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Convex duality and the Skorokhod Problem. II*

Paul Dupuis[1], Kavita Ramanan[2]

[1]Lefschetz Center for Dynamical Systems, Division of Applied Mathematics, Brown University, Providence, RI 02912, USA (e-mail: dupuis@cfm.brown.edu)
[2]Bell Laboratories, Lucent Technologies, 600 Mountain Avenue, Murray Hill, New Jersey 07974, USA (e-mail: kavita@research.bell-labs.com)

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Abstract. In this paper we consider Skorokhod Problems on polyhedral domains with a constant and possibly oblique constraint direction specified on each face of the domain, and with a corresponding cone of constraint directions at the intersection of faces. In part one of this paper we used convex duality to develop new methods for the construction of solutions to such Skorokhod Problems, and for proving Lipschitz continuity of the associated Skorokhod Maps. The main alternative approach to Skorokhod Problems of this type is the reflection mapping technique introduced by Harrison and Reiman [8]. In this part of the paper we apply the theory developed in part one to show that the reflection mapping technique of [8] is restricted to a slight generalization of the class of problems originally considered in [8]. We further illustrate the power of the duality approach by applying it to two other classes of Skorokhod Problems – those with normal directions of constraint, and a new class that arises from a model of processor sharing in communication networks. In particular, we prove existence of solutions to and Lipschitz continuity of the Skorokhod Maps associated with each of these Skorokhod Problems.

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1. Introduction

Many Skorokhod Problems (SPs) that arise as models of physical processes have polyhedral domains with a constant, possibly oblique direction of constraint on each face of the domain, and a corresponding cone of constraint directions at the intersection of faces [6, Section 1]. SPs in this class can be represented by a finite collection of triplets $\{(d_i, n_i, c_i), i = 1, \ldots, N\}$ where the domain is $G = \cap_{i=1,\ldots,N}\{x : \langle x, n_i\rangle \geq c_i\}$, d_i is the direction of constraint associated with the face $\{x : \langle x, n_i\rangle = c_i\}$ and $\langle d_i, n_i\rangle = 1$. For a precise definition of the SP and a more detailed description of SPs in this class, see [6, Definition 1.1 and Section 2.2]. New methods for proving existence of solutions to SPs of this class, and for establishing Lipschitz continuity of the associated Skorokhod Maps (SMs) were developed in [6]. In this paper we apply the techniques of [6] to analyze concrete SPs.

In [4, Theorem 2.2] the following sufficient geometric condition was derived for Lipschitz continuity of the SM associated with a SP in this class. Let C° denote the interior of a set C. Given a convex set $C \subset I\!R^n$ and $x \in \partial C$, define the set of inward normals to C at x by

$$\nu(x) \doteq \{\gamma : \|\gamma\| = 1, \text{ and } \langle \gamma, x - y\rangle \leq 0 \ \forall y \in C\}. \tag{1.1}$$

Consider a SP with representation $\{(d_i, n_i, c_i), i = 1, \ldots, N\}$.

Assumption 1.1 (Set B). *There exists a compact, convex set B with $0 \in B^\circ$, such that if $\nu(z)$ denotes the set of inward normals to B at $z \in \partial B$, then there exists $\delta > 0$ such that for $i = 1, \ldots, N$,*

$$\left\{\begin{array}{l} z \in \partial B \\ |\langle z, n_i\rangle| < \delta \end{array}\right\} \Rightarrow \langle \nu, d_i\rangle = 0 \textit{ for all } \nu \in \nu(z). \tag{1.2}$$

While this work of [4] was interesting in that it reduced the problem of regularity of the SM (an infinite dimensional dynamic problem) to the existence of a certain set (an infinite dimensional static problem), it was not fully satisfactory because it did not provide any methodology for verifying this condition for a given SP. In [6, Theorem 3.3] an alternate *dual* sufficient condition for Lipschitz continuity of the SM was derived. This condition, stated below as Assumption 1.2, is expressed in terms of the collection $\{L_i^*, i = 1, \ldots, N\}$ of adjoint projection operators associated with the SP $\{(d_i, n_i, c_i), i = 1, \ldots, N\}$, where

$$L_i^* x \doteq x - \langle x, d_i\rangle n_i, \tag{1.3}$$

for $i = 1, \ldots, N$.

Assumption 1.2 (Set B*). *There exists a finite set of vertices* $v_1, v_2, \ldots, v_J$ *with* $\mathrm{span}(\{v_j, j = 1, \ldots, J\}) = I\!R^n$, *such that if* $B^* \doteq \mathrm{conv}[\pm v_j, j = 1, \ldots, J]$, *then for every* $i \in \{1, \ldots, N\}$ *and* $j \in \{1, \ldots, J\}$,

$$\textit{either } L_i^* v_j = v_j \qquad \textit{or } L_i^* v_j \in (B^*)^\circ. \tag{1.4}$$

Although this dual condition is also geometric, it lends itself more easily to verification. Indeed, methods for deriving algebraic conditions that guarantee the existence of a set B^* that satisfies (1.4) were also proposed in [6]. In the present paper we use these methods to verify property (1.4) for specific classes of SPs, and thereby establish Lipschitz continuity of the associated SMs. More specifically, we analyze every SP for which regularity is known, prove some interesting new results for these SPs, and then proceed to examine a new class of Skorokhod Problems for which there are no existing results.

The first general result on SPs of this type was derived by Harrison and Reiman [8]. They used contraction mapping techniques to prove existence of solutions to a certain class of SPs, which we refer to as the Harrison–Reiman (H–R) class. The techniques used in [8] also automatically established Lipschitz continuity of the associated SMs. Limitations of the contraction mapping approach were discussed in [6, Section 2.4]. In this paper (see Section 2.2), we use the duality approach to obtain a sharp result in this direction. We prove in Lemma 2.3 that the contraction mapping techniques of [8] are in fact limited to what we call the generalized Harrison–Reiman (gH–R) class of SPs. SPs in the gH–R class have directions of constraint that are linearly independent and satisfy a certain spectral radius condition (see Theorem 2.1). In Section 2.4 we use the techniques of [6] to show that this spectral radius condition is not necessary for regularity of the SM.

In Section 3, we analyze a new SP that arises from a generalized processor sharing (GPS) model. Once again, we establish Lipschitz continuity of the SM by constructing a set B^* that satisfies the dual condition Assumption 1.2 for the SP. In Section 3.4, we also prove existence of solutions to this SP for a certain class of paths, using tools that were developed in [6, Section 4]. SPs with normal directions of constraint have the representation $\{(n_i, n_i, c_i), i = 1, \ldots, N\}$ for some $N < \infty$. Lipschitz continuity of the SMs associated with this class of SPs was established in [4]. The proof, which entailed a direct verification of property (1.2), was rather tedious. In Section 4 we use the convex duality approach of [6] to provide a simpler proof of Lipschitz continuity.

The paper concludes with Section 5, which discusses the issues of finite algebraic characterizations for Lipschitz continuity of SMs (in Section 5.1), singularity of the structure of the set B across different classes of SPs (in

Section 5.2) and non-uniqueness of the representation for a SP (in Section 5.3).

2. The generalized Harrison–Reiman Skorokhod Problem

2.1. Introduction

In [8] Harrison and Reiman derived sufficient algebraic conditions for Lipschitz continuity of a family of SMs. The class considered in [8] corresponds to SPs on the positive n-dimensional orthant $\mathbb{R}^n_+$ with n linearly independent oblique directions of constraint. They define the map $\psi \to \phi$ in terms of what we term a reflection map rather than the SM. However, as discussed in [6, Section 2.4], the two concepts coincide for the setup used in [8]. The technique of the proof used in [8] is also discussed in [6, Section 2.4]. In [4] a slight generalization of the Harrison–Reiman condition given in [8] was derived by directly verifying Assumption 1.1. This generalization can also be obtained by adapting the techniques used in [8].

In Section 2.2 we use the dual condition in Assumption 1.2 to establish the sufficiency of the generalized Harrison–Reiman (gH–R) condition for Lipschitz continuity of the SM. We also show that this condition is necessary if one wishes to use the technique of [8] to prove the Lipschitz property. The gH–R problems are precisely those for which there exists a set B that satisfies Assumption 1.1 with exactly $2n$ faces. Since B must have at least $2n$ faces in $\mathbb{R}^n$ for it to be symmetric, bounded and contain 0 in its interior, the gH–R class of SPs uses a set that is, in a sense, the simplest possible. The lack of any additional algebraic results after that of [8] is likely to be due to this increase in the complexity of the set required for SPs outside the gH–R class. This leads naturally to the following question. Suppose we consider a SP which satisfies all the other conditions assumed in [8] (that is $G = \mathbb{R}^n_+$ with n linearly independent directions of constraint), but not the gH–R condition. Such SPs still have an interpretation in terms of a reflection map. If a more complicated set is used (i.e., one with more than $2n$ sides), is it still possible to satisfy Assumption 1.1 even though the gH–R condition is violated? The answer is yes, and a 3-dimensional example is given in Section 2.4.

2.2. The generalized Harrison–Reiman condition

Here we construct a polytope B^* that satisfies (1.4) in order to establish the Lipschitz continuity of the SM for SPs in the gH–R class. This class of problems helps to illustrate how fundamental vertex directions are constructed and how the techniques developed in [6, Section 3.3] can be used to obtain algebraic characterizations for a given class of SPs. Since the fundamental vertex directions are specified in terms of the directions of constraint, it is easiest to consider the case where the directions of constraint have the maximum symmetry. Although the original H–R SP considered in [8] had the simplifying property that the n_i's were the basis vectors, for the geometric approach it is more convenient to assume that the d_i's take a simple form. Thus we first assume that $d_i = e_i, i = 1, \ldots, n$. In this case the adjoint operators are

$$L_i^* x = x - \langle x, e_i \rangle n_i,$$

and $\mathscr{V} = \{\pm e_i, i = 1, \ldots, n\}$, where $\mathscr{V}$ is the fundamental set of vertices defined in [6, Section 3.3]. Note that L_i^* leaves all the vectors e_j, $j \neq i$ invariant, which greatly simplifies the verification of (1.4) for a set B^* of the form conv$[\pm a_i e_i, i = 1, \ldots, n]$. This property motivates the use of simple d_i and complicated n_i rather than the reverse. Then linear transformations can be used to determine the d_i's for which Assumption 1.2 holds when the inward normals are mapped to the basis vectors. The same principle is followed in Section 3 while analyzing the SP arising from a generalized processor sharing model.

Theorem 2.1. *Consider the SP in $\mathbb{R}^n$ specified by $\{(e_i, n_i, c_i), i = 1, \ldots, n\}$, where the directions of constraint $\mathscr{D} \doteq \{e_i, i = 1, \ldots, n\}$ are the standard orthonormal basis for $\mathbb{R}^n$. Assume the normal directions n_i have been normalized so that $\langle e_i, n_i \rangle = 1$. Let the matrix V be defined by*

$$V \doteq [v_{ij}] = [\langle e_i, n_j \rangle],$$

and let $Q \doteq |I - V|$, where I is the $n \times n$ identity matrix. Then there exists a set B^* satisfying Assumption 1.2 *if $\sigma(Q) < 1$.*

Proof. Let $Q = [q_{ij}] = [|\delta_{ij} - \langle e_i, n_j \rangle|]$, where $\delta_{ij} = 1$ for $i = j$ and equals zero otherwise. We are given that $\sigma(Q) < 1$. We follow the construction outlined in [6, Section 3.3] to build a set that satisfies property (1.4) for the given SP. Let $\mathscr{D} = \{e_i, i = 1, \ldots, n\}$ denote the set of directions of constraint. The set $\mathscr{V}$ of fundamental directions is $\{\pm e_i, i = 1, \ldots, n\}$, and thus the dual set is assumed to be of the form $B^* = \text{conv}[\pm a_i e_i, i = 1, \ldots, n]$ for some constants $a_i > 0$. We now identify a certain subset of SPs for which the set B^* with the choice of $a_i = 1$ for every i satisfies property (1.4). Recall that the adjoint operators are given by

$$L_j^* x = x - \langle x, e_j \rangle n_j.$$

For $i \neq j$, L_j^* leaves e_i invariant since $\langle e_i, e_j\rangle = 0$, and for $j = i$, $L_j^* e_j = e_j - n_j$. Since $\{e_i\}$ is an orthonormal basis for $\mathbb{R}^n$, $L_j^* e_j$ can be written as a unique linear combination of the vertices of B^*,

$$\begin{aligned} L_j^* e_j = \sum_{i=1}^n \langle L_j^* e_j, e_i\rangle e_i &= \Sigma_{i=1}^n \langle e_j - n_j, e_i\rangle e_i \\ = e_j - \Sigma_{i=1}^n \langle n_j, e_i\rangle e_i &= -\Sigma_{i=1, i\neq j}^n \langle n_j, e_i\rangle e_i, \end{aligned} \tag{2.5}$$

where the last equality follows because $\langle n_j, e_j\rangle = 1$. Note that $q_{ij} = 0$ if $i = j$ and $q_{ij} = |\langle e_i, n_j\rangle|$ if $i \neq j$. First consider the case where $\Sigma_{j=1}^n q_{ij} < 1$ for all $i = 1, \ldots, n$. In this case (2.5) implies that $L_j^* e_j$ lies in the interior of conv$[\pm e_i, i = 1, \ldots, n]$ and the existence of B^* is established. Next suppose that $\Sigma_{j=1}^n q_{ij} \geq 1$ for at least one i. Here we follow [8] and observe that $\sigma(Q) < 1$ implies [17, Lemma 3] the existence of positive scalars u_j such that $\Sigma_{j=1}^n q_{ij} u_j / u_i < 1$ for $i = 1, \ldots, n$. This corresponds to applying the similarity transform

$$A = (a_{ij}) = \begin{cases} u_j & i = j, \\ 0 & i \neq j, \end{cases}$$

to the operators L_j^*. Thus the set $S = \text{conv}[\pm e_i, i = 1, \ldots, n]$ satisfies property (1.4) for the operators $A^{-1} L_j^* A$. By [6, Lemma 3.7], the set

$$B^* \doteq AS = \text{conv}[\pm u_i e_i, i = 1, \ldots, n]$$

satisfies property (1.4) for the collection of operators $\{L_j^*, j = 1, \ldots, n\}$. □

The sets B and B^* that correspond to the SP considered in Theorem 2.1 are shown in Figure 1. In the next theorem we use the transformation result obtained in [6, Theorem 3.7] to generalize the condition derived in Theorem 2.1 to SPs with n arbitrary linearly independent constraint directions.

