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Homogenization of random semilinear PDEs

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Abstract. We prove a homogenization result for system of semilinear parabolic PDEs of the type

$$\partial_t u^\varepsilon = \frac{1}{2} e^{2V(x/\varepsilon)} \mathrm{div}\left(e^{-2V(x/\varepsilon)} a(x/\varepsilon) \nabla u^\varepsilon\right) + h(x, u^\varepsilon, \nabla u^\varepsilon) \ ,$$

where V and a are random ergodic fields. We extend to the random case, results of Buckdahn, Hu & Peng [4] for periodic structures. The same method involving stability results is applied. Our main tool is an L^p estimate for the gradient of the solution of the auxiliary problems. The same type of results is obtained for systems of semilinear elliptic PDEs.

1. Introduction

The problem considered in this paper is the homogenization of systems of semilinear parabolic PDEs with random coefficients,

$$\begin{aligned} \partial_t u_k^\varepsilon + \mathcal{L}^{\varepsilon,\omega} u_k^\varepsilon + h_k(x, u^\varepsilon, \nabla u^\varepsilon) = 0, \quad t \in [0, T],\ k \in \{1, \cdots, m\} \\ u^\varepsilon(T, x) = H(x), \quad x \in \mathbb{R}^d \end{aligned} \tag{1}$$

where $\mathcal{L}^{\varepsilon,\omega}$ is the second order partial differential operator with random stationary ergodic coefficients

$$\mathcal{L}^{\varepsilon,\omega} f = \frac{1}{2} e^{2V(x/\varepsilon,\omega)} \mathrm{div}\left(a(x/\varepsilon, \omega) e^{-2V(x/\varepsilon,\omega)} \nabla f\right) \ . \tag{2}$$

Note that when a and V are smooth enough, $\mathcal{L}^{\varepsilon,\omega}$ can be rewritten as

$$\mathcal{L}^{\varepsilon,\omega} f = \frac{1}{2} \sum_{i,j} a_{ij}(x/\varepsilon, \omega) \partial^2_{ij} f + \frac{1}{\varepsilon} \sum_{i,j} \left(\frac{1}{2} \partial_i a_{ij}(x/\varepsilon, \omega) - \partial_i V(x/\varepsilon, \omega)\right) \partial_j f \ ,$$

and exhibits an exploding term.

The aim of homogenization is to prove the "convergence" of u^ε (when $\varepsilon \longrightarrow 0$) to the solution of systems of semilinear PDEs, where the inhomogeneous coefficients have been replaced by "constant" ones. This type of problems have been

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studied in a lot of papers (see [1], [11] for monographs on the subject). A common classification of the methods used, is to distinguish between “analytical” methods, and “probabilistic” ones. For the system (1) with periodic coefficients, an analytical proof of homogenization can be found in [1].

This paper belongs to the second set of methods. Roughly speaking, the probabilistic method begins with representing the quantities of interest by means of stochastic processes, and then tries to prove convergence in laws of these processes.

For the linear case ($h = 0$), the problem often reduces to transforming the underlying Markov process into a martingale, whose quadratic variation has a deterministic limit thanks to the ergodic theorem. This transformation requires to find harmonic functions (the so-called “auxiliary” problem). This method has been introduced by Freidlin [6] in the periodic framework. Since then, it has been extended in various directions ([5] for Brownian motion in a random potential, [26] for Brownian motion in a random perforated domain, [15] for diffusions with coefficients depending on time, [20] [19] and [16] for Markov processes associated to Dirichlet forms...)

The probabilistic way of handling nonlinear terms, is to work with the backward stochastic differential equations (BSDEs) introduced by Pardoux & Peng [24], and to prove convergence in laws of the stochastic processes involved. This method has been successfully applied for rapidly oscillating non linear terms not depending of the gradient (more precisely, the non linear terms are of the type $h(t, x/\varepsilon, x, u^\varepsilon) + \nabla u^\varepsilon \hat{h}(t, x/\varepsilon, x, u^\varepsilon))$ (see [23] for the periodic case, and [16] for Markov processes associated to a Dirichlet form), or for non linear terms of type $h(x, u^\varepsilon) + \hat{h}(x, u^\varepsilon)\,\|\nabla u^\varepsilon\|^2$ (see [8]). Another method which allows more general non-linearity in the gradient, is to exploit the stability results of Hu & Peng [10]. This strategy has been employed in the periodic case in [4], [9], or [2]. This paper extends these results to the random case. The main result (see theorem 9 for a precise statement) says that for some $p \in]1, 2]$, for any bounded open domain G of $\mathbb{R}^d$, and all $t \in [0, T]$,

$$\left\langle \int_G \left\| u^\varepsilon(t,x) - u^0(t,x) \right\|^p \, dx \right\rangle \underset{\varepsilon \to 0}{\longrightarrow} 0 \ ,$$

where $\langle . \rangle$ denotes the expectation over the randomness of a and V, and u^0 is the solution of a semilinear system

$$\begin{aligned} \partial_t u^0 + \frac{1}{2} \bar{a}_{i,j} \partial^2_{i,j} u^0_t + \bar{h}(x, u^0_t, \nabla u^0_t) = 0 \ , \ \ t \in [0,T] \, , \\ u^0_T = H \ . \end{aligned}$$

Apart from stationarity and ergodicity, this theorem is proved under uniform ellipticity of a and boundedness of a and V.

One key point in the proof of theorem 9, is to get L^p-estimates ($p > 2$) of the gradient of the solution of the auxiliary problem. These are obtained using results of Meyers [17] in the deterministic case, and the ergodic theorem.

The paper is organized as follows. Section 2 states the problem and the assumptions made throughout the paper. Section 3 deals with the auxiliary problems and the L^p-estimates of the gradient. In section 4, the homogenization result for systems of parabolic PDEs is proved, and section 5 treats the case of elliptic PDEs.

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2. Standing assumptions and notations

Let us now be more precise on the assumptions made on the system (1). All the results stated in this section are already proved elsewhere, so they are often recalled without justification. The reader is referred to [11] or [18] for the description of the random media, and to [16] for the construction and the properties of the environment viewed from the particle in the case of the Markov process associated to a Dirichlet form.

Random media. The description of the random media involves a probability space $(\Omega, \mathcal{A}, \mu)$, on which acts a group of transformations $(\tau_x, x \in \mathbb{R}^d)$. As is now usual in this kind of problems, we assume that

(RM 1) μ is invariant under the action of $(\tau_x, x \in \mathbb{R}^d)$;
(RM 2) $(\Omega, \mathcal{A}, \mu, (\tau_x)_{x\in\mathbb{R}^d})$ is ergodic;
(RM 3) if $\mathbf{f}$ is any measurable function on $(\Omega, \mathcal{A})$, $(x, \omega) \mapsto \mathbf{f}(\tau_x \omega)$ is measurable on $(\mathbb{R}^d \times \Omega, \mathcal{B}(\mathbb{R}^d) \otimes \mathcal{A})$;
(RM 4) $(\tau_x, x \in \mathbb{R}^d)$ is stochastically continuous, i.e. $\forall \mathbf{f} \in L^2(\Omega, \mathcal{A}, \mu)$, $\forall \varepsilon > 0$,

$$\mu\left(|\mathbf{f}(\tau_h\omega) - \mathbf{f}(\omega)| \geq \varepsilon\right) \underset{\|h\|\to 0}{\longrightarrow} 0 \ .$$

Let T_x be the operator on $L^2(\Omega, \mu)$ defined by

$$T_x\mathbf{f}(\omega) \stackrel{\triangle}{=} \mathbf{f}(\tau_x\omega) \ .$$

$(T_x, x \in \mathbb{R}^d)$ is a strongly continuous unitary group on $L^2(\Omega, \mu)$. For all $i \in \{1, \cdots, d\}$, it is then possible to define the infinitesimal generator D_i of the group $(T_{he_i}, h \in \mathbb{R})$ ($(e_1, \cdots, e_d)$ is the canonical basis of $\mathbb{R}^d$). That is,

$$\mathrm{dom}(D_i) = \left\{ \mathbf{f} \in L^2(\Omega, \mu), \lim_{h\to 0} \frac{T_{he_i}\mathbf{f} - \mathbf{f}}{h} \text{ exists in } L^2(\Omega, \mu) \right\} \ ,$$

$$\forall \mathbf{f} \in \mathrm{dom}(D_i), \quad D_i\mathbf{f} = L^2 - \lim_{h\to 0} \frac{T_{he_i}\mathbf{f} - \mathbf{f}}{h} \ .$$

It is clear that if $\mathbf{f} \in \bigcap_{i=1}^{d} \mathrm{dom}(D_i)$, then μ-a.s., $x \mapsto f(x, \omega) \stackrel{\triangle}{=} \mathbf{f}(\tau_x\omega)$ is differentiable and that $\nabla f(x, \omega) = D\mathbf{f}(\tau_x\omega)$.

As a consequence of stationarity, one gets moreover the following integration by parts formula

Lemma 1. *Under assumptions* **(RM 1..4)**,

$$\forall \mathbf{f}, \mathbf{g} \in dom(D_i), \quad \int \mathbf{f}(\omega)\, D_i\mathbf{g}(\omega)\, d\mu = -\int \mathbf{g}(\omega)\, D_i\mathbf{f}(\omega)\, d\mu \ .$$

About the linear part. The matrix $a(x,\omega)$ and the function $V(x,\omega)$ are respectively given by a measurable function $\mathbf{a}$ on Ω with values in the space of symmetric matrices $\mathcal{S}_d$ and a measurable real-valued function $\mathbf{V}$, in such a way that

$$a(x,\omega) = \mathbf{a}(\tau_x\omega)\ ,\ \ V(x,\omega) = \mathbf{V}(\tau_x\omega)\ . \tag{3}$$

It is assumed that there exist strictly positive constants $\underline{a}$, $\bar{A}$, and Λ , such that

(L 1) μ-a.s., for all $y \in \mathbb{R}^d$,

$$\underline{a}\ \|y\|^2 \le (\mathbf{a}(\omega)y, y) \le \bar{A}\ \|y\|^2\ , \tag{4}$$

where $\|.\|$ is the euclidean norm, and $(.,.)$ the corresponding scalar product.

(L 2) $|\mathbf{V}(\omega)| \le \Lambda$, μ a.s.,
(L 3) $\int e^{-2\mathbf{V}(\omega)}\, d\mu(\omega) = 1$.

The measure $d\pi \stackrel{\triangle}{=} e^{-2V(\omega)}\, d\mu$ is then a probability measure on Ω. The scalar product on $L^2(\Omega,\mu)$ (respectively $L^2(\Omega,\pi)$) will be denoted by $\langle .,.\rangle_\mu$ (respectively $\langle .,.\rangle_\pi$).

The Markov process in fixed environment. For each $\omega \in \Omega$, let π^ω be the positive measure on $\mathbb{R}^d$ defined by $d\pi^\omega(x) \stackrel{\triangle}{=} e^{-2V(x,\omega)}\, dx$. Let us consider the Dirichlet form on $L^2(\mathbb{R}^d,\pi^\omega)$, with domain $H^1(\mathbb{R}^d,\pi^\omega)$,

$$\mathcal{E}^\omega(f,g) \stackrel{\triangle}{=} \frac{1}{2}\int a_{i,j}(x,\omega)\partial_i f(x)\ \partial_j g(x)\, d\pi^\omega(x)\ , \tag{5}$$

where repeated indices are summed from 1 to d, and $\partial_i \stackrel{\triangle}{=} \frac{\partial}{\partial x_i}$. $\mathcal{E}^\omega$ is a regular strongly local Dirichlet form (with $\mathcal{C}_c^\infty(\mathbb{R}^d)$ as a core). $(\mathcal{L}^\omega, \mathrm{dom}(\mathcal{L}^\omega))$ will denote the self-adjoint operator associated to $(\mathcal{E}^\omega, \mathrm{dom}(\mathcal{E}^\omega))$; and (X, P_x^ω) the associated continuous and conservative Markov process.

Let us recall now the generalized Itô's formula (theorems 5.5.1, 5.5.2, and corollary 5.5.4 of [7]).

Proposition 2. *Let v be a function in $H^1_{loc}(\mathbb{R}^d)$, and $\tilde{v}$ be a quasi-continuous version of v. Then μ-a.s., and q.e. in $x \in \mathbb{R}^d$,*

$$\tilde{v}(X_t) - \tilde{v}(X_0) = M_t^{\omega,v} + N_t^{\omega,v}\ ,\ P_x^\omega\text{-p.s.},$$

where $N_t^{\omega,v}$ is a continuous additive functional locally of zero energy, and $M_t^{\omega,v}$ is a martingale continuous additive functional (MAF) locally of finite energy, whose quadratic variation is

$$\left\langle M^{\omega,v}\right\rangle_t \stackrel{\triangle}{=} \int_0^t (a(X_s,\omega)\nabla v(X_s), \nabla v(X_s))\ ds\ ,\ P_x^\omega\text{-p.s., q.e. in } x \in \mathbb{R}^d\ ,$$

Moreover, if there exists a function u in $L^1_{loc}(\mathbb{R}^d)$ such that

$$\forall f \in \mathcal{C}_c^\infty(\mathbb{R}^d)\ ,\quad \mathcal{E}^\omega(v,f) = -\int u(x)f(x)\,dx\ ,$$

then $N_t^{\omega,v} = \int_0^t u(X_s)e^{2V(X_s,\omega)}\ ds$, P_x^ω-p.s., q.e. in $x \in \mathbb{R}^d$.

Applying this to the functions $x \in \mathbb{R}^d \mapsto x_i \in \mathbb{R}$, we get the decomposition $X = M^X + N^X$, where M^X is a local continuous martingale with values in $\mathbb{R}^d$, and with quadratic variations given by

$$\left\langle M_i^X, M_j^X \right\rangle_t = \int_0^t a_{i,j}(X_s, \omega)\, ds \ , \ P_x^\omega\text{-p.s., q.e. in } x \in \mathbb{R}^d \ .$$

Since a is bounded, M^X is actually a martingale additive functional. Moreover, theorem 5.6.3 in [7] allows one to express $M^{\omega,v}$ of proposition 2 as

$$M_t^{\omega,v} = \int_0^t \nabla v(X_s)\, dM_s^X \ , \ P_x^\omega\text{-p.s., q.e. in } x \in \mathbb{R}^d \ .$$

Finally for each $\varepsilon > 0$ and ω, let $d\pi^{\varepsilon,\omega}(x) \stackrel{\triangle}{=} e^{-2V(\frac{x}{\varepsilon},\omega)}\, dx$, and $\mathcal{E}^{\varepsilon,\omega}$ be the regular strongly local Dirichlet form defined on $H^1(\mathbb{R}^d, \pi^{\varepsilon,\omega})$ by

$$\mathcal{E}^{\varepsilon,\omega}(f,g) \stackrel{\triangle}{=} \frac{1}{2} \int a_{i,j}\left(\frac{x}{\varepsilon}, \omega\right) \partial_i f(x)\ \partial_j g(x) e^{-2V(\frac{x}{\varepsilon},\omega)}\, dx \ . \tag{6}$$

In accordance with what precedes, $(\mathcal{L}^{\varepsilon,\omega}, \text{dom}(\mathcal{L}^{\varepsilon,\omega}))$ will denote the generator of $(\mathcal{E}^{\varepsilon,\omega}, H^1(\mathbb{R}^d, \pi^{\varepsilon,\omega}))$, and $(X, P_x^{\varepsilon,\omega})$ the associated Markov process. Obviously, $(X_t, t \geq 0)$ has the same law under $P_x^{\varepsilon,\omega}$ as $(\varepsilon X_{t/\varepsilon^2}, t \geq 0)$ under $P_{x/\varepsilon}^\omega$. Moreover, it follows from stationarity that $(X_t - y, t \geq 0)$ has the same law under $P_{x+y}^{\varepsilon,\omega}$ as $(X_t, t \geq 0)$ under $P_x^{\varepsilon,\tau_{y/\varepsilon}\omega}$.

