring of H_ϕ). After performing the above coloring operations for every clause α_k, where $1 \leq k \leq m$, we obtain a well defined coloring χ_ϕ of all vertices of H_ϕ. Note that in this coloring χ_ϕ, all copies of the gadget of Fig. 3c are properly colored with 3 colors. Furthermore, by construction this coloring χ_ϕ is also a proper 3-coloring of $G_{n,m}$. Therefore χ_ϕ is a proper 3-coloring of H_ϕ.

($\Leftarrow$) Suppose now that H_ϕ is 3-colorable and let χ_ϕ be a proper 3-coloring of H_ϕ. Assume without loss of generality that vertex v_0 is colored red in χ_ϕ. We will construct a satisfying assignment τ of ϕ. Consider an index $i \in \{1, 2, \ldots, n+5m\}$ and the rows ℓ_i and ℓ'_i of the matrix. Suppose that ℓ_i has at least one vertex u_{i,j_1} that is colored red and at least one vertex u_{i,j_2} that is colored blue in χ_ϕ. Then clearly all vertices of ℓ'_i, except possibly of w_{i,j_1} and w_{i,j_2}, are colored green in χ_ϕ, since they are adjacent to both u_{i,j_1} and u_{i,j_2}. Therefore all vertices of $\{v_1, v_2, \ldots, v_{8m}\} \setminus \{v_{j_1}, v_{j_2}\}$ are colored blue in χ_ϕ. Thus, all vertices of $(U \cup W) \setminus (U_{j_1} \cup U_{j_2} \cup W_{j_1} \cup W_{j_2})$ are colored either green or red in χ_ϕ . However there exists at least one copy of the gadget of Fig. 3c on these vertices, by the construction of H_ϕ. That is, this gadget has a proper coloring (induced by χ_ϕ) with the colors green and red. This is a contradiction by Observation 2. Thus, there exists no row ℓ_i with at least one vertex colored red and another one colored blue in χ_ϕ . Similarly we can prove that there exists no row ℓ_i (resp. ℓ'_i) with at least one vertex colored red and another one colored blue or green in χ_ϕ. That is, if at least one vertex of a row ℓ_i (resp. ℓ'_i) is colored red in χ_ϕ, then all vertices of ℓ_i (resp. ℓ'_i) are colored red in χ_ϕ.

We will now prove that for any $i \in \{1, 2, \ldots, n + 5m\}$, at least one vertex of ℓ_i or at least one vertex of ℓ'_i is red in χ_ϕ. Suppose otherwise that every vertex of the rows ℓ_i and ℓ'_i is colored either green or blue in χ_ϕ. Then, since the vertices of ℓ_i and of ℓ'_i induce a connected bipartite graph, it follows that all vertices of ℓ_i are colored green and all vertices of ℓ'_i are colored blue in χ_ϕ, or vice versa. Thus, in particular, for every $j = 1, 2, \ldots, 8m$, vertex v_j is adjacent to one blue and to one green vertex (one from ℓ_i and the other one from ℓ'_i). Thus, since χ_ϕ is a proper 3-coloring of H_ϕ, it follows that v_j is colored red in χ_ϕ. This is a contradiction, since $v_0 \in N(v_j)$ and v_0 is colored red in χ_ϕ by assumption. Therefore, for any $i \in \{1, 2, \ldots, n + 5m\}$, at least one vertex of ℓ_i or at least one vertex of ℓ'_i is colored red in χ_ϕ.

Summarizing, for every $i \in \{1, 2, \ldots, n + 5m\}$, either all vertices of the row ℓ_i or all vertices of the row ℓ'_i are colored red in χ_ϕ. We define now the truth assignment τ of ϕ as follows. For every $i \in \{1, 2, \ldots, n + 5m\}$, we set $x_i = 0$ in τ if all vertices of ℓ_i are colored red in χ_ϕ; otherwise, if all vertices of ℓ'_i are colored red in χ_ϕ, then we set $x_i = 1$ in τ. We will prove that τ is a satisfying assignment of ϕ. Consider a clause $\alpha_k = (l_{k,1} \vee l_{k,2} \vee l_{k,3})$ of ϕ, where $l_{k,p} \in \{x_{i_{k,p}}, \overline{x_{i_{k,p}}}\}$ for $p \in \{1, 2, 3\}$. By the construction of the graph H_ϕ, this clause corresponds to a copy of the gadget of Fig. 3c in H_ϕ. Thus, since χ_ϕ is a proper 3-coloring of H_ϕ by assumption, the vertices of this gadget are colored with three colors by Observation 2. Furthermore, not all three vertices $g_{k,1}, g_{k,2}, g_{k,3}$ have the same color in χ_ϕ by Observation 3. Moreover, note by the construction of H_ϕ and by the definition of the truth assignment τ, that $l_{k,p} = 0$ in τ if and only if vertex $g_{k,p}$ is colored red in χ_ϕ, where $p \in \{1, 2, 3\}$. Thus, since the vertices $g_{k,1}, g_{k,2}, g_{k,3}$ are not all red in χ_ϕ, it follows that the literals $l_{k,1}, l_{k,2}, l_{k,3}$ of clause α_k are not all false in τ. Therefore α_k is satisfied by τ, and thus τ is a satisfying truth assignment of ϕ. This completes the proof of the theorem. □

The next theorem follows by Lemma 4 and Theorem 7.

Theorem 8 *The* 3*-coloring problem is NP-complete on irreducible and triangle-free graphs with diameter* 3 *and radius* 2.

4.3 Time Complexity Lower Bounds and General NP-Completeness Results

In this section we present our three different amplification techniques of the reduction of Theorem 7. In particular, using these three amplifications we extend for every $\varepsilon \in [0, 1)$ the result of Theorem 8 (by providing both NP-completeness and time complexity lower bounds) to irreducible triangle-free graphs with diameter 3 and radius 2 and minimum degree $\delta = \Theta(n^{\varepsilon})$. We use our first amplification technique in Theorems 9 and 11, our second one in Theorems 10 and 13, and our third one in Theorem 12.

