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Large deviations and concentration properties for $\nabla\varphi$ interface models

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Abstract. We consider the massless field with zero boundary conditions outside $D_N \equiv D \cap (\mathbb{Z}^d/N)$ ($N \in \mathbb{Z}^+$), D a suitable subset of $\mathbb{R}^d$, i.e. the continuous spin Gibbs measure $\mathbb{P}_N$ on $\mathbb{R}^{\mathbb{Z}^d/N}$ with Hamiltonian given by $H(\varphi) = \sum_{x,y:|x-y|=1} V(\varphi(x)-\varphi(y))$ and $\varphi(x) = 0$ for $x \in D_N^{\complement}$. The interaction V is taken to be strictly convex and with bounded second derivative. This is a standard effective model for a $(d+1)$-dimensional interface: φ represents the height of the interface over the *base* D_N. Due to the choice of scaling of the base, we scale the height with the same factor by setting $\xi_N = \varphi/N$.

We study various concentration and relaxation properties of the family of random surfaces $\{\xi_N\}$ and of the induced family of gradient fields $\{\nabla^N \xi_N\}$ as the discretization step $1/N$ tends to zero ($N \to \infty$). In particular, we prove a large deviation principle for $\{\xi_N\}$ and show that the corresponding rate function is given by $\int_D \sigma(\nabla u(x))\mathrm{d}x$, where σ is the surface tension of the model. This is a multidimensional version of the sample path large deviation principle. We use this result to study the concentration properties of $\mathbb{P}_N$ under the volume constraint, i.e. the constraint that $(1/N^d)\sum_{x\in D_N} \xi_N(x)$ stays in a neighborhood of a fixed volume $v > 0$, and the hard–wall constraint, i.e. $\xi_N(x) \geq 0$ for all x. This is therefore a model for a droplet of volume v lying above a hard wall. We prove that under these constraints the field $\{\xi_N\}$ of rescaled heights concentrates around the solution of a variational problem involving the surface tension, as it would be predicted by the phenomenological theory of phase boundaries. Our principal result, however, asserts local relaxation properties of the gradient field $\{\nabla^N \xi_N(\cdot)\}$ to the corresponding extremal Gibbs states. Thus, our approach has little in common with traditional large deviation techniques and is closer in spirit to hydrodynamic limit type of arguments. The proofs have both probabilistic and analytic aspects. Essential analytic tools are $\mathbb{L}_p$ estimates for elliptic equations and the theory of Young measures. On the side of probability tools, a central role is played by the Helffer–Sjöstrand

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[31] PDE representation for continuous spin systems which we rewrite in terms of *random walk in random environment* and by recent results of T. Funaki and H. Spohn [25] on the structure of gradient fields.

1. Introduction and main results

1.1. Problems of phase separation

In various models of Statistical Mechanics pure phases are expected to be separated on the macroscopic scale along a deterministic surface of minimal energy, that is along a solution to a certain constrained isoperimetric type variational problem. A thermodynamical formulation of this fact was developed by Wulff [49]: the equilibrium shape $\mathbf{K}_v^*$ of a crystal of the prescribed volume v should minimize the value of the integral surface tension functional

$$\mathbf{K} \mapsto \int_{\partial \mathbf{K}} \tau\big(n(s)\big)\mathrm{d}s \quad , \tag{1.1}$$

under the fixed volume constraint $\mathrm{vol}(\mathbf{K}) = v$, where τ is the direction dependent surface tension between the crystal and its vapour (and $n(s)$ is the outward normal to $\partial\mathbf{K}$ at s).

In probabilistic terms such statements should correspond to very peculiar limit results as the size (number of random variables) of the statistical mechanical system tends to infinity. In a sense these results lie beyond the framework of the theory of large deviations ([23], [40], [19]) for Gibbsian random fields, and not only for merely technical reasons: the phenomenon in question is not a *bulk* one and all the key issues have to be settled in the regime of zero specific relative entropy. Moreover the very notion of the bulk entropy is irrelevant here, since phase separation manifests itself precisely in the breaking of translation invariance.