The environment viewed from the particle. Related to X^ω, one can define a Markov process on Ω, the so-called "environment viewed from the particle". For this purpose, let us introduce further notations. Let $\mathcal{E}^\pi$ be the bilinear form defined on $H^1(\mu) \stackrel{\triangle}{=} \cap_{i=1}^d \text{dom}(D_i)$ by

$$\mathcal{E}^\pi(\mathbf{f},\mathbf{g}) \stackrel{\triangle}{=} \frac{1}{2} \int \mathbf{a}_{i,j}(\omega) D_i \mathbf{f}(\omega)\ D_j \mathbf{g}(\omega)\, d\pi(\omega) \ . \tag{7}$$

$\mathcal{E}^\pi$ is a closed bilinear symetric form densely defined on $L^2(\Omega, \mu)$. Consequently, there exists a self-adjoint operator $(\mathcal{L}^\pi, \text{dom}(\mathcal{L}^\pi))$ such that

$$\mathcal{E}^\pi(\mathbf{f},\mathbf{g}) = -\left\langle \mathcal{L}^\pi \mathbf{f}, \mathbf{g} \right\rangle_\pi \ , \ \forall \mathbf{f} \in \text{dom}(\mathcal{L}^\pi)\, ,\ \forall \mathbf{g} \in H^1(\mu)\, .$$

Lemma 3. *(see for instance [16], chapter 2).*
Let $\omega_t \stackrel{\triangle}{=} \tau_{X_t^\omega}\omega$. Under **(RM 1..4)** *and* **(L 1..3)***, ω_t is a Markov process on Ω with generator $(\mathcal{L}^\pi, \text{dom}(\mathcal{L}^\pi))$, and Dirichlet form $(\mathcal{E}^\pi, H^1(\mu))$. ω_t is ergodic, with invariant probability π.*

Some remarks on the assumptions. First of all, note that any stationary random fields $a(x,\omega)$ and $V(x,\omega)$, can be represented by formula (3). It suffices to choose for Ω the canonical space $\mathcal{S}_d^{\mathbb{R}^d} \times \mathbb{R}^{\mathbb{R}^d}$, equipped with the σ-field generated by the cylinder sets. μ is then the law of the stationary fields a and V; $(\tau_x, x \in \mathbb{R}^d)$ is the group of translations; the r.v. **a** is the function which maps $\omega = (\omega_1, \omega_2) \in \Omega$ to $\omega_1(0)$, and $\mathbf{V}(\omega) = \omega_2(0)$.

Moreover, adding a strictly positive constant to **V**, we can always assume that **(L 3)** is satisfied.

3. The auxiliary problems

This section is devoted to the study of the auxiliary problems

$$\forall i \in \{1, \cdots, d\}\,, \quad \text{find functions } v^i(x,\omega) \text{ such that } \mathcal{L}^\omega(x_i + v^i(x,\omega)) = 0\ , \quad (8)$$

which enable us to get rid of the highly oscillating terms in the linear part. Note that if v^i is a stationary random field (i.e $v^i(x,\omega) = \mathbf{v}^i(\tau_x\omega)$), then (8) can be written as

$$\forall \mathbf{f} \in H^1(\mu)\,,\ \mathcal{E}^\pi(\mathbf{f}, \mathbf{v}^i) = -\frac{1}{2}\left\langle \mathbf{a}_{i,j} D_j \mathbf{f}\right\rangle_\pi\ . \quad (9)$$

But v^i is not in general a stationary field. It is however possible to construct its gradient as a stationary field satisfying (9). That's the aim of this section, which begins with some notations and results borrowed from [11], and [16]. The construction of the gradient of v^i in $L^2(\Omega,\mu)$, is made with whe point of view of [11]. The only new result in this section is proposition 7 which gives L^p-estimates ($p > 2$) of the gradient.

3.1. The space $L^2(\Omega, \mu)$

For any $\mathbf{f} \in L^2(\Omega,\mu)$ and any $\varphi \in \mathcal{C}_c^\infty(\mathbb{R}^d)$, let $\varphi \star \mathbf{f}$ be the convolution

$$\varphi \star \mathbf{f}(\omega) \stackrel{\triangle}{=} \int \mathbf{f}(\tau_x\omega)\,\varphi(x)\,dx\ . \quad (10)$$

$\varphi \star \mathbf{f}$ is an element of $L^2(\Omega,\mu)$. Let

$$\mathcal{S} = \text{Span}\left\{\varphi \star \mathbf{f}\,,\ \mathbf{f} \in L^2(\Omega,\mu)\,, \varphi \in \mathcal{C}_c^\infty(\mathbb{R}^d)\right\} \quad (11)$$

be the space of smooth functionals. The main property of $\mathcal{S}$ is given in the next lemma

Lemma 4. *Assume* **(RM 1..4)**. *$\mathcal{S}$ is dense in $L^2(\Omega,\mu)$. Moreover, $\mathcal{S}$ is a subset of $\cap_{i=1}^d dom(D_i)$, and*

$$\forall \mathbf{f} \in L^2(\Omega,\mu)\,,\ \forall \varphi \in \mathcal{C}_c^\infty(\mathbb{R}^d)\,,\ D_i(\varphi \star \mathbf{f}) = -\partial_i\varphi \star \mathbf{f}\ .$$

The space of vortex-free stationary fields is then defined by

$$\mathcal{V}^2_{pot} = \left\{\mathbf{f} \in L^2(\Omega,\mu)^d \,\middle|\, \begin{array}{l} \forall i \in \{1,\cdots,d\}, \int \mathbf{f}_i\,d\mu = 0 \\ \forall i,j \in \{1,\cdots,d\}, \forall \varphi \in \mathcal{C}_c^\infty(\mathbb{R}^d),\ \ \partial_j\varphi \star \mathbf{f}_i = \partial_i\varphi \star \mathbf{f}_j \end{array}\right\}$$

If $\mathbf{f}$ is a function in $\mathcal{V}^2_{pot}$, then μ-a.e., the function $x \mapsto \mathbf{f}(\tau_x\omega)$ is vortex free.

Lemma 5. *Assume* **(RM 1..4)**. *$\mathcal{V}^2_{pot}$ is a closed subset of $L^2(\Omega,\mu)^d$ and*

$$\mathcal{V}^2_{pot} = \overline{\{(D_1\mathbf{f},\cdots,D_d\mathbf{f}), \mathbf{f} \in \mathcal{S}\}}^{\|\cdot\|_{L^2(\Omega,\mu)^d}}\ .$$

This lemma is proved in [11] and in [16].

3.2. Solution of the auxiliary problem in L^2

On the Hilbert space $(\mathcal{V}^2_{pot}, \langle .,.\rangle_\pi)$, let us define the bilinear form

$$\mathcal{G}(\mathbf{f},\mathbf{g}) = \frac{1}{2}\left\langle \mathbf{a}_{i,j}\mathbf{f}_i\mathbf{g}_j\right\rangle_\pi \ .$$

Due to **(L 1)**, this is a coercive quadratic form on $\mathcal{V}^2_{pot}$. On the other hand, for all $i \in \{1,\cdots,d\}$, $\mathbf{g} \in \mathcal{V}^2_{pot} \mapsto -\frac{1}{2}\left\langle \mathbf{a}_{i,j}\ \mathbf{g}_j\right\rangle_\pi$ is a linear form on $\mathcal{V}^2_{pot}$. By the Lax-Milgram theorem, for all $i \in \{1,\cdots,d\}$, there exists a unique $\phi^i \in \mathcal{V}^2_{pot}$ such that for all $\mathbf{g} \in \mathcal{V}^2_{pot}$,

$$\mathcal{G}(\phi^i,\mathbf{g}) = -\frac{1}{2}\left\langle \mathbf{a}_{i,j}\ \mathbf{g}_j\right\rangle_\pi \ . \tag{12}$$

Since $\phi^i \in \mathcal{V}^2_{pot}$, μ-a.s. $x \in \mathbb{R}^d \mapsto \phi^i(\tau_x\omega)$ is vortex free. Therefore, we can choose a function $f^i(x,\omega)$ such that μ-a.s., $\nabla f^i(x,\omega) = \phi^i(\tau_x\omega)$. Take for instance

$$f^i(x,\omega) = \int_0^1 x_j\ \phi^i_j(\tau_{tx}\omega)\,dt \ . \tag{13}$$

Lemma 6. *μ-a.s., $g^i(x,\omega) \stackrel{\triangle}{=} x_i + f^i(x,\omega)$ is a function in $H^1_{loc}(\mathbb{R}^d)$ which is a weak solution of the equation*

$$\mathcal{L}^\omega g^i(x,\omega) = 0 \ . \tag{14}$$

Proof of Lemma 6. Let $\psi \in \mathcal{C}^\infty_c(\mathbb{R}^d)$ and $\mathbf{G} \in L^2(\Omega,\mu)$, and let $\mathbf{g} \stackrel{\triangle}{=} D(\psi \star \mathbf{G})$. Then on one hand,

$$\begin{aligned}\mathcal{G}(\phi^i,\mathbf{g}) &= -\tfrac{1}{2}\int \mathbf{a}_{k,j}(\omega)(\partial_j\psi \star \mathbf{G})(\omega)\,\phi^i_k(\omega)\,d\pi(\omega)\\ &= -\tfrac{1}{2}\int_{\mathbb{R}^d} dx\ \partial_j\psi(x)\ \int_\Omega \mathbf{a}_{k,j}(\omega)\mathbf{G}(\tau_x\omega)\phi^i_k(\omega)d\pi(\omega)\\ &= -\tfrac{1}{2}\int_{\mathbb{R}^d} dx\ \partial_j\psi(x)\int_\Omega \mathbf{a}_{k,j}(\tau_{-x}\omega)\mathbf{G}(\omega)\phi^i_k(\tau_{-x}\omega)e^{-2V(\tau_{-x}\omega)}d\mu(\omega)\\ &= -\tfrac{1}{2}\left\langle \mathbf{G}, \int_{\mathbb{R}^d}\partial_j\psi(-x)\mathbf{a}_{k,j}(x,\omega)\phi^i_k(x,\omega)d\pi^\omega(x)\right\rangle_\mu\\ &= \tfrac{1}{2}\left\langle \mathbf{G}, \int_{\mathbb{R}^d}\partial_j\hat{\psi}(x)\mathbf{a}_{k,j}(x,\omega)\phi^i_k(x,\omega)d\pi^\omega(x)\right\rangle_\mu\end{aligned}$$

where $\hat{\psi}(x) \stackrel{\triangle}{=} \psi(-x)$. On the other hand,

$$\begin{aligned}-\tfrac{1}{2}\left\langle \mathbf{a}_{i,j}\mathbf{g}_j\right\rangle_\pi &= \tfrac{1}{2}\int \mathbf{a}_{i,j}(\omega)\,(\partial_j\psi \star \mathbf{G})(\omega)\,d\pi(\omega)\\ &= \tfrac{1}{2}\int_{\mathbb{R}^d} dx\ \partial_j\psi(x)\int \mathbf{a}_{i,j}(\omega)\mathbf{G}(\tau_x\omega)d\pi(\omega)\\ &= \tfrac{1}{2}\int_{\mathbb{R}^d} dx\ \partial_j\psi(x)\int \mathbf{a}_{i,j}(\tau_{-x}\omega)\mathbf{G}(\omega)e^{-2V(\tau_{-x}\omega)}d\mu(\omega)\\ &= -\tfrac{1}{2}\left\langle \mathbf{G}, \int_{\mathbb{R}^d}\partial_j\hat{\psi}(x)\mathbf{a}_{i,j}(x,\omega)d\pi^\omega(x)\right\rangle_\mu\end{aligned}$$

Applying (12) to functions $\mathbf{g} = D(\psi \star \mathbf{G})$, for $\psi \in \mathcal{C}_c^\infty(\mathbb{R}^d)$ and $\mathbf{G} \in L^2(\Omega, \mu)$, yields then $\forall \mathbf{G} \in L^2(\Omega, \mu)$, $\forall \psi \in \mathcal{C}_c^\infty(\mathbb{R}^d)$, $\forall i \in \{1, \cdots, d\}$,

$$\left\langle \mathbf{G}, \int a_{k,j}(x,\omega)\partial_j \psi(x)\phi_k^i(x,\omega)\, d\pi^\omega(x) \right\rangle_\mu = -\left\langle \mathbf{G}, \int a_{i,j}(x,\omega)\partial_j \psi(x)\, d\pi^\omega(x) \right\rangle_\mu$$

Therefore, μ-a.s., for all $\psi \in \mathcal{C}_c^\infty(\mathbb{R}^d)$ and all $i \in \{1, \cdots, d\}$,

$$\mathcal{E}^\omega(f^i(.,\omega), \psi) = -\mathcal{E}^\omega(x_i, \psi) \tag{15}$$

□

3.3. L^p-estimates of the gradient

The aim of this section is to prove

Proposition 7. *Assume* **(RM 1..4)** *and* **(L 1..3)**. *Let us denote*

$$\underline{V} = essinf \mathbf{V}\,, \;\; \bar{V} = esssup \mathbf{V}\,, \;\; \Delta V = \bar{V} - \underline{V}\,,$$

$$\bar{A} \text{ and } \underline{a} \text{ the best constants in (4)}, \;\; b = e^{-2\Delta V}\frac{\underline{a}}{\bar{A}} (\in [0,1])\,.$$

Then, there exists $Q(b,d) > 2$ such that for all $p \in [2, Q(b,d)[$, for all $i \in \{1, \cdots, d\}$, $\phi^i \in L^p(\Omega, \mu)$. Moreover, $Q(b,d) \underset{b\to 1}{\longrightarrow} +\infty$, and $Q(b,d) \underset{b\to 0}{\longrightarrow} 2$.

Proof of Proposition 7. Since $g^i(.,\omega)$ is a weak solution of (14), for all $R > 0$, the function $g_R^i(x,\omega) \stackrel{\triangle}{=} g^i(x,\omega) - \int_{\mathcal{B}(0,2R)} g^i(x,\omega)\, dx$ is again a weak solution of (14) ($\mathcal{B}(0, 2R)$ is the ball of $\mathbb{R}^d$ centered at 0, of radius $2R$). Consider

$$\mathbf{H}(\omega) = \frac{1}{\bar{A}}\mathbf{a}(\omega)e^{-2(\mathbf{V}(\omega)-\underline{V})}\;.$$

μ-a.s., for all $x, \xi \in \mathbb{R}^d$,

$$b\,\|\xi\|^2 \le (\mathbf{H}(\tau_x\omega)\xi, \xi) \le \|\xi\|^2 \;\; ; \tag{16}$$

i.e., $H(.,\omega)$ satisfy assumptions (10) and (11) of [17] (p 192). Moreover, for all $R > 0$ and all $i \in \{1, \cdots, d\}$, g_R^i is a solution in $H^1_{loc}(\mathbb{R}^d)$ of

$$\mathrm{div}(\mathbf{H}(\tau_x\omega)\nabla g_R^i(x,\omega)) = 0\;. \tag{17}$$

Applying theorem 2 of [17] p 200, we get the existence of constants $Q(b,d) > 2$ and $C(b,p,d) > 0$ such that for all $p \in [2, Q(b,d)[$, μ-a.s.,

$$\left\|\nabla g_R^i(.,\omega)\right\|_{p,\mathcal{B}(0,R)} \le C(b,p,d)R^{d/p-d/2-1}\left\|g_R^i(.,\omega)\right\|_{2,\mathcal{B}(0,2R)}\;\;,$$

where $\|.\|_{p,\mathcal{B}(0,R)}$ is the norm in $L^p(\mathcal{B}(0,R),dx)$. Since $\int_{\mathcal{B}(0,2R)} g_R^i(x,\omega)\,dx = 0$, Poincaré inequality then yields

$$\left\| g_R^i(.,\omega) \right\|_{2,\mathcal{B}(0,2R)} \leq C(d)R \left\| \nabla g_R^i(.,\omega) \right\|_{2,\mathcal{B}(0,2R)} .$$

We have thus obtained that μ-a.s., for all $p \in [2, Q(b,d)[$ and all $R > 0$,

$$R^{-d/p} \left\| \nabla g_R^i(.,\omega) \right\|_{p,\mathcal{B}(0,R)} \leq C(b,p,d) R^{-d/2} \left\| \nabla g_R^i(.,\omega) \right\|_{2,\mathcal{B}(0,2R)} . \tag{18}$$

But $\nabla g_R^i(x,\omega) = e_i + \phi^i(\tau_x\omega)\mu$-a.s. By the ergodic theorem,

$$R^{-d} \left\| \nabla g_R^i(.,\omega) \right\|^2_{2,\mathcal{B}(0,2R)} = \frac{1}{R^d} \int_{B(0,2R)} \left\| e_i + \phi^i(\tau_x\omega) \right\|^2 dx$$

$$\underset{R\to\infty}{\overset{\mu-a.s.,L^1(\mu)}{\longrightarrow}} |\mathcal{B}(0,2)| \int \left\| e_i + \phi^i(\omega) \right\|^2 d\mu .$$

Therefore, μ-a.s.,

$$\limsup_{R\to\infty} R^{-d} \left\| e_i + \phi^i(\tau_.\omega) \right\|^p_{p,\mathcal{B}(0,R)} \leq C(b,p,d) \left\| e_i + \phi^i \right\|^p_{L^2(\mu)} .$$

But, for all $K > 0$,

$$\begin{aligned} &\limsup_{R\to\infty} R^{-d} \left\| e_i + \phi^i(\tau_.\omega) \right\|^p_{p,\mathcal{B}(0,R)} \\ &\quad \geq \limsup_{R\to\infty} \frac{1}{R^d} \int_{\mathcal{B}(0,R)} \left[\left\| e_i + \phi^i(\tau_x\omega) \right\| \wedge K \right]^p dx \\ &\quad = |\mathcal{B}(0,1)| \int \left[\left\| e_i + \phi^i(\omega) \right\| \wedge K \right]^p d\mu \text{ , by ergodic theorem} \end{aligned}$$

The Beppo Levi monotone convergence theorem allows now to conclude that

$$\left\| e_i + \phi^i \right\|_{L^p(\mu)} \leq C(b,p,d) \left\| e_i + \phi^i \right\|_{L^2(\mu)} .$$

□

Remark 1. One can take $Q(b,1) = +\infty$ for all $b \in [0;1]$. Indeed, in the one-dimensional case, equation (12) can be solved explicitly. Using ergodicity, one easily get that $\phi = \frac{e^{2\mathbf{V}}/\mathbf{a}}{\langle e^{2\mathbf{V}}/\mathbf{a}\rangle_\mu} - 1$, which is actually in $\mathcal{V}^2_{pot}$, since in the 1-dimensional case, $\mathcal{V}^2_{pot}$ reduces to the elements in $L^2(\mu)$ with zero mean.