Theorem 9 *Let $G = (V, E)$ be an irreducible and triangle-free graph with diameter 3 and radius 2. If the minimum degree of G is $\delta(G) = \Theta(|V|^{\varepsilon})$, where $\varepsilon \in [\frac{1}{2}, 1)$, then it is NP-complete to decide whether G is 3-colorable.*

Proof Let ϕ be a boolean formula with n variables and m clauses. Using the reduction of Sect. 4.2, we construct from the formula ϕ the irreducible and triangle-free graph $H_{\phi} = (V_{\phi}, E_{\phi})$ where H_{ϕ} has diameter 3 and radius 2. Furthermore $|V_{\phi}| = \Theta(nm)$ by the construction of H_{ϕ}. Then, ϕ is satisfiable if and only if H_{ϕ} is 3-colorable, by Theorem 7.

Let now $\varepsilon \in [\frac{1}{2}, 1)$. Define $\varepsilon_0 = \frac{\varepsilon}{1-\varepsilon}$ and $k_0 = m^{\varepsilon_0}$. Since $\varepsilon \in [\frac{1}{2}, 1)$ by assumption, it follows that $\varepsilon_0 \geq 1$. We construct now from the graph H_{ϕ} the irreducible graph $H_1(\phi, \varepsilon)$ with diameter 3 and radius 2, as follows. First we add $8m$ new vertices $v_{0,1}, v_{0,2}, \ldots, v_{0,8m}$. For every vertex v_j in H_{ϕ}, where $j \in \{1, 2, \ldots, 8m\}$, we remove v_j and introduce $2k_0$ new vertices $A_j = \{v'_{j,1}, v'_{j,2}, \ldots, v'_{j,k_0}\}$ and $B_j = \{v''_{j,1}, v''_{j,2}, \ldots, v''_{j,k_0}\}$. The vertices of A_j and of B_j induce two independent sets in $H_1(\phi, \varepsilon)$. We then add between the vertices of A_j and of B_j all possible edges, except those of $\{v'_{j,p}v''_{j,p} : 1 \leq p \leq k_0\}$. That is, we add $k_0^2 - k_0$ edges between the vertices of A_j and B_j, such that they induce a complete bipartite graph without a perfect matching between A_j and B_j. Furthermore we add all k_0 edges between v_0 and the vertices of A_j. Moreover we add all $2(n + 5m) \cdot k_0$ edges between the $2(n + 5m)$ vertices of $U_j \cup W_j$ (i.e. the jth column of the matrix arrangement of the vertices of $U \cup V$) and the k_0 vertices of A_j. Finally we add all k_0 edges between $v_{0,j}$ and the vertices of B_j, as well as all $(8m - 1)k_0$ edges between $v_{0,j}$ and the vertices of $A_{j'}$, where $j' \in \{1, 2, \ldots, 8m\} \setminus \{j\}$. Denote the resulting graph by $H_1(\phi, \varepsilon)$. The replacement of vertex v_j by the vertex sets A_j and B_j in $H_1(\phi, \varepsilon)$ is illustrated in Fig. 4.

Observe that, by this construction, for every $j \in \{1, 2, \ldots, 8m\}$, all neighbors of vertex v_j in the graph H_{ϕ} are included in the neighborhood of every vertex $v'_{j,p}$ of A_j in the graph $H_1(\phi, \varepsilon)$, where $1 \leq p \leq k_0$. In particular, H_{ϕ} is an induced subgraph of $H_1(\phi, \varepsilon)$: if we remove from $H_1(\phi, \varepsilon)$ the vertices of $(\{v_{0,j}\} \cup B_j \cup A_j) \setminus \{v'_{j,1}\}$, for every $j \in \{1, 2, \ldots, 8m\}$, we obtain a graph isomorphic to H_{ϕ}, where $v'_{j,1}$ of $H_1(\phi, \varepsilon)$ corresponds to vertex v_j of H_{ϕ}, for every $j \in \{1, 2, \ldots, 8m\}$. Note that, similarly to H_{ϕ}, the graph $H_1(\phi, \varepsilon)$ has radius 2, since $d(v_0, u) \leq 2$ in $H_1(\phi, \varepsilon)$ for every vertex u of $H_1(\phi, \varepsilon) - \{v_0\}$.

We now prove that $H_1(\phi, \varepsilon)$ has diameter 3. First note that the distance between any two vertices of $\cup_{j=1}^{8m} A_j$ is at most 2, since they all have v_0 as common neighbor.

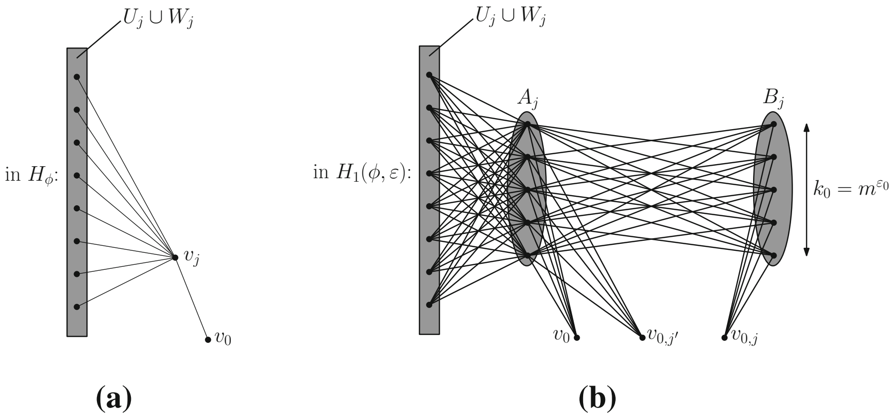

Fig. 4 **a** The vertex v_j with its neighbors $N(v_j) = \{v_0\} \cup U_j \cup W_j$ in the graph H_ϕ and **b** the vertex sets A_j and B_j that replace vertex v_j in the graph $H_1(\phi, \varepsilon)$; here j' is an arbitrary index from $\{1, 2, \ldots, 8m\} \setminus \{j\}$