A rigorous probabilistic approach to the problems of phase separation was developed by Dobrushin, Kotecký and Shlosman around 10 years ago in the monograph [22], where it was also brilliantly and comprehensively implemented in the context of the two-dimensional low temperature ($\beta \gg 1$) Ising model.

The results of [22] triggered a wave of investigations which, however, have been confined to the original two-dimensional DKS-setting and to the attempts to relax their formidable proofs and to push their results all the way up to the critical temperature ([4], [42], [32], [33], [46], [17], [43]). Only recently the issue was completely resolved in the whole of the phase transition region [34].

The results of [22] and [34] have been obtained directly under the canonical constraint, that is on the level of local limit theorems with sharp finite volume corrections. Roughly speaking, the DKS approach of [22] is to split the problem into two:

1. Study the statistics of the phase boundaries;
2. Give refined local estimates on the fluctuations of the order parameters inside and outside these phase boundaries, that is in large but still finite volumes and in various metastable regimes.

The solution to both of these problems was strongly linked in [22] and [34] to the particular structure of the 2D Ising model, and it is not immediately clear how to extend it not only to higher dimensions but even to other ferromagnetic two-dimensional models with a more complex structure of interactions, e.g. next nearest neighbour Ising model.

In the physically more interesting case of higher dimensions, rigorous results have been so far obtained only on the level of weak integral limit theorems: a Gaussian one-droplet model with prescribed wetted region has been worked out in [6], and a sort of deterministic Winterbottom construction [48] of a small equilibrated particle placed on a foreign substrate has been obtained in the scaling limit of the gas of Gaussian droplets [10]. The Gaussian framework of these works, however, was so specific, that the corresponding results could be obtained essentially without shedding much light on how to deal with the intrinsic issues of phase separation in a generic situation.

In the context of Kac models , the solution to the isoperimetric problem (liquid drop) has been recovered in [3], [7] and [8]. In the latter works the renormalization estimates have been linked to the analytic tools of geometric measure theory, and, as a result, an interesting and robust procedure of proving integral large deviation estimates has been developed. Since, however, the authors were able to recover the surface tension only in the Lebowitz-Penrose limit, these estimates remained imprecise at each finite value of the interaction length $1/\gamma$.

In a recent remarkable work [15] a version of the above mentioned approach has been combined with a relaxed definition of the surface tension and with profound coarse graining techniques of [41], and then used to prove a form of the 3D Wulff construction for the supercritical independent Bernoulli bond percolation. The representation [15] of the surface tension and the general renormalization philosophy [7], [8], again implemented via Pisztora's coarse graining procedures [41], have been adopted in [9] for the proof of a similar result for the nearest neighbour Ising model in any dimension $d \geq 3$ at sufficiently low temperatures. Finally, the results of [9] have been extended, in the percolation context of [15], to all temperatures below the slab percolation threshold in [16].

Our approach is different from that of [9] and [15] and yields more information on local relaxation properties of interfaces, i.e. the local response of the measure describing the interface to external perturbations, in the less realistic context of effective interface models. In order to explain it in informal terms recall that on the macroscopic scale the surface tension is produced locally and does not depend on the global setup of the ensemble around. In particular, the statistical properties of the 2D Ising interface could be completely recovered in the simplified setting that we now explain.

Consider the Ising Gibbs measure in the strip

$$\mathbf{S}_N = \left\{ (x, y) \in \mathbb{Z}^2 : \ -N < x < N, \ y \in \mathbb{Z} \right\} \ . \tag{1.2}$$

at a value of inverse temperature $\beta > \beta_c$ and $\pm$-*boundary conditions* on $\partial\mathbf{S}_N$;

$$\eta(-N, y) = \eta(N, y) = \text{sign}(y) \ . \tag{1.3}$$

Any configuration of spins on $\mathbf{S}_N$ with such boundary conditions contains the unique crossing $\pm$-*contour* going from the point $(-N, 1/2)$ to the point $(N, 1/2)$ in the dual lattice (see e.g. [22]). Essentially this contour is viewed as the graph of the random $\pm$ interface. One way then to perform a renormalization analysis of this interface is to start perturbing the model with magnetic fields $h(x, y)$ which are constant in the vertical direction and slowly varying in the horizontal direction, that is

$$h(x, y) = h\left(\frac{x}{N}\right) \quad , \tag{1.4}$$

with, for example, $h \in \mathbb{C}_0^\infty([-1, 1])$.