Remark 2. The constants $Q(b,d)$ can be described in the following way. Consider the problem

$$\begin{cases} \triangle u = \operatorname{div} f \, , \\ u \in W_0^{1,p}(\mathcal{B}_d(0,1)) \, , \\ f \in L^p(\mathcal{B}_d(0,1)) \, . \end{cases}$$

It gets a unique solution in $W_0^{1,p}(\mathcal{B}_d(0,1))$, and one has the estimate

$$\|\nabla u\|_{L^p(\mathcal{B}_d(0,1))} \leq K_p(d)\, \|f\|_{L^p(\mathcal{B}_d(0,1))} \ .$$

Let $K_p(d)$ be the best constant in the preceding inequality. Then

$$Q(b,d) = \sup\left\{ p \text{ such that } \frac{1}{K_p(d)} > 1-b \right\} \ . \tag{19}$$

4. Homogenization of parabolic systems

4.1. The problem and assumptions made on the non-linear term and the final condition

This section is concerned with the homogenization of the parabolic system (1). Apart from assumptions **(RM 1..4)** and **(L 1..3)** on the linear part, system (1) is studied under the following assumptions on the non-linear term h, and the final condition H:

(P 1) **(i)** $h : \mathbb{R}^d \times \mathbb{R}^m \times \mathbb{R}^{m\times d} \to \mathbb{R}^m$ is uniformly continuous and bounded.
(ii) $\exists K > 0$ such that $\forall (x,y,z,z') \in \mathbb{R}^d \times \mathbb{R}^m \times \mathbb{R}^{m\times d} \times \mathbb{R}^{m\times d}$,

$$\left\| h(x,y,z) - h(x,y,z') \right\| \leq K \left\| z - z' \right\| \ ;$$

(iii) $\exists \lambda \in \mathbb{R}$ such that $\forall (x,y,y',z) \in \mathbb{R}^d \times \mathbb{R}^m \times \mathbb{R}^m \times \mathbb{R}^{m\times d}$,

$$\left(h(x,y,z) - h(x,y',z), y - y' \right) \leq \lambda \left\| y - y' \right\|^2 \ ;$$

(iv) $\exists C > 0$ and $g \in L^2(\mathbb{R}^d)$, such that $\forall (x,y,z) \in \mathbb{R}^d \times \mathbb{R}^m \times \mathbb{R}^{m\times d}$,

$$\|h(x,y,z)\| \leq C\left[g(x) + \|y\| + \|z\| \right] \ .$$

(P 2) $H : \mathbb{R}^d \to \mathbb{R}^m$ is continuous and $H \in L^2(\mathbb{R}^d)^m$.

Assumptions **(L 1..3)** and **(P 1-2)** are enough to ensure that μ-a.s., there exists a unique weak solution to the system (1), i.e a solution in the space

$$\mathcal{W}_2^1\left(0,T,H^1(\mathbb{R}^d),L^2(\mathbb{R}^d)\right) \triangleq \Big\{ u(t,x) \in L^2([0,T];H^1(\mathbb{R}^d)^m) \text{ such that } \partial_t u(t,x) \in L^2([0,T],H^{-1}(\mathbb{R}^d)^m) \Big\} \ .$$

Moreover, this space is continuously embedded in the space $\mathcal{C}\left([0,T];L^2(\mathbb{R}^d)^m\right)$ (see for instance chapter 30 of [25]).

4.2. *The homogenized equation*

Let us define the homogenized coefficients

$$\bar{a} \stackrel{\triangle}{=} \left\langle (\mathrm{Id} + \phi)\mathbf{a}(Id + \phi)^* \right\rangle_\pi \quad , \tag{20}$$

where $\phi_{i,j} \stackrel{\triangle}{=} \phi^i_j$.

$$\bar{h}(x, y, z) \stackrel{\triangle}{=} \langle h(x, y, z(Id + \phi)) \rangle_\pi \tag{21}$$

Lemma 8. *Assume* **(RM 1..4)**, **(L 1..3)**, **(P 1-2)** *are satisfied. Then, $\bar{a}$ is a strictly positive symmetric matrix, and $\bar{h}$ satisfies* **(P 1)** *with constants $K \langle \|Id + \phi\| \rangle_\pi$ and λ in* **(P 1 (ii))** *and* **(P 1 (iii))**. *Thus equation*

$$\begin{cases} \partial_t u^0 + \frac{1}{2}\bar{a}_{i,j}\partial^2_{i,j} u^0_t + \bar{h}(x, u^0_t, \nabla u^0_t) = 0\,, \ \ t \in [0, T]\,, \\ u^0_T = H \end{cases} \tag{22}$$

has a unique solution $u^0 \in \mathcal{W}^1_2 \left(0, T, H^1(\mathbb{R}^d)^m, L^2(\mathbb{R}^d)^m\right)$.

Proof of Lemma 8. $\bar{a}$ is clearly a non-negative symmetric matrix. We just have to check that it is non-degenerate. Assume that there exists $\xi \in \mathbb{R}^d$ such that $(\xi, \bar{a}\xi) = 0$. Then μ-a.s., $\|\xi + \phi^*\xi\|^2_{\mathbf{a}(\omega)} = 0$. $\mathbf{a}(\omega)$ being uniformly elliptic, this implies that μ-a.s., $\xi + \phi^*\xi = 0$. Integrating this equation over μ, we get $\xi = 0$, since for all $i \in \{1, \cdots, d\}$, $\phi^i \in \mathcal{V}^2_{pot}$, and therefore $\forall i, j \in \{1, \cdots, d\}$, $\int \phi^i_j \, d\mu = 0$.

The assertion concerning $\bar{h}$ is an easy consequence of **(P 1)** and $\phi \in L^2(\Omega, \mu)$. Now the existence and uniqueness of a weak solution to (22) is a consequence of theorem 30.A in [25]. □

4.3. *The homogenization result in parabolic case*

We are now able to state the main result of this section

Theorem 9. *Assume that* **(RM 1..4)**, **(L 1..3)** *and* **(P 1-2)** *are satisfied. Then for all $p \in]1, \frac{1}{2}Q(b, d)[$, $p \leq 2$, for all bounded domain G of $\mathbb{R}^d$, for all $t \in [0, T]$,*

$$\left\langle \int_G \left\| u^\varepsilon(t, x) - u^0(t, x) \right\|^p \, dx \right\rangle_\mu \underset{\varepsilon \to 0}{\longrightarrow} 0\ . \tag{23}$$

4.4. *Proof of theorem 9*

(a) The non-linear Feynman-Kac formula. Let t_0 be any time in [0,T], and $(X, P^{\varepsilon,\omega}_{t_0,x})$ be the process X starting from x at time t_0. $\mathcal{F}^X$ is the minimal admissible filtration generated by X. Since a martingale representation theorem is valid with respect to the martingale part M^X of X (because of uniform ellipticity), it is proved for instance in [22] or [16] that under **(P 1-2)**, there exists a unique pair

$(Y_t, Z_t)_{t\in[t_0,T]}$ of $\mathcal{F}^X$-progressively measurable processes satisfying for all $x \in \mathbb{R}^d$ and all t_0,

$$Y_t = H(X_T) + \int_t^T h(X_r, Y_r, Z_r)\, dr - \int_t^T Z_r\, dM_r^X\ ,\ \ t \in [t_0, T]\,,\ \ P_{t_0,x}^{\varepsilon,\omega} - \text{p.s.}\ , \tag{24}$$

$$E_{t_0,x}^{\varepsilon,\omega}\left[\sup_{t_0\le t\le T} \|Y_t\|^2 + \int_{t_0}^T \|Z_r\|^2\ dr\right] < \infty\ . \tag{25}$$

Moreover, it has been proved in [3] (theorem 20) that $P_{t_0,x}^{\varepsilon,\omega}$-p.s., for all $t \in [t_0, T]$, $Y_t = u^\varepsilon(t, X_t)$, where u^ε is a continuous version of the weak solution of system (1). Therefore $Y_{t_0} = u^\varepsilon(t_0, x)$ and we are going to prove that

$$\left\langle \int_G \left\| Y_{t_0} - u^0(t_0, x)\right\|^p\ dx\right\rangle_\mu \underset{\varepsilon\to 0}{\longrightarrow} 0\,.$$

To avoid heavy notations, we will take in all the sequel $t_0 = 0$.

(b) Change on X. Let g^i be the functions defined in lemma 6 and recall from lemma 6 that μ-a.s., for all $\psi \in \mathcal{C}_c^\infty(\mathbb{R}^d)$, $\mathcal{E}^\omega(g^i, \psi) = 0$. Thus, if $g_\varepsilon^i(x, \omega) \overset{\triangle}{=} \varepsilon g^i(\frac{x}{\varepsilon}, \omega) = x^i + \varepsilon f^i(\frac{x}{\varepsilon}, \omega)$, for all $\psi \in \mathcal{C}_c^\infty(\mathbb{R}^d)$ and all $\varepsilon > 0$,

$$\mathcal{E}^{\varepsilon,\omega}(g_\varepsilon^i, \psi) = 0\ . \tag{26}$$

Applying proposition 2 to the functions $g_\varepsilon^i(., \omega)$, we get then that μ-a.s.,

$$\tilde{g}_\varepsilon(X_t, \omega) - \tilde{g}_\varepsilon(X_0, \omega) = M_t^{\varepsilon,\omega}\ ,\ P_x^{\varepsilon,\omega}\text{-p.s., q.e. in } x \in \mathbb{R}^d\ , \tag{27}$$

where

$$M_t^{\varepsilon,\omega} = \int_0^t (\mathrm{Id} + \phi)\left(\frac{X_s}{\varepsilon}, \omega\right) dM_s^X\ ,\ P_x^{\varepsilon,\omega} - \text{p.s., q.e. in } x \in \mathbb{R}^d\ .$$

Lemma 10. *Assume that* **(RM 1..4)** *and* **(L 1..3)** *are satisfied. Let* $\hat{X}_t \overset{\triangle}{=} X_0 + \int_0^t (Id + \phi)\left(\frac{X_s}{\varepsilon}, \omega\right) dM_s^X$. *Then,* μ*-a.s, q.e in* x*, the law of* $(\hat{X}_t;\, t \ge 0)$ *under* $P_x^{\varepsilon,\omega}$ *converges weakly in* $\mathcal{C}_x(\mathbb{R}^+; \mathbb{R}^d)$ *(space of continuous paths starting from* x*, endowed with the uniform convergence on compact subsets) to the law of* $(B_t;\, t \ge 0)$*, where* B *is a Brownian motion with covariance matrix* $\bar{a}$*, starting from* x*. Moreover,* μ*-a.s., q.e in* x*, for all* $T > 0$ *and all* $\delta > 0$*,*

$$P_x^{\varepsilon,\omega}\left[\sup_{0\le t\le T} \left\| X_t - \hat{X}_t\right\| \ge \delta\right] \underset{\varepsilon\to 0}{\longrightarrow} 0\ .$$

As a consequence, the law of $((X_t, \hat{X}_t);\, t \ge 0)$ *under* $P_x^{\varepsilon,\omega}$ *converges weakly in* $\mathcal{C}_x(\mathbb{R}^+; \mathbb{R}^d) \times \mathcal{C}_x(\mathbb{R}^+; \mathbb{R}^d)$ *to the law of* $((B_t, B_t);\, t \ge 0)$

Proof of Lemma 10. This lemma is proved in [16]. □

(c) Change on Y. We are first going to prove theorem 9 in the case where the coefficients h and H are smooth. To this end, we introduce assumptions **(P 1')** and **(P 2')**.

(P 1') h is bounded and Lipschitz, and $h(x, 0, 0) \in L^2(\mathbb{R}^d)^m$.
(P 2') $H : \mathbb{R}^d \mapsto \mathbb{R}^m$ is a $\mathcal{C}^{(2,\delta)}$-function for some $\delta \in]0; 1[$ (i.e. bounded, with bounded derivatives up to order 2, and with second derivatives which are Hölder continuous of order δ), and $H \in L^2(\mathbb{R}^d)^m$.

Under assumptions **(P 1'-2')**, the solution u_0 of system (22) is in the space $\mathcal{C}^{(1,\delta/2)(2,\delta)}$ of functions with one derivative in time which is Hölder continuous of order $\delta/2$, and with derivatives in space up to order two, Hölder continuous of order δ (see for instance chapter VII of [13]).

Lemma 11. *Assume that* **(RM 1..3)**, **(L 1..3)** *and* **(P 1-2')** *hold. Let*

$$\begin{cases} \hat{Y}_t \overset{\triangle}{=} Y_t - u^0(t, \hat{X}_t) \\ \hat{Z}_t \overset{\triangle}{=} Z_t - \nabla u^0(t, \hat{X}_t)(Id + \phi)\left(\frac{X_t}{\varepsilon}, \omega\right) \end{cases} \tag{28}$$

Then, for q.e. $x \in \mathbb{R}^d$, $P_x^{\varepsilon,\omega}$*-p.s.,*

$$\hat{Y}_t = \eta + \int_t^T F(r, \hat{Y}_r, \hat{Z}_r)\, dr - \int_t^T \hat{Z}_r\, dM_r^X\ , \tag{29}$$

where

$$\begin{aligned} \eta &\overset{\triangle}{=} H(X_T) - H(\hat{X}_T) \\ F(r, y, z) &\overset{\triangle}{=} h\left(X_r, y + u^0(r, \hat{X}_r), z + \nabla u^0(r, \hat{X}_r)\left(Id + \phi(\tfrac{X_r}{\varepsilon}, \omega)\right)\right) \\ &\quad -\bar{h}(\hat{X}_r, u^0(r, \hat{X}_r), \nabla u^0(r, \hat{X}_r)) \\ &\quad +\tfrac{1}{2}\left[(Id + \phi)\mathbf{a}(Id + \phi)^* - \bar{a}\right]_{i,j}\left(\tfrac{X_r}{\varepsilon}, \omega\right)\, \partial_{i,j}^2 u^0(r, \hat{X}_r) \end{aligned} \tag{30}$$

Proof of Lemma 11. It is an easy consequence of Itô's formula applied to the smooth function u^0, of equation (22) satisfied by u^0, and of definitions of the processes Y, $\hat{Y}$ and $\hat{Z}$. □

(d) An ergodic lemma.

Lemma 12. *Assume that* **(RM 1..3)**, **(L 1..3)** *hold. Let* $G : [0, T] \times \mathbb{R}^d \times \Omega \to \mathbb{R}$ *be a measurable function such that*

(i) *There exists* $\xi \in L^1(\Omega, \mu)$ *such that for all* $\delta > 0$*, for all* K *compact subsets of* $\mathbb{R}^d$, $\exists \alpha > 0$ *such that* $\forall r, r' \in [0, T]$, $|r - r'| \le \alpha$, $\forall x, x' \in K$, $\|x - x'\| \le \alpha$, μ*-a.s.,*

$$|G(r, x, \omega) - G(r', x', \omega)| \le \xi(\omega)\delta\ ,$$

(ii) *For all* $r \in [0,T]$, *and all* $x \in \mathbb{R}^d$, $\omega \mapsto G(r,x,\omega)$ *is in* $L^1(\Omega,\mu)$ *and* $\int G(r,x,\omega)\,d\pi = 0$.