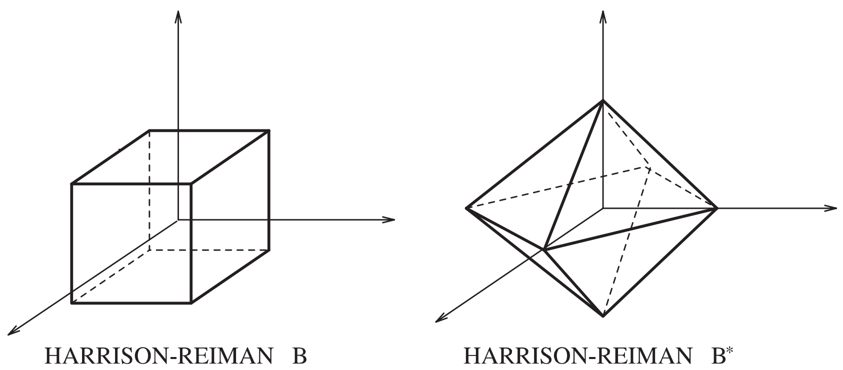

Fig. 1. The sets B and B^* for the H–R case

Theorem 2.2. *Consider the SP in* $I\!R^n$ *specified by* $\{(d_i, n_i, c_i), i = 1, \ldots, n\}$, *where the directions of constraint* $\{d_i\}$ *are linearly independent. Let the matrix* $\tilde{V}$ *be defined by*

$$\tilde{V} = [\tilde{v}_{ij}] \doteq [\langle d_i, n_j \rangle],$$

and let $Q \doteq |I - \tilde{V}|$, *where* I *is the* $n \times n$ *identity matrix. If* $\sigma(Q) < 1$, *then the SM is Lipschitz continuous.*

Proof. Define the collection $\mathscr{C}$ of SPs to be $\{(e_i, w_i, k_i), i = 1, \ldots, n\}$ such that $\sigma(|I - V|) < 1$, where $V = [v_{ij}] = [\langle e_i, w_j \rangle]$. Then by Theorem 2.1 every SP in $\mathscr{C}$ satisfies Assumption 1.2. Let D be the matrix of the directions of constraint $[d_1, \ldots, d_n]$. Then $\det D \neq 0$ since $\mathrm{span}(\mathscr{D}) = I\!R^n$ for $\mathscr{D} = \{d_i, i = 1, \ldots, n\}$. Since

$$\tilde{v}_{ij} = \langle d_i, n_j \rangle = \langle De_i, n_j \rangle = \langle e_i, D^* n_j \rangle$$

and $\sigma(|I - \tilde{V}|) < 1$, it follows that the SP $\{(e_i, D^* n_j, c_i), i = 1, \ldots, n\}$ belongs to $\mathscr{C}$. Choosing the invertible transformation $A = D^{-1}$ in [6, Theorem 3.7], it follows that the SP $\{(d_i, n_i, c_i), i = 1, \ldots, n\}$ satisfies Assumption 1.2. Hence the associated SM is Lipschitz continuous. □

Theorem 2.2 is proved by constructing a symmetric set with $2n$ faces that is a "tilted" version of a level set of the sup norm in $I\!R^n$. It is natural to ask if a set of this form can satisfy (1.4) under any weakening of the condition $\sigma(Q) < 1$. Lemma 2.3 shows that this is not possible, and thus the condition $\sigma(Q) < 1$ is also *necessary* for the existence of a set B *of the prescribed form*. Thus the gH–R class of SPs is the precise class for which a simple form of the set B ($2n$ faces) suffices.

Lemma 2.3. *Consider the SP and the matrices* V *and* Q *defined in Theorem* 2.2. *If* $\sigma(Q) \geq 1$ *then there does not exist a set* B^* *of the form* $\mathrm{conv}[\pm v_i, i = 1, \ldots, n]$ *that satisfies Assumption* 1.2.

Proof. It suffices to consider the case when $d_i = e_i, i = 1, \ldots, n$, since all other cases can be reduced to this case by using [6, Theorem 3.7] and an invertible change of coordinates. By [6, Theorem 3.4] any set of the form $\mathrm{conv}[\pm v_i, i = 1, \ldots, n]$ that satisfies (1.4) must have the particular form $B^* = \mathrm{conv}[\pm e_i/a_i, i = 1, \ldots, n]$, where a_i are positive scalars. Suppose such a set does exist. Then by [6, Theorem 2.4] B^* is also an invariant set for the collection of operators $\mathscr{L}^*$. Fix any value $j \in \{1, \ldots, n\}$. From (2.5)

$$Qe_j = \Sigma_{i=1, i \neq j}^n |\langle e_i, n_j \rangle| e_i \quad \text{and} \quad L_j^* e_j = -\Sigma_{i=1, i \neq j}^n \langle e_i, n_j \rangle e_i.$$

A vector $v = \sum_{i=1}^{n} \beta_i e_i$ is in B^* if and only if $\sum_{i=1}^{n} |a_i \beta_i| \leq 1$. Thus for any $j \in \{1, \ldots, n\}$, $L_j^*(e_j/a_j) \in B^*$ (respectively $(B^*)^\circ$) if and only if $Q(e_j/a_j) \in B^*$ (respectively $(B^*)^\circ$).

Now suppose that $\sigma(Q) \geq 1$ and that an invariant set of the proposed form satisfying Assumption 1.2 exists. Then for each $j \in \{1, \ldots, n\}$, $L_j^*(e_j/a_j) \in (B^*)^\circ$ implies $Q(e_j/a_j) \in (B^*)^\circ$. This implies $Q^n \to 0$ as $n \to \infty$, which contradicts the fact that $\sigma(Q) \geq 1$. □

We next show that in the two-dimensional case, one can without loss of generality assume that the set B satisfying (1.2) has 4 faces. Thus in two dimensions $\sigma(Q) < 1$ serves as both a necessary and sufficient condition for Assumption 1.1 to hold. There is also an obvious extension to the case where $N = 2$ and $n \geq 2$.

Lemma 2.4. *Consider any SP such that $N = n = 2$ and such that n_1 is not a scalar multiple of n_2. Define Q as in Theorem* 2.2. *Then $\sigma(Q) < 1$ is a necessary and sufficient condition for Assumption* 1.1 *to be satisfied.*

Proof. Theorem 2.2 shows that $\sigma(Q) < 1$ is a sufficient condition for Assumption 1.1 to hold. Consider a two dimensional SP $\{(d_i, n_i, c_i), i = 1, 2\}$. Suppose it is true that whenever there exists B satisfying Assumption 1.1, there also exists a set $\tilde{B}$ that satisfies this assumption and the additional requirement that it be the intersection of 4 halfspaces. In this case, the necessity of the condition $\sigma(Q) < 1$ follows from Lemma 2.3. We now show that such a set $\tilde{B}$ can always be found.

Let B satisfy Assumption 1.1, and for $i = 1, 2$ let $d_i^\perp$ and $n_i^\perp$ represent unit vectors that are perpendicular to d_i and n_i, respectively. Recall that Assumption 1.1 implies B is symmetric. Thus for each $i = 1, 2$ there is a scalar $\alpha_i \in (0, \infty)$ such that $\pm\alpha_i n_i^\perp \in \partial B$. B satisfies Assumption 1.1 if and only if there are open neighborhoods $\pm N_i$, $i = 1, 2$ around the points $\pm\alpha_i n_i^\perp$ such that the inward normals to ∂B at points $z \in \pm N_i \cap \partial B$ are orthogonal to d_i. Thus there exist $a_i \in (0, \infty)$, $i = 1, 2$ such that all points in the sets $\pm N_i \cap \partial B$ satisfy $|\langle x, d_i^\perp \rangle| = a_i$ for $i = 1, 2$. Setting $\tilde{B} \doteq \{x : |\langle x, d_i^\perp \rangle| \leq a_i, i = 1, 2\}$, we observe that $\pm N_i \cap \partial B = \pm N_i \cap \partial \tilde{B}$ for $i = 1, 2$. Thus $\tilde{B}$ satisfies Assumption 1.1, and the proof is complete. □

Open feedforward networks

In the original Harrison–Reiman framework, the normals were chosen to be the orthonormal basis, $I - V$ was required to be nonnegative, where V is as defined in the proof of Theorem 2.2, and the spectral radius condition

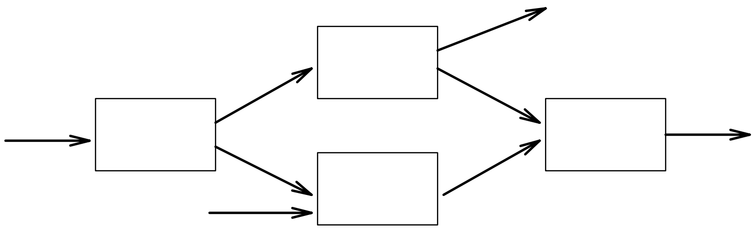

Fig. 2. Open feedforward network

$\sigma(I - V) < 1$ was shown to guarantee existence and Lipschitz continuity of solutions to the SP. In that case $\sigma(I - V) = \sigma(|I - V|)$, and hence Theorem 2.2 is indeed a generalization of the H–R condition. Here we illustrate the utility of this generalization with an example of a SP that arises in open feedforward networks. This SP cannot be analyzed by the original H–R condition but does fall into the gH–R class. For open feedforward networks, an example of which is shown in Figure 2, there exists a natural ordering of the buffers in the network and the map that takes the inputs to the buffer contents can be shown to be Lipschitz continuous using an inductive argument on the buffers. However, the problem can also be formulated as a SP on the nonnegative orthant $\mathbb{R}^n_+$ (where n represents the number of buffers in the network) with an upper triangular "reflection matrix" V, as usual suitably normalized to have ones along the diagonal. Then $I - V$ is upper triangular with zeros along the diagonal and consequently has spectral radius zero. However, the H–R condition is not applicable since $I - V$ is not necessarily non-negative. On the other hand, the generalized H–R condition derived in Theorem 2.2 can be used since $Q \doteq |I - V|$ also has spectral radius zero. SPs arising from single class networks with feedback also fall into the gH–R category. These SP representations were used to derive heavy traffic limits for multi-class open feedforward and single class networks with feedback by [14] and [13] respectively.

2.3. *Existence of solutions for the gH–R class of SPs*

Recall the definition of the class of generalized Harrison–Reiman SPs described in Section 2.2. For such SPs existence of solutions on $C([0, T] : \mathbb{R}^n)$ was established in [8] using contraction mapping techniques. In [8] it was also shown that the gH–R SP gives rise to a semimartingale RBM. It is known that [6, Assumption 4.1] is also a necessary condition for the existence of a semimartingale Brownian motion in this framework [18]. Thus the gH–R condition implies [6, Assumption 4.1] (which is equivalent in this context to the completely-$\mathscr{S}$ condition) and so by [6, Theorem 4.2] solutions to the gH–R SP exist on all of $D_G([0,\infty) : I\!R^n)$, the set of right continuous functions with left limits on $[0,\infty)$ that start in G.

2.4. A Lipschitz continuous Skorokhod Map on $I\!R^3_+$ that violates the generalized Harrison–Reiman condition

In the last section it was established that for SPs on $I\!R^n_+$ with n linearly independent directions of constraint, $\sigma(Q) < 1$ is a necessary and sufficient condition for the existence of a set with $2n$ faces that satisfies Assumption 1.1. It was also shown in Lemma 2.4 that in the two dimensional case, $\sigma(Q) < 1$ is both a necessary and sufficient condition for Assumption 1.1 to be fulfilled. However in higher dimensions it is possible to satisfy Assumption 1.1 (and therefore obtain Lipschitz continuity of the SM) even when $\sigma(Q) > 1$. In this section we provide a 3-dimensional example of such a SP. As required by Lemma 2.3, the set B that satisfies (1.2) in this case has $8 > 2n = 6$ faces.

Example: Consider the SP $\{(d_i, n_i, 0), i = 1, 2, 3\}$, where

$$\begin{aligned} d_1 &= (1,0,0)^T,\ n_1 = (1, -.5-\varepsilon, .5+\varepsilon)^T, \\ d_2 &= (0,1,0)^T,\ n_2 = (.5-a\varepsilon, 1, -.5+a\varepsilon)^T, \\ d_3 &= (0,0,1)^T,\ n_3 = (.5-a\varepsilon, -.5+a\varepsilon, 1)^T, \end{aligned} \tag{2.6}$$

where the superscript T denotes transpose and a and ε are positive scalars. The matrix Q defined in Theorem 2.2 is then given by

$$\begin{pmatrix} 0 & .5-a\varepsilon & .5-a\varepsilon \\ .5+\varepsilon & 0 & .5-a\varepsilon \\ .5+\varepsilon & .5-a\varepsilon & 0 \end{pmatrix}.$$

For $\varepsilon = 0$, $\sigma(Q) = 1$. If $a > 0$ is sufficiently small then $\sigma(Q) > 1$ for all sufficiently small values of $\varepsilon > 0$. Since $\mathrm{span}(\{d_1, d_2, d_3\}) = I\!R^3$, it follows from [6, Theorem 3.4] that the set of vertex directions of B^* must include $\mathscr{V} \doteq \{\pm e_1, \pm e_2, \pm e_3\}$, which suggests

$$C^* \doteq \mathrm{conv}[\pm e_1, \pm e_2, \pm e_3]$$

as a candidate set for Assumption 1.2. However, Lemma 2.3 implies that C^* cannot satisfy (1.4). In order to guess at the correct form for the dual set, it is instructive to see why C^* does not satisfy (1.4). We begin by observing that $L_i^* e_j = e_j$ if $i \neq j$. Also, since $a\varepsilon > 0$

$$L_2^* e_2 = -(.5-a\varepsilon)e_1 + (.5-a\varepsilon)e_3 \in (C^*)^\circ,$$

and similarly

$$L_3^* e_3 = -(.5 - a\varepsilon)e_1 + (.5 - a\varepsilon)e_2 \in (C^*)^\circ.$$

Since

$$L_1^* e_1 = (.5 + \varepsilon)e_2 - (.5 + \varepsilon)e_3 = (.5 + \varepsilon)(e_2 - e_3)$$

lies outside C^*, we consider for some $k > 0$ the set

$$B^* = \mathrm{conv}[\pm e_1, \pm e_2, \pm e_3, \pm k(e_2 - e_3)]. \tag{2.7}$$

Note that $L_1^* e_1 \in (B^*)^\circ$ if $.5 + \varepsilon < k$ and $L_2^* e_2$ and $L_3^* e_3$ also lie in $(B^*)^\circ$ since $C^* \subset B^*$. It remains to check where the new vertex $k(e_2 - e_3)$ is mapped under the projection operators. It is clearly left invariant by L_1^*, and its image under L_2^* is

$$-k(.5 - a\varepsilon)e_1 + [k(.5 - a\varepsilon) - k]e_3.$$

Rewriting this sum as

$$-(.5 - a\varepsilon)ke_1 - (.5 + a\varepsilon)ke_3,$$

we find that it lies in $\mathrm{conv}[e_1, -e_3]^\circ \subset (B^*)^\circ$ when $k < 1$. An analogous calculation shows that the same is true for the image under L_3^*. Thus the set B^* in (2.7) satisfies (1.4) for the SP for $k \in (.5 + \varepsilon, 1)$.