Of course, in order to ensure a competition between the bulk effect of the magnetic field and the surface effect of the crossing contour on the scale of $\mathbf{S}_N$, one should multiply the corresponding magnetic fields by the factor of $1/N$.

It can be shown, using, for example, a much more detailed analysis of [34], that for each $h \in \mathbb{C}_0^\infty([-1, 1])$ the corresponding distribution of the scaled crossing $\pm$-contour is concentrated near the solution $u \in \mathbb{H}_0^1([-1, 1])$ to the following semi-linear Dirichlet boundary value problem:

$$\frac{\mathrm{d}}{\mathrm{d}t}\sigma'_\beta\Big(\frac{\mathrm{d}u}{\mathrm{d}t}\Big) = -2m^*(\beta)h(t) \quad , \tag{1.5}$$

where $\sigma_\beta(u) = \tau_\beta(1, u)$, τ_β is the surface tension of the Ising model and $m^*(\beta)$ is the spontaneous magnetization.

The equations (1.5) give rise to a large deviation principle for the scaled crossing $\pm$-interface with the rate function Σ_β having the effective domain in $\mathbb{H}_0^1([-1, 1])$

$$\Sigma_\beta(u) = \int_{-1}^{1} \sigma_\beta\big(u'(t)\big)\mathrm{d}t = \int_\gamma \tau_\beta(n(s))\mathrm{d}s \quad , \tag{1.6}$$

where γ is the graph of u considered as a curve in $\mathbb{R}^2$, s is the length parameter along γ and $n(s)$ is the direction of the normal at s. In other words Σ_β is just the integral surface tension of γ. In particular, the distribution of the scaled $\pm$ crossing interface concentrates, under a canonical constraint, around the appropriate portion of the Wulff curve.

If one tries to model the pinned Ising interface above by a path of a one-dimensional random walk conditioned to return to the origin in $2N$ step, then the result which we have informally sketched above becomes much easier to prove and, in fact, happens to be nothing but a form of the sample path large deviation principle [11].

In this work we try to derive a higher dimensional analog of (1.5), and hence to prove the corresponding concentration properties, for what would be a higher dimensional counterpart of the random walk path approximation of the Ising interface, namely for a class of effective interface models of the gradient type. These results could be reformulated in terms of multi-dimensional sample path large deviations principles. The reader should be aware, though, that traditional large deviations techniques and ideas play a very marginal role in our considerations. The essential part of the analysis is to identify the surface tension via an investigation of local relaxation properties of effective interfaces under slowly varying magnetic fields with a subsequent derivation of formulas similar to (1.5).

In the next subsection we systematically describe the class of models we consider and set up the key notation. Main results are stated and briefly discussed in Subsection 1.3 and a rough outline of the proof is given in Subsection 1.4. Finally Subsection 1.5 contains a guide to the rest of the paper and is designed to facilitate the orientation of the reader.

1.2. The model and the notation

Let D be an open bounded subset of $\mathbb{R}^d$. We discretize D as

$$D_N \equiv D \cap \mathbb{Z}^d_N \quad , \tag{1.7}$$

where $\mathbb{Z}^d_N \equiv (1/N)\mathbb{Z}^d$. A scaled random interface ξ_N over D_N is always either a scalar lattice field, $\xi_N \in \mathbb{R}^{D_N}$ with zero boundary conditions outside D_N,

$$\xi_N\big|_{\mathbb{Z}^d_N \setminus D_N} \equiv 0 \quad , \tag{1.8}$$

or its continuous interpolation $\tilde{\xi}_N$ which we define in (1.17) below. Our main objective is to study concentration properties of ξ_N as $N \to \infty$ for a class of nearest neighbour effective interface models which we proceed to describe: the formal Hamiltonian $\mathscr{H}_N$ on $\mathbb{R}^{\mathbb{Z}^d_N}$ is given by