Let $P_x^{\varepsilon,\mu} \stackrel{\triangle}{=} \int_\Omega d\mu(\omega) P_x^{\varepsilon,\omega}$. *Then, for q.e.* $x \in \mathbb{R}^d$, *and all* $s \in [0,T]$, *for all* $\beta > 0$,

$$P_x^{\varepsilon,\mu}\left[\left|\int_0^s G\left(r,\hat{X}_r,\tau_{\frac{X_r}{\varepsilon}}\omega\right)\,dr\right| \geq \beta\right] \underset{\varepsilon\to 0}{\longrightarrow} 0 \ .$$

Proof of Lemma 12. It follows from lemma 10 that the family of laws $(\hat{X}, P_x^{\varepsilon,\mu})$ is tight in the space $\mathcal{C}_x([0,T],\mathbb{R}^d)$ of continuous functions starting from x. This enables to replace the process $\hat{X}$ by a fixed trajectory in $\mathcal{C}_x$. Indeed, let $\delta > 0$ and K_δ a compact subset of $\mathcal{C}_x$ such that $\inf_{\varepsilon>0} \mathbb{P}_x^{\varepsilon,\mu}\left[\hat{X} \in K_\delta\right] \geq 1-\delta$.

$$C_\delta \stackrel{\triangle}{=} \left\{y \in \mathbb{R}^d \text{ such that } \exists r \in [0,T], \exists x \in K_\delta, \|y - x(r)\| \leq 1\right\}$$

is a compact subset of $\mathbb{R}^d$. By (i), one can choose $\alpha \in]0,1]$ such that for all $x, x' \in C_\delta$, $\|x - x'\| \leq \alpha$, $\left|G(r,x,\omega) - G(r,x',\omega)\right| \leq \xi(\omega)\delta$. $\mathcal{C}_x$ being separable (with countable basis (x_i)), K_δ can be covered by a finite number N_α of balls of $\mathcal{C}_x$, say $\mathcal{B}(x_i,\alpha)$; i.e.

$$K_\delta \subset \bigcup_{i=1}^{N_\alpha} \mathcal{B}(x_i,\alpha) = \bigcup_{i=1}^{N_\alpha} \tilde{\mathcal{B}}_i \ ,$$

where $\tilde{\mathcal{B}}_i \subset \mathcal{B}(x_i,\alpha)$ are disjoints. Now,

$$\begin{aligned}
&P_x^{\varepsilon,\mu}\left[\left|\textstyle\int_0^s G\left(r,\hat{X}_r,\tau_{\frac{X_r}{\varepsilon}}\omega\right)\,dr\right| \geq \beta\right] \\
&\leq P_x^{\varepsilon,\mu}\left[\hat{X} \notin K_\delta\right] + \sum_{i=1}^{N_\alpha} P_x^{\varepsilon,\mu}\left[\left|\int_0^s G\left(r,x_i(r),\tau_{\frac{X_r}{\varepsilon}}\omega\right)\,dr\right| \geq \frac{\beta}{2}\right] \\
&\quad + \sum_{i=1}^{N_\alpha} P_x^{\varepsilon,\mu}\left[\int_0^T \left|G\left(r,\hat{X}_r,\tau_{\frac{X_r}{\varepsilon}}\omega\right) - G\left(r,x_i(r),\tau_{\frac{X_r}{\varepsilon}}\omega\right)\right|\,dr \geq \frac{\beta}{2}; \hat{X} \in \tilde{\mathcal{B}}_i \cap K_\delta\right]
\end{aligned}$$

But, when $\hat{X} \in \tilde{\mathcal{B}}_i$, for all $r \in [0,T]$, $\left\|\hat{X}_r - x_i(r)\right\| \leq \alpha$, and $\hat{X}_r \in C_\delta$. Therefore, μ-a.s., $\left|G\left(r,\hat{X}_r,\tau_{\frac{X_r}{\varepsilon}}\omega\right) - G\left(r,x_i(r),\tau_{\frac{X_r}{\varepsilon}}\omega\right)\right| \leq \xi\left(\tau_{\frac{X_r}{\varepsilon}}\omega\right)\delta$. Hence,

$$\begin{aligned}
&P_x^{\varepsilon,\mu}\left[\left|\textstyle\int_0^T G\left(r,\hat{X}_r,\tau_{\frac{X_r}{\varepsilon}}\omega\right)\,dr\right| \geq \beta\right] \\
&\leq \delta + P_x^{\varepsilon,\mu}\left[\int_0^T \xi(\tau_{\frac{X_r}{\varepsilon}}\omega)dr \geq \frac{\beta}{2\delta}\right] + \sum_{i=1}^{N_\alpha} P_x^{\varepsilon,\mu}\left[\left|\int_0^s G\left(r,x_i(r),\tau_{\frac{X_r}{\varepsilon}}\omega\right)dr\right| \geq \frac{\beta}{2}\right] \\
&\leq \delta + Te^{2\triangle V}\tfrac{2\delta}{\beta}\langle\xi\rangle_\mu + \textstyle\sum_{i=1}^{N_\alpha} P_x^{\varepsilon,\mu}\left[\left|\int_0^s G\left(r,x_i(r),\tau_{\frac{X_r}{\varepsilon}}\omega\right)dr\right| \geq \frac{\beta}{2}\right]
\end{aligned}$$

It is then sufficient to show that for all $x(.) \in \mathcal{C}_x$,

$$E_x^{\varepsilon,\mu}\left[\left|\int_0^s G\left(r,x(r),\tau_{\frac{X_r}{\varepsilon}}\omega\right)\,dr\right|\right] \underset{\varepsilon\to 0}{\longrightarrow} 0 \, .$$

Let then $x(.) \in \mathcal{C}_x$ and $\delta > 0$ be fixed. Let K be the compact subset of $\mathbb{R}^d$ defined by $K \stackrel{\triangle}{=} \left\{y \in \mathbb{R}^d \text{ such that } \exists r \in [0,T], \|y - x(r)\| \leq 1\right\}$. Let $\alpha \in]0,1]$ be such that for all $r, r' \in [0,T]$, $|r - r'| \leq \alpha$, for all $x, x' \in K$, $\left\|x - x'\right\| \leq \alpha$,

$$\left|G(r,x,\omega) - G(r',x',\omega)\right| \leq \xi(\omega)\delta\,.$$

Finally, let $\beta \in]0,\alpha]$ be such that for all $r, r' \in [0,T]$, $\left|r - r'\right| \leq \beta$ implies $\left\|x(r) - x(r')\right\| \leq \alpha$. If (t_i) is a subdivision of $[0,s]$ with step β, we get then

$$\begin{aligned}
&E_x^{\varepsilon,\mu}\left[\left|\int_0^s G\left(r, x(r), \tau_{\frac{X_r}{\varepsilon}}\omega\right) dr\right|\right] \\
&\quad\leq \sum_i E_x^{\varepsilon,\mu}\left[\left|\int_{t_i}^{t_{i+1}} G\left(t_i, x(t_i), \tau_{\frac{X_r}{\varepsilon}}\omega\right) dr\right|\right] \\
&\qquad + \sum_i E_x^{\varepsilon,\mu}\left[\int_{t_i}^{t_{i+1}} \left|G\left(r, x(r), \tau_{\frac{X_r}{\varepsilon}}\omega\right) - G\left(t_i, x(t_i), \tau_{\frac{X_r}{\varepsilon}}\omega\right)\right| dr\right] \\
&\quad\leq e^{2\triangle V}\sum_i E_0^{1,\mu}\left[\left|\varepsilon^2\int_0^{\frac{t_{i+1}-t_i}{\varepsilon^2}} G(t_i, x(t_i), \tau_{\frac{X_r}{\varepsilon}}\omega)\, dr\right|\right] + \delta E_x^{\varepsilon,\mu}\left[\int_0^T \xi\left(\tau_{\frac{X_r}{\varepsilon}}\omega\right) dr\right]
\end{aligned}$$

But, for all i, $G(t_i, x(t_i), \omega) \in L^1(\Omega,\mu)$, and $\int G(t_i, x(t_i), \omega)\, d\pi = 0$. Ergodicity of $\tau_{\frac{X_r}{\varepsilon}}\omega$ implies now that for all $\delta > 0$,

$$\limsup_{\varepsilon\to 0} E_x^{\varepsilon,\mu}\left[\left|\int_0^s G\left(r, x(r), \tau_{\frac{X_r}{\varepsilon}}\omega\right) dr\right|\right] \leq C\delta T \left\langle \xi \right\rangle_\mu . \qquad \square$$

(e) Some L^p-estimates.

Lemma 13. *Assume that* **(RM 1..3)**, **(L 1..3)** *and* **(P 1'-2')** *hold. Let η and F be as in lemma 11. Then, for any bounded domain G of $\mathbb{R}^d$, for all $s \in [0,T]$,*

$$\forall p > 1, \qquad \left\langle \int_G E_x^{\varepsilon,\omega}\left[\|\eta\|^p\right] dx \right\rangle_\mu \underset{\varepsilon\to 0}{\longrightarrow} 0\ ; \tag{31}$$

$$\forall p \in]1, \frac{1}{2}Q(b,d)[, \qquad \left\langle \int_G E_\omega^{\varepsilon,x}\left[\left\|\int_0^s F(r,0,0)\, dr\right\|^p\right] dx \right\rangle_\mu \underset{\varepsilon\to 0}{\longrightarrow} 0\ . \tag{32}$$

Proof of Lemma 13. (31) is a straightforward consequence of lemma 10, and of the fact that the map

$$(x,\hat{x}) \in \mathcal{C}(\mathbb{R}^+;\mathbb{R}^d) \times \mathcal{C}(\mathbb{R}^+,\mathbb{R}^d) \mapsto \left\|H(x(T)) - H(\hat{x}(T))\right\|^p$$

is bounded and continuous by **(P 2')**.

Let us prove (32). $F(r,0,0) = F_1(r) + F_2(r) + F_3(r)$, where

$$\begin{aligned}
F_1(r) &= h\left(X_r, u^0(r,\hat{X}_r), \nabla u^0(r,\hat{X}_r)(\mathrm{Id}+\phi)\left(\tfrac{X_r}{\varepsilon},\omega\right)\right) \\
&\quad - h\left(\hat{X}_r, u^0(r,\hat{X}_r), \nabla u^0(r,\hat{X}_r)(\mathrm{Id}+\phi)\left(\tfrac{X_r}{\varepsilon},\omega\right)\right) \\
F_2(r) &= h\left(\hat{X}_r, u^0(r,\hat{X}_r), \nabla u^0(r,\hat{X}_r)(\mathrm{Id}+\phi)\left(\tfrac{X_r}{\varepsilon},\omega\right)\right) \\
&\quad - \bar{h}\left(\hat{X}_r, u^0(r,\hat{X}_r), \nabla u^0(r,\hat{X}_r)\right) \\
F_3(r) &= \tfrac{1}{2}\left[(\mathrm{Id}+\phi)a(\mathrm{Id}+\phi)^* - \bar{a}\right]_{i,j}\left(\tfrac{X_r}{\varepsilon},\omega\right)\partial^2_{i,j}u^0(r,\hat{X}_r)
\end{aligned}$$

Let $P_G^{\varepsilon,\mu} \triangleq \int_G \frac{dx}{|G|} P_x^{\varepsilon,\mu}$, and $E_G^{\varepsilon,\mu}$ the expectation under $P_G^{\varepsilon,\mu}$. By **(P 1')**, for all $\delta > 0$,

$$E_G^{\varepsilon,\mu}\left[\left\|\int_0^s F_1(r)\,dr\right\|^p\right] \le T^{p-1} E_G^{\varepsilon,\mu}\left[\int_0^T \|F_1(r)\|^p\,dr\right] \\ \le (2T)^p \|h\|_\infty^p P_G^{\varepsilon,\mu}\left[\sup_{0\le t\le T}\left\|X_t - \hat{X}_t\right\| \ge \delta\right] + CT^p\delta^p.$$

Letting ε go to 0, then δ go to 0, lemma 10 leads to

$$\forall p > 1, \qquad E_G^{\varepsilon,\mu}\left[\left\|\int_0^s F_1(r)\,dr\right\|^p\right] \underset{\varepsilon\longrightarrow 0}{\longrightarrow} 0\ .$$

$F_2(r) = G_2(r, \hat{X}_r, \tau_{\frac{X_r}{\varepsilon}}\omega)$, with

$$G_2(r,x,\omega) = h(x, u^0(r,x), \nabla u^0(r,x)(\mathrm{Id}+\phi)(\omega)) - \bar{h}(x, u^0(r,x), \nabla u^0(r,x))\ .$$

By boundedness of h and definition of $\bar{h}$, G_2 is a bounded measurable function which satisfies (ii) of lemma 12. Moreover, by **(P 1')**,

$$\left\|G_2(r,x,\omega) - G_2(r',x',\omega)\right\| \le C\left[\|x-x'\| + \left\|u^0(r,x) - u^0(r',x')\right\|\right] \\ + C\left[\left\|\nabla u^0(r,x) - \nabla u^0(r',x')\right\|(1+\|\phi(\omega)\|)\right],$$

and (i) of lemma 12 is satisfied thanks to the smoothness properties of u^0. Thus, $\int_0^s F_2(r)\,dr$ converges to 0 in $P_G^{\varepsilon,\mu}$-probability. Since G_2 is bounded, this convergence takes also place in all $L^p(P_G^{\varepsilon,\mu})$.

In the same way, $F_3(r) = G_3(r, \hat{X}_r, \tau_{\frac{X_r}{\varepsilon}}\omega)$, with

$$G_3(r,x,\omega) = \frac{1}{2}\left[(\mathrm{Id}+\phi)\mathbf{a}(\mathrm{Id}+\phi)^* - \bar{a}\right]_{i,j}(\omega)\partial^2_{i,j}u^0(r,x)\ .$$

Since $\partial^2_{i,j}u^0$ are bounded and Hölder, G_3 satisfies the assumptions of lemma 12. Moreover, for all $p \in [2, Q(b,d)[$,

$$\sup_{\varepsilon>0} E_G^{\varepsilon,\mu}\left[\left\|\int_0^s G_3(r,\hat{X}_r,\tau_{\frac{X_r}{\varepsilon}}\omega)\,ds\right\|^{p/2}\right] \le CT^{p/2}\max_{i,j}\left\|\partial^2_{i,j}u^0\right\|_\infty^{p/2}\left(1+\left\langle\|\phi\|^p\right\rangle_\mu\right)$$

Therefore, $\int_0^s F_3(r)\,dr$ converges to zero in $L^p(P_G^{\varepsilon,\mu})$ for all $p < \frac{1}{2}Q(b,d)$. □

(f) Conclusion in the case of smooth *h* and *H*.

Lemma 14. *Assume that* **(RM 1..3)**, **(L 1..3)** *and* **(P 1'-2')** *hold. Then, for any bounded domain G of $\mathbb{R}^d$, for all $p \in]1, \frac{1}{2}Q(b,d)[$, $p \le 2$,*

$$\left\langle\int_G \left\|u^\varepsilon(0,x) - u^0(0,x)\right\|^p dx\right\rangle_\mu \underset{\varepsilon\to 0}{\longrightarrow} 0\ .$$

Proof of Lemma 14. Let

$$\begin{aligned}\tilde{Y}_t &\stackrel{\triangle}{=} \hat{Y}_t + \int_0^t F(r,0,0)\,dr \\ &= \eta + \int_0^T F(r,0,0)\,dr + \int_t^T \Big(F(r,\hat{Y}_r,\hat{Z}_r) - F(r,0,0)\Big)\,dr - \int_t^T \hat{Z}_r\,dM_r^X\end{aligned}$$

We are going to prove that $\left\langle \int_G E_x^{\varepsilon,\omega}\left[\left\|\tilde{Y}_0\right\|^p\right] dx\right\rangle_\mu \underset{\varepsilon\to 0}{\longrightarrow} 0$.