Consider two arbitrary vertices $a \in A_j$ and $b \in B_{j'}$, where $j, j' \in \{1, 2, \ldots, 8m\}$. If $j = j'$ then either a and b are adjacent or there exists another vertex $a' \in A_j \setminus \{a\}$ such that a and b are connected with the path (b, a', v_0, a) of length 3. If $j \neq j'$ then a and b have $v_{0,j}$ as a common neighbor, i.e. their distance is 2. Consider now two vertices $b \in B_j$ and $b' \in B_{j'}$. If $j = j'$ then b and b' have $v_{0,j}$ as a common neighbor, i.e. their distance is 2. If $j \neq j'$ then there exists at least one vertex $a' \in A_{j'}$ such that b and b' are connected with the path $(b, v_{0,j}, a', b')$ of length 3. That is, the distance between any two vertices of $\cup_{j=1}^{8m} A_j \cup_{j=1}^{8m} B_j$ is at most 3.

Consider an arbitrary vertex $a \in \cup_{j=1}^{8m} A_j$ and an arbitrary vertex $u_{i,j} \in U$ (resp. $w_{i,j} \in W$). If a and $u_{i,j}$ (resp. $w_{i,j}$) are not adjacent in $H_1(\phi, \varepsilon)$, there exists the path $(a, v_0, v'_{j,1}, u_{i,j})$ (resp. the path $(a, v_0, v'_{j,1}, w_{i,j})$) of length 3 between a and $u_{i,j}$ (resp. $w_{i,j}$). Consider an arbitrary vertex $b' \in B_{j'}$ and an arbitrary vertex $u_{i,j} \in U$ (resp. $w_{i,j} \in W$). If $j = j'$ then b' and $u_{i,j}$ (resp. $w_{i,j}$) have at least one common neighbor $a' \in A_{j'}$, i.e. their distance is 2. If $j \neq j'$ then b' and $u_{i,j}$ (resp. $w_{i,j}$) are connected with the path $(b', v_{0,j'}, v'_{j,1}, u_{i,j})$ (resp. the path $(b', v_{0,j'}, v'_{j,1}, w_{i,j})$) of length 3. That is, the distance between any vertex of $\cup_{j=1}^{8m} A_j \cup_{j=1}^{8m} B_j$ and any vertex of $U \cup W$ in $H_1(\phi, \varepsilon)$ is at most 3.

Consider now an arbitrary vertex $v_{0,j}$. This vertex has every vertex of $A_{j'}$, where $j' \in \{1, 2, \ldots, 8m\} \setminus \{j\}$, as common neighbors with v_0, i.e. $d(v_{0,j}, v_0) = 2$. Furthermore, for every $j' \in \{1, 2, \ldots, 8m\} \setminus \{j\}$, there exists the path $(v_{0,j}, v''_{j,2}, v'_{j,1}, v_{0,j'})$ of length 3 between the vertices $v_{0,j}$ and $v_{0,j'}$. Note that, by construction of $H_1(\phi, \varepsilon)$, the distance between $v_{0,j}$ and every vertex of $\cup_{j=1}^{8m} A_j \cup_{j=1}^{8m} B_j$ is at most 2. Thus, since every vertex of $U \cup W$ is adjacent to at least one vertex of $\cup_{j=1}^{8m} A_j$, it follows that the distance between $v_{0,j}$ and every vertex of $U \cup W$ is at most 3. Finally, since H_ϕ is an induced subgraph of $H_1(\phi, \varepsilon)$, it follows by Lemma 2 that also $d(z, z') \leq 3$ in $H_1(\phi, \varepsilon)$, for every pair of vertices $z, z' \in U \cup W$. Therefore $H_1(\phi, \varepsilon)$ has diameter 3.

Recall now that for every $j \in \{1, 2, \ldots, 8m\}$, vertex v_j of H_ϕ has been replaced by the vertices of $A_j \cup B_j$ in $H_1(\phi, \varepsilon)$. Furthermore, recall that the vertices of

$A_j \cup B_j$ induce in $H_1(\phi, \varepsilon)$ a complete bipartite graph without a perfect matching between A_j and B_j. Therefore there exists no pair of sibling vertices in $A_j \cup B_j$, for every $j \in \{1, 2, \ldots, 8m\}$. Similarly there exists no pair of sibling vertices among $\{v_0, v_{0,1}, v_{0,2}, \ldots, v_{0,8m}\}$. Furthermore it can be easily checked that $H_1(\phi, \varepsilon)$ has no triangles, and thus it has no diamonds. Thus, since H_ϕ is irreducible by Lemma 4, it follows that $H_1(\phi, \varepsilon)$ is irreducible as well.

We now prove that $H_1(\phi, \varepsilon)$ is 3-colorable if and only if H_ϕ is 3-colorable. Suppose first that $H_1(\phi, \varepsilon)$ is 3-colorable. Then, since H_ϕ is an induced subgraph of $H_1(\phi, \varepsilon)$, it follows immediately that H_ϕ is also 3-colorable. Now suppose that H_ϕ is 3-colorable, and let χ be a proper 3-coloring of H_ϕ. Assume without loss of generality that v_0 is colored red in χ. We will extend χ into a proper 3-coloring of $H_1(\phi, \varepsilon)$ as follows. First we color all vertices $\{v_{0,1}, v_{0,2}, \ldots, v_{0,8m}\}$ red. Consider the vertex v_j of H_ϕ, where $1 \leq j \leq 8m$. Since v_0 is colored red in χ, it follows that v_j is colored either blue or green in χ. If v_j is colored green in χ, then we color in $H_1(\phi, \varepsilon)$ all vertices of A_j green and all vertices of B_j blue. Otherwise, if v_j is colored blue in χ, then we color in $H_1(\phi, \varepsilon)$ all vertices of A_j blue and all vertices of B_j green. It is now straightforward to check that the resulting 3-coloring of $H_1(\phi, \varepsilon)$ is proper, i.e. that $H_1(\phi, \varepsilon)$ is 3-colorable. That is, $H_1(\phi, \varepsilon)$ is 3 -colorable if and only if H_ϕ is 3-colorable. Therefore Theorem 7 implies that the formula ϕ is satisfiable if and only if $H_1(\phi, \varepsilon)$ is 3-colorable.