$$\mathscr{H}_N(\varphi) = \sum_{x\sim y}^{N} V(\varphi(x) - \varphi(y)) \quad , \tag{1.9}$$

where the sum $\sum^N$ is over $\mathbb{Z}^d_N$ or products of it and $x \sim y$ denotes the restriction of the sum over nearest neighbor sites x and y of $\mathbb{Z}^d_N$. The corresponding finite volume Gibbs state D_N is then defined via

$$\mathbb{P}_N(\mathrm{d}\varphi) \equiv \mathbb{P}_{D_N}(\mathrm{d}\varphi) = \frac{e^{-\mathscr{H}_N(\varphi)}}{\mathscr{Z}_N} \mathrm{m}_N(\mathrm{d}\varphi) \quad , \tag{1.10}$$

where the reference measure m_N is given by,

$$\mathrm{m}_N(\mathrm{d}\varphi) \stackrel{\Delta}{=} \prod_{x\in D_N} \mathrm{d}\varphi(x) \prod_{x\in\mathbb{Z}^d_N\setminus D_N} \delta_0(\mathrm{d}\varphi(x)) \quad . \tag{1.11}$$

Our scaled version ξ_N is then defined as

$$\xi_N(x) \stackrel{\Delta}{=} \frac{\varphi(x)}{N}, \qquad x \in \mathbb{Z}^d_N \quad . \tag{1.12}$$

Notice that under $\mathbb{P}_N$ the random interface ξ_N automatically satisfies zero boundary conditions (1.8). The important conditions are those imposed on the interaction potential V: we assume that $V : \mathbb{R} \to \mathbb{R}^+$ is sufficiently smooth ($V \in C^{2,\delta}(\mathbb{R}, \mathbb{R})$,

$\delta \in (0,1)$, where $C^{2,\delta}$ is the space of functions with Hölder continuous second derivative, is enough – see Section 2 for more details), even and that there is $c_V \geq 1$ such that

$$c_V^{-1} \leq V''(r) \leq c_V \tag{1.13}$$

for all $r \in \mathbb{R}$. These are severe limitations which rule out, for example, continuous versions of the solid-on-solid models (for which $V(r) = |r|$, and $\mathscr{H}_N$ is then the area of the random interface $\varphi(\cdot)$). On the other hand, we stress that no restriction on the constant c_V is imposed (except $c_V < \infty$), that is our analysis is a fully non-perturbative one.

The concentration properties of ξ_N are intimately related to the notion of surface tension of the model. The latter is well defined for formal Hamiltonians $\mathscr{H}_N$ we consider [25], [37]. Let us briefly recall the corresponding construction: given a vector $v \in \mathbb{R}^d$, use $\mathscr{Z}_N(v)$ to denote the partition function of the random surface with the Hamiltonian (1.9) and v-wired boundary conditions outside the discretization V_N of the unit cube $V_N = (0,1)^d \cap \mathbb{Z}_N^d$. By this we mean that the heights $\varphi(x)$ of the interface outside V_N are pinned as

$$x \in \mathbb{Z}_N^d \backslash V_N \implies \varphi(x) = N(v,x) \ , \tag{1.14}$$

where $(\cdot,\cdot)$ is the usual scalar product in $\mathbb{R}^d$. Then,

$$\sigma(v) \stackrel{\Delta}{=} -\lim_{N\to\infty} \frac{1}{N^d} \log \frac{\mathscr{Z}_N(v)}{\mathscr{Z}_N(0)} \ , \tag{1.15}$$

see [37] for a proof of the existence of this limit. Eventually, we are going to prove that the scaled interface ξ_N satisfies, under the $\{\mathbb{P}_N\}$ family of measures, a large deviation principle with the rate function being the integrated surface tension Σ (see 1.41 for the precise formula),

$$\Sigma(u) \stackrel{\Delta}{=} \int_D \sigma(\nabla u(x))\mathrm{d}x \ . \tag{1.16}$$