First of all, note that for all $p \in]1, \frac{1}{2}Q(b,d)[$, μ-a.s., q.e. in x,

$$E_x^{\varepsilon,\omega}\left[\int_0^T |F(r,0,0)|^p\,dr\right] \leq C E_x^{\varepsilon,\omega}\left[\int_0^T \left(1+\|\phi\|^{2p}\right)\left(\frac{X_r}{\varepsilon},\omega\right) dr\right] < \infty$$

since $\phi \in L^{2p}(\mu)$. Therefore, using estimate (25), and the boundedness of u^0, for all $p \in]1, \frac{1}{2}Q(b,d)[$, $p \leq 2$, μ-a.s, q.e in x

$$E_x^{\varepsilon,\omega}\left[\sup_{0\leq t\leq T}\left\|\tilde{Y}_t\right\|^p\right] < \infty\ . \tag{33}$$

Moreover, using estimate (25), the boundedness of ∇u^0, and the fact that $\phi \in L^2(\mu)$, we also get that μ-a.s, q.e in x,

$$E_x^{\varepsilon,\omega}\left[\int_0^T \left\|\hat{Z}_r\right\|^2 dr\right] < \infty\,. \tag{34}$$

For all $n \in \mathbb{N}^*$, let T_n be the stopping time

$$T_n = \inf\{t \geq 0, \left\|\tilde{Y}_t\right\| \leq \frac{1}{n}\}\ .$$

For all p, $1 < p < \frac{1}{2}Q(b,d)$, $p \leq 2$, Itô's formula yields that for all $n \in \mathbb{N}^*$, q.e. $x \in \mathbb{R}^d$, $P_x^{\varepsilon,\omega}$-p.s.,

$$\begin{aligned}&\left\|\tilde{Y}_{t\wedge T_n}\right\|^p + \frac{p}{2}\int_{t\wedge T_n}^{T\wedge T_n}\left\|\tilde{Y}_r\right\|^{p-2}\mathrm{Trace}\left[(\hat{Z}_r a(X_r/\varepsilon,\omega)\hat{Z}_r^*)(P_{\tilde{Y}_r^\perp} + (p-1)P_{\tilde{Y}_r})\right] dr \\ &\quad = \left\|\tilde{Y}_{T\wedge T_n}\right\|^p + \int_{t\wedge T_n}^{T\wedge T_n} p\left\|\tilde{Y}_r\right\|^{p-2}\Big(\tilde{Y}_r, F(r,\hat{Y}_r,\hat{Z}_r) - F(r,0,0)\Big)\,dr \\ &\qquad - \int_{t\wedge T_n}^{T\wedge T_n} p\left\|\tilde{Y}_r\right\|^{p-2}\Big(\tilde{Y}_r, \hat{Z}_r\,dM_r^X\Big)\,,\end{aligned} \tag{35}$$

where for all $x \in \mathbb{R}^d$, P_x denotes the projection onto Span$\{x\}$, and $P_{x^\perp}$ the projection onto $(\mathrm{Span}\{x\})^\perp$.

The local martingale $\int_0^t \left\|\tilde{Y}_r\right\|^{p-2}\left(\tilde{Y}_r, \hat{Z}_r\, dM_r^X\right)$ is actually a martingale. Indeed,

$$
\begin{aligned}
&E_x^{\varepsilon,\omega}\left[\sup_{0\le t\le T}\left|\int_0^t \left\|\tilde{Y}_r\right\|^{p-2}\left(\tilde{Y}_r, \hat{Z}_r\, dM_r^X\right)\right|\right]\\
&\quad\le C E_x^{\varepsilon,\omega}\left[\left(\int_0^T \left\|\tilde{Y}_r\right\|^{2(p-2)}\left(\tilde{Y}_r, \hat{Z}_r\, a(\frac{X_r}{\varepsilon},\omega)\, \hat{Z}_r^* \tilde{Y}_r\right) dr\right)^{1/2}\right]\\
&\quad\le C E_x^{\varepsilon,\omega}\left[\sup_{0\le t\le T}\left\|\tilde{Y}_t\right\|^{p-1}\left(\int_0^T \left\|\hat{Z}_r\right\|^2 dr\right)^{1/2}\right]\\
&\quad\le C E_x^{\varepsilon,\omega}\left[\sup_{0\le t\le T}\left\|\tilde{Y}_t\right\|^{p}+\left(\int_0^T \left\|\hat{Z}_r\right\|^2 dr\right)^{p/2}\right] \text{ by Young inequality,}\\
&\quad\le C E_x^{\varepsilon,\omega}\left[\sup_{0\le t\le T}\left\|\tilde{Y}_t\right\|^{p}\right]+C\left(E_x^{\varepsilon,\omega}\left[\int_0^T \left\|\hat{Z}_r\right\|^2 dr\right]\right)^{p/2} \text{ since } 2/p\ge 1\\
&\quad< \infty \qquad \mu\text{-a.s., by (33) and (34) .}
\end{aligned}
$$

Taking the expectation in (35) and using the fact that $F(r, y, z)$ is uniformly Lipschitz in (y, z),

$$
\begin{aligned}
&E_x^{\varepsilon,\omega}\left[\left\|\tilde{Y}_{t\wedge T_n}\right\|^p+\tfrac{p}{2}\int_{t\wedge T_n}^{T\wedge T_n}\left\|\tilde{Y}_r\right\|^{p-2}\mathrm{Trace}\left[\left(\hat{Z}_r a \hat{Z}_r^*\right)\left(P_{\tilde{Y}_r^{\perp}}+(p-1)P_{\tilde{Y}_r}\right)\right]dr\right]\\
&\le E_x^{\varepsilon,\omega}\left[\left\|\tilde{Y}_{T\wedge T_n}\right\|^p+pK\int_{t\wedge T_n}^{T\wedge T_n}\left\|\tilde{Y}_r\right\|^{p-1}\left(\left\|\hat{Y}_r\right\|+\left\|\hat{Z}_r\right\|\right)dr\right]\\
&\le C E_x^{\varepsilon,\omega}\left[\left\|\tilde{Y}_{T\wedge T_n}\right\|^p+\int_{t\wedge T_n}^{T\wedge T_n}\left\|\tilde{Y}_r\right\|^{p-1}\left(\left\|\tilde{Y}_r\right\|+\left\|\int_0^r F(u,0,0)\,du\right\|+\left\|\hat{Z}_r\right\|\right)dr\right]
\end{aligned}
$$

Young inequality gives

$$
\begin{aligned}
\left\|\tilde{Y}_r\right\|^{p-1}\left\|\int_0^r F(u,0,0)\,du\right\| &\le \tfrac{1}{q}\left\|\tilde{Y}_r\right\|^{(p-1)q}+\tfrac{1}{p}\left\|\int_0^r F(u,0,0)\,du\right\|^p\\
&\le \tfrac{1}{q}\left\|\tilde{Y}_r\right\|^{p}+\tfrac{1}{p}\left\|\int_0^r F(u,0,0)\,du\right\|^p
\end{aligned}
$$

and for all $\delta > 0$,

$$
\begin{aligned}
\left\|\tilde{Y}_r\right\|^{p-1}\left\|\hat{Z}_r\right\| &= \left\|\tilde{Y}_r\right\|^{p/2}\left\|\tilde{Y}_r\right\|^{p/2-1}\left\|\hat{Z}_r\right\|\\
&\le \tfrac{1}{2\delta}\left\|\tilde{Y}_r\right\|^{p}+\tfrac{\delta}{2}\left\|\tilde{Y}_r\right\|^{p-2}\left\|\hat{Z}_r\right\|^2
\end{aligned}
$$

Moreover,

$$\begin{aligned}
&\text{Trace}\left[\left(\hat{Z}_r\, a\left(\tfrac{X_r}{\varepsilon},\omega\right)\, \hat{Z}_r^*\right)\left(P_{\tilde{Y}_r^\perp} + (p-1)P_{\tilde{Y}_r}\right)\right] \\
&= \text{Trace}\left(P_{\tilde{Y}_r^\perp}\hat{Z}_r\, a\left(\tfrac{X_r}{\varepsilon},\omega\right)\hat{Z}_r^* P_{\tilde{Y}_r^\perp}\right) + (p-1)\text{Trace}\left(P_{\tilde{Y}_r}\hat{Z}_r\, a\left(\tfrac{X_r}{\varepsilon},\omega\right)\hat{Z}_r^* P_{\tilde{Y}_r}\right) \\
&\geq (p-1)\text{Trace}\left(P_{\tilde{Y}_r^\perp}\hat{Z}_r\, a\left(\tfrac{X_r}{\varepsilon},\omega\right)\hat{Z}_r^* P_{\tilde{Y}_r^\perp} + P_{\tilde{Y}_r}\hat{Z}_r\, a\left(\tfrac{X_r}{\varepsilon},\omega\right)\hat{Z}_r^* P_{\tilde{Y}_r}\right) \\
&\geq (p-1)\text{Trace}\left(\hat{Z}_r\, a\left(\tfrac{X_r}{\varepsilon},\omega\right)\hat{Z}_r^*\right) \\
&\geq \underline{a}(p-1)\left\|\hat{Z}_r\right\|^2 \text{ by ellipticity of } a\ .
\end{aligned}$$

Putting all together, we obtain that for all $\delta > 0$, and all n,

$$\begin{aligned}
&E_x^{\varepsilon,\omega}\left[\left\|\tilde{Y}_{t\wedge T_n}\right\|^p + C(1-\delta)\int_{t\wedge T_n}^{T\wedge T_n}\left\|\tilde{Y}_r\right\|^{p-2}\left\|\hat{Z}_r\right\|^2 dr\right] \\
&\leq C E_x^{\varepsilon,\omega}\left[\left\|\tilde{Y}_{T\wedge T_n}\right\|^p + (1+\tfrac{1}{\delta})\int_{t\wedge T_n}^{T\wedge T_n}\left\|\tilde{Y}_r\right\|^p dr + \int_{t\wedge T_n}^{T\wedge T_n}\left\|\int_0^r F(u,0,0)du\right\|^p dr\right]
\end{aligned}$$

Choosing δ sufficiently small so that $1-\delta > 0$, the preceding inequality gives for all $t \in [0,T]$,

$$\begin{aligned}
E_G^{\varepsilon,\mu}\left[\left\|\tilde{Y}_{t\wedge T_n}\right\|^p\right] &+ E_G^{\varepsilon,\mu}\left[\int_{t\wedge T_n}^{T\wedge T_n}\left\|\tilde{Y}_r\right\|^{p-2}\left\|\hat{Z}_r\right\|^2 dr\right] \\
&\leq C E_G^{\varepsilon,\mu}\left[\left\|\tilde{Y}_{T\wedge T_n}\right\|^p\right] + C\int_0^T E_G^{\varepsilon,\mu}\left[\left\|\int_0^r F(u,0,0)\,du\right\|^p\right] dr \\
&\quad + C\int_t^T E_G^{\varepsilon,\mu}\left[\left\|\tilde{Y}_{r\wedge T_n}\right\|^p\right] dr
\end{aligned}$$

For all $t \in [0,T]$, Gronwall lemma then leads to

$$E_G^{\varepsilon,\mu}\left[\left\|\tilde{Y}_{t\wedge T_n}\right\|^p\right] \leq C(T)\left[E_G^{\varepsilon,\mu}\left[\left\|\tilde{Y}_{T\wedge T_n}\right\|^p\right] + \int_0^T E_G^{\varepsilon,\mu}\left[\left\|\int_0^r F(u,0,0)\,du\right\|^p\right] dr\right].$$

Let $T_\infty = \lim \nearrow T_n = \inf\{t \geq 0, \tilde{Y}_t = 0\}$. For all $t \in [0,T]$, $\tilde{Y}_{t\wedge T_n}$ converges a.s. to $\tilde{Y}_{t\wedge T_\infty}$ when n tends to ∞, and is dominated by $\sup_{0\leq t\leq T}\left\|\tilde{Y}_t\right\|$ which is in L^p by (33). By dominated convergence, letting n go to infinity in the preceding inequality, yields

$$E_G^{\varepsilon,\mu}\left[\left\|\tilde{Y}_{t\wedge T_\infty}\right\|^p\right] \leq C(T)E_G^{\varepsilon,\mu}\left[\left\|\tilde{Y}_{T\wedge T_\infty}\right\|^p + \int_0^T\left\|\int_0^r F(u,0,0)\,du\right\|^p dr\right]$$

But,

$$\left\|\tilde{Y}_{T\wedge T_\infty}\right\| = \left\|\tilde{Y}_T\right\| \ \mathbb{1}_{T\leq T_\infty} \leq \left\|\eta + \int_0^T F(u,0,0)\,du\right\|\ ,$$

so that for all $t \in [0,T]$,

$$\begin{aligned}
E_G^{\varepsilon,\mu}\left[\left\|\tilde{Y}_{t\wedge T_\infty}\right\|^p\right] &\leq C(T)E_G^{\varepsilon,\mu}\left[\left\|\eta + \int_0^T F(u,0,0)\,du\right\|^p\right] \\
&\quad + C(T)E_G^{\varepsilon,\mu}\left[\int_0^T\left\|\int_0^r F(u,0,0)\,du\right\|^p dr\right]
\end{aligned} \tag{36}$$

By lemma 13, the first term in the right-hand side converges to zero, and for all $r \in [0,T]$, $E_G^{\varepsilon,\mu}\left[\left\|\int_0^r F(u,0,0)\,du\right\|^p\right] \underset{\varepsilon\to 0}{\longrightarrow} 0$. Moreover,

$$\begin{aligned} E_G^{\varepsilon,\mu}\left[\left\|\int_0^r F(u,0,0)\,du\right\|^p\right] &\le T^{p-1} E_G^{\varepsilon,\mu}\left[\int_0^T \|F(u,0,0)\|^p\,du\right] \\ &\le C T^{p-1}\left(T + E_G^{\varepsilon,\mu}\left[\int_0^T \|\phi\|^{2p}\left(\frac{X_u}{\varepsilon},\omega\right)du\right]\right) \\ &\le C T^p\left(1+\left\langle \|\phi\|^{2p}\right\rangle_\mu\right) \end{aligned}$$

By dominated convergence, we conclude then that the second term in the right hand side of (36) converges also to zero. We have thus proved that for all $t \in [0,T]$,

$$E_G^{\varepsilon,\mu}\left[\left\|\tilde{Y}_{t\wedge T_\infty}\right\|^p\right] \underset{\varepsilon\to 0}{\longrightarrow} 0\ .$$

Taking $t = 0$, we get the desired conclusion.

(g) Regularization procedure. Let us assume now that h and H satisfy **(P 1-2)**. Since $H \in L^2(\mathbb{R}^d)^m$, we can approximate H in L^2-norm by a sequence H_n of functions in $\mathcal{C}_c^\infty(\mathbb{R}^d)^m$ (i.e. infinitely differentiable functions with compact support). Let $\rho : \mathbb{R}^d \times \mathbb{R}^m \times \mathbb{R}^{m\times d} \mapsto \mathbb{R}^+$ be a $\mathcal{C}_c^\infty$ function with support in $\mathcal{B}(0,1)$, and such that $\int \rho(x,y,z)\,dx\,dy\,dz = 1$. For all $n \in \mathbb{N}^*$, let $\rho_n(x,y,z) \triangleq n^{d+m+dm}\rho(n(x,y,z))$, and $h_n(x,y,z) \triangleq \rho_n \star h$. Then h_n is infinitely differentiable with bounded derivatives. The functions h_n satisfy assumptions **(P 1-(ii))** and **(P 1-(iii))** with the same constants K and λ as h. Moreover, for all $(x,y,z) \in \mathbb{R}^d \times \mathbb{R}^m \times \mathbb{R}^{m\times d}$

$$\begin{aligned} \|h_n(x,y,z) - h(x,y,z)\| &\le \sup_{\|(x,y,z)-(x',y',z')\|\le 1/n} \left\|h(x,y,z) - h(x',y',z')\right\| \\ &\triangleq \omega_n(h) \end{aligned} \tag{37}$$

Let (Y^n, Z^n) be the solution of the BSDE

$$Y_t^n = H_n(X_T) + \int_t^T h_n(X_r, Y_r^n, Z_r^n)\,dr - \int_t^T Z_r^n\,dM_r^X\ ,\ \ t \in [0,T]\ ,\ \ P_x^{\varepsilon,\omega}\text{–p.s.}\ ,$$

$$E_x^{\varepsilon,\omega}\left[\sup_{0\le t\le T}\left\|Y_t^n\right\|^2 + \int_0^T \left\|Z_r^n\right\|^2 dr\right] < \infty\ .$$

Usual computations (see for instance theorem 2.3 in [22]) lead to

$$\begin{aligned} &E_x^{\varepsilon,\omega}\left[\sup\nolimits_{0\le t\le T}\left\|Y_t^n - Y_t\right\|^2\right] \\ &\le C E_x^{\varepsilon,\omega}\left[\|H_n(X_T) - H(X_T)\|^2 + \int_0^T \|h_n(X_r,Y_r,Z_r) - h(X_r,Y_r,Z_r)\|^2\,dr\right], \end{aligned} \tag{38}$$

where the constant C depends only on the constants λ and K, and on T.