By construction, the graph $H_1(\phi, \varepsilon)$ has $N = 2(n + 5m)8m + 8m \cdot 2k_0 + 8m + 1$ vertices, where $k_0 = m^{\varepsilon_0}$. Thus, since $m = \Omega(n)$ and $\varepsilon_0 \geq 1$, it follows that $N = \Theta(m^{1+\varepsilon_0})$. Therefore $m = \Theta(N^{\frac{1}{1+\varepsilon_0}})$, where N is the number of vertices in $H_1(\phi, \varepsilon)$. Furthermore, the degree of each of the vertices $\{v_0, v_{0,1}, v_{0,2}, \ldots, v_{0,8m}\}$ in $H_1(\phi, \varepsilon)$ is $\Theta(m \cdot k_0) = \Theta(m^{1+\varepsilon_0})$, the degree of every vertex $v'_{j,p}$ in $H_1(\phi, \varepsilon)$ is $\Theta(n + m + k_0) = \Theta(m^{\varepsilon_0})$, the degree of every vertex $v''_{j,p}$ in $H_1(\phi, \varepsilon)$ is $\Theta(k_0) = \Theta(m^{\varepsilon_0})$, and the degree of every vertex $u_{i,j}$ (resp. $w_{i,j}$) in $H_1(\phi, \varepsilon)$ is $\Theta(m + k_0) = \Theta(m^{\varepsilon_0})$. Therefore the minimum degree of $H_1(\phi, \varepsilon)$ is $\delta = \Theta(m^{\varepsilon_0})$. Thus, since $m = \Theta(N^{\frac{1}{1+\varepsilon_0}})$, it follows that $\delta = \Theta(N^{\frac{\varepsilon_0}{1+\varepsilon_0}})$, i.e. $\delta = \Theta(N^{\varepsilon})$.

Summarizing, for every $\varepsilon \in [\frac{1}{2}, 1)$ and for every formula ϕ with n variables and m clauses, we can construct in polynomial time a graph $H_1(\phi, \varepsilon)$ with $N = \Theta(m^{1+\varepsilon_0})$ (i.e. $N = \Theta(m^{\frac{1}{1-\varepsilon}})$) vertices and minimum degree $\delta = \Theta(m^{\varepsilon_0})$ (i.e. $\delta = \Theta(N^{\varepsilon})$), such that $H_1(\phi, \varepsilon)$ is 3-colorable if and only if ϕ is satisfiable. Moreover, the constructed graph $H_1(\phi, \varepsilon)$ is irreducible and triangle-free, and it has diameter 3 and radius 2. This completes the proof of the theorem. □

In the next theorem we provide an amplification of the reduction of Theorem 7 to the case of graphs with minimum degree $\delta = \Theta(|V|^{\varepsilon})$, where $\varepsilon \in [0, \frac{1}{2})$.

Theorem 10 *Let $G = (V, E)$ be an irreducible and triangle-free graph with diameter 3 and radius 2. If the minimum degree of G is $\delta(G) = \Theta(|V|^{\varepsilon})$, where $\varepsilon \in [0, \frac{1}{2})$, then it is NP-complete to decide whether G is 3-colorable.*

Proof Let $G_1 = (V_1, E_1)$ be an arbitrary irreducible and triangle-free graph with diameter 3 and radius 2, such that G_1 has n vertices and minimum degree $\delta(G_1) = \Theta(\sqrt{n})$. Note that such a graph G_1 exists by the construction of H_ϕ in Sect. 4.2 (see also Theorems 7 and 9). For simplicity, we arbitrarily enumerate the vertices of G_1 as $v_1, v_2, \ldots, v_n$. Let $\varepsilon \in [0, \frac{1}{2})$. Define $\varepsilon_0 = \frac{\varepsilon}{1-\varepsilon}$ and $k_0 = n^{\varepsilon_0}$. Since $\varepsilon \in [0, \frac{1}{2})$ by assumption, it follows that $\varepsilon_0 \in [0, 1)$. We now construct from the graph G_1 the irreducible graph $G_2(\varepsilon)$ with diameter 3 and radius 2 as follows. First we introduce $n+1$ new vertices $v_0, v_{0,1}, v_{0,2}, \ldots, v_{0,n}$. For every $i \in \{1, 2, \ldots, n\}$, we add $2k_0$ new vertices $A_i = \{v'_{i,1}, v'_{i,2}, \ldots, v'_{i,k_0}\}$ and $B_i = \{v''_{i,1}, v''_{i,2}, \ldots, v''_{i,k_0}\}$. The vertices of A_i and of B_i induce two independent sets in $G_2(\varepsilon)$. We then add between the vertices of A_i and of B_i all possible edges, except those of $\{v'_{i,\ell}v''_{i,\ell} : 1 \leq \ell \leq k_0\}$. That is, we add $k_0^2 - k_0$ edges between the vertices of A_i and B_i, such that they induce a complete bipartite graph without a perfect matching between A_i and B_i. Moreover we add all k_0 edges between v_i and the vertices of A_i, as well as all k_0 edges between v_0 and the vertices of A_i. Finally we add all k_0 edges between $v_{0,i}$ and the vertices of B_i, as well as all $(n-1)k_0$ edges between $v_{0,i}$ and the vertices of $A_{i'}$, where $i' \in \{1, 2, \ldots, n\}\backslash\{i\}$. Denote the resulting graph by $G_2(\varepsilon)$.