The above SP is illustrative of a case when Assumption 1.1 holds but no set B^* having vertices only along the fundamental vertex directions will satisfy (1.4). Thus additional vertex directions need to be added in order to obtain an appropriate set B^*. While the choice of additional directions is not ad-hoc, there is as yet no systematic method of selecting them. For SPs arising from real models (such as open feedforward networks and the GPS model discussed in Sections 2 and 3 respectively) the physical constraints of the process seem to suggest the appropriate vertex directions. Finally we observe that from the monotonicity property stated in [6, Theorem 3.5] it is clear that a set B^* of the form constructed in (2.7) (or appropriate linear transforms of it) also satisfies (1.4) for SPs that satisfy the gH–R condition. Thus by using a more complicated set, one obtains Lipschitz continuity under strictly weaker conditions (i.e. $\sigma(Q) > 1$) on the problem data. It is thus conceivable that by making the set increasingly more complicated, one can obtain Lipschitz continuity under increasingly weaker algebraic conditions. This seems to suggest that no tight algebraic condition for Lipschitz continuity may be possible, which is in keeping with the conjecture made in [9] that there do not exist semi-algebraic criteria to determine uniqueness of solutions to certain SPs. A perspective that is put forth in [6], and supported by the examples found in this paper, is that one must group SPs according to the structure required of the corresponding set B^*, and then search for algebraic characterizations of regularity within the individual groups. A compelling example of this approach is given in Section 3.

2.5. Existence of solutions for the three-dimensional example

In this section we show that the three-dimensional SP considered in the last section, which we refer to as P, satisfies [6, Assumption 4.1]. As discussed in Section 4.3.2 of [6], this is a necessary and sufficient condition for the existence of solutions on $D_G([0,\infty) : \mathbb{R}^n)$ for standard SPs. We first consider the associated SP P_0 which is obtained by setting $\varepsilon = 0$ in the definition of the SP P in (2.6). In other words let $P_0 \doteq \{(d_i, n_i^0, 0), i = 1, 2, 3\}$, where

$$\begin{aligned} d_1 &= (1,0,0)^T,\ n_1^0 = (1,-.5,.5)^T, \\ d_2 &= (0,1,0)^T,\ n_2^0 = (.5,1,-.5)^T, \\ d_3 &= (0,0,1)^T,\ n_3^0 = (.5,-.5,1)^T. \end{aligned}$$

Then it follows from the properties stated below that Assumption 4.1 of [6] is satisfied for P_0. Consider $i \in \{1,2,3\}$ and $j \in \{1,2,3\} \setminus \{i\}$.

1. For $x \in \text{rel int}\,(\partial G \cap \partial G_i)$, consider $d = d_i$. With the choice $\tilde{n} \doteq n_i \in n(x)$,

$$\langle \tilde{n}, d_i \rangle = 1 > 0.$$

2. For $x \in \text{rel int}\,(\partial G \cap \partial G_i \cap \partial G_j)$, consider any d of the form $\alpha_i d_i + \alpha_j d_j$ with $\alpha_i \geq 0$, $\alpha_j \geq 0$, $\alpha_i + \alpha_j = 1$. With the choice $\tilde{n} \doteq \frac{1}{\sqrt{2}}\left(n_i + n_j\right) \in n(x)$,

$$\langle \tilde{n}, d \rangle \geq \frac{1}{\sqrt{2}} > 0.$$

3. For $x = 0$ consider d of the form $\alpha_1 d_1 + \alpha_2 d_2 + \alpha_3 d_3$ where each $\alpha_i \geq 0$ and $\Sigma_{i=1}^3 \alpha_i = 1$. With the choice $\tilde{n} \doteq \frac{1}{\sqrt{3}}(1,1,1) \in n(x)$

$$\langle \tilde{n}, d \rangle \geq \frac{1}{\sqrt{3}} > 0.$$

Thus P^0 satisfies Assumption 4.1 of [6], and therefore the same is true of the SP P for all sufficiently small $\varepsilon > 0$.

3. The generalized processor sharing Skorokhod Problem

3.1. Introduction

Generalized processor sharing (GPS) is a policy that has been proposed for distributing processing in a fair manner between different data classes in high-speed networks. The processor sharing model, which is shown in Figure 3, is defined in terms of a probability vector $\rho = (\rho_1, \ldots, \rho_n)$. The total number of classes present is n and each $\rho_i > 0$ represents the minimal fraction of the overall processing capacity that is guaranteed to class i. The queue length process for a model of GPS with fluid absolutely continuous inputs was characterized in [5] as the solution to a constrained ordinary differential equation (CODE). This CODE was then recast in terms of a Skorokhod Problem (which we refer to as the GPS SP below), and the queue length process expressed as the associated Skorokhod Map applied to a suitably centered input process. In Section 3.2 we give a description of the Skorokhod Problem that arises from the GPS fluid model and derive a suitable representation $\{(d_i, n_i, c_i), i = 1, \ldots, N\}$ for it. Like the Harrison–Reiman SP considered in the last section, the GPS SP has the n-dimensional orthant $\mathbb{R}_+^n$ as its domain. However, due to the structure of its constraint directions, it falls outside the gH–R framework and therefore eludes analysis by existing techniques. The study of the SM arising from the GPS model thus serves as a good motivation for the techniques developed in [6, Section 3].

In Section 3.3 we describe how a dual set B^* that satisfies (1.4) can be constructed for this SP. The construction of B^* here is similar to but

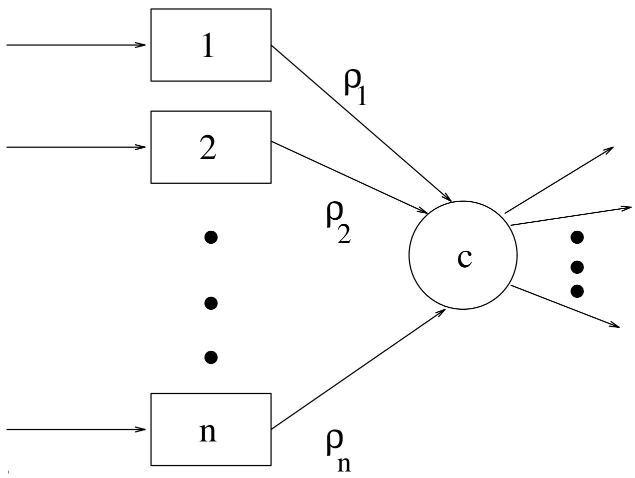

Fig. 3. Generalized processor sharing

more complicated than that in the gH–R case. As in that case, we first consider the GPS SP that has the simplest directions of constraint, which we denote by $\{z_i\}$, and construct the fundamental set of vertices of the form $\mathscr{V} = \{\pm w_j, j = 1, \ldots, K\}$ associated with it. This construction, which is greatly facilitated by the simple structure of the constraint directions, leads to the conjectured internal representation $\mathrm{conv}[\pm a_j w_j, j = 1, \ldots, K]$ for the set B^*, where the $a_j > 0$ are to be chosen so that B^* satisfies (1.4). The structure of the $\{z_i\}$ is such that each vertex $\pm a_j w_j$ of the conjectured set B^* is left invariant by most of the projection operators. This greatly simplifies the problem of determining the normals $\{n_i\}$ for which there exists B^* of the given form that satisfies (1.4).

A crucial step in obtaining the algebraic condition is the derivation of an external representation for B^* of the form $B^* = \cap_{\nu \in \mathscr{K}} \{x : \langle x, \nu \rangle \leq c_\nu\}$ for some constants c_ν and a finite set of vectors $\mathscr{K}$. Though this is a highly non-trivial task in general, it becomes feasible here due to the symmetry of the fundamental vertex directions $\{w_j\}$ associated with the simplest directions of constraint $\{z_i\}$. Thus the verification of (1.4) is reduced to checking a finite number of linear inequalities. This algebraic condition is generalized using the transformation techniques discussed in [6, Section 3.4] to determine the constraint directions d_i for which B^* exists when the normals $\{n_i\}$ are mapped to the vectors that define the GPS domain. This is in analogy with the method used in Theorem 2.2 for the gH–R SP. Theorem 3.5 shows that the transformed algebraic conditions are fulfilled by the directions of constraint associated with any GPS model (i.e., for any dimension and any ρ), and consequently establishes Lipschitz continuity of the corresponding SMs. The Lipschitz continuity of the SM is very useful in studying the GPS model, as illustrated in [5]. Prior work on the processor sharing model that does not make use of a Skorokhod formulation is restricted to either the two-dimensional case or to just bounds for higher dimensional cases, and includes [10], [11], [12], [19].

3.2. Description of the generalized processor sharing SP

The domain of the GPS SP is the non-negative orthant $I\!R^n_+ = \cap_{i=1,\ldots,n}\{x : \langle x, e_i \rangle \geq 0\}$. Given a probability vector $\rho = (\rho_1, \rho_2, \ldots, \rho_n)$, the direction of constraint on the face $\{x : \langle x, e_i \rangle = 0\}$ is

$$d_i = (e_i - \rho)/(1 - \rho_i) \tag{3.8}$$

for $i = 1, \ldots, n$. There is also an additional direction of constraint d_{n+1} pointing along $\Sigma_{i=1}^n e_i$ at the origin. The structure of the SP for 2 dimensions is shown in Figure 4. The constraint directions $d_i, i = 1, \ldots, n$ effect a

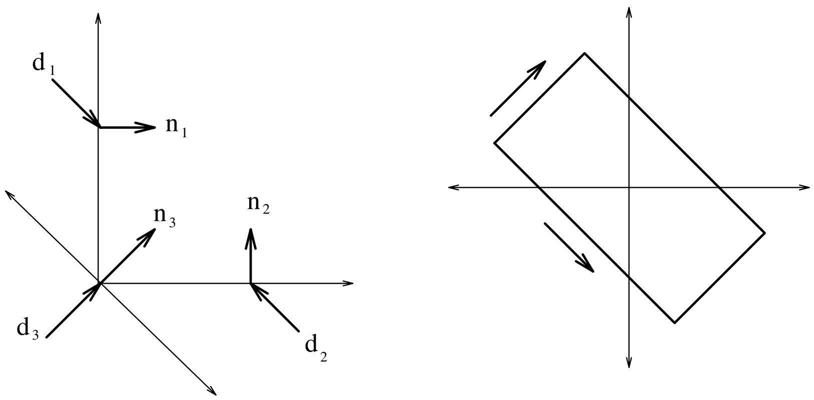

Fig. 4. The two-dimensional GPS SP and the associated set B

"fair" (as measured by ρ) redistribution of any processing capacity that is not used by one set of users among all the remaining users, while d_{n+1} is used to maintain the non-negativity of the buffers if at any time there is more than enough processing capacity to service both the data that is currently being input by the users as well as any data that is buffered. We now look for a convenient representation of the SP as a finite collection of triplets. It is apparent that the representation of the GPS SP must include the triplets $(d_i, e_i, 0)$ for $i = 1, \ldots, n$. The domain associated with this collection of triplets is $\mathbb{R}^n_+$. However, it is not possible to represent the GPS SP with just these triplets, since $\langle d_{n+1}, d_i \rangle = 0$ implies that d_{n+1} would not be in the set of directions of constraint at the origin. In order to properly describe the SP, it is necessary to add a face $\{x : \langle x, n_{n+1} \rangle = 0\}$ passing through the origin with d_{n+1} as the associated direction of constraint. Since the domain must equal $\mathbb{R}^n_+$ even after we include this new triplet, this additional face must be a supporting hyperplane to $\mathbb{R}^n_+$ at the origin. As we note in Section 5.3, this face is "fictitious" in the sense that it does not actually form a face of the domain, but it is introduced to properly enlarge the cone of allowed directions of constraint at the origin. A proper choice for the normal is $n_{n+1} \doteq d_{n+1}$ which, by the normalization convention $\langle d_{n+1}, n_{n+1} \rangle = 1$, implies that

$$n_{n+1} \doteq d_{n+1} \doteq \frac{\Sigma_{i=1}^n e_i}{\sqrt{n}}. \tag{3.9}$$

Thus we obtain the representation $\{(d_i, n_i, 0), i = 1, \ldots, n+1\}$ for the GPS SP, where $n_i = e_i$ and d_i is as defined in (3.8) for $i = 1, \ldots, n$ and $n_{n+1} = d_{n+1}$ are as defined in (3.9).

The 2-dimensional GPS SP can be analyzed in a rather straightforward manner. The Lipschitz continuity of the mapping Γ is easily established via a direct construction of a set B that satisfies property (1.2) for the SP, as shown in Figure 4 (cf. also [4]). In higher dimensions, however, non-trivial interactions between the buffers take place and the methods developed in this paper are needed to construct B. In order to make the problem tractable, we first characterize the special structure present in the GPS SP. Lemma 3.1 shows that for any probability vector ρ and dimension n, the directions of constraint defined in (3.8) always span a fixed hyperplane in $\mathbb{R}^n$.

Lemma 3.1. *Let $\rho = (\rho_1, \rho_2, \dots, \rho_n)$ with $\rho_i > 0$ and $\Sigma_{i=1}^n \rho_i = 1$, and let the associated constraint directions $d_i = (e_i - \rho)/(1 - \rho_i)$ for $i = 1, \dots, n$ be given. Define $\mathscr{E} \doteq \{d_i, i = 1, \dots, n\}$. Then* span$(\mathscr{E})$ *is the hyperplane normal to $d_{n+1} = \frac{1}{\sqrt{n}} \Sigma_{i=1}^n e_i$ and the constraint directions satisfy the relation*

$$\Sigma_{i=1}^n \rho_i (1 - \rho_i) d_i = 0. \tag{3.10}$$

Furthermore, any subset of $n-2$ vectors in $\mathscr{E}$ spans an $(n-2)$-dimensional space.