Uniform ellipticity of a implies that the law of X_T under $P_x^{\varepsilon,\omega}$ has a density $p_T^{\varepsilon,\omega}(x,y)$ with respect to Lebesgue measure. Moreover, $p_T^{\varepsilon,\omega}(x,y)$ satisfies the Aronson estimate

$$p_T^{\varepsilon,\omega}(x,y) \le \frac{M}{T^{d/2}} \exp\left(-\frac{\|x-y\|^2}{MT}\right) . \tag{39}$$

Therefore,

$$\begin{aligned} E_x^{\varepsilon,\omega}\left[\|H_n(X_T) - H(X_T)\|^2\right] &= \int \|H_N(y) - H(y)\|^2 \, p_T^{\varepsilon,\omega}(x,y)\,dy \\ &\le C\,\|H_n - H\|^2_{L^2(\mathbb{R}^d)} \ . \end{aligned}$$

It follows then from (38), (37) that there exists a constant C such that for all $\varepsilon > 0$ and all $n \in \mathbb{N}^*$,

$$E_x^{\varepsilon,\omega}\left[\sup_{0\le t\le T} \left\|Y_t^n - Y_t\right\|^2\right] \le C\left(\|H_n - H\|^2_{L^2(\mathbb{R}^d)} + \omega_n(h)\right) . \tag{40}$$

In the same way, let $\bar{h}_n(x,y,z) \stackrel{\triangle}{=} \langle h_n(x,y,z(\mathrm{Id}+\phi))\rangle_\phi$. Then $\bar{h}_n$ satisfies **(P 1-(ii))** and **(P 1-(iii))** with constants independent of n, and for all x, y, z,

$$\left\|\bar{h}_n(x,y,z) - \bar{h}(x,y,z)\right\| \le \omega_n(h) \ .$$

Therefore, if E_x^0 denotes the expectation under the law of a Brownian motion with covariance matrix $\bar{a}$, if $(\bar{Y}_n, \bar{Z}_n)$ is the solution of the BSDE

$$\bar{Y}_t^n = H_n(X_T) + \int_t^T \bar{h}_n(X_r, \bar{Y}_r^n, \bar{Z}_r^n)\,dr - \int_t^T \bar{Z}_r^n\,dM_r^X \ , \ \ t \in [0,T] \, , \ \ P_x^0 - \text{p.s.} \, ,$$

and if $(\bar{Y}, \bar{Z})$ is the solution of the BSDE

$$\bar{Y}_t = H(X_T) + \int_t^T \bar{h}(X_r, \bar{Y}_r, \bar{Z}_r)\,dr - \int_t^T \bar{Z}_r\,dM_r^X \ , \ \ t \in [0,T] \, , \ \ P_x^0 - \text{p.s.} \, ,$$

we get

$$E_x^0\left[\sup_{0\le t\le T} \left\|\bar{Y}_t^n - \bar{Y}_t\right\|^2\right] \le C\left(\|H_n - H\|^2_{L^2(\mathbb{R}^d)} + \omega_n(h)\right) . \tag{41}$$

Theorem 9 follows now from lemma 14, and estimates (40) and (41). □

5. Homogenization of elliptic systems

5.1. The problem and assumptions made on the non-linear term and the boundary condition

Let G be an open bounded domain of $\mathbb{R}^d$ with boundary ∂G, which is $\mathcal{C}^{2,\delta}$ for some $\delta \in]0,1]$. We are interested in this section in the homogenization of the elliptic system

$$\begin{cases} \mathcal{L}^{\varepsilon,\omega} v_k^\varepsilon + h_k(x, v^\varepsilon, \nabla v^\varepsilon) = 0\,, & x \in G\,,\ \ k \in \{1, \cdots, m\} \\ v^\varepsilon = H\,, & x \in \partial G \end{cases} \tag{42}$$

Assumptions made on the linear part are the same as for the parabolic problem, i.e. **(RM 1-4)** and **(L 1-3)**. For the non linear part, we assume that

(E 1) **(i)** $h : \mathbb{R}^d \times \mathbb{R}^m \times \mathbb{R}^{m\times d} \mapsto \mathbb{R}^m$ is uniformly continuous and bounded.

(ii) $\exists K > 0$ such that $\forall (x, y, z, z') \in \bar{G} \times \mathbb{R}^m \times \mathbb{R}^{m\times d} \times \mathbb{R}^{m\times d}$,

$$\left\| h(x, y, z) - h(x, y, z') \right\| \le K \left\| z - z' \right\| \ ;$$

(iii) $\exists \lambda \in \mathbb{R}$ such that for all $x \in \bar{G}$, all $y, y' \in \mathbb{R}^d$ and all $z \in \mathbb{R}^{m\times d}$,

$$(h(x, y, z) - h(z, y', z), y - y') \le \lambda \left\| y - y' \right\|^2$$

(iv) $\underline{a}\lambda \,[Q(b, d) \wedge 4 - 2] + K^2 < 0$.

Note that **(E-1-(iv))** implies that $\lambda < 0$.

(E 2) $H : \mathbb{R}^d \mapsto \mathbb{R}^m$ is a uniformly continuous function, which is in $H^1(\bar{G})$.

Under **(L 1-3)** and **(E 1-2)**, μ-a.s., and for all $\varepsilon > 0$, system (42) has a unique weak solution v^ε, i.e. a solution such that $v^\varepsilon \in H^1(G)$ and for all $\psi \in \mathcal{C}_c^\infty(G)$,

$$-\int_G a_{ij}(\frac{x}{\varepsilon}, \omega)\partial_i \psi \partial_j v^\varepsilon e^{-2V(x/\varepsilon,\omega)}\, dx + \int_G v(x) h(x, v^\varepsilon(x), \nabla v^\varepsilon(x))\, dx = 0\ .$$

Indeed, let $\mathcal{A}^{\varepsilon,\omega} : H_0^1(G; d\pi^{\varepsilon,\omega})^m \mapsto H^{-1}(G)^m$ be the operator defined by

$$\mathcal{A}^{\varepsilon,\omega}(u) = -\mathcal{L}^{\varepsilon,\omega} u - h(x, u, \nabla u)\ .$$

It follows from **(L 1)** and **(E 1)** that for all $u, v \in H_0^1(\mathbb{R}^d, d\pi^{\varepsilon,\omega})^m$,

$$\begin{aligned} &\langle \mathcal{A}^{\varepsilon,\omega}(u - v), u - v\rangle \\ &\quad = \tfrac{1}{2} \textstyle\int_G \mathbf{a}_{i,j} \left(\tfrac{x}{\varepsilon}, \omega\right) \partial_i (u - v)(x) \partial_j (u - v)(x)\, d\pi^{\varepsilon,\omega}(x) \\ &\qquad - \textstyle\int_G \left(h(x, u(x), \nabla u(x)) - h(x, v(x), \nabla v(x)), u(x) - v(x)\right)\, d\pi^{\varepsilon,\omega}(x) \\ &\quad \ge \tfrac{\underline{a}}{2} \left\|\nabla(u - v)\right\|^2_{L^2(G,d\pi^{\varepsilon,\omega})} - \lambda \left\|u - v\right\|^2_{L^2(G,d\pi^{\varepsilon,\omega})} \\ &\qquad - K \left\|u - v\right\|_{L^2(G,d\pi^{\varepsilon,\omega})} \left\|\nabla(u - v)\right\|_{L^2(G,d\pi^{\varepsilon,\omega})}\ . \end{aligned}$$

The matrix $\begin{pmatrix} -\lambda & -K/2 \\ -K/2 & \underline{a}/2 \end{pmatrix}$ is strictly positive, as soon as $2\underline{a}\lambda + K^2 < 0$. Since $2\underline{a}\lambda + K^2 \le \underline{a}\lambda\,[Q(b, d) \wedge 4 - 2] + K^2$, assumption **(E 1-(iv))** ensures existence of a constant $C > 0$, such that μ-a.s., $\forall \varepsilon > 0$,

$$\left\langle \mathcal{A}^{\varepsilon,\omega}(u - v), u - v\right\rangle \ge C \left\|u - v\right\|^2_{H_0^1(G,d\pi^{\varepsilon,\omega})}\ . \tag{43}$$

The existence and uniqueness of a weak solution is then a well-known fact (see for instance theorem 26.A of [25]).

Remark 3. Note that **(E 1.(iv))** is a too stringent assumption to ensure existence and uniqueness of weak solutions of system (42) on one hand, and of BSDE's with random terminal time on the other hand. A more natural assumption should be $2\underline{a}\lambda + K^2 < 0$, as it appears precedingly. We are however unable to prove the homogenization result under this assumption.

5.2. *The homogenized equation*

Let $\bar{a}$ and $\bar{h}$ be defined by (20) and (21).

Lemma 15. *Assume that* **(RM 1..4)**, **(L 1..3)**, *and* **(E 1-2)** *are satisfied. Then, the system*

$$\begin{cases} \frac{1}{2}\bar{a}_{i,j}\partial^2_{i,j}v^0 + \bar{h}(x, v^0, \nabla v^0) = 0\,, & x \in G\,, \\ v^0 = H\,, & x \in \partial G\,. \end{cases} \tag{44}$$

has a unique weak solution v^0 in $H^1(G)$. If we assume moreover

(E 1') $h : \bar{G} \times \mathbb{R}^d \times \mathbb{R}^{m\times d} \mapsto \mathbb{R}^m$ *is Lipschitz.*
(E 2') $H : \bar{G} \mapsto \mathbb{R}^m$ *is of class $\mathcal{C}^{2,\delta}$.*

then, v^0 is itself of class $\mathcal{C}^{2,\delta}$.

Proof of Lemma 15. Let $\mathcal{A}^0 : H^1_0(G)^m \mapsto H^{-1}(G)^m$ be the non linear operator defined by

$$\mathcal{A}^0(u) = -\frac{1}{2}\bar{a}_{i,j}\partial^2_{i,j}u - \bar{h}(x, u(x), \nabla u(x))\ .$$

From the definition of $\overline{a}$ and $\overline{h}$, we get that

$$\begin{aligned} &\left\langle \mathcal{A}^0(u-v), u-v \right\rangle \\ &= \tfrac{1}{2}\left\langle \int_G \mathrm{Trace}\left[\nabla(u-v)(x)(\mathrm{Id}+\phi)\mathbf{a}(\mathrm{Id}+\phi)^*\nabla(u-v)(x)^*\right]\,dx\right\rangle_\pi \\ &\quad -\left\langle \int_G \left(h(x,u,\nabla u(\mathrm{Id}+\phi)) - h(x,v,\nabla v(\mathrm{Id}+\phi)), u-v\right)\,dx\right\rangle_\pi \\ &\geq \tfrac{\underline{a}}{2}\,\|\nabla(u-v)(x)(\mathrm{Id}+\phi)(\omega)\|^2_{L^2(G\times\Omega,dx\otimes d\pi)} - \lambda\,\|u-v\|^2_{L^2(G,dx)} \\ &\quad -K\,\|u-v\|_{L^2(G,dx)}\,\|\nabla(u-v)(x)(\mathrm{Id}+\phi)(\omega)\|_{L^2(G\times\Omega,dx\otimes d\pi)} \text{ by } \textbf{(E 1)}, \\ &\geq C\left(\|u-v\|^2_{L^2(G,dx)} + \|\nabla(u-v)(x)(\mathrm{Id}+\phi)(\omega)\|^2_{L^2(G\times\Omega,dx\otimes d\pi)}\right), \end{aligned}$$

as in the proof of (43).

But,

$$\begin{aligned} &\|\nabla(u-v)(x)(\mathrm{Id}+\phi)(\omega)\|^2_{L^2(G\times\Omega,dx\otimes d\pi)} \\ &\qquad = \mathrm{Trace}\left[\left(\int_G \nabla(u-v)(x)^*\nabla(u-v)(x)\,dx\right)\langle(\mathrm{Id}+\phi)(\mathrm{Id}+\phi)^*\rangle_\pi\right]\ . \end{aligned}$$

It has already been proved in lemma 8 that the matrix $\langle(\mathrm{Id}+\phi)(\mathrm{Id}+\phi)^*\rangle_\pi$ is strictly positive. Therefore,

$$\begin{aligned}&\|\nabla(u-v)(x)(\mathrm{Id}+\phi)(\omega)\|^2_{L^2(G\times\Omega,dx\otimes d\pi)}\\ &\quad\geq \lambda_{min}\left(\langle(\mathrm{Id}+\phi)(\mathrm{Id}+\phi)^*\rangle_\pi\right)\mathrm{Trace}\left(\int_G \nabla(u-v)(x)^*\nabla(u-v)(x)\,dx\right)\\ &\quad= \lambda_{min}\left(\langle(\mathrm{Id}+\phi)(\mathrm{Id}+\phi)^*\rangle_\pi\right)\|\nabla(u-v)\|^2_{L^2(G,dx)}\end{aligned}$$

Thus, $\mathcal{A}^0$ is strongly monotone and the existence and uniqueness of weak solutions is proved. The regularity of v^0 under **(E -1')** and **(E 2')** follows then from the regularity of the data (see theorem 5.1, chapter 8 of [14]). □

5.3. The homogenization result in elliptic case

Theorem 16. *Assume that* **(RM 1..4)**, **(L 1..3)** *and* **(E 1-2)** *are satisfied. There exists* $p\in]1,2[$ *such that*

$$\left\langle\int_G \left\|v^\varepsilon(x)-v^0(x)\right\|^p\,dx\right\rangle_\mu \underset{\varepsilon\to 0}{\longrightarrow} 0\ . \tag{45}$$

5.4. Proof of Theorem 16

It follows the same lines as the proof of theorem 9.

(a) The non-linear Feynman-Kac formula. For all x continuous path from $\mathbb{R}^+$ to $\mathbb{R}^d$, let

$$\theta_G(x)\overset{\triangle}{=}\inf\{t\geq 0, x(t)\notin G\}\ . \tag{46}$$

Since **a** is uniformly elliptic, and G is bounded, it is well-known that μ-a.s., for all $\varepsilon>0$ and all $x\in\bar{G}$,

$$P_x^{\varepsilon,\omega}[\theta_G(X)<\infty]=1\ .$$

Moreover, the regular points of X for G are the same as those of Brownian motion, i.e. all points are regular, since ∂G is assumed to be smooth enough.

It is proved in [22] that under **(E 1-2)**, there is a unique pair $(Y_t,Z_t)_{t\in\mathbb{R}^+}$ of $\mathcal{F}^X$-progressively measurable processes such that μ-a.s., for all $\varepsilon>0$, for q.e. $x\in\bar{G}$,

$$P_x^{\varepsilon,\omega}\left[Z_t\,\mathbb{1}_{t>\theta_G(X)}=0\right]=1\ ; \tag{47}$$

$$P_x^{\varepsilon,\omega}\left[Y_t\,\mathbb{1}_{t\geq\theta_G(X)}=H(X_{\theta_G(X)})\,\mathbb{1}_{t\geq\theta_G(X)}\right]=1\ ; \tag{48}$$

for all t,T such that $0\leq t\leq T$, $P_x^{\varepsilon,\omega}$-p.s.,

$$Y_t=Y_T+\int_{t\wedge\theta_G(X)}^{T\wedge\theta_G(X)} h(X_r,Y_r,Z_r)\,dr-\int_{t\wedge\theta_G(X)}^{T\wedge\theta_G(X)} Z_r\,dM_r^X\ ; \tag{49}$$

$$E_x^{\varepsilon,\omega}\left[\sup_{t\geq 0}\|Y_t\|^2+\int_0^\infty \|Z_r\|^2\,dr\right]<\infty\ . \tag{50}$$

Moreover, using the results of [3] or [16] in the parabolic case, it can be proved that $P_x^{\varepsilon,\omega}$-p.s., $Y_0 = v^\varepsilon(x)$, and we are going to prove that

$$\left\langle \int_G \left\| Y_0 - v^0(x) \right\|^p \, dx \right\rangle_\mu \underset{\varepsilon \to 0}{\longrightarrow} 0 \, .$$

(b) Change on Y. We are first going to prove theorem 16 in the case where the coefficients h and H are smooth; i.e. under additional assumptions **(E 1')** and **(E 2')**.

Lemma 17. *Assume that* **(RM 1..3)**, **(L 1..3)**, **(E 1-2)** *and* **(E 1'-2')** *hold. Let* $\theta \triangleq \theta_G(X)$ *and* $\hat\theta \triangleq \theta_G(\hat X)$, *where* $\hat X$ *is the process defined in lemma 10. Let*

$$\begin{cases} \hat Y_t \triangleq Y_t - v^0(\hat X_{t\wedge\theta\wedge\hat\theta}) \\ \hat Z_t \triangleq \left[Z_t - \mathbb{1}_{t\le\hat\theta} \nabla v^0(\hat X_t)(Id+\phi)\left(\frac{X_t}{\varepsilon}, \omega\right)\right] \mathbb{1}_{t\le\theta} \end{cases} \tag{51}$$

Then, μ*-a.s., for q.e.* $x \in \mathbb{R}^d$, *for all* $\varepsilon > 0$, *for all* $0 \le t \le T$,

$$P_x^{\varepsilon,\omega}\left[\hat Z_t \, \mathbb{1}_{t>\theta} = 0\right] = 1 \; ; \tag{52}$$

$$P_x^{\varepsilon,\omega}\left[\hat Y_t \, \mathbb{1}_{t\ge\theta} = \zeta \, \mathbb{1}_{t\ge\theta}\right] = 1 \; . \tag{53}$$

$$\hat Y_t = \hat Y_T + \int_{t\wedge\theta}^{T\wedge\theta} G(r, \hat Y_r, \hat Z_r)\, dr - \int_{t\wedge\theta}^{T\wedge\theta} \hat Z_r \, dM_r^X \; , \tag{54}$$

where

$$\begin{cases} \zeta & \triangleq \; H(X_\theta) - v^0(\hat X_{\theta\wedge\hat\theta}) \\ G(r,y,z) & \triangleq \; h\left(X_{r\wedge\theta}, y + v^0(\hat X_{r\wedge\hat\theta}), z + \mathbb{1}_{r\le\hat\theta}\nabla v^0(\hat X_r)\left(Id + \phi(\frac{X_r}{\varepsilon},\omega)\right)\right) \\ & \quad + \frac{1}{2}\, \mathbb{1}_{r\le\hat\theta}\left[(Id+\phi)\mathbf{a}(Id+\phi)^*\right]_{i,j}\left(\frac{X_r}{\varepsilon},\omega\right) \partial^2_{i,j} v^0(\hat X_r) \end{cases} \tag{55}$$

Proof of Lemma 17. It follows from Itô's formula applied to the smooth function v^0, and of definitions of the processes $\hat Y$ and $\hat Z$. □

(c) An ergodic lemma.