Observe that, by construction, G_1 is an induced subgraph of $G_2(\varepsilon)$. Furthermore, $G_2(\varepsilon)$ has $N = n + n \cdot 2k_0 + n + 1$ vertices, and thus $N = \Theta(n^{1+\varepsilon_0})$. Therefore $n = \Theta(N^{\frac{1}{1+\varepsilon_0}})$, where N is the number of vertices in $G_2(\varepsilon)$. Furthermore, the degree of each of the vertices $\{v_0, v_{0,1}, v_{0,2}, \ldots, v_{0,n}\}$ in $G_2(\varepsilon)$ is $\Theta(n \cdot k_0) = \Theta(n^{1+\varepsilon_0})$, the degree of every vertex $v'_{i,\ell}$ in $G_2(\varepsilon)$ is $\Theta(n + k_0) = \Theta(n)$, the degree of every vertex $v''_{i,\ell}$ in $G_2(\varepsilon)$ is $\Theta(k_0) = \Theta(n^{\varepsilon_0})$, and the degree of every vertex $v_i \in V_1$ in $G_2(\varepsilon)$ is at least $\delta(G_1) + k_0 = \Theta(\sqrt{n} + n^{\varepsilon_0})$. Therefore, for every $\varepsilon_0 \in [0, 1)$, the minimum degree of $G_2(\varepsilon)$ is $\delta = \Theta(n^{\varepsilon_0})$. Thus, since $n = \Theta(N^{\frac{1}{1+\varepsilon_0}})$, it follows that $\delta = \Theta(N^{\frac{\varepsilon_0}{1+\varepsilon_0}})$, i.e. $\delta = \Theta(N^{\varepsilon})$. Note that the graph $G_2(\varepsilon)$ has radius 2, since $d(v_0, u) \leq 2$ in $G_2(\varepsilon)$ for every vertex u of $G_2(\varepsilon) - \{v_0\}$.

We now prove that $G_2(\varepsilon)$ has diameter 3. First note that the distance between any two vertices of $\cup_{i=1}^{n} A_i$ is at most 2, since they all have v_0 as common neighbor. Consider two arbitrary vertices $a \in A_i$ and $b \in B_{i'}$, where $i, i' \in \{1, 2, \ldots, n\}$. If $i = i'$ then either a and b are adjacent or there exists another vertex $a' \in A_i\backslash\{a\}$ such that a and b are connected with the path (b, a', v_0, a) of length 3. If $i \neq i'$ then a and b have $v_{0,i}$ as a common neighbor, i.e. their distance is 2. Consider now two vertices $b \in B_i$ and $b' \in B_{i'}$. If $i = i'$ then b and b' have $v_{0,i}$ as a common neighbor, i.e. their distance is 2. If $i \neq i'$ then there exists at least one vertex $a' \in A_{i'}$ such that b and b' are connected with the path $(b, v_{0,i}, a', b')$ of length 3. That is, the distance between any two vertices of $\cup_{i=1}^{n} A_i \cup_{i=1}^{n} B_i$ is at most 3.

Consider an arbitrary vertex $v_i \in V_1$ and an arbitrary vertex $a \in A_i$. If $i = i'$ then v_i is adjacent with a in $G_2(\varepsilon)$. Otherwise, if $i \neq i'$, there exists the path $(a, v_0, v'_{i,1}, v_i)$ of length 3 between a and v_i. Consider an arbitrary vertex $v_i \in V_1$ and an arbitrary vertex $b' \in B_{i'}$. If $i = i'$ then b' and v_i have at least one common neighbor $a' \in A_{i'}$, i.e. their distance is 2. If $i \neq i'$ then b' and v_i are connected with the path $(b', v_{0,i'}, v'_{i,1}, v_i)$ of length 3. That is, the distance between any vertex of $\cup_{i=1}^{n} A_i \cup_{i=1}^{n} B_i$ and any vertex of V_1 in $G_2(\varepsilon)$ is at most 3.

Consider now an arbitrary vertex $v_{0,i}$. Similarly to the proof of Theorem 9, $v_{0,i}$ has every vertex of $A_{i'}$ as a common neighbor with v_0, where $i' \in \{1, 2, \ldots, n\}\backslash\{i\}$. That is, $d(v_{0,i}, v_0) = 2$. Furthermore, for every $i' \in \{1, 2, \ldots, n\}\backslash\{i\}$, there exists the path $(v_{0,i}, v''_{i,2}, v'_{i,1}, v_{0,i'})$ of length 3 between the vertices $v_{0,i}$ and $v_{0,i'}$. Note that, by construction of $G_2(\varepsilon)$, the distance between $v_{0,i}$ and every vertex of $\cup_{i=1}^n A_i \cup_{i=1}^n B_i$ is at most 2. Thus, since every vertex of V_1 is adjacent to at least one vertex of $\cup_{i=1}^n A_i$, it follows that the distance between $v_{0,i}$ and every vertex of V_1 is at most 3. Finally, since G_1 is an induced subgraph of $G_2(\varepsilon)$ and G_1 has diameter 3 by assumption, it follows that also $d(z, z') \leq 3$ in $G_2(\varepsilon)$ for every pair of vertices $z, z' \in V_1$. Therefore $G_2(\varepsilon)$ has diameter 3.