Proof. Since

$$\langle d_{n+1}, d_i \rangle = \left\langle d_{n+1}, \frac{e_i - \rho}{1 - \rho_i} \right\rangle = \frac{1 - \Sigma_{j=1}^n \rho_j}{\sqrt{n}(1 - \rho_i)} = 0,$$

d_{n+1} is normal to span$(\mathscr{E})$. Moreover,

$$\Sigma_{i=1}^n \rho_i (1 - \rho_i) d_i = \Sigma_{i=1}^n (\rho_i e_i - \rho_i \rho) = \Sigma_{i=1}^n \rho_i e_i - \rho \Sigma_{i=1}^n \rho_i = \rho - \rho = 0,$$

which establishes (3.10). Thus the directions of constraint $\{d_i, i = 1, \dots, n\}$ span a space of dimension at most $n-1$. To prove that the span actually has dimension $n-1$, it is necessary to establish that some subset of $n-1$ directions of $\mathscr{E}$ is linearly independent. We shall choose the set $\{d_1, d_2, \dots, d_{n-1}\}$, but the analogous argument works for every subset of $n-1$ directions and thus establishes the slightly stronger result that any set of $n-1$ directions in $\mathscr{E}$ is linearly independent. Suppose there exist α_i such that $\Sigma_{i=1}^{n-1} \alpha_i d_i = 0$. To prove that the $n-1$ constraint directions are linearly independent, we need to show that $\alpha_i = 0$ for $i = 1, \dots, n-1$. Now for any $i < n$, equating the ith and nth components of $\Sigma_{i=1}^{n-1} \alpha_i d_i$ to zero yields

$$-\frac{\alpha_1 \rho_i}{1 - \rho_1} - \frac{\alpha_2 \rho_i}{1 - \rho_2} - \cdots \quad -\alpha_i \quad \cdots - \frac{\alpha_{n-1} \rho_i}{1 - \rho_{n-1}} = 0$$

$$-\frac{\alpha_1 \rho_n}{1 - \rho_1} - \frac{\alpha_2 \rho_n}{1 - \rho_2} - \cdots - \frac{\alpha_i \rho_n}{1 - \rho_i} \cdots - \frac{\alpha_{n-1} \rho_n}{1 - \rho_{n-1}} = 0.$$

Since $\rho_i > 0$ for every i, we can multiply the first equation by ρ_n/ρ_i and subtract it from the second to obtain

$$-\frac{\alpha_i \rho_n}{1-\rho_i} + \frac{\alpha_i \rho_n}{\rho_i} = 0 \quad \Rightarrow \quad \alpha_i(1-2\rho_i) = 0. \tag{3.11}$$

This implies that $\alpha_i = 0$ unless $\rho_i = 1/2$. When $n = 2$, $\rho_1 = \rho_2 = 1/2$, $d_1 = (1, -1)$ and $d_2 = (-1, 1)$, and span($\mathscr{D}$) is the line perpendicular to $d_{n+1} = \frac{1}{\sqrt{2}}(1, 1)$. In all other cases, there is at most one component of ρ that is equal to $1/2$ since by assumption all ρ_i are strictly positive and $\Sigma_{i=1}^{n} \rho_i = 1$. Suppose $\rho_j = 1/2$ for some $j \leq n-1$. If $i \neq j$, then $\rho_i \neq 1/2$ and thus by (3.11), $\alpha_i = 0$. However, this implies that α_j is also zero since $-\alpha_j d_j = \Sigma_{i=1, i\neq j}^{n-1} \alpha_i d_i = 0$. Thus $\alpha_i = 0$ for $1 \leq i \leq n-1$ and span($\mathscr{E}$) is the $(n-1)$-dimensional space normal to d_{n+1}. Finally suppose there exists any subset of $\mathscr{E}$ of cardinality $n-2$ that spans a k-dimensional space for $k < n-2$. Then there exists a subset of $n-1$ vectors of $\mathscr{E}$ that spans an m-dimensional space with $m < n-1$. This contradicts the argument given above and thereby establishes the last assertion of the lemma. □

3.3. Construction of the dual set B^* for the generalized processor sharing SP

In Figure 4 it was shown that for the two-dimensional case, Lipschitz continuity of the SM could be established by directly constructing a set B that satisfies Assumption 1.1. In higher dimensions Lipschitz continuity is obtained by constructing the dual set that satisfies (1.4). In this section we describe how this is done. The construction of the dual set, as outlined in [6, Section 3.3], first involves the identification of the fundamental vertex directions and then the determination of suitable multiplicative constants that make the convex hull of the scaled vertices satisfy property (1.4). Following the methodology outlined there, we let $\mathscr{D} \doteq \{d_i, i = 1, \ldots, n+1\}$ and $\mathscr{E} \doteq \{d_i, i = 1, \ldots, n\}$. Since by Lemma 3.1 d_{n+1} is orthogonal to span($\mathscr{E}$), the subsets of $\mathscr{D}$ which span $(n-1)$-dimensional spaces are easily calculated to be $\mathscr{E} = \{d_i, i = 1, \ldots, n\}$ and sets which are formed by the union of d_{n+1} with subsets of $\mathscr{E}$ that span $(n-2)$-dimensional spaces. Lemma 3.1 identifies the latter subsets to be exactly those that have cardinality $n-2$. Each such subset of $\mathscr{E}$ can therefore be identified by the two constraint directions in $\mathscr{E}$ that it does not contain, and thus the subsets that determine the fundamental vertex directions are given by $\mathscr{E}$ and

$$\mathscr{S}_{ij} \doteq \mathscr{D} \setminus \{d_i, d_j\}$$

for $i > j, i, j \in \{1, \ldots, n\}$. If we define $d_{n+1}^* \doteq d_{n+1}$ and d_{ij}^* to be a unit vector orthogonal to span($\mathscr{S}_{ij}$), then the fundamental vertex directions are given by

$$\mathscr{V} = \{\pm d_{n+1}^*, \pm d_{ij}^* : j > i, i, j \in \{1, \ldots, n\}\}.$$

$\mathscr{V}$ has cardinality $n^2 - n + 2$, which is also equal to the number of faces of the dual set B. As was discussed in [6, Section 3.1], this is a measure of the complexity of the set B associated with the SP. Recall that the gH–R SP required a set with only $2n$ faces, which turns out to be the simplest possible. The conjectured form of the set B^* for the GPS SP is then

$$B^* = \mathrm{conv}\left[\pm a_{n+1} d_{n+1}^*, \pm a_{ij} d_{ij}^* : j > i, i, j \in \{1, \ldots, n\}\right]. \tag{3.12}$$

We first consider a special case in which there is sufficient symmetry to make the problem tractable. Recall that a similar approach was taken in Section 2.2, where the orthonormal basis $\{e_i, i = 1, \ldots, n\}$ served as the simplest set of constraint directions within the gH–R framework. The analogous directions for the processor sharing model are obtained by considering $\rho = (1/n, \ldots, 1/n)$, and we refer to this as the equal sharing case.

A. Equal sharing case ($\rho_i = 1/n$)

We now construct the dual set B^* for the equal sharing SP which has weight vector $\rho = (\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n})$. Here the direction vectors are given by

$$z_i \doteq \frac{n}{n-1} e_i - \frac{1}{n-1} \Sigma_{j=1}^n e_j \tag{3.13}$$

for $i = 1, \ldots, n$ and $z_{n+1} \doteq \Sigma_{i=1}^n e_i / \sqrt{n}$. By Lemma 3.1 the $\{z_i\}$ span the hyperplane with normal $z_{n+1}^* \doteq z_{n+1}$. The analysis here is greatly simplified by the symmetry of the direction vectors which is evident from the following relations when i, j and k are in $\{1, \ldots, n\}$.

$$\begin{aligned}
\langle z_k, z_k \rangle &= \frac{n}{n-1}, \\
\langle z_k, z_j \rangle &= -\frac{n}{(n-1)^2} \quad \text{for } j \neq k, \\
\langle z_k, z_k - z_j \rangle &= \frac{n^2}{(n-1)^2} \quad \text{for } j \neq k, \\
\langle z_k, z_i - z_j \rangle &= 0 \qquad\quad \text{for } k \neq \{i, j\}.
\end{aligned} \tag{3.14}$$

Consider the sets $\mathscr{S}_{ij} = \mathscr{E} \setminus \{z_i, z_j\}$ whose spans are orthogonal to the fundamental vertex directions z_{ij}^*. Since the last relation in (3.14) implies that $\langle x, z_i - z_j \rangle = 0$ for $x \in \mathrm{span}(\mathscr{S}_{ij})$, we obtain the simple expression

$$z_{ij}^* = z_i - z_j = \frac{n}{n-1}(e_i - e_j). \tag{3.15}$$

Thus a conjectured internal representation for the dual set can be expressed succinctly in terms of the constraint directions as

$$B^* = \mathrm{conv}\left[\pm a_{n+1}z_{n+1}, \pm a_{ij}(z_i - z_j) : i, j \in \{1, \ldots, n\}, j > i\right], \quad (3.16)$$

where $a_{n+1}, a_{ij} > 0$.

Our objective is to develop algebraic conditions that determine the range of normals $\{n_i\}$ in the equal sharing case for which (1.4) holds. Transformation techniques can then be applied, as shown in Theorem 3.5, to solve the general GPS problem for arbitrary ρ and dimension n. The verification of property (1.4) for the set B^* requires an efficient way of determining if a point lies in the interior of the set. We will make use of the facts that $L_i^* z_{n+1}^* = z_{n+1}^*$ for $i \in \{1, \ldots, n\}$ and that $L_{n+1}^* z_{n+1}^* = 0$. These properties imply that regardless of the choice of the coefficient a_{n+1}, the points $\pm a_{n+1} z_{n+1}^*$ will always satisfy the conditions required in (1.4). Let $M \doteq \mathrm{span}(\{z_{ij}^*, i, j \in \{1, \ldots, n\}, j > i\})$ and observe that since z_{n+1}^* is orthogonal to M, $M^\perp \doteq \mathrm{span}(\{z_{n+1}^*\})$. A necessary and sufficient for any point of the form $\pm a_{ij} z_{ij}^*$ to be projected into the interior of B^* (for sufficiently large a_{n+1}) is that it be projected into the interior of

$$\mathrm{conv}\left[\pm a_{ij} z_{ij}^* : i, j \in \{1, \ldots, n\}, j > i\right] \times M^\perp.$$

We shall first consider the case when all $a_{ij} = 1$, and define

$$C^* \doteq \mathrm{conv}\left[\pm z_{ij}^* : i, j \in \{1, \ldots, n\}, j > i\right] \times M^\perp.$$

We next derive an external representation of the form

$$C^* = \cap_{\nu \in \mathscr{K}} \{x : |\langle x, \nu\rangle| \le c_\nu\}$$

for some $\gamma > 0$ and a finite set of vectors $\mathscr{K}$. The external representation reduces the problem of verifying (1.4) to checking the linear inequalities $|\langle L_i^* z_{ij}^*, \nu\rangle| < \gamma$ for $\nu \in \mathscr{K}$.

The geometry of the SP for the three-dimensional equal sharing case and the structure of the associated sets B and B^* are illustrated in Figures 5 and 6 respectively. In this case it is not hard to show that the set of normals to the corresponding set C^* is $K = \{\pm z_1, \pm z_2, \pm z_3\}$.

In Theorem 3.3 we show that the structure of the normals can be generalized to the n-dimensional equal sharing case in the following way. The set of normals to C^* always has a simple representation in terms of the original directions of constraint as $\pm z_i$, $\pm(z_i + z_j)$, $\pm(z_i + z_j + z_k)$, …, and so on. (Recall that when dealing with external representations, the normals need not necessarily be normalized to have norm 1.) For any $p \in I\!N$ let a sum

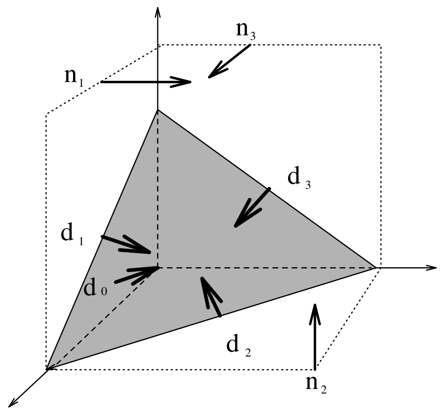

Fig. 5. The three-dimensional equal sharing SP

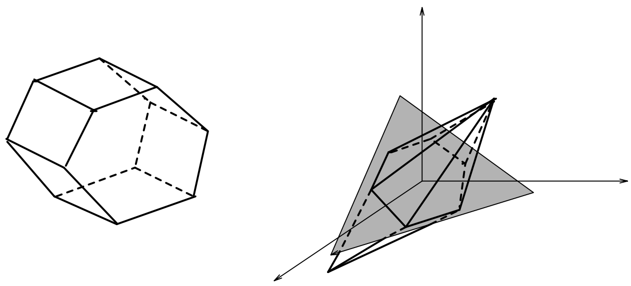

Fig. 6. The sets B and B^* for the equal sharing GPS SP

of p distinct vectors be called a p-sum. In dimension n the normals to C^* are all possible p-sums of the constraint directions $\mathscr{D} = \{d_1, \ldots, d_n\}$ for $p < n$. This indicates that the number of normals increases with dimension as $2^n - 2$.

Remark 3.2. It is worthwhile to note the remarkable connection between the geometry of the particular class of SPs arising from equal sharing GPS models that is considered here and the theory of root systems that arises in the study of simple Lie Groups [1, p. 188]. It turns out that the set $[e_i - e_j, i, j \in \{1, \ldots, n\}, i \neq j]$ of vertices of a scaled version of C^* is the root system for the Lie Algebra A_{n-1} associated with the Lie Group sl_n [1, p. 250]. A knowledge of root system theory then allows one to directly guess that the set K of normals to C^* can be obtained as Weyl transformations of the fundamental weights of the dual root system. The fact that A_{n-1} is self-dual, the associated Weyl group is the set of permutations on $\{1, \ldots, n\}$ and the fundamental weights of A_{n-1} are given by $\Sigma_{i=1}^{j} e_i$, $j = 1, \ldots, n-1$ can then be used to arrive at the general structure of K given above.