Lemma 18. *Assume that* **(RM 1..3)**, **(L 1..3)** *hold. Let* $F : \bar G \times \Omega \to \mathbb{R}$ *be a measurable function such that*

(i) *There exists* $\xi \in L^1(\Omega, \mu)$ *such that* μ*-a.s.,*
- *for all* $x \in \bar G$, $|F(x,\omega)| \le \xi(\omega)$;
- *for all* $\delta > 0$, $\exists \alpha > 0$ *such that* $\forall x, x' \in \bar G$,

$$\left\| x - x' \right\| \le \alpha \Rightarrow |F(x,\omega) - F(x',\omega)| \le \xi(\omega)\delta \; ;$$

(ii) *For all* $x \in \bar G$, $\int F(x,\omega)\, d\pi = 0$.

Then, for all $T > 0$, for all $\beta > 0$,

$$P_G^{\varepsilon,\mu}\left[\left|\int_0^{T\wedge\theta} F\left(\hat{X}_{r\wedge\hat{\theta}}, \tau_{\frac{X_r}{\varepsilon}}\omega\right) dr\right| \geq \beta\right] \underset{\varepsilon\longrightarrow 0}{\longrightarrow} 0.$$

Proof of Lemma 18. First of all, note that since all points are regular for $(X, P_G^{\varepsilon,\mu})$, the map $x \in \mathcal{C}(\mathbb{R}^+, \mathbb{R}^d) \mapsto \theta_G(x)$ is $P_G^{\varepsilon,\mu}$-a.s. continuous. It follows then from lemma 10 that under $P_G^{\varepsilon,\mu}$, $(X, \hat{X}, \theta, \hat{\theta})$ converges in law (in $\mathcal{C}(\mathbb{R}^+, \mathbb{R}^d) \times \mathcal{C}(\mathbb{R}^+, \mathbb{R}^d) \times \mathbb{R}^+ \times \mathbb{R}^+$) to the law of $(B, B, \theta_G(B), \theta_G(B))$, where B is a d-dimensional Brownian motion with covariance matrix $\bar{a}$. Thus, $((\hat{X}_{t\wedge\hat{\theta}}, t \in [0, T]), \theta)$ converges in law under $P_G^{\varepsilon,\mu}$.

Let $\delta > 0$ be fixed. Let K_δ be a compact subset of $\mathcal{C}([0, T], \bar{G})$ and $T_\delta \in \mathbb{R}^+$ be such that

$$\inf_{\varepsilon>0} P_G^{\varepsilon,\mu}\left[\hat{X}_{.\wedge\hat{\theta}} \in K_\delta; \theta \leq T_\delta\right] \geq 1 - \delta\ .$$

Let us cover $K_\delta \times [0, T_\delta]$ by balls of radius α, where $\alpha \in]0, 1]$ is such that $\|x - x'\| \leq \alpha$ imply $|F(x, \omega) - F(x', \omega)| \leq \xi(\omega)\delta$.

$$K_\delta \times [0, T_\delta] \subset \bigcup_{i=1}^{N_\alpha} \tilde{\mathcal{B}}_i \times \bigcup_{j=1}^{M_\alpha} I_j\ ,$$

where $\tilde{\mathcal{B}}_i \subset \mathcal{B}(x_i, \alpha)$ are disjoints, and $I_j \stackrel{\triangle}{=} [t_j; t_{j+1}[$ are disjoints intervals of length α. Now,

$$\begin{aligned}
&P_G^{\varepsilon,\mu}\left[\left|\int_0^{T\wedge\theta} F\left(\hat{X}_{r\wedge\hat{\theta}}, \tau_{\frac{X_r}{\varepsilon}}\omega\right) dr\right| \geq \beta\right]\\
&\leq P_G^{\varepsilon,\mu}\left[\hat{X}_{.\wedge\hat{\theta}} \notin K_\delta \text{ or } \theta \geq T_\delta\right] + \sum_{i,j} P_G^{\varepsilon,\mu}\left[\left|\int_0^{T\wedge t_j} F\left(x_i(r), \tau_{\frac{X_r}{\varepsilon}}\omega\right) dr\right| \geq \frac{\beta}{2}\right]\\
&+ \sum_{i,j} P_G^{\varepsilon,\mu}\left[\left|\int_0^{T\wedge\theta} F\left(\hat{X}_{r\wedge\hat{\theta}}, \tau_{\frac{X_r}{\varepsilon}}\omega\right) dr - \int_0^{T\wedge t_j} F\left(x_i(r), \tau_{\frac{X_r}{\varepsilon}}\omega\right) dr\right| \geq \frac{\beta}{2}\ ;\right.\\
&\qquad\qquad\qquad\qquad\qquad\qquad \left.\hat{X}_{.\wedge\hat{\theta}} \in \tilde{\mathcal{B}}_i \cap K_\delta\ ;\ \theta \in I_j\right]\\
&\leq \delta + \sum_{i,j} P_G^{\varepsilon,\mu}\left[\left|\int_0^{T\wedge t_j} F\left(x_i(r), \tau_{\frac{X_r}{\varepsilon}}\omega\right) dr\right| \geq \frac{\beta}{2}\right]\\
&\quad + P_G^{\varepsilon,\mu}\left[\int_0^T \xi\left(\tau_{\frac{X_r}{\varepsilon}}\omega\right) \geq \frac{\beta}{4\delta}\right] + \sum_{j; t_j\leq T} P_G^{\varepsilon,\mu}\left[\int_{t_j\wedge T}^{t_{j+1}\wedge T} \xi\left(\tau_{\frac{X_r}{\varepsilon}}\omega\right) dr \geq \frac{\beta}{4}\right]\\
&\leq \delta + \sum_{i,j} P_G^{\varepsilon,\mu}\left[\left|\int_0^{T\wedge t_j} F\left(x_i(r), \tau_{\frac{X_r}{\varepsilon}}\omega\right) dr\right| \geq \frac{\beta}{2}\right]\\
&\quad + C\frac{\delta T}{\beta}\langle\xi\rangle_\mu + \sum_{j; t_j\leq T} P_G^{\varepsilon,\mu}\left[\int_{t_j\wedge T}^{t_{j+1}\wedge T} \xi\left(\tau_{\frac{X_r}{\varepsilon}}\omega\right) dr \geq \frac{\beta}{4}\right]
\end{aligned}$$

Using ergodic theorem and lemma 12, we get

$$\limsup_{\varepsilon\to 0} P_G^{\varepsilon,\mu}\left[\left|\int_0^{T\wedge\theta} F\left(\hat{X}_{r\wedge\hat{\theta}}, \tau_{\frac{X_r}{\varepsilon}}\omega\right) dr\right| \geq \beta\right] \leq \delta + C\frac{\delta T}{\beta}\langle\xi\rangle_\mu + C\frac{T}{\alpha}\mathbb{1}_{\alpha\langle\xi\rangle_\pi \geq \frac{\beta}{4}}.$$

Taking $\alpha < \frac{\beta}{4\langle\xi\rangle_\pi}$, and letting δ go to 0, yield the result of Lemma 18 . □

(d) Some L^p-estimates.

Lemma 19. *Assume that* **(RM 1..3)**, **(L 1..3)**, **(E 1-2)** *and* **(E 1'-2')** *hold. Let* ζ *and* G *be as in lemma 17. Then, for all* $0 < t < T$,

$$\forall p > 1, \qquad \left\langle \int_G E_x^{\varepsilon,\omega}\left[\|\zeta\|^p\right] dx \right\rangle_\mu \underset{\varepsilon\to 0}{\longrightarrow} 0 \; ; \tag{56}$$

$$\forall p \in]1, \frac{1}{2} Q(b,d)[, \qquad \left\langle \int_G E_\omega^{\varepsilon,x}\left[\left\| \int_{t\wedge\theta}^{T\wedge\theta} G(r,0,0)\, dr \right\|^p\right] dx \right\rangle_\mu \underset{\varepsilon\to 0}{\longrightarrow} 0 \; . \tag{57}$$

Proof of Lemma 19. Since $(X_\theta, \hat{X}_{\theta\wedge\hat\theta})$ converges in law under $P_G^{\varepsilon,\mu}$ to the law of $(B_{\theta_G(B)}, B_{\theta_G(B)})$, continuity and boundedness of H and v^o imply that

$$E_G^{\varepsilon,\mu}\left[\|\zeta\|^p\right] \underset{\varepsilon\to 0}{\longrightarrow} \int_G E_x\left[\left\| H(B_{\tau_G(B)}) - v^0(B_{\tau_G(B)})\right\|^p\right] \frac{dx}{|G|} = 0 \; ,$$

since $v^0 = H$ on ∂G.

Let us now prove (57). Using equation (44) satisfied by v^0, $G(r,0,0)$ can be rewritten as $G(r,0,0) = \sum_{i=1}^5 G_i(r)$, where

$$\begin{aligned}
G_1(r) &\stackrel{\triangle}{=} h\left(X_{r\wedge\theta}, v^0(\hat{X}_{r\wedge\hat\theta}),\; \mathbb{1}_{r\le\hat\theta}\nabla v^0(\hat{X}_r)(\mathrm{Id}+\phi)(\tfrac{X_r}{\varepsilon},\omega)\right) \\
&\qquad -h\left(\hat{X}_{r\wedge\hat\theta}, v^0(\hat{X}_{r\wedge\hat\theta}),\; \mathbb{1}_{r\le\hat\theta}\nabla v^0(\hat{X}_r)(\mathrm{Id}+\phi)(\tfrac{X_r}{\varepsilon},\omega)\right) \; , \\
G_2(r) &\stackrel{\triangle}{=} h\left(\hat{X}_{r\wedge\hat\theta}, v^0(\hat{X}_{r\wedge\hat\theta}),\; \mathbb{1}_{r\le\hat\theta}\nabla v^0(\hat{X}_r)(\mathrm{Id}+\phi)(\tfrac{X_r}{\varepsilon},\omega)\right) \\
&\qquad -h\left(\hat{X}_{r\wedge\hat\theta}, v^0(\hat{X}_{r\wedge\hat\theta}), \nabla v^0(\hat{X}_{r\wedge\hat\theta})(\mathrm{Id}+\phi)(\tfrac{X_r}{\varepsilon},\omega)\right) \; , \\
G_3(r) &\stackrel{\triangle}{=} h\left(\hat{X}_{r\wedge\hat\theta}, v^0(\hat{X}_{r\wedge\hat\theta}), \nabla v^0(\hat{X}_{r\wedge\hat\theta})(\mathrm{Id}+\phi)(\tfrac{X_r}{\varepsilon},\omega)\right) \\
&\qquad -\bar{h}\left(\hat{X}_{r\wedge\hat\theta}, v^0(\hat{X}_{r\wedge\hat\theta}), \nabla v^0(\hat{X}_{r\wedge\hat\theta})\right) \; , \\
G_4(r) &\stackrel{\triangle}{=} \tfrac{1}{2}\left[(\mathrm{Id}+\phi)a(\mathrm{Id}+\phi)^* - \bar{a}\right]_{i,j}(\tfrac{X_r}{\varepsilon},\omega)\; \partial^2_{i,j} v^0(\hat{X}_{r\wedge\hat\theta}) \; , \\
G_5(r) &\stackrel{\triangle}{=} -\tfrac{1}{2}\left[(\mathrm{Id}+\phi)a(\mathrm{Id}+\phi)^*\right]_{i,j}(\tfrac{X_r}{\varepsilon},\omega)\; \partial^2_{i,j} v^0(\hat{X}_{r\wedge\hat\theta})\, \mathbb{1}_{\hat\theta<r} \; .
\end{aligned}$$

Treatment of G_1: It follows from the inequality

$$\left\|h(x,y,z) - h(x',y,z)\right\| \le C\left(\left\|x-x'\right\| \wedge \|h\|_\infty\right) \; , \forall x, x', y, z \; ,$$

and from the convergence in law of $X_{.\wedge\theta} - \hat{X}_{.\wedge\hat\theta}$ to 0, that

$$E_G^{\varepsilon,\mu}\left[\left\|\int_{t\wedge\theta}^{T\wedge\theta} G_1(r)\, dr\right\|^p\right] \le C T^p E_G^{\varepsilon,\mu}\left[\sup_{r\in[0,T]}\left\|X_{r\wedge\theta} - \hat{X}_{r\wedge\hat\theta}\right\|^p \wedge \|h\|_\infty^p\right] \underset{\varepsilon\to 0}{\longrightarrow} 0.$$

Treatment of G_2: Since $\|G_2(r)\| \le 2\|h\|_\infty\; \mathbb{1}_{\hat\theta<r}$, and $(\theta,\hat\theta)$ converges in law to $(\theta_G(B), \theta_G(B))$,

$$E_G^{\varepsilon,\mu}\left[\left\|\int_{t\wedge\theta}^{T\wedge\theta} G_2(r)\, dr\right\|^p\right] \le C\,\|h\|_\infty^p\, E_G^{\varepsilon,\mu}\left[\left|\theta\wedge T - \hat\theta\wedge T\right|^p\right] \underset{\varepsilon\to 0}{\longrightarrow} 0 \; .$$

Treatment of G_3: $G_3(r) = \tilde{G}_3(\hat{X}_{r\wedge\hat{\theta}}, \tau_{\frac{X_r}{\varepsilon}}\omega)$, where

$$\tilde{G}_3(x,\omega) = h(x, v^0(x), \nabla v^o(x)(\mathrm{Id}+\phi)(\omega)) - \bar{h}(x, v^0(x), \nabla v^0(x))$$

satisfies condition (i) and (ii) of lemma 18 thanks to the smoothness properties of h and v^0, and boundedness of h. By lemma 18, $\int_{t\wedge\theta}^{T\wedge\theta} G_3(r)\,dr$ converges to 0 in $P_G^{\varepsilon,\mu}$-probability, when $\varepsilon \longrightarrow 0$. Moreover, $\int_{t\wedge\theta}^{T\wedge\theta} G_3(r)\,dr$ is uniformly bounded in ε, so that this convergence takes also place in all $L^p(P_G^{\varepsilon,\mu})$.
Treatment of G_4: $G_4(r) = \tilde{G}_4(\hat{X}_{r\wedge\hat{\theta}}, \tau_{\frac{X_r}{\varepsilon}}\omega)$, where

$$\tilde{G}_4(x,\omega) = \frac{1}{2}\left[(\mathrm{Id}+\phi)\mathbf{a}(\mathrm{Id}+\phi)^* - \bar{a}\right]_{i,j}(\omega)\ \partial^2_{i,j}v^0(x)$$

satisfies condition (i) and (ii) of lemma 18 since $\partial^2_{i,j}v^0$ are Hölder and bounded. By lemma 18, $\int_{t\wedge\theta}^{T\wedge\theta} G_4(r)\,dr$ converges to 0 in $P_G^{\varepsilon,\mu}$-probability, when $\varepsilon \longrightarrow 0$. Moreover, since $\phi \in L^{2p}(\mu)$ for all $p \in]1, \frac{1}{2}Q(b,d)[$, $\int_{t\wedge\theta}^{T\wedge\theta} G_4(r)\,dr$ is uniformly bounded in $L^p(P_G^{\varepsilon,\mu})$ for all $p \in]1, \frac{1}{2}Q(b,d)[$, so that this convergence in probability takes also place in $L^p(P_G^{\varepsilon,\mu})$, for all $p \in]1, \frac{1}{2}Q(b,d)[$.
Treatment of G_5: For all $p \in]1, \frac{1}{2}Q(b,d)[$,

$$\begin{aligned}
&E_G^{\varepsilon,\mu}\left[\left\|\int_{t\wedge\theta}^{T\wedge\theta} G_5(r)\,dr\right\|\right]\\
&\le C E_G^{\varepsilon,\mu}\left[\int_0^T \left\|\mathrm{Id}+\phi(\tau_{\frac{X_r}{\varepsilon}}\omega)\right\|^{2p} dr\right]^{1/p} E_G^{\varepsilon,\mu}\left[\int_0^T \mathbb{1}_{\hat{\theta}<r\le\theta}\,dr\right]^{1/q}\\
&\le C\left\langle\|\mathrm{Id}+\phi\|^{2p}\right\rangle_\mu^{1/p} E_G^{\varepsilon,\mu}\left[\left|T\wedge\theta - T\wedge\hat{\theta}\right|\right]^{1/q}\\
&\underset{\varepsilon\to 0}{\longrightarrow} 0 \text{ since } (\theta,\hat{\theta}) \text{ converges in law to } (\theta_G(B), \theta_G(B)).
\end{aligned}$$

Moreover,

$$\begin{aligned}
E_G^{\varepsilon,\mu}\left[\left\|\int_{t\wedge\theta}^{T\wedge\theta} G_5(r)\,dr\right\|^p\right] &\le C E_G^{\varepsilon,\mu}\left[\int_0^T \left\|\mathrm{Id}+\phi(\tau_{\frac{X_r}{\varepsilon}})\right\|^{2p} dr\right]\\
&\le C\left\langle\|\mathrm{Id}+\phi\|^{2p}\right\rangle_\mu .
\end{aligned}$$

Therefore, $\int_{t\wedge\theta}^{T\wedge\theta} G_5(r)\,dr$ converges to 0 in $L^p(P_G^{\varepsilon,\mu})$ for all $p \in]1, \frac{1}{2}Q(b,d)[$. □

(e) Conclusion in the case of smooth *h* and *H*.