Recall that for every $i \in \{1, 2, \ldots, n\}$ the vertices of $A_i \cup B_i$ induce in $G_2(\varepsilon)$ a complete bipartite graph without a perfect matching between A_i and B_i. Therefore there exists no pair of sibling vertices in $A_i \cup B_i$, for every $i \in \{1, 2, \ldots, n\}$. Similarly there exists no pair of sibling vertices among $\{v_0, v_{0,1}, v_{0,2}, \ldots, v_{0,n}\}$. Furthermore it can be easily checked that $G_2(\varepsilon)$ has no triangles, and thus it has no diamonds. Thus, since G_1 is irreducible by assumption, it follows that $G_2(\varepsilon)$ is irreducible as well.

We now prove that $G_2(\varepsilon)$ is 3-colorable if and only if G_1 is 3-colorable. If $G_2(\varepsilon)$ is 3-colorable, then clearly G_1 is also 3-colorable, since G_1 is an induced subgraph of $G_2(\varepsilon)$. Now suppose that G_1 is 3-colorable, and let χ be a proper 3-coloring of G_1 that uses the colors red, blue, and green. We will extend χ into a proper 3-coloring χ' of $G_2(\varepsilon)$ as follows. Consider the vertex $v_i \in V_1$, where $1 \leq i \leq n$. If v_i is colored red or blue in χ, then we color all vertices of A_i green and all vertices of B_i blue in χ'. Otherwise, if v_i is colored green in χ, then we color all vertices of A_i blue and all vertices of B_i green in χ'. Finally, we color all vertices $\{v_0, v_{0,1}, v_{0,2}, \ldots, v_{0,n}\}$ red in χ'. It is now straightforward to check that the resulting 3-coloring χ' of $G_2(\varepsilon)$ is proper, i.e. that $G_2(\varepsilon)$ is 3-colorable. That is, $G_2(\varepsilon)$ is 3-colorable if and only if G_1 is 3-colorable.

Summarizing, for every irreducible graph G_1 with diameter 3 and radius 2, such that G_1 has n vertices and minimum degree $\delta(G_1) = \Theta(\sqrt{n})$, we can construct in polynomial time a graph $G_2(\varepsilon)$ with $N = \Theta(n^{1+\varepsilon_0})$ (i.e. $N = \Theta(n^{\frac{1}{1-\varepsilon}})$) vertices and minimum degree $\delta = \Theta(N^{\varepsilon})$, such that $G_2(\varepsilon)$ is 3-colorable if and only if G_1 is 3-colorable. Moreover, the constructed graph $G_2(\varepsilon)$ is irreducible and has diameter 3 and radius 2. This completes the proof of the theorem, since it is NP-complete to decide whether G_1 is 3-colorable by Theorem 9. □

Therefore, Theorems 9 and 10 imply that, for every $\varepsilon \in [0, 1)$, the 3-coloring problem remains NP-complete for irreducible and triangle-free graphs $G = (V, E)$ with diameter 3 and radius 2, where the minimum degree $\delta(G)$ is $\Theta(|V|^{\varepsilon})$. However, Theorems 9 and 10 do not provide any information about how efficiently (although not polynomially, assuming P≠NP) we can decide 3-coloring on such graphs. We provide in the next three theorems subexponential lower bounds for the time complexity of 3-coloring on irreducible and triangle-free graphs with diameter 3 and radius 2. Moreover, the lower bounds provided in Theorem 11 are asymptotically almost tight, due to the algorithm of Theorem 6.

Theorem 11 *Let* $\varepsilon \in [\frac{1}{2}, 1)$. *Assuming ETH, there exists no algorithm with running time* $2^{o\left(\frac{N}{\delta}\right)} = 2^{o(N^{1-\varepsilon})}$ *for 3-coloring on irreducible and triangle-free graphs* G *with diameter* 3*, radius* 2*, and* N *vertices, where the minimum degree of* G *is* $\delta(G) = \Theta(N^{\varepsilon})$.

Proof Let $\varepsilon \in [\frac{1}{2}, 1)$ and define $\varepsilon_0 = \frac{\varepsilon}{1-\varepsilon}$. In the reduction of Theorem 9, given the value of ε and a boolean formula ϕ with n variables and m clauses, we constructed a graph $H_1(\phi, \varepsilon)$ with $N = \Theta(m^{1+\varepsilon_0})$ vertices and minimum degree $\delta = \Theta(m^{\varepsilon_0})$. Therefore $\delta = \Theta(N^{\frac{\varepsilon_0}{1+\varepsilon_0}})$, i.e. $\delta = \Theta(N^{\varepsilon})$. Furthermore the graph $H_1(\phi, \varepsilon)$ is by construction irreducible and triangle-free, and it has diameter 3 and radius 2. Moreover ϕ is satisfiable if and only if $H_1(\phi, \varepsilon)$ is 3-colorable by Theorem 9.

Suppose now that there exists an algorithm $\mathcal{A}$ that, given an irreducible triangle-free graph G with N vertices, diameter 3, radius 2, and minimum degree $\delta = \Theta(N^{\varepsilon})$ for some $\varepsilon \in [\frac{1}{2}, 1)$, decides 3-coloring on G in time $2^{o\left(\frac{N}{\delta}\right)}$. Then $\mathcal{A}$ decides 3-coloring on input $G = H_1(\phi, \varepsilon)$ in time $2^{o\left(\frac{N}{\delta}\right)} = 2^{o(m)}$. However, since the formula ϕ is satisfiable if and only if $H_1(\phi, \varepsilon)$ is 3-colorable, algorithm $\mathcal{A}$ can be used to decide the 3SAT problem in time $2^{o(m)}$, where m is the number of clauses in the given boolean formula ϕ. Thus $\mathcal{A}$ can decide 3SAT in time $2^{o(m)}$. This is a contradiction by Theorem 1, assuming ETH. This completes the proof of the theorem. □

Theorem 12 *Let $\varepsilon \in [\frac{1}{3}, \frac{1}{2})$. Assuming ETH, there exists no algorithm with running time $2^{o(\delta)} = 2^{o(N^{\varepsilon})}$ for 3-coloring on irreducible and triangle-free graphs G with diameter 3, radius 2, and N vertices, where the minimum degree of G is $\delta(G) = \Theta(N^{\varepsilon})$.*