Let $T \doteq \{1, \ldots, n\}$, $U \doteq \{e_i - e_j, i > j\}$ and for $S \subset T$, $S \notin \{\emptyset, T\}$, let

$$\gamma_S \doteq \frac{n-1}{n} \Sigma_{i \in S} z_i = \left\{ \Sigma_{i \in S} \left(e_i - \frac{1}{n} \Sigma_{j=1}^n e_j \right) \right\}.$$

Define $\mathcal{Q} \doteq \{\gamma_S : S \subset T, S \notin \{\emptyset, T\}\}$ and recall $M = \{x : \langle x, z_{n+1}^* \rangle = 0\}$. Note that since $U = \frac{n-1}{n}\{\pm z_{ij}^* : i, j \in \{1, \ldots, n\}, j > i\}$, $U \perp M$. For simplicity, we first find a convenient external representation for a scaled version of the projection of C^* onto the subspace M.

Theorem 3.3. *Let* $A \doteq \mathrm{conv}[U]$. *Then* $\cap_{\gamma \in \mathcal{Q}} \{x \in M : \langle x, \gamma \rangle \leq 1\}$ *is an external representation for the set* A.

Proof. Define $\tilde{A} \doteq \cap_{\gamma \in \mathcal{Q}} \{x \in M : \langle x, \gamma \rangle \leq 1\}$. We first show that $A \subset \tilde{A}$. Observe that for every $S \subset T$, $S \notin \{\emptyset, T\}$

$$\langle e_i - e_j, \gamma_S \rangle = \begin{cases} 1 & \text{if } S \cap \{i, j\} = \{i\}, \\ -1 & \text{if } S \cap \{i, j\} = \{j\}, \\ 0 & \text{otherwise}. \end{cases} \tag{3.17}$$

Thus for every vertex $e_i - e_j \in U$, $i \neq j$, and every $\gamma \in \mathcal{Q}$, $\langle e_i - e_j, \gamma \rangle \leq 1$. Since A is convex, it follows that for every $x \in A$, $\langle x, \gamma \rangle \leq 1$ for each $\gamma \in \mathcal{Q}$ and consequently that $A \subset \tilde{A}$.

We now establish that $\tilde{A} \subset A$. Let $U^* \doteq \{\gamma \in M : \max_{u \in U} \langle u, \gamma \rangle \leq 1\}$. Since $A = \mathrm{conv}[U]$, for every $x \in M$ $\max_{u \in U} \langle u, x \rangle = \max_{u \in A} \langle u, x \rangle$. Therefore, as stated in [6, Equation (3.7)], the fact that $A = (A^*)^*$ yields the following external representation for A :

$$A = \left\{ x \in M : \max_{\gamma \in U^*} \langle x, \gamma \rangle \leq 1 \right\}. \tag{3.18}$$

Now fix any $\gamma \in U^*$ and consider its unique representation in terms of the standard orthonormal basis $\{e_i, i = 1, \ldots, n\}$:

$$\gamma = \Sigma_{i=1}^n \lambda_i e_i.$$

Given any $\gamma \in U^*$, define $\hat{\gamma}$ by

$$\begin{aligned} \hat{\gamma} &= \gamma - \left(\min_{j \in T} \lambda_j\right) \Sigma_{i=1}^n e_i \\ &= \Sigma_{i=1}^n \left(\lambda_i - \min_{j \in T} \lambda_j\right) e_i \\ &= \Sigma_{i=1}^n \hat{\lambda}_i e_i, \end{aligned} \tag{3.19}$$

where the last equality defines the coefficients $\hat{\lambda}_i$. With this definition, $\hat{\lambda}_i \geq 0$ for every $i \in T$ and $\min_{i \in T} \hat{\lambda}_i = 0$.

Define

$$\hat{\mathscr{Q}} \doteq \{w_S : S \subset T, S \notin \{\emptyset, T\}\},$$

where for $S \subset T$, $S \notin \{\emptyset, T\}$,

$$w_S = \Sigma_{i \in S} e_i. \tag{3.20}$$

We now state a claim that is the key step in obtaining a finite external representation for A.

Claim . $\{\hat{\gamma} : \gamma \in U^*\} \subset \text{conv}[\hat{\mathscr{Q}}]$.

Suppose the claim were true. Since $\Sigma_{i=1}^n e_i$ is orthogonal to M, if γ is replaced by $\hat{\gamma}$ in the description (3.18) for A the set A remains unaltered. Thus

$$A = \left\{ x \in M : \max_{\gamma \in U^*} \langle x, \hat{\gamma} \rangle \leq 1 \right\}. \tag{3.21}$$

Similarly, since $\mathscr{Q}$ and $\hat{\mathscr{Q}}$ are equivalent sets modulo $\Sigma_{i=1}^n e_i$ and $\tilde{A} \subset M$, we can rewrite

$$\tilde{A} = \cap_{\gamma \in \hat{\mathscr{Q}}} \{x \in M : \langle x, \gamma \rangle \leq 1\}. \tag{3.22}$$

Then it follows from the last two displays and the claim that $\tilde{A} \subset A$.

We now prove the claim. By definition, for any $\gamma \in U^*$, $\hat{\gamma}$ has a representation $\hat{\gamma} = \Sigma_{i=1}^n \hat{\lambda}_i e_i$, where $\hat{\lambda}_i \geq 0$ for every $i \in T$ and $\min_{i \in T} \hat{\lambda}_i = 0$. Now suppose that the representation for $\hat{\gamma}$ is such that $\hat{\lambda}_1 \geq \hat{\lambda}_2 \geq \cdots \geq \hat{\lambda}_n = 0$. Then we claim that $\hat{\gamma}$ can be written as a convex combination of $\{w_{\{1,\ldots,j\}}, j = 1, \ldots, n-1\}$, where w_S is defined in (3.20). Rewriting $\hat{\gamma}$ in terms of the w's yields

$$\hat{\gamma} = \Sigma_{i=1}^n \hat{\lambda}_i e_i = \Sigma_{i=1}^{n-1} \beta_i w_{\{1,\ldots,i\}},$$

where $\beta_{n-1} \doteq \hat{\lambda}_{n-1}$ and $\beta_{n-k} \doteq \hat{\lambda}_{n-k} - \hat{\lambda}_{n-k+1}$ for $k = 2, \ldots, n-1$.

The monotonicity assumption on the $\hat{\lambda}_i$'s implies that $\beta_i \geq 0$ for all i. Since $\max_{u \in U} \langle u, \gamma \rangle \leq 1$ and $\hat{\lambda}_n = 0$, we have in particular

$$\langle e_1 - e_n, \hat{\gamma} \rangle = \hat{\lambda}_1 - \hat{\lambda}_n = \hat{\lambda}_1 \leq 1$$

This shows that $\hat{\gamma}$ is a convex combination of the w's since

$$\Sigma_{k=1}^{n-1} \beta_k = \hat{\lambda}_{n-1} + \Sigma_{k=1}^{n-2} (\hat{\lambda}_k - \hat{\lambda}_{k+1}) = \hat{\lambda}_1 \leq 1.$$

Now for $\hat{\gamma} = \Sigma_{i=1}^{n} \hat{\lambda}_i e_i$, we consider the general case where

$$0 = \min_{i=1,\dots,n-1} \hat{\lambda}_i \leq \max_{i=1,\dots,n-1} \hat{\lambda}_i \leq 1.$$

Note that the last inequality is obtained once again using the fact that $\max_{u \in U} \langle u, \hat{\gamma} \rangle \leq 1$. If $j \doteq \operatorname{argmin}_{i=1,\dots,n-1} \hat{\lambda}_i$ and $k \doteq \operatorname{argmax}_{i=1,\dots,n-1} \hat{\lambda}_i$, then u is chosen to be $e_k - e_j$ to obtain the desired result. The appropriate subset of vectors in $\hat{\mathcal{Q}}$ that includes $\hat{\gamma}$ in its convex hull can now be identified via the following procedure. Define σ to be a permutation on $\{1, \dots, n\}$ such that $\hat{\lambda}_{\sigma(1)} \geq \hat{\lambda}_{\sigma(2)} \cdots \geq \hat{\lambda}_{\sigma(n)} = 0$. Then

$$\hat{\gamma} = \Sigma_{i=1}^{n-1} \hat{\lambda}_{\sigma(i)} e_{\sigma(i)}$$

and the above argument can once again be used to show that $\hat{\gamma}$ lies in the convex hull of the subset of $\hat{\mathcal{Q}}$ given by $\{w_j^{\sigma}, j = 1, \dots, n-1\}$, where $w_j^{\sigma} \doteq \Sigma_{k=1}^{j} e_{\sigma(k)} \in \hat{\mathcal{Q}}$. This establishes the claim, from which it follows that $\tilde{A} \subset A$. Along with the earlier conclusion that $A \subset \tilde{A}$, we infer that $\tilde{A}$ is indeed a valid finite external representation for A.

We now show that this external representation is minimal in the sense that every vector in $\mathcal{Q}$ is normal to a legitimate face of $A \subset M$. Given any $S \subset T$, $S \notin \{\emptyset, T\}$, fix $i_0 \in S$ and $j_0 \in T \setminus S$. Then from (3.17) it can be seen that all $n-1$ vertices in the set $F \doteq \{e_i - e_{j_0}, e_{i_0} - e_j, i \in S, j \notin S\}$ lie on the hyperplane $\{x : \langle x, \gamma_S \rangle = 1\}$. (Note that F has $n-1$ vertices since there are $|S|$ vectors in the set $\{e_i - e_{j_0}, i \in S\}$, $|S^c|$ vectors in the set $\{e_{i_0} - e_j, j \notin S\}$, $|S \cup S^c| = |T| = n$ and $e_{i_0} - e_{j_0}$ belongs to both sets.) Moreover, these vertices also lie on the hyperplane M and thus span at most an $(n-2)$-dimensional space. However, since the $n-2$ vertices in $F \setminus \{e_{i_0} - e_{j_0}\}$ are linearly independent, $\dim[\operatorname{span}(F)] = n-2$. Thus every vector in $\mathcal{Q}$ is normal to an $(n-2)$-dimensional face of A and so the given external representation is minimal. $\square$

Corollary 3.4. *The set C^* with representation*

$$\operatorname{conv}\left[\pm z_{ij}^* : i, j \in \{1, \dots, n\}, j > i\right] \times M^{\perp}$$

is equal to

$$\bigcap_{S \subset T, S \notin \{\emptyset, T\}} \left\{ x \in I\!R^n : \langle x, \nu_S \rangle \leq \frac{n^2}{(n-1)^2} \right\},$$

where

$$\nu_S \doteq \Sigma_{i \in S} z_i. \tag{3.23}$$

Proof. Let A be such that span(A) is a hyperplane in $\mathbb{R}^n$. If $\mathrm{conv}[A] = \cap_{\gamma\in\mathscr{K}}\{x \in \mathrm{span}(A) : \langle x, \gamma\rangle \leq c_\gamma\}$, for some set $\mathscr{K} \subset \mathrm{span}(A)$ and $c_\gamma \in \mathbb{R}$, the external representation for $\mathrm{conv}[A] \times \mathrm{span}(A)^\perp$ is $\cap_{\gamma\in\mathscr{K}}\{x \in \mathbb{R}^n : \langle x, \gamma\rangle \leq c_\gamma\}$. Thus applying the affine transformation $x \to \dfrac{n}{n-1}x$ in M and noting that the transformation takes γ_S to ν_S, the corollary follows directly from Theorem 3.3. □

We now use the minimal representation given in Corollary 3.4 to derive algebraic conditions on the normals under which the equal GPS SP is Lipschitz continuous. We recall the definition $\nu_S \doteq \Sigma_{i\in S} z_S$. Recall that $\{n_i\}$ denotes the set of inward normals in the description of the SP.

Theorem 3.5. *Let* z_i *and* z^*_{ij} *be as defined in* (3.13) *and* (3.15) *respectively. Let a set of normals* n_i *satisfying the standard normalization* $\langle n_i, z_i\rangle = 1$ *for* $i = 1, \ldots, n+1$ *be given. Suppose for every* $i = 1, \ldots, n$ *and* $S \subset T \setminus \{i\}$ *satisfying* $1 \leq |S| \leq n-2$ *that*

$$|\langle n_i, \nu_S\rangle| < 1. \tag{3.24}$$

Then there exists $a_{n+1} \in (0, \infty)$ *such that the set*

$$B^* = \mathrm{conv}[\pm a_{n+1} z^*_{n+1}, \pm z^*_{ij} : i, j \in \{1, \ldots, n\}, j > i]$$

satisfies property (1.4) *for the SP* $\{(z_i, n_i, 0), i = 1, \ldots, n+1\}$.

Proof. Note that B^* is symmetric and that L^*_{n+1} maps $\pm a_{n+1} z^*_{n+1}$ to the origin and leaves all other vertices z^*_{ij} invariant. Also note that L^*_i leaves all vertices other than $\pm z^*_{ij}$, $j \neq i$ and $\pm z^*_{ji}$, $j \neq i$ alone. Thus to establish property (1.4) for the set B^*, it is enough to show that for every $1 \leq i \leq n$ and every $j \neq i$, $L^*_i z^*_{ij}$ lies in the interior of the set B^*. It is necessary and sufficient to show that these points lie in the interior of $C^* = \mathrm{conv}[\pm z^*_{ij} : i, j \in \{1, \ldots, n\}, j > i] \times M^\perp$, where we recall the definition $M = \mathrm{span}(\{\pm z^*_{ij} : i, j \in \{1, \ldots, n\}, j > i\})$. If this is true, then one can clearly choose a_{n+1} large enough so that all the projected points lie in the interior of B^*. Using the external representation for C^* derived in Corollary 3.4, observe that (1.4) is satisfied if and only if for every $i \in T$, $j \in T \setminus \{i\}$, and $S \subset T$ such that $1 \leq |S| \leq n-1$,

$$|\langle L^*_i z^*_{ij}, \nu_S\rangle| < \frac{n^2}{(n-1)^2}. \tag{3.25}$$

From (3.14) we see that

$$\begin{aligned} L^*_i z^*_{ij} &= z_i - z_j - \langle z_i - z_j, z_i\rangle n_i \\ &= z_i - z_j - \frac{n^2}{(n-1)^2} n_i. \end{aligned}$$

To show (3.25), we first observe that due to (3.10) in Lemma 3.1, $\Sigma_{i=1}^n z_i = 0$ and thus for any $S \subset T$, $S \notin \{\emptyset, T\}$, $\nu_{S^c} = -\nu_S$. We now fix $i \in T$ and $j \in T \setminus \{i\}$ and consider the four types of sets $S \subset T$ such that $1 \leq |S| \leq n-1$.