which, thanks to (3.63) and the differentiability of σ [25], is well defined (and finite), for every $\nabla u \in \mathbb{L}_2\big(D\big)$. Of course, ξ_N was defined so far only on the vertices of the lattice $\mathbb{Z}_N^d$, and, in order to make the statement of the corresponding theorem meaningful, we should interpolate it to the whole of $\mathbb{R}^d$. There are several natural ways to do so, the simplest one being just the plaquette reconstruction of ξ_N. It happens, however, to be more convenient to work with the following Sobolev space oriented polilinear interpolation: for $x \in \mathbb{R}^d$ define

$$\tilde{\xi}_N(x) = \sum_{v\in\{0,1\}^d} \left[\prod_{i=1}^d \big(v_i\{Nx_i\} + (1-v_i)(1-\{Nx_i\}\big)\right] \xi_N\left(\frac{[Nx]+v}{N}\right) \ , \tag{1.17}$$

where $[\cdot]$ and $\{\cdot\}$ denote the integer and the fractional parts respectively.

Notice that the interpolation is consistent with the values of ξ_N on the vertices of $\mathbb{Z}_N^d$. Moreover, at a generic point $x \in \mathbb{R}^d$, $\xi_N(x)$ defined above is a convex combination of the values of ξ_N at the vertices of the $\mathbb{Z}_N^d$-*plaquette* containing x.

The interplay between the lattice quantities and their interpolations is very important for us. Given a scalar lattice field $u : \mathbb{Z}_N^d \mapsto \mathbb{R}$, we use $\nabla^N u$ to denote the discrete approximate gradient

$$\begin{aligned}\nabla^N u(x) &= \left(\nabla_1^N u(x), \ldots, \nabla_d^N u(x)\right)\\ &\triangleq N\left(u\left(x+\tfrac{e_1}{N}\right)-u(x), \ldots, u\left(x+\tfrac{e_d}{N}\right)-u(x)\right) \quad .\end{aligned} \tag{1.18}$$

Similarly, given a vector lattice field $g : \mathbb{Z}_N^d \mapsto \mathbb{R}^d$, we use $\mathrm{div}_N g$ to denote its discrete approximate divergence

$$\mathrm{div}_N g(x) \triangleq N \sum_{i=1}^d \left(g_i(x) - g_i(x - \frac{e_i}{N})\right) \quad . \tag{1.19}$$

Notice that in the interpolation formula (1.17), the (continuous) gradient $\nabla\tilde{\xi}_N$ is almost everywhere defined, and its i-th entry is given by

$$\begin{aligned}\frac{\partial}{\partial x_i}\tilde{\xi}_N(x) = \; &\frac{1}{2}\sum_{v\in\{0,1\}^d}\left[\prod_{j\neq i}\left(v_j\{Nx_j\}+(1-v_j)(1-\{Nx_j\}\right)\right]\\ &\times\nabla_i^N \xi_N\left(\frac{[Nx]+\hat{v}_i}{N}\right) \quad ,\end{aligned} \tag{1.20}$$

where

$$\hat{v}_i = (v_1, \ldots, v_{i-1}, 0, v_{i+1}, \ldots, v_d) \quad . \tag{1.21}$$

A look at (1.20) reveals that $\partial/\partial x_i \tilde{\xi}_N(x)$ is again the convex combination of $\nabla_i^N \xi_N$ over the corresponding $\mathbb{Z}_N^d$-lattice points. Using Jensen's inequality and elementary estimates, we then immediately infer that for every power $p > 1$ and every finitely supported scalar $\mathbb{Z}_N^d$ lattice field u, there exists a constant $c_1(d, p) > 0$, such that

$$c_1(d,p)\frac{1}{N^d}\sum_x^N |\nabla^N u|^p \le \int_{\mathbb{R}^d} |\nabla\tilde{u}|^p \mathrm{d}x \le \frac{1}{N^d}\sum_x^N |\nabla^N u|^p \quad . \tag{1.22}$$