Proof We provide for the purposes of the proof an amplification of the reduction of Theorem 7. In Sect. 4.2, given a boolean formula ϕ with n variables and m clauses, we constructed the graph H_ϕ, which is irreducible and triangle-free by Lemma 4. By its construction in Sect. 4.2, the graph H_ϕ has $\Theta(m^2)$ vertices. Furthermore, the degree of vertex v_0 in H_ϕ is $\Theta(m)$. The degree of every vertex v_j in H_ϕ is $\Theta(n+m) = \Theta(m)$, where $j \in \{1, 2, \ldots, 8m\}$. Finally, the degree of every vertex $u_{i,j}$ (resp. $w_{i,j}$) in H_ϕ is $\Theta(m)$, where $i \in \{1, 2, \ldots, n+5m\}$ and $j \in \{1, 2, \ldots, 8m\}$. Therefore the minimum degree of H_ϕ is $\delta = \Theta(m)$.

Let $\varepsilon \in [\frac{1}{3}, \frac{1}{2})$ and define $\varepsilon_0 = \frac{1}{\varepsilon} - 2$. Note that $\varepsilon_0 \in (0, 1]$, since $\varepsilon \in [\frac{1}{3}, \frac{1}{2})$. We construct now from H_ϕ a graph $H_2(\phi, \varepsilon)$ as follows. Consider the rows ℓ_i and ℓ'_i of the matrix arrangement of the vertices $U \cup W$ of H_ϕ, where $i \in \{1, 2, \ldots, n+5m\}$. For every $i \in \{1, 2, \ldots, n+5m\}$, we extend row ℓ_i by new $(m^{1+\varepsilon_0} - 8m)$ vertices $\{u_{i,8m+1}, u_{i,8m+2}, \ldots, u_{i,m^{1+\varepsilon_0}}\}$. Denote the resulting row of the matrix with the $m^{1+\varepsilon_0}$ vertices $\{u_{i,1}, u_{i,2}, \ldots, u_{i,m^{1+\varepsilon_0}}\}$ by $\widehat{\ell_i}$. Similarly, we extend row ℓ'_i by new $(m^{1+\varepsilon_0} - 8m)$ vertices $\{w_{i,8m+1}, w_{i,8m+2}, \ldots, w_{i,m^{1+\varepsilon_0}}\}$. Denote the resulting row of the matrix with the $m^{1+\varepsilon_0}$ vertices $\{w_{i,1}, w_{i,2}, \ldots, w_{i,m^{1+\varepsilon_0}}\}$ by $\widehat{\ell'_i}$. Furthermore, add all necessary edges between vertices of $\widehat{\ell_i}$ and of $\widehat{\ell'_i}$, such that they induce a complete bipartite graph without a perfect matching. For simplicity of notation, denote by $U' = \{u_{i,j} : 1 \leq i \leq n+5m, 1 \leq j \leq m^{1+\varepsilon_0}\}$ and $W' = \{w_{i,j} : 1 \leq i \leq n+5m, 1 \leq j \leq m^{1+\varepsilon_0}\}$ the vertex sets that extend the sets U and W, respectively. Moreover, similarly to the notation of Sect. 4.2, denote $U'_j = \{u_{1,j}, u_{2,j}, \ldots, u_{n+5m,j}\}$ and $W'_j = \{w_{1,j}, w_{2,j}, \ldots, w_{n+5m,j}\}$, for every $j \in \{1, 2, \ldots, m^{1+\varepsilon_0}\}$. Then, the vertices of $U'_j \cup W'_j$ contain the vertices of the jth column in the (updated) matrix arrangement of the vertices of $U' \cup W'$.

Similarly to the construction of the graph H_ϕ (cf. Sect. 4.2), we add for every $j \in \{8m+1, 8m+2, \ldots, m^{1+\varepsilon_0}\}$ a vertex v_j that is adjacent to $U'_j \cup W'_j \cup \{v_0\}$. Denote the resulting graph by $H_2(\phi, \varepsilon)$.

Now, following exactly the same argumentation as in the proof of Lemma 4 and Theorem 7, we can prove that: (a) $H_2(\phi, \varepsilon)$ has diameter 3 and radius 2, (b) $H_2(\phi, \varepsilon)$ is irreducible and triangle-free, and (c) the formula ϕ is satisfiable if and only if $H_2(\phi, \varepsilon)$ is 3-colorable.

By the above construction, the graph $H_2(\phi, \varepsilon)$ has $N = 2(n+5m)\cdot m^{1+\varepsilon_0} + m^{1+\varepsilon_0} + 1$ vertices. Thus, since $m = \Omega(n)$, it follows that $N = \Theta(m^{2+\varepsilon_0})$. Furthermore, the degree of vertex v_0 in $H_2(\phi, \varepsilon)$ is $\Theta(m^{1+\varepsilon_0})$. The degree of every vertex v_j in $H_2(\phi, \varepsilon)$ is $\Theta(n+m) = \Theta(m)$, where $j \in \{1, 2, \ldots, m^{1+\varepsilon_0}\}$. Finally, the degree of every vertex $u_{i,j}$ (resp. $w_{i,j}$) in $H_2(\phi, \varepsilon)$ is $\Theta(m^{1+\varepsilon_0})$, where $i \in \{1, 2, \ldots, n+5m\}$ and $j \in \{1, 2, \ldots, m^{1+\varepsilon_0}\}$. Thus the minimum degree of $H_2(\phi, \varepsilon)$ is $\delta = \Theta(m)$. Therefore, since $N = \Theta(m^{2+\varepsilon_0})$, it follows that $\delta = \Theta(N^{\frac{1}{2+\varepsilon_0}}) = \Theta(N^{\varepsilon})$, where N is the number of vertices in the graph $H_2(\phi, \varepsilon)$.