Case 1: $S \cap \{i, j\} = \emptyset$, $S \neq \emptyset$.
From (3.14) we have $\langle z_i - z_j, \nu_S\rangle = 0$. Hence for such S, $1 \leq |S| \leq n-2$ and

$$\left|\langle L_i^* z_{ij}^*, \nu_S\rangle\right| = \left|-\frac{n^2}{(n-1)^2}\langle n_i, \nu_S\rangle\right| < \frac{n^2}{(n-1)^2}, \tag{3.26}$$

where the last inequality follows from (3.24).

Case 2: $S \cap \{i, j\} = \{i, j\}$, $S \neq T$.
Then $S^c \cap \{i, j\} = \emptyset$ and $S^c \neq \emptyset$. Thus (3.26) applies to S^c and since $\nu_{S^c} = -\nu_S$ we obtain

$$|\langle L_i^* z_{ij}^*, \nu_S\rangle| = |\langle L_i^* z_{ij}^*, \nu_{S^c}\rangle| < \frac{n^2}{(n-1)^2}.$$

Case 3: $S \cap \{i, j\} = \{i\}$.
Then from (3.14) $\langle z_i - z_j, \nu_S\rangle = \dfrac{n^2}{(n-1)^2}$ and hence

$$|\langle L_i^* z_{ij}^*, \nu_S\rangle| = \left|\frac{n^2}{(n-1)^2} - \frac{n^2}{(n-1)^2}\langle n_i, \nu_S\rangle\right| = \frac{n^2}{(n-1)^2}\, |1 - \langle n_i, \nu_S\rangle| \,. \tag{3.27}$$

If $S = \{i\}$ (so that $\nu_S = z_i$), then $\langle n_i, z_i\rangle = 1$ implies that the condition (3.25) is satisfied automatically. Otherwise $j \notin S$, $i \in S$, and $2 \leq |S| \leq n-1$. If this is the case, we define $S' \doteq S \setminus \{i\}$ so that $S' \cap \{i, j\} = \emptyset$ and $1 \leq |S| \leq n-2$. Then using (3.27) and applying (3.26) to S' we get

$$|\langle L_i^* z_{ij}^*, \nu_S\rangle| = \frac{n^2}{(n-1)^2}\, |1 - \langle n_i, z_i\rangle - \langle n_i, \nu_{S'}\rangle| = \frac{n^2}{(n-1)^2}\, |\langle n_i, \nu_{S'}\rangle| < 1. \tag{3.28}$$

Case 4: $S \cap \{i, j\} = \{j\}$
Here once again we use the facts that $S^c \cap \{i, j\} = \{i\}$, and $\nu_{S^c} = -\nu_S$ to conclude from (3.28) that $|\langle L_i^* z_{ij}^*, \nu_S\rangle| < 1$.

Since the four cases considered above exhaust all sets $S \subset T$ for which $1 \leq |S| \leq n-1$, the theorem is established. □

Corollary 3.6. *The map* $\Gamma : \psi \to \phi$ *for the equal sharing case with* $\rho = (\frac{1}{n}, \ldots, \frac{1}{n})$ *is Lipschitz continuous on its domain of definition.*

Proof. When $n = 2$, the mapping was shown in Section 3.2 to be Lipschitz continuous. For the equal sharing case, we verify the conditions of Theorem 3.5 for normals $n_i = e_i, i = 1, \ldots, n$. Since $\langle e_i, z_j \rangle = -\frac{1}{n-1} < 0$ whenever $j \neq i$,

$$\max_{S \subset T \setminus \{i\}: 1 \le |S| \le n-2} |\langle e_i, \nu_S \rangle| = \frac{n-2}{n-1} < 1$$

and so (3.24) holds. Thus by Theorem 3.5 the set B^* satisfies property (1.4), and therefore the mapping is Lipschitz continuous. □

B. The general case (arbitrary ρ)

Theorem 3.7. *For any given ρ and n, consider the GPS SP $\{(d_i, n_i, 0), i = 1, \ldots, n+1\}$, where $n_i = e_i$, $d_i = (e_i - \rho)/(1-\rho_i)$ for $i = 1, \ldots, n$, and $e_{n+1} = d_{n+1} = \Sigma_{i=1}^n e_i/\sqrt{n}$. The associated SM is Lipschitz continuous.*

Proof. Let $\{z_i, i = 1, \ldots, n+1\}$ be the directions of constraint associated with the equal sharing GPS SP which has $\rho = (1/n, \ldots, 1/n)$. Let $\mathscr{C}$ be the collection of SPs $\{(z_i, w_i, k_i), i = 1, \ldots, n+1\}$ that satisfy (3.24). Then by Theorem 3.5, Assumption 1.1 is satisfied for all SPs in $\mathscr{C}$. For any probability vector ρ consider the associated GPS SP $\{(d_i, e_i, 0), i = 1, \ldots, n+1\}$. Recall from Lemma 3.1 that

$$\Sigma_{i=1}^n \rho_i (1-\rho_i) d_i = 0. \tag{3.29}$$

In particular for the equal sharing case,

$$\Sigma_{i=1}^n \frac{n-1}{n^2} z_i = 0. \tag{3.30}$$

For $i = 1, \ldots, n$ define

$$\tilde{d}_i \doteq \rho_i(1-\rho_i) d_i \quad \text{and} \quad \tilde{n}_i \doteq \frac{1}{\rho_i(1-\rho_i)} e_i, \tag{3.31}$$

and $\tilde{d}_{n+1} = \tilde{n}_{n+1} = d_{n+1}$. Then $\langle \tilde{d}_i, \tilde{n}_i \rangle = 1$, and it is clear that $\{(\tilde{d}_i, \tilde{n}_i, 0), i = 1, \ldots, n+1\}$ satisfies Assumption 1.1 if and only if $\{(d_i, e_i, 0), i = 1, \ldots, n+1\}$ does, since they both represent the same SP. We first show that there exists an invertible matrix A such that $A\tilde{d}_i = z_i$ for $i = 1, \ldots, n+1$. By Lemma 3.1 $\{\tilde{d}_i, i = 1, \ldots, n-1\}$ and $\{z_i, i = 1, \ldots, n-1\}$ are bases for the space orthogonal to $\tilde{d}_{n+1} = z_{n+1}$. Suppose A is the unique invertible matrix such that

$$A\tilde{d}_i = z_i, \qquad i = 1, \ldots, n-1,$$
$$A\tilde{d}_{n+1} = z_{n+1}.$$

Then by the definition in (3.31) and the relations (3.29) and (3.30),

$$\begin{aligned}A\tilde{d}_n &= A[\rho_n(1-\rho_n)d_n]\\&= -A[\Sigma_{i=1}^{n-1}\rho_i(1-\rho_i)d_i]\\&= -\Sigma_{i=1}^{n-1}A\tilde{d}_i\\&= -\Sigma_{i=1}^{n-1}z_i\\&= z_n.\end{aligned}$$

Thus $A\tilde{d}_i = z_i, i = 1, \ldots, n+1$. Consider the SP $\{(z_i, (A^T)^{-1}\tilde{n}_i, 0), i = 1, \ldots, n+1\}$. Since $\rho_i \in (0,1)$ for any $i \in \{1, \ldots, n\}$ and $j \in \{1, \ldots, n\} \setminus \{i\}$,

$$\begin{aligned}\langle (A^T)^{-1}\tilde{n}_i, z_j\rangle = \langle (A^T)^{-1}\tilde{n}_i, A\tilde{d}_j\rangle = \langle \tilde{n}_i, \tilde{d}_j\rangle &= \frac{\rho_j(1-\rho_j)}{\rho_i(1-\rho_i)}\langle e_i, d_j\rangle\\&= \frac{\rho_j(1-\rho_j)}{\rho_i(1-\rho_i)}(-\frac{\rho_i}{1-\rho_j})\\&= -\frac{\rho_j}{1-\rho_i}\\&< 0.\end{aligned}$$

If $\tilde{\nu}_S \doteq \Sigma_{i\in S}\tilde{z}_i$ then the maximum value of $\langle (A^T)^{-1}\tilde{n}_i, \tilde{\nu}_S\rangle$ for $S \subset N \setminus \{i\}$ with $1 \le |S| \le n-2$ is

$$\max_{j\neq i}\left| -\Sigma_{k\in N\setminus\{i,j\}}\frac{\rho_k}{1-\rho_i}\right|. \tag{3.32}$$

Since $\rho_j > 0$ and $\Sigma_{j=1}^n \rho_j = 1$, for every $i \in N$ and $j \neq i$, we have

$$1-\rho_i = \Sigma_{k\in N\setminus\{i\}}\rho_k > \Sigma_{k\in N\setminus\{i,j\}}\rho_k > 0,$$

from which it directly follows that the quantity in (3.32) is strictly less than 1. Thus $\{(z_i, (A^T)^{-1}\tilde{n}_i, 0), i = 1, \ldots, n\}$ satisfies (3.24) and hence belongs to $\mathscr{C}$. Then by [6, Theorem 3.7] the transformed SP $\{(A^{-1}z_i, A^T(A^T)^{-1}\tilde{n}_i, 0), i = 1, \ldots, n+1\} = \{(\tilde{d}_i, \tilde{n}_i, 0), i = 1, \ldots, n+1\}$ satisfies Assumption 1.1. Thus so does the GPS SP $\{(d_i, e_i, 0), i = 1, \ldots, n+1\}$ and consequently by [6, Theorems 3.2 and 2.1], the associated SM is Lipschitz continuous. □

3.4. Existence of solutions to the generalized processor sharing SP

The terminology and tools used here to establish existence of solutions to the GPS SP are taken from [6, Section 4]. Recall that the GPS SP described in Section 3 takes the form $\{(d_i, n_i, 0), i = 1, \ldots, n+1\}$, where for $i = 1, \ldots, n$,

$$d_i = \frac{e_i - \rho}{1 - \rho_i}, \quad n_i = e_i,$$

and $d_{n+1} = n_{n+1} = \Sigma_{i=1}^n e_i/\sqrt{n}$.

Theorem 3.8. *Consider the GPS SP specified above. The SM is well defined and Lipschitz continuous on $\mathscr{F}_G$ and has a Lipschitz continuous extension to $D_G([0, \infty) : I\!\!R^n)$.*

Proof. Let $\tilde{I} = \{1, \ldots, n+1\}$ and

$$c_i^k \doteq \begin{cases} -\frac{1}{k} & i \leq n, \\ & \\ 0 & i = n+1. \end{cases}$$

Then the GPS SP described above can be approximated by the sequence of simple SPs $P^k \doteq \{(d_i, n_i, c_i^k), i = 1, \ldots, n+1\}$, $k \in I\!\!N$, each of which has n vertices given by

$$v_j^k = \left(\cap_{i \in I_j} \{x : \langle x, n_i \rangle = -1/k\}\right) \cap \{x : \langle x, n_{n+1} \rangle = 0\},$$

where $I_j = \{1, \ldots, n\} \setminus \{j\}$. The standard SPs associated with P^k are

$$P_j^k \doteq \{(d_i, n_i, c_i^k), i \in \tilde{I} \setminus \{j\}\}.$$

The approximating simple SPs and the associated standard SPs corresponding to the two-dimensional GPS SP are shown in Figure 7.

We now show that every P_j^k for $j = 1, \ldots, n$ and $k \in I\!\!N$ is of the gH–R class. Define $V = [v_{ij}] = [\langle d_i, e_j \rangle]$ to be the $n \times n$ matrix that is associated with the standard SP P_n^k. Note that the matrix is independent of the value of k. We have

$$v_{ij} = \begin{cases} 1 & \text{if } i = j \leq n \\ -\dfrac{\rho_i}{1 - \rho_j} & \text{if } i, j < n, i \neq j, \\ 0 & \text{if } i < j = n \\ \frac{1}{\sqrt{n}} & \text{if } j < i = n. \end{cases} \tag{3.33}$$

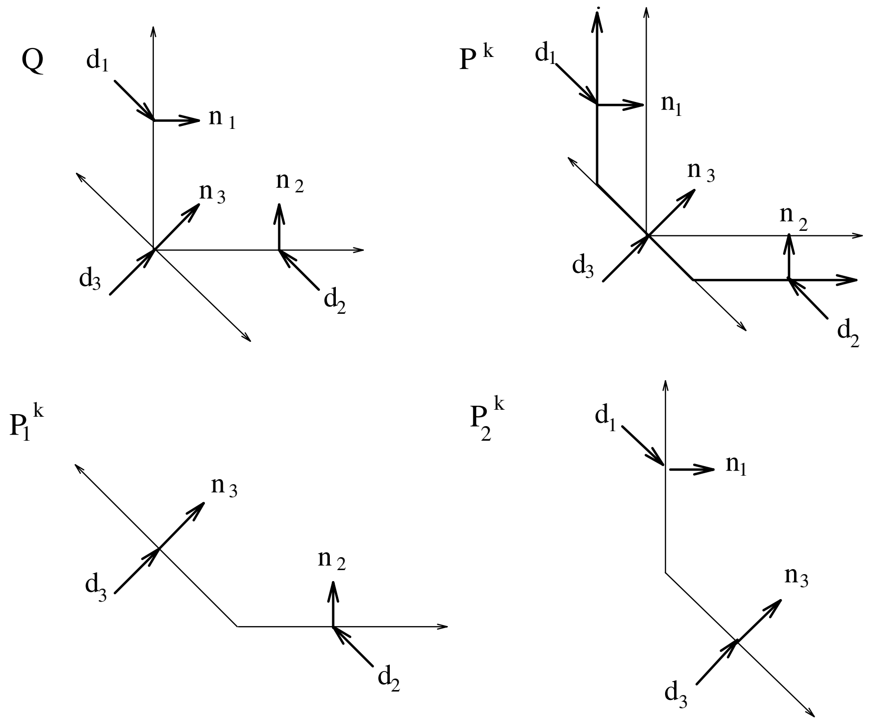

Fig. 7. The two-dimensional GPS SP, an approximating simple SP and its associated standard SPs

Then $Q = |I - V|$ has the form

$$\begin{bmatrix} Q' & 0 \\ A & 0 \end{bmatrix},$$

where Q' is an $(n-1) \times (n-1)$ substochastic matrix whose ith row sums to $(1 - \rho_i - \rho_n)/(1 - \rho_i) < 1$, and A is the $1 \times (n-1)$ row vector $[1/\sqrt{n}, \ldots, 1/\sqrt{n}]$. Thus $\sigma(Q') < 1$. Since the spectrum of Q is equal to zero plus the spectrum of Q', it follows that $\sigma(Q) < 1$. As shown in Section 2.3 solutions exist on $D([0, \infty) : I\!R^n)$ for standard SPs satisfying the gH–R condition. Thus in particular a global projection exists for every such standard SP. The matrix V for P_j^k is analogous to that for P_n^k which was defined above, and in fact the same spectral radius condition holds for P_j^k for every $j = 1, \ldots, n$ and $k > 0$. Thus a projection exists for each P_j^k.