As far as the norms are considered, we shall use the generic notation $\|\cdot\|_{1,p}$ both for

$$\left(\frac{1}{N^d}\sum_x^N |\nabla^N u|^p\right)^{1/p} \quad \text{and} \quad \left(\int |\nabla\tilde{u}|^p \mathrm{d}x\right)^{1/p} \quad , \tag{1.23}$$

respectively in the case of finitely supported scalar lattice fields and in the case of their continuous counterparts. Similarly, for a lattice field u we use $\|\cdot\|_p$ both for

$$\left(\frac{1}{N^d}\sum_x^N |u|^p\right)^{1/p} \quad \text{and} \quad \left(\int |\tilde{u}|^p \mathrm{d}x\right)^{1/p} \quad . \tag{1.24}$$

The very same notation and convention remain valid also for the $\mathbb{R}^d$-valued vector lattice fields, in the latter case we shall interpolate $g : \mathbb{Z}_N^d \mapsto \mathbb{R}^d$ according to

(1.20): the i-th entry of the interpolation $\tilde{g}_i(x)$ at a generic point $x \in \mathbb{R}^d$ is given now by

$$\tilde{g}_i(x) = \frac{1}{2} \sum_{v \in \{0,1\}^d} \left[\prod_{j \neq i} \bigl(v_j \{Nx_j\} + (1 - v_j)(1 - \{Nx_j\}) \bigr) \right] g_i \left(\frac{[Nx] + \hat{v}_i}{N} \right) \tag{1.25}$$

As it was indicated in Subsection 1.1 one of the focal points of our work is to study local relaxation properties of the scaled random interface ξ_N (or of its continuous interpolation $\tilde{\xi}_N$ given by (1.17)) under a slowly varying external field $h \in \mathbb{C}_0^\infty(D)$. To set up the appropriate notation let us define the *tilted* Hamiltonian $\mathscr{H}_{N,h}$ as

$$\mathscr{H}_{N,h}(\varphi) = \sum_{x \sim y}^{N} V(\varphi(x) - \varphi(y)) \ - \frac{1}{N} \sum_{x}^{N} h(x)\varphi(x) \ . \tag{1.26}$$

Then the tilted measure $\mathbb{P}_{N,h}$ is given by

$$\mathbb{P}_{N,h}(d\varphi) \ \equiv \ \mathbb{P}_{D_N,h}(d\varphi) = \frac{e^{-\mathscr{H}_{N,h}(\varphi)}}{\mathscr{Z}_{N,h}} \mathrm{m}_N(d\varphi) \ , \tag{1.27}$$

where the reference measure m_N was defined in (1.11).

Alternatively, one could think of $\mathbb{P}_N$ (or of $\mathbb{P}_{N,h}$) as of the probability distributions on the field of height differences η:

$$\eta(x) \stackrel{\Delta}{=} \nabla^N \xi_N(x) = (\varphi(x + e_1/N) - \varphi(x), \ldots, \varphi(x + e_d/N) - \varphi(x)) \ . \tag{1.28}$$

The field η of height differences is the random object to be studied. In this respect our investigation was motivated and prompted by the recent work [25], where the study of the thermodynamics of the shift invariant fields of bound differences was essentially initiated. Let us, therefore recall some of their results and notations:

It is also possible to view the field of height differences η as being defined on the nearest neighbour bonds of $\mathbb{Z}_N^d$: the i-th coordinate $\eta_i(x)$ corresponds in this way to the value $\eta(b)$ of η on the bond $b = \langle x, x + e_i/N \rangle$. The orientation of the bonds is then reflected in the convention $\eta(\langle x, y \rangle) = -\eta(\langle y, x \rangle)$. We are going to employ both the site and the bond notation for η without further comment. Due to the symmetry of V, the formal Hamiltonian (1.9) can be rewritten as

$$\mathscr{H}_N(\eta) = \sum_{x}^{N} \sum_{i=1}^{d} V\bigl(\eta_i(x)\bigr) = \frac{1}{2} \sum_{b}^{N} V\bigl(\eta(b)\bigr) \ . \tag{1.29}$$