Suppose now that there exists an algorithm $\mathcal{A}$ that, given an irreducible triangle-free graph G with N vertices, diameter 3, radius 2, and minimum degree $\delta = \Theta(N^{\varepsilon})$ for some $\varepsilon \in [\frac{1}{3}, \frac{1}{2})$, decides 3-coloring on G in time $2^{o(\delta)}$. Then $\mathcal{A}$ decides 3-coloring on input $G = H_2(\phi, \varepsilon)$ in time $2^{o(\delta)} = 2^{o(m)}$. However, since the formula ϕ is satisfiable if and only if $H_2(\phi, \varepsilon)$ is 3-colorable, algorithm $\mathcal{A}$ can be used to decide the 3SAT problem in time $2^{o(m)}$, where m is the number of clauses in the given boolean formula ϕ. Therefore $\mathcal{A}$ can decide 3SAT in time $2^{o(m)}$. This is a contradiction by Theorem 1, assuming ETH. This completes the proof of the theorem. □

Theorem 13 *Let $\varepsilon \in [0, \frac{1}{3})$. Assuming ETH, there exists no algorithm with running time $2^{o\left(\sqrt{\frac{N}{\delta}}\right)} = 2^{o\left(N^{\left(\frac{1-\varepsilon}{2}\right)}\right)}$ for 3-coloring on irreducible and triangle-free graphs G with diameter 3, radius 2, and N vertices, where the minimum degree of G is $\delta(G) = \Theta(N^{\varepsilon})$.*

Proof Let $\varepsilon \in [0, \frac{1}{3})$ and define $\varepsilon_0 = \frac{\varepsilon}{1-\varepsilon}$. In the reduction of Theorem 10, given the value of ε and an irreducible graph G_1 with diameter 3 and radius 2 such that G_1 has n vertices and minimum degree $\delta(G_1) = \Theta(\sqrt{n})$, we constructed an irreducible graph $G_2(\varepsilon)$ with $N = \Theta(n^{1+\varepsilon_0})$ vertices and minimum degree $\delta = \Theta(n^{\varepsilon_0})$. Therefore $\delta = \Theta(N^{\frac{\varepsilon_0}{1+\varepsilon_0}})$, i.e. $\delta = \Theta(N^{\varepsilon})$. Furthermore the graph $G_2(\varepsilon)$ has by construction diameter 3 and radius 2. Moreover G_1 is 3-colorable if and only if $G_2(\varepsilon)$ is 3-colorable by Theorem 10.

Suppose now that there exists an algorithm $\mathcal{A}$ that, given an irreducible triangle-free graph G with N vertices, diameter 3, radius 2, and minimum degree $\delta = \Theta(N^{\varepsilon})$ for some $\varepsilon \in [0, \frac{1}{3})$, decides 3-coloring on G in time $2^{o\left(\sqrt{\frac{N}{\delta}}\right)}$. Then $\mathcal{A}$ decides 3-coloring on input $G = G_2(\varepsilon)$ in time $2^{o\left(\sqrt{\frac{N}{\delta}}\right)} = 2^{o(\sqrt{n})}$. However, since G_1 is 3-colorable if and only if $G_2(\varepsilon)$ is 3-colorable, algorithm $\mathcal{A}$ can be used to decide 3-coloring of G_1 in time $2^{o(\sqrt{n})}$, where n is the number of vertices in the given graph G_1. This is a contradiction by Theorem 11, assuming ETH. This completes the proof of the theorem. □

5 Concluding Remarks

In this paper we investigated graphs with small diameter, i.e. with diameter at most 2, and at most 3. For graphs with diameter at most 2, we provided the first subexponential algorithm for 3-coloring, with complexity $2^{O(\sqrt{n \log n})}$. This time complexity is asymptotically the same as the currently best known complexity for the graph isomorphism (GI) problem [4]. Thus, as the 3-coloring problem on graphs with diameter 2 has been neither proved to be polynomially solvable nor to be NP-complete, it would be worthwhile to investigate whether this problem is polynomially reducible to/from the GI problem. Furthermore we presented a subclass of graphs with diameter 2 that admits a polynomial algorithm for 3-coloring. An interesting open problem for further research is to establish the time complexity of 3-coloring on arbitrary graphs with diameter 2. Moreover, the complexity of 3-coloring remains open also for triangle-free graphs of diameter 2, or equivalently, on maximal triangle-free graphs.

For graphs with diameter at most 3, we established the complexity of 3-coloring, even for triangle-free graphs, which has been an open problem. Namely we proved that for every $\varepsilon \in [0, 1)$, 3-coloring is NP-complete on triangle-free graphs of diameter 3 and radius 2 with n vertices and minimum degree $\delta = \Theta(n^{\varepsilon})$. Moreover, assuming the ETH, we provided three different amplification techniques of our hardness results, in order to obtain for every $\varepsilon \in [0, 1)$ subexponential asymptotic lower bounds for the complexity of 3-coloring on triangle-free graphs with diameter 3, radius 2, and minimum degree $\delta = \Theta(n^{\varepsilon})$. Finally, we provided a 3-coloring algorithm with running time $2^{O\left(\min\{\delta\Delta,\ \frac{n}{\delta}\log\delta\}\right)}$ for arbitrary graphs with diameter 3, where n is the number of vertices and δ (resp. Δ) is the minimum (resp. maximum) degree of the input graph. Due to our lower bounds, the running time of this algorithm is asymptotically almost tight, when the minimum degree if the input graph is $\delta = \Theta(n^{\varepsilon})$, where $\varepsilon \in [\frac{1}{2}, 1)$. An interesting problem for further research is to find asymptotically matching lower bounds for the complexity of 3-coloring on graphs with diameter 3, for all values of minimum degree δ.

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