Since the standard SPs P_j^k, $j = 1, \ldots, n$ are derived from the simple SP P^k, it is easy to see that they form a consistent collection of SPs, whose composite SP is P^k. For every $j = 1, \ldots, n$,

$$\overline{\partial G^k \setminus \left(\partial G^k \cap \partial G_j^k\right)} = \{x : \langle x, n_j \rangle = -1/k\} \cap \partial G^k.$$

Since

$$\bigcap_{j=1}^{n}\{x : \langle x, n_j\rangle = -1/k\} = \left\{-\Sigma_{i=1}^{n} e_i/k\right\}$$

is not contained in G^k, we conclude that

$$\bigcap_{j=1}^{n} \overline{\partial G^k \setminus \left(\partial G^k \cap \partial G_j^k\right)} = \emptyset.$$

Thus the consistent collection $\{P_j^k, j = 1, \ldots, n\}$ satisfies the condition given in Theorem 4.5 of [6] for every $k \in I\!N$. It was proved in Theorem 3.7 that Assumption 1.1 is satisfied for the GPS SP. Consequently it is also satisfied for each approximating simple SP P^k since Assumption 1.1 is independent of the values c_i in the representation of the SP. Hence each P^k, $k \in I\!N$, satisfies the conditions of [6, Theorem 4.5] and therefore possesses a global projection. [6, Theorem 4.10] then guarantees the existence of a projection for the GPS SP. By [6, Theorem 4.2] the SM is well defined and Lipschitz continuous on $\mathscr{F}_G$, (which includes all functions of bounded variation in $D_G([0, \infty) : I\!R^n)$) and has a Lipschitz continuous extension to $D_G([0, \infty) : I\!R^n)$. $\square$

Remark 3.9. Since the GPS SP does not satisfy [6, Assumption 4.1], we cannot apply [6, Theorem 4.2] to conclude that η has bounded variation for all ψ. However, it turns out that a principal functional of interest is the component of η that lies along n_{n+1} and this is of bounded variation. This functional represents the idle time process which measures the amount of time that the buffer is empty. Finally we note that the analysis of the SP arising from the GPS model considered in this section would also be useful in establishing the existence of a corresponding non-semimartingale Brownian motion on the n-dimensional orthant.

4. SPs with normal directions of constraint

We consider the class of SPs on polyhedral domains with normal directions of constraint $d_i = n_i$. Any problem in this class has the representation $\{(n_i, n_i, c_i), i = 1, \ldots, N\}$ for some $N < \infty$. Lipschitz continuity of the SM for this class of SPs was first established in [4]. The proof presented there, which involves the construction of a set B satisfying property (1.2) for the associated SP, is rather long and complicated. Here we exploit the dual formulation developed in [6, Section 3.2] to provide a simpler and more succinct proof of the same result. We show that any SP with normal directions of constraint satisfies Assumption 4.1 stated below. Since this property was shown to be equivalent to Assumption 1.2 in [7, Theorem 3.2], this establishes Lipschitz continuity of the map.

Assumption 4.1. *Let L_i^*, $i = 1, \ldots, N$, be the adjoint operators defined in* (1.3) *and let H_i be the corresponding invariant spaces. Then there exists a symmetric compact set S with* $\mathrm{span}(S) = I\!R^n$ *such that for $B^* \doteq \mathrm{conv}[S]$ and each $i = 1, \ldots, N$,*

$$\mathrm{cl}\left(L_i^*[S \setminus H_i]\right) \subset (B^*)^\circ. \tag{4.34}$$

We first introduce some notation that will be used in the proof. For $r > 0$, define $S_r \doteq \{x \in I\!R^n : ||x|| = r\}$ and $Q_r \doteq \{x \in I\!R^n : ||x|| \leq r\}$. The set $\mathscr{D} \doteq \{n_1, n_2, \ldots, n_N\}$ is used to denote the set of directions of constraint, where the usual normalization implies that $\|n_i\| = 1$. For each $j = 1, \ldots, n-1$, we define the collection of sets

$$\mathscr{D}^j \doteq \{A \subset \mathscr{D} : \dim[\mathrm{span}(A)] = n - j \text{ and}$$
$$\forall w \in \mathscr{D} \setminus A, \dim[\mathrm{span}(A \cup \{w\})] = n - j + 1\}.$$

Thus $\mathscr{D}^j$ is the set of subsets of $\mathscr{D}$ that span $(n-j)$-dimensional subspaces of $I\!R^n$, where each subset is maximal in the sense that it contains all vectors of $\mathscr{D}$ that lie in its span. Let

$$H_A \doteq \{x \in I\!R^n : \langle x, n_i \rangle = 0 \text{ for } n_i \in A\}.$$

For $A \in \mathscr{D}^j$, H_A is the j-dimensional subspace orthogonal to $\mathrm{span}(A)$. If $A \in \mathscr{D}^{n-1}$, then A contains just one element, say n_i, so that $H_A = H_{\{n_i\}}$ is the hyperplane orthogonal to n_i. For notational convenience we denote $H_{\{n_i\}}$ by H_i. If $A \subset B$, then clearly $H_B \subset H_A$. The δ-fattening of the linear space H_A is defined to be

$$H_A^\delta \doteq \{x \in I\!R^n : d(x, H_A) < \delta\},$$

where $d(x, H_A) = \inf_{y \in H_A} ||x - y||$, with $\| \cdot \|$ representing the usual Euclidean norm.

Theorem 4.1. *Consider any SP $\{(n_i, n_i, c_i), i = 1, \ldots, N\}$, where $N < \infty$. Then the associated SM is Lipschitz continuous.*

Proof. To establish Lipschitz continuity of the SM it suffices to construct a set $B^* \in \mathscr{S}$ that satisfies Assumption 4.1. We now describe the construction of B^*, which is built as the convex hull of a sequence of flattened spheres of decreasing radius and increasing dimension. The set is defined in terms of the parameters $\delta_k > 0$, $k = 2, \ldots, n-1$ and $r_1 > r_2 > \cdots > r_n > 0$ as follows.

$$B^* \doteq \operatorname{conv}\left[\cup_{i=1}^n C_i\right], \tag{4.35}$$

where

$$C_1 \doteq S_{r_1} \cap \left(\cup_{A\in\mathscr{D}^1} H_A\right), \tag{4.36}$$

for $k = 2, \ldots, n-1$

$$C_k \doteq S_{r_k} \cap \left[\cup_{A\in\mathscr{D}^k}\left(H_A \setminus \cup_{\{j:n_j\in\mathscr{D}\setminus A\}} H_j^{\delta_k}\right)\right], \tag{4.37}$$

and

$$C_n \doteq S_{r_n}. \tag{4.38}$$

The set B^* is clearly convex and symmetric. For any $r > 0$ and $\delta > 0$, S_r, H_A and $H_A \setminus \cup_{\{j:n_j\in\mathscr{D}\setminus A\}} H_j^\delta$ are closed. Thus C_i is closed for every $i \in \{1, \ldots, n\}$ and since $B^* \subset S_{r_1}$ we conclude B^* is compact. Moreover, $0 \in (C_n)^\circ = (S_{r_n})^\circ \subset (B^*)^\circ$ and thus $B^* \in \mathscr{S}$.

For $j = 1, \ldots, n$ let

$$\tilde{\rho}_j \doteq \max_{i=1,\ldots,N} \max_{x\in C_j\setminus H_i} \|L_i^* x\| \quad \text{and} \quad \rho_j \doteq \max_{l=1,\ldots,j} \tilde{\rho}_l. \tag{4.39}$$

Consider the adjoint operators

$$L_i^* x = L_i x = x - \langle x, n_i\rangle n_i.$$

The bound

$$\|L_i^* x\| = \left(\|x\|^2 - |\langle x, n_i\rangle|^2\right)^{1/2} \leq \|x\|$$

shows that these operators have norm no greater than one. Let $r \in (0, \infty)$ be given. Since $S_r \setminus H_i^\delta$ is a compact set and $|\langle x, n_i\rangle|$ is a non-negative continuous function that is zero only on H_i, given any $\delta \in (0, r)$ we have $\delta' \doteq \min_{x\in S_r\setminus H_i^\delta} |\langle x, n_i\rangle| \in (0, r)$. Hence

$$\max_{x\in S_r\setminus H_i^\delta} \|L_i^* x\| = (r^2 - \delta'^2)^{1/2} < r. \tag{4.40}$$

This fact, that the points on a sphere that lie a certain minimum distance away from a hyperplane are mapped strictly into the interior of the same sphere by the associated projection operator, is central to the construction of the set B^*.

Claim. There exist $\delta_k > 0$, $k = 2, \ldots, n-1$ and $r_1 > r_2 > \cdots > r_n > 0$ such that B^* defined in (4.35) satisfies (4.34). In other words, we claim that

$$\bigcup_{i=1}^{N} \left[\mathrm{cl}\left(L_i^* \left[\cup_{k=1}^{n} C_k \setminus H_i \right] \right) \right] \subset (B^*)^\circ. \tag{4.41}$$

To prove the claim we consider the increasing sequence of sets $A_j \doteq \cup_{k=1}^{j} C_k$, $j = 1, \ldots, n$. Each A_{j+1} is chosen so that every projection operator L_i^* maps $A_j \setminus H_i$ into the interior of the convex hull of A_{j+1}. The procedure terminates with $A_n = \cup_{k=1}^{n} C_k$, since (as we will show) the interior of the convex hull of A_n contains $L_i^*[A_n \setminus H_i]$ for every $i = 1, \ldots, N$.

First set $r_1 = 1$ and let C_1 be as defined in (4.36). For every $A \in \mathscr{D}^1$, $S_1 \cap H_A$ is a pair of points on S_1 which lie a finite distance away from every hyperplane $H_i = H_{\{n_i\}}$ for which $n_i \in \mathscr{D} \setminus A$. Thus from (4.39) and (4.40) (with $r = r_1$) we obtain $\rho_1 = \tilde{\rho}_1 < r_1$. We then choose r_2 so that $\rho_1 < r_2 < r_1$.

We claim that there is $\delta_2 > 0$ so that for every $A \in \mathscr{D}^2$

$$S_{r_2} \cap H_A \subset \mathrm{conv}\left[S_{r_2} \cap \left(H_A \setminus \cup_{\{j : n_j \in \mathscr{D} \setminus A\}} H_j^{\delta_2} \right), S_{r_1} \cap \left(\cup_{\{B \in \mathscr{D}^1 : A \subset B\}} H_B \right) \right] \tag{4.42}$$

Note that if $A \in \mathscr{D}^2$ (so that H_A is a two-dimensional linear space) and $n_j \in \mathscr{D} \setminus A$ (so that H_j is an $(n-1)$-dimensional linear space that does not contain H_A), then $H_A \setminus H_j$ is a plane with a line removed. Therefore $S_{r_2} \cap \left(H_A \setminus \cup_{\{j : n_j \in \mathscr{D} \setminus A\}} H_j^{\delta_2} \right)$ is a circle of radius r_2 with a finite number of arcs excised, each of whose chords has width $2\delta_2$. Clearly $\delta_2 > 0$ can always be chosen small enough to ensure that the chords do not intersect. Now for $B \in \mathscr{D}^1$ such that $A \subset B$, B has the form $A \cup \{j\}$ for some $n_j \in \mathscr{D} \setminus A$. Thus $S_{r_1} \cap \left(\cup_{\{B \in \mathscr{D}^1 : A \subset B\}} H_B \right)$ is a collection of points lying a distance $r_1 > r_2$ from the origin. Each point p in the collection lies on the line in the plane H_A that is orthogonal to the chord K of an excised arc of $S_{r_2} \cap \left(H_A \setminus \cup_{\{j : n_j \in \mathscr{D} \setminus A\}} H_j^{\delta_2} \right)$. Thus, as shown in Figure 8, for the entire circle $S_{r_2} \cap H_A$ to be contained in the convex hull defined on the right hand side of (4.42), it is necessary that each point p lie above the intersection of the tangents to the circle at the end-points of the corresponding chord K. An elementary trigonometric calculation shows that this is satisfied if $0 < \delta_2 < \frac{r_2}{r_1}\sqrt{r_1^2 - r_2^2}$, in which case (4.42) is established. Note that this argument does not rely on the fact that we are dealing with the two-dimensional case, and can therefore be used to show the existence of a δ_j that satisfies (4.44).