A fruitfull idea of T. Funaki and H. Spohn was to consider the η-field in its own right (for the moment we refer either to [25] or to Section 4 for the exact DLR setting). They were able to prove that for every $v \in \mathbb{R}^d$ there is unique infinite volume ergodic square integrable Gibbs state $\mathbb{P}_v^{\mathrm{FS}}$ on the space of height differences, such that,

$$\bigl\langle |\eta(x)|^2 \bigr\rangle_{\mathbb{P}_v^{\mathrm{FS}}} \ < \ \infty \qquad \text{and} \qquad \bigl\langle \eta(x) \bigr\rangle_{\mathbb{P}_v^{\mathrm{FS}}} = v \ . \tag{1.30}$$

Moreover, for each $v \in \mathbb{R}^d$ the Funaki–Spohn state $\mathbb{P}_v^{\mathrm{FS}}$ enjoys the following crucial relation to the surface tension σ:

$$\big\langle V'(\eta(0))\big\rangle_{\mathbb{P}_v^{\mathrm{FS}}} = \nabla\sigma(v) \ . \tag{1.31}$$

where, for any function $\rho : \mathbb{R} \mapsto \mathbb{R}$, we use the shortcut notation

$$\rho(\eta) \stackrel{\Delta}{=} \big(\rho(\eta_1), \ldots, \rho(\eta_d)\big) \in \mathbb{R}^d \ . \tag{1.32}$$

Here, and in the rest of the paper, by $\langle\cdot\rangle_{\mathbb{P}}$ we mean the expectation with respect to the probability measure $\mathbb{P}$. Also we use the shortcut notations $\langle\cdot\rangle_N$ and $\langle\cdot\rangle_{N,h}$ for the expectations under $\mathbb{P}_N$ and $\mathbb{P}_{N,h}$ respectively.

1.3. Main results

All of our results remain true whenever ∂D is sufficiently regular to ensure the validity of certain Poincaré–Sobolev type inequalities used in the proof of Lemma 3.4. In particular, ∂D of Lipschitz regularity is enough and it will be assumed throughout the paper. For a test function $h \in \mathbb{C}_0^\infty\big(D\big)$ set

$$\Lambda_D(h) = \lim_{N\to\infty} \frac{1}{N^d} \log\left\langle \exp\left(\frac{1}{N}\sum_x^N \varphi(x)h(x)\right)\right\rangle_N \ . \tag{1.33}$$

Our principle result relates the log-moment generating function Λ_D to the surface tension σ defined in (1.15). Let $\mathbb{H}_0^1(D)$ be the closure of $\mathbb{C}_0^\infty(D)$ with respect to the Hilbert norm $\|\cdot\|_{1,2}$. We have the following:

Theorem 1.1. *Assume that D has a Lipschitz boundary ∂D. Then Λ_D in (1.33) is well defined for any $h \in \mathbb{C}_0^\infty(D)$. Moreover,*

$$\Lambda_D(h) = \int_D \int_0^1 u_{[th]}(x)h(x)\mathrm{d}t\,\mathrm{d}x \tag{1.34}$$

where, for each $f \in \mathbb{L}_2(D)$, we use $u_{[f]} \in \mathbb{H}_0^1(D)$ to denote the unique weak solution in $\mathbb{H}_0^1(D)$ to the semi-linear elliptic equation

$$\mathrm{div}\big(\nabla\sigma(\nabla u)\big) = -f \ . \tag{1.35}$$

An alternative way to describe Λ_D is:

$$\Lambda_D(h) = \max_{u\in\mathbb{H}_0^1}\left\{\int_D u(x)h(x)\mathrm{d}x - \Sigma(u)\right\} = \int_D u_{[h]}(x)h(x)\mathrm{d}x - \Sigma(u_{[h]}) \ , \tag{1.36}$$

where $\Sigma(u) = \int_D \sigma\big(\nabla u(x)\big)\,\mathrm{d}x$.

The equation (1.35) is the promised multidimensional counterpart of the equation (1.5) of the previous subsection. Moreover, in the heart of the proof of Theorem 1.1 lies the following statement, which describes the relaxation of the averaged interface $u_{N,f}$,

$$u_{N,f} \stackrel{\Delta}