Lemma 20. *Assume that* **(RM 1..3)**, **(L 1..3)**, **(E 1-2)** *and* **(E 1'-2')** *hold. Then, for all* $p \in]1, \frac{1}{2}Q(b,d)\wedge 2[$,

$$\left\langle \int_G \left\|v^\varepsilon(x) - v^0(x)\right\|^p dx \right\rangle_\mu \underset{\varepsilon\to 0}{\longrightarrow} 0\ .$$

Proof of Lemma 20. Let

$$\begin{aligned}\tilde{Y}_t &\stackrel{\triangle}{=} \hat{Y}_t + \int_0^{t\wedge\theta} G(r,0,0)\,dr \\ &= \tilde{Y}_T + \int_{t\wedge\theta}^{T\wedge\theta} \left(G(r,\hat{Y}_r,\hat{Z}_r) - G(r,0,0)\right) dr - \int_{t\wedge\theta}^{T\wedge\theta} \hat{Z}_r\, dM_r^X \ .\end{aligned}$$

We are going to prove that $E_G^{\varepsilon,\mu}\left[\left\|\tilde{Y}_0\right\|^p\right] \underset{\varepsilon\to 0}{\longrightarrow} 0$.

As a first step in this direction, we establish some uniform estimates on the process $\hat{Y}$. Standard results on BSDE's with random terminal time (see for instance theorem 4.1 in [22]) give

$$E_x^{\varepsilon,\omega}\left[\sup_{0\le t}\left\|\hat{Y}_t\right\|^2\right] \le C E_x^{\varepsilon,\omega}\left[\|H(X_\theta)\|^2 + \int_0^\theta \|h(X_r,0,0)\|^2\, dr\right] ,$$

where the constant C does not depend on ε (it depends only on $\underline{a}$, λ and K). Since v^0,H, and h are bounded, this leads to

$$\sup_{\varepsilon>0} E_G^{\varepsilon,\mu}\left[\sup_{t\ge 0}\left\|\hat{Y}_t\right\|^2\right] \le C\left(1 + \sup_{\varepsilon>0} E_G^{\varepsilon,\mu}(\theta)\right) .$$

Using uniform ellipticity of $\mathbf{a}$, it can be proved that

$$\sup_{\varepsilon>0} E_G^{\varepsilon,\mu}(\theta) < \infty \ , \tag{58}$$

so that

$$\sup_{\varepsilon>0} E_G^{\varepsilon,\mu}\left[\sup_{t\ge 0}\left\|\hat{Y}_t\right\|^2\right] < \infty \ . \tag{59}$$

Now, exactly the same computations as in the proof of lemma 14 yield for all $p < \frac{1}{2}Q(b,\delta)$, $1 < p \le 2$, for all $0 < t < T$,

$$\begin{aligned}&E_G^{\varepsilon,\mu}\left[\left\|\tilde{Y}_{t\wedge T_n}\right\|^p + \frac{\underline{a}p(p-1)}{2}\int_{t\wedge T_n\wedge\theta}^{T\wedge T_n\wedge\theta}\left\|\tilde{Y}_r\right\|^{p-2}\left\|\hat{Z}_r\right\|^2 dr\right] \\ &\le E_G^{\varepsilon,\mu}\left[\left\|\tilde{Y}_{T\wedge T_n}\right\|^p + p\int_{t\wedge T_n\wedge\theta}^{T\wedge T_n\wedge\theta}\left\|\tilde{Y}_r\right\|^{p-2}\left(\tilde{Y}_r, G(r,\hat{Y}_r,\hat{Z}_r) - G(r,0,0)\right) dr\right]\end{aligned} \tag{60}$$

But,

$$\begin{aligned}&\left(\tilde{Y}_r, G(r,\hat{Y}_r,\hat{Z}_r) - G(r,0,0)\right) \\ &= \left(\tilde{Y}_r, G(r,\hat{Y}_r,\hat{Z}_r) - G\left(r, -\int_0^{r\wedge\theta} G(u,0,0)\,du, \hat{Z}_r\right)\right) \\ &\quad + \left(\tilde{Y}_r, G\left(r, -\int_0^{r\wedge\theta} G(u,0,0)\,du, \hat{Z}_r\right) - G(r,0,0)\right) \\ &\le \lambda\left\|\tilde{Y}_r\right\|^2 + K\left\|\tilde{Y}_r\right\|\left\|\hat{Z}_r\right\| + K_y\left\|\tilde{Y}_r\right\|\left\|\int_0^{r\wedge\theta} G(u,0,0)\,du\right\|\end{aligned}$$

where K_y denotes the Lipschitz constant of G in the variable y. Young inequality then yields for all $\alpha_1, \alpha_2 > 0$, for all $0 < t < T$ and all $p < \frac{1}{2}Q(b,\delta)$, $1 < p \leq 2$,

$$E_G^{\varepsilon,\mu}\left[\left\|\tilde{Y}_{t\wedge T_n}\right\|^p\right] + \left(\frac{\underline{a}p(p-1)}{2} - \frac{pK\alpha_1}{2}\right) E_G^{\varepsilon,\mu}\left[\int_{t\wedge T_n\wedge\theta}^{T\wedge T_n\wedge\theta}\left\|\tilde{Y}_r\right\|^{p-2}\left\|\hat{Z}_r\right\|^2 dr\right]$$
$$\leq E_G^{\varepsilon,\mu}\left[\left\|\tilde{Y}_{T\wedge T_n}\right\|^p\right] + \left(p\lambda + \frac{pK}{2\alpha_1} + \frac{pK_y\alpha_2^q}{q}\right) E_G^{\varepsilon,\mu}\left[\int_{t\wedge T_n\wedge\theta}^{T\wedge T_n\wedge\theta}\left\|\tilde{Y}_r\right\|^p dr\right]$$
$$+\frac{K_y}{\alpha_2^p} E_G^{\varepsilon,\mu}\left[\int_{t\wedge T_n\wedge\theta}^{T\wedge T_n\wedge\theta}\left\|\int_0^{r\wedge\theta} G(u,0,0)\,du\right\|^p\right]$$

By **(E 1-(iv))**, $1 - \frac{K^2}{2\underline{a}\lambda} < \frac{1}{2}Q(b,d) \wedge 2$. Therefore, for all $p \in]1 - \frac{K^2}{2\underline{a}\lambda}; \frac{1}{2}Q(b,d)[$, $p \leq 2$, we get $2(p-1)\underline{a}\lambda + K^2 < 0$. For such a p, one can then choose $\alpha_1, \alpha_2 > 0$ such that

$$\begin{cases} (p-1)\underline{a} - K\alpha_1 > 0 \ , \\ \lambda + \frac{K}{2\alpha_1} + \frac{K_y\alpha_2^p}{q} < 0 \ . \end{cases}$$

For such a choice of p, α_1, α_2, we have thus obtained that there exists a constant $C > 0$, such that for all $T > 0$, for all $t \in [0,T]$, for all n

$$E_G^{\varepsilon,\mu}\left[\left\|\tilde{Y}_{t\wedge T_n}\right\|^p\right] \leq E_G^{\varepsilon,\mu}\left[\left\|\tilde{Y}_{T\wedge T_n}\right\|^p + C\int_0^T\left\|\int_0^{r\wedge\theta} G(u,0,0)\,du\right\|^p dr\right] \ .$$

Letting n go to infinity, and using the fact that $E_G^{\varepsilon,\mu}\left[\sup_{t\geq 0}\left\|\tilde{Y}_t\right\|^p\right] < \infty$, we get

$$E_G^{\varepsilon,\mu}\left[\left\|\tilde{Y}_{t\wedge T_\infty}\right\|^p\right] \leq E_G^{\varepsilon,\mu}\left[\left\|\tilde{Y}_{T\wedge T_\infty}\right\|^p + C\int_0^T\left\|\int_0^{r\wedge\theta} G(u,0,0)\,du\right\|^p dr\right] \ .$$

Note that

$$\left\|\tilde{Y}_{T\wedge T_\infty}\right\| = \left\|\tilde{Y}_T\right\| \ \mathbb{1}_{T\leq T_\infty} \leq \left\|\hat{Y}_T\right\| + \left\|\int_0^{T\wedge\theta} G(r,0,0)\,dr\right\|$$
$$\leq \|\zeta\| + \left\|\hat{Y}_T\right\| \ \mathbb{1}_{T<\theta} + \left\|\int_0^{T\wedge\theta} G(r,0,0)\,dr\right\| \ .$$

Therefore, for all $p \in]1 - \frac{K^2}{2\underline{a}\lambda}; \frac{1}{2}Q(b,d) \wedge 2[$, there is a constant C such that for all $T > 0$, for all $t \in [0,T]$, for all $\varepsilon > 0$,

$$E_G^{\varepsilon,\mu}\left[\left\|\tilde{Y}_{t\wedge T_\infty}\right\|^p\right] \leq C E_G^{\varepsilon,\mu}\left[\|\zeta\|^p + \left\|\int_0^{T\wedge\theta} G(r,0,0)\,dr\right\|^p\right.$$
$$\left. + \int_0^T\left\|\int_0^{r\wedge\theta} G(u,0,0)\,du\right\|^p dr\right]$$
$$+ C E_G^{\varepsilon,\mu}\left[\left\|\hat{Y}_T\right\|^2\right]^{2/p} P_G^{\varepsilon,\mu}[\theta > T]^{1-p/2} \ .$$

Let $\delta > 0$ be fixed. By (58) and (59), one can choose T such that for all $\varepsilon > 0$,

$$C E_G^{\varepsilon,\mu}\left[\left\|\hat{Y}_T\right\|^2\right]^{2/p} P_G^{\varepsilon,\mu}[\theta > T]^{1-p/2} \leq \delta \ .$$

For such a T, lemma 19 leads to

$$\limsup_{\varepsilon\to 0} E_G^{\varepsilon,\mu}\left[\left\|\tilde{Y}_{t\wedge T_\infty}\right\|^p\right] \le \delta \ ,$$

and lemma 20 is proved by taking $t = 0$ and letting δ go to 0.

(f) Regularization procedure. It is performed in the same way as for the parabolic case. Let $(\rho_n)_n$ be a sequence of mollifiers. We approximate h (respectively H) by $h_n \stackrel{\triangle}{=} \rho_n \star h$ (respectively $H_n \stackrel{\triangle}{=} \rho_n \star H$), so that for all n, h_n and H_n satisfy **(E 1-2) (E 1'-2')**. Note moreover that h_n satisfies **(E 1-(ii))** and **(E 1-(iii))** with the same constants K and λ as for h. Let (Y^n, Z^n) be the solution to

$$Y^n_{t\wedge\theta} = H_n(X_\theta) + \int_{t\wedge\theta}^{\theta} h_n(X_r, Y^n_r, Z^n_r)\, dr - \int_{t\wedge\theta}^{\theta} Z^n_r\, dM^X_r \ , P^{\varepsilon,\omega}_x - \text{p.s.} \ .$$

Standard estimates on BSDE's with random terminal time give that $\exists C > 0$ such that for all $n \in \mathbb{N}^*$ and all $\varepsilon > 0$,

$$\begin{aligned}
&E_G^{\varepsilon,\mu}\left[\sup_{t\geq 0}\left\|Y_t - Y^n_t\right\|^2\right] \\
&\leq C E_G^{\varepsilon,\mu}\left[\|H_n(X_\theta) - H(X_\theta)\|^2\right] \\
&\quad + C E_G^{\varepsilon,\mu}\left[\int_0^\theta \|h_n(X_r, Y_r, Z_r) - h(X_r, Y_r, Z_r)\|^2\, dr\right] \\
&\leq C\left[\omega_n(H)^2 + \omega_n(h)^2 E_G^{\varepsilon,\mu}(\theta)\right] \ ;
\end{aligned} \tag{61}$$

where $\omega_n(H) \stackrel{\triangle}{=} \sup_{\|x-x'\|\leq 1/n}\left\|H(x) - H(x')\right\|$, and $\omega_n(h)$ is defined in (37). Let $\bar{h}_n(x, y, z) \stackrel{\triangle}{=} \langle h_n(x, y, z(\mathrm{Id} + \phi))\rangle_\pi$, and let $(\bar{Y}^n, \bar{Z}^n)$ be the solution to the BSDE with random terminal time

$$\bar{Y}^n_{t\wedge\theta} = H_n(X_\theta) + \int_{t\wedge\theta}^{\theta} \bar{h}_n(X_r, \bar{Y}^n_r, \bar{Z}^n_r)\, dr - \int_{t\wedge\theta}^{\theta} \bar{Z}^n_r\, dM^X_r \ , P^0_x - \text{p.s.} \ ,$$

where P^0_x is the law of a Brownian motion with covariance matrix $\bar{a}$. Note that existence and uniqueness of $(\bar{Y}^n, \bar{Z}^n)$ is not ensured by usual criterions. Indeed, $\bar{H}_n$ satisfies **(E 1-(iii))** with constant λ, and **(E 1-(ii))** with constant $K\lambda_{max}(\langle(\mathrm{Id} + \phi)(\mathrm{Id} + \phi)^*\rangle_\pi)^{1/2}$. Moreover, $\bar{a}$ satisfies **(L 1)** with constant $\underline{a}\lambda_{min}(\langle(\mathrm{Id} + \phi)(\mathrm{Id} + \phi)^*\rangle_\pi)$, so that it is possible that

$$2\underline{a}\lambda_{min}(\left\langle(\mathrm{Id} + \phi)(\mathrm{Id} + \phi)^*\right\rangle_\pi)\lambda + K^2\lambda_{max}(\left\langle(\mathrm{Id} + \phi)(\mathrm{Id} + \phi)^*\right\rangle_\pi) \geq 0 \ .$$

However, using the same type of arguments as in the proof of lemma 15, and following the proof of theorem 4.1 in [22], it is possible to prove existence and uniqueness of $(\bar{Y}^n, \bar{Z}^n)$. Now, if $(\bar{Y}, \bar{Z})$ is the solution to

$$\bar{Y}_{t\wedge\theta} = H(X_\theta) + \int_{t\wedge\theta}^{\theta} \bar{h}(X_r, \bar{Y}_r, \bar{Z}_r)\, dr - \int_{t\wedge\theta}^{\theta} \bar{Z}_r\, dM^x_r \ , P^0_x - \text{p.s.} \ ,$$

we get easily that there exists $C > 0$ such that

$$
\begin{aligned}
E_G^{\varepsilon,\mu}\left[\sup_{t\geq 0}\left\|\bar{Y}_t-\bar{Y}_t^n\right\|^2\right] &\leq C E_G^{\varepsilon,\mu}\left[\|H_n(X_\theta)-H(X_\theta)\|^2\right] \\
&+C E_G^{\varepsilon,\mu}\left[\int_0^\theta \left\|\bar{h}_n(X_r,\bar{Y}_r,\bar{Z}_r)-\bar{h}(X_r,\bar{Y}_r,\bar{Z}_r)\right\|^2 dr\right] \\
&\leq C\left[\omega_n(H)^2+\omega_n(\bar{h})^2 E_G^{\varepsilon,\mu}(\theta)\right] \\
&\leq C\left[\omega_n(H)^2+\omega_n(h)^2 E_G^{\varepsilon,\mu}(\theta)\right] \; ; \qquad (62)
\end{aligned}
$$

Theorem 16 follows now from lemma 20, and estimates (58), (61) and (62).

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