We now use induction to construct the set B^* with the required property. Assume that for some $k < n-1$, there exist $r_j > 0$, $\delta_j > 0$, $j = 2, \ldots, k$ such that

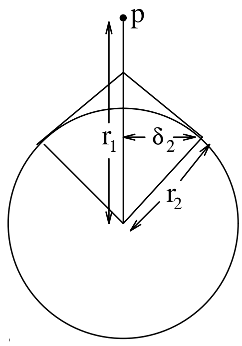

Fig. 8. The choice of δ_2 in the construction of B^*

$$\rho_{j-1} < r_j < r_{j-1}, \tag{4.43}$$

and for every $A \in \mathscr{D}^j$,

$$S_{r_j} \cap H_A \subset \operatorname{conv}\left[S_{r_j} \cap \left(H_A \setminus \cup_{\{l:n_l \in \mathscr{D}\setminus A\}} H_l^{\delta_j}\right), S_{r_{j-1}} \cap \left(\cup_{\{B\in\mathscr{D}^{j-1}:A\subset B\}} H_B\right)\right]. \tag{4.44}$$

Then, as we now show, the conditions (4.43) and (4.44) also hold for $j = k+1$. Fix $i \in \{1,\ldots,N\}$, $k \leq n-1$ and $\delta > 0$. Note that $n_i \in A$ implies $H_A \subset H_i$, and that for every $A \in \mathscr{D}^k$ such that $n_i \notin A$, $H_i^\delta \subset \cup_{\{j:n_j\in\mathscr{D}\setminus A\}} H_j^\delta$. Therefore

$$\begin{aligned}
&\bigcup_{A\in\mathscr{D}^k} \left(H_A \setminus \cup_{\{j:n_j\in\mathscr{D}\setminus A\}} H_j^\delta\right) \\
&\quad = \left[\cup_{\{A\in\mathscr{D}^k:n_i\in A\}} \left(H_A \setminus \cup_{\{j:n_j\in\mathscr{D}\setminus A\}} H_j^\delta\right)\right] \\
&\qquad\qquad \bigcup \left[\cup_{\{A\in\mathscr{D}^k:n_i\notin A\}} \left(H_A \setminus \cup_{\{j:n_j\in\mathscr{D}\setminus A\}} H_j^\delta\right)\right] \\
&\quad \subset H_i \cup \left[\left(\cup_{\{A\in\mathscr{D}^k:n_i\notin A\}} H_A\right) \setminus H_i^\delta\right].
\end{aligned}$$

Thus using (4.37) and the last display we obtain

$$\begin{aligned}
C_k \setminus H_i &= \left(S_{r_k} \cap \left[\cup_{A\in\mathscr{D}^k} \left(H_A \setminus \cup_{\{j:n_j\in\mathscr{D}\setminus A\}} H_j^{\delta_k}\right)\right]\right) \setminus H_i \\
&\subset S_{r_k} \cap \left[\left(\cup_{\{A\in\mathscr{D}^k:n_i\notin A\}} H_A\right) \setminus H_i^{\delta_k}\right] \\
&\subset S_{r_k} \setminus H_i^{\delta_k}.
\end{aligned} \tag{4.45}$$

Now define $\tilde{\rho}_k$ by (4.39). Then (4.40) along with (4.45) ensures that $\tilde{\rho}_k < r_k$, which in turn implies that $\rho_k = \max(\tilde{\rho}_k, \rho_{k-1}) < r_k$ since $\rho_{k-1} < r_k$ by

(4.43). Choose r_{k+1} so that $\rho_k < r_{k+1} < r_k$ so that (4.43) is satisfied for $j = k+1$. Using an argument analogous to the one used for the case $k = 2$, there exists $\delta_{k+1} > 0$ that satisfies (4.44) for $j = k + 1$. Let C_{k+1} be as defined in (4.37) with the chosen values of r_{k+1} and δ_{k+1}. Since r_2 and δ_2 were chosen earlier to satisfy (4.43) and (4.44), by induction there exist positive decreasing r_j, non-decreasing ρ_j, and $\delta_j > 0$ for $j \le n - 1$ that also satisfy the same conditions. Since $\rho_{n-1} < r_{n-1}$, there exists r_n such that $\rho_{n-1} < r_n < r_{n-1}$. By the definition $C_n \doteq S_{r_n}$ in (4.38) it follows that $\tilde{\rho}_n$ as defined by (4.39) equals r_n, and consequently $\rho_n = \max[\tilde{\rho}_n, \rho_{n-1}] = r_n$. The properties stated in the last two sentences imply

$$\max_i \max_{x \in \left(\cup_{k=1}^n C_k\right) \setminus H_i} ||L_i^* x|| = \rho_n = r_n < r_{n-1}.$$

Define $Q_r \doteq \{x \in I\!R^n : \|x\| \le r\}$. Since $L_i^*(I\!R^n) \subset H_i$ for every i, the last display implies

$$\cup_{i=1}^N L_i^* \left(\cup_{k=1}^n C_k \setminus H_i\right) \subset \cup_{i=1}^N \left(Q_{r_n} \cap H_i\right) \subset \left(\operatorname{conv}\left[\cup_{i=1}^N \left(S_{r_{n-1}} \cap H_i\right)\right]\right)^\circ . \tag{4.46}$$

Recall from (4.41) that our objective is to show that

$$\cup_{i=1}^N \left[\operatorname{cl}\left(L_i^* \left(\cup_{k=1}^n C_k \setminus H_i\right)\right)\right] \subset \left(B^*\right)^\circ = \left(\operatorname{conv}\left[\cup_{k=1}^n C_k\right]\right)^\circ .$$

This will follow from (4.46) if we prove that

$$\cup_{i=1}^N \left(S_{r_{n-1}} \cap H_i\right) \subset \cup_{k=1}^{n-1} C_k. \tag{4.47}$$

To show (4.47) we use the fact that (4.44) is satisfied for $j = 1, \ldots, n-1$. Taking the union over all $A \in \mathscr{D}^j$ in (4.44), using the definition (4.37) for C_j, and noting that $\cup_{A \in \mathscr{D}^j}\{B \in \mathscr{D}^{j-1} : A \subset B\} = \mathscr{D}^{j-1}$, we see that for any $j \le n - 1$,

$$\bigcup_{A \in \mathscr{D}^j} \left[S_{r_j} \cap H_A\right] \subset \operatorname{conv}\left[C_j, \bigcup_{B \in \mathscr{D}^{j-1}} \left[S_{r_{j-1}} \cap H_B\right]\right].$$

Substituting $j = 2$ in the above display and using definition (4.36) for C_1, we see that

$$\bigcup_{A \in \mathscr{D}^2} \left[S_{r_2} \cap H_A\right] \subset \operatorname{conv}\left[C_2 \cup C_1\right].$$

Thus from the last two displays we infer that

$$\bigcup_{A \in \mathscr{D}^{n-1}} \left[S_{r_{n-1}} \cap H_A\right] \subset \operatorname{conv}\left[\cup_{j=1}^{n-1} C_j\right].$$

If $A \in \mathscr{D}^{n-1}$, then A is single valued and equal to $\{n_i\}$ for some $i = 1, \ldots, N$ and so $H_A = H_{\{n_i\}} = H_i$. Thus the last display is equivalent to (4.47), which completes the proof. Thus the set B^* satisfies [6, Assumption 3.1] for the SP. Consequently, by [6, Theorems 3.2], it also satisfies Assumption 1.2, which establishes Lipschitz continuity of the SM [6, Theorem 3.3]. □

Existence of solutions. Existence of solutions for the normal case was proved in [4]. For the notation used in the rest of this paragraph we refer the reader to [6, Section 4]. Since the domain G is convex, for normal directions of constraint the projection $\pi(x)$ is defined as the unique point in G that is closest to x. As shown above, Assumption 1.1 always holds for this class of SPs and thus the SM is Lipschitz continuous wherever defined. In [4] it was shown that [6, Assumption 4.1] is also satisfied when $G^\circ \neq \emptyset$. By [6, Theorem 4.2] we therefore conclude that for any SP in this class, the SM is well-defined and Lipschitz continuous on $\mathscr{F}_G$. If in addition, $G^\circ \neq \emptyset$, then $\mathscr{F}_G = D_G([0, \infty) : \mathbb{R}^n)$.

5. Conclusion

In this paper we have demonstrated how the techniques of [6] can be used to derive algebraic conditions for Lipschitz continuity of the Skorokhod Map for classes of Skorokhod Problems. The examples described, in particular the successful analysis of the processor sharing model, indicate that the convex duality approach has some power. In this section we discuss some relatively subtle features of the approach. Section 5.1 postulates why tight algebraic characterizations of Lipschitz continuity for SPs on polyhedral domains may not be possible. Section 5.2 discusses the singularity of the structure of the set B^* across classes of Skorokhod Problems. Section 5.3 deals with the non-uniqueness of representations for a SP and indicates how the choice of representation is linked with the structure of the set B^*, which in turn depends on the complexity of the underlying Skorokhod Problem.

5.1. Finite algebraic characterizations

We believe that a finite algebraic characterization (that is a characterization in terms of a finite set of polynomial equations) of those SPs for which the Skorokhod Map is Lipschitz continuous is not possible. This is based in part on a conjecture in [9], which asserts that there is no semi-algebraic characterization of the SPs on $\mathbb{R}^n_+$ with n directions of constraint for which solutions are unique. However, it is also based on an examination of three-

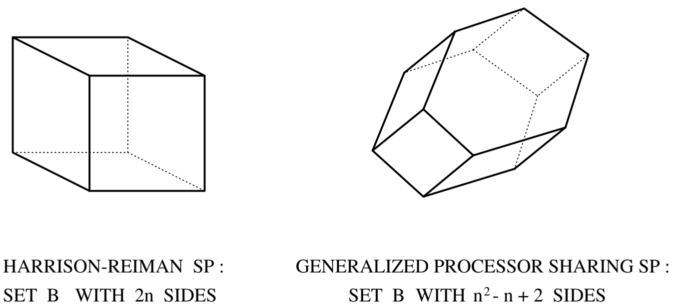

Fig. 9. Structure of the simplest B for two classes of Skorokhod Problems

dimensional SPs in this class – in particular the example considered in Section 2.4. This example shows that if one uses a more complicated set B^* than that required for the gH–R class, (i.e. if B^* is assumed to have 8 vertices rather than 6) then one carves out a *strictly larger* class of SPs for which there exists a set B^* that satisfies Assumption 1.2. In particular, the Skorokhod Map is found to be Lipschitz continuous for some SPs for which the gH–R spectral radius condition fails. The fact that the gH–R class is included in the class for which more complicated sets are required follows automatically from the monotonicity property stated in Theorem 3.5 of [6]: the set of SPs for which Lipschitz continuity can be proved with a set of a given complexity always includes those for which a simpler set suffices. The important point is that one can deal with a strictly larger collection when the number of faces of B is increased. This property seems to persist as more faces are added to B, which supports the conjecture made in [9].

5.2. Singularities in the structure of the set

In this paper, we have shown how Lipschitz continuity of the Skorokhod Map associated with a Skorokhod Problem P can be established by constructing a set B^* that satisfies Assumption 1.2 for P. To facilitate the derivation of an algebraic condition for the existence of B^*, it is desirable to construct a set that has the smallest number of vertices possible. Thus the objective is to find the "simplest" polytope B^* for which Assumption 1.2 is satisfied. Note that this corresponds to finding a set B that satisfies Assumption 1.1 with the least number of faces.

In Section 2 we saw that the simplest set for the gH–R class of SPs has $2n$ vertices in $\mathbb{R}^n$. This is the least number of vertices that any set in $\mathscr{S}$ (i.e. any bounded, symmetric set with 0 in its interior) possesses. On the other hand, the simplest set for the GPS class of SPs, as shown in Section 3, has $n^2 - n + 2$ vertices in $\mathbb{R}^n$. The two sets are contrasted in Figure 9. Note that for any SP of the GPS class, one can find a sequence of SPs from the gH–R class (with domain $\mathbb{R}^n_+$) which converge to the processor sharing model, in the sense that the directions of constraint converge for each point in ∂G. This gives a particularly striking illustration of the singular behaviours associated with the SP, in that one can analyze all the prelimit SPs using sets with $2n$ faces, but in the limit one must use a set with $n^2 - n + 2$ faces. Thus the structure of the simplest set B^* is preserved within a class of SPs but can be discontinuous across classes. Observe that the condition in Theorem 3.5 identifies a class of SPs for which a set B^* having $n^2 - n + 2$ vertices can be used. This class not only includes all the GPS SPs, but also a class of SPs arising from single class closed networks.

5.3. Representations

In [6, Section 2.2] it was shown that Skorokhod Problems on polyhedral domains with a constant direction of constraint associated with each face can be represented by a finite collection of triplets $\{(d_i, n_i, c_i), i = 1, \ldots, N\}$. It is important to observe that this description is not unique. The two different representations

$$P_1 \doteq \{(n_1, n_1, 0), (n_2, n_2, 0)\} \quad \text{and}$$

$$P_2 \doteq \{(n_1, n_1, 0), (n_2, n_2, 0), (n_3, n_3, 0)\},$$

with $n_1 = (1, 0)$, $n_2 = (0, 1)$ and $n_3 = \frac{1}{\sqrt{2}}(1, 1)$ give rise to the same Skorokhod Problem in the sense that they have the same domain $G_1 = G_2$ and directions of constraint $d(x)$. As illustrated in Figure 10, this is due to the fact that the extra direction of constraint n_3 specified at the origin 0 in the second description is already contained in the set of allowed directions at the origin $d(0)$ specified by the first description. Moreover, the extra face $\{x : \langle x, n_3\rangle = 0\}$ in the second description is a supporting hyperplane to the domain G_1 at the origin. We refer to the face $\{x : \langle x, n_3\rangle = 0\}$ as a "fictitious face", since it is not an actual face of the domain of the SP.

As illustrated by the example in Figure 10, it is often possible to add certain faces and associated directions to a given representation without changing the nature of the underlying SP. However in some cases like the GPS SP, it becomes *necessary* to introduce "fictitious faces" in order to obtain a proper description of the SP. This usually arises when the SP specifies an additional constraint d at the intersection $\mathscr{I}$ of two or more faces. If d lies outside the convex cone generated by the constraint directions associated with those faces, an additional face has to be introduced with d as its

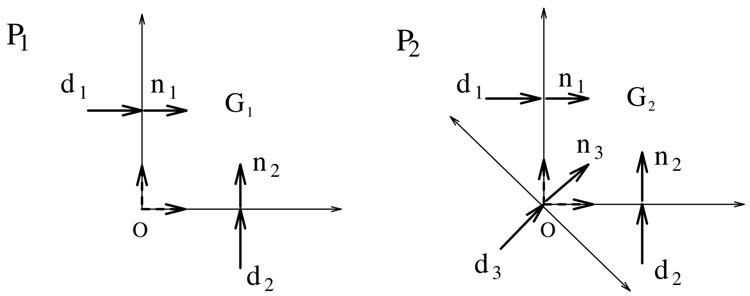

Fig. 10. Non-uniqueness of description of the Skorokhod Problem

constraint direction in order for the allowable constraint directions on all parts of the domain to properly represent the SP. This "fictitious face" must clearly be a supporting hyperplane to the domain at $\mathscr{I}$.

The need for fictitious faces is not restricted to the GPS SP, and in fact several other important classes of SPs require them as well. These classes include SPs associated with single class closed networks, certain multi-class single queue models, and also some multi-class networks. The first two classes are similar to the GPS SP, while the third combines features of the gH–R and GPS models and in fact requires more than one fictitious face. All three classes indicate important directions for further research into the SP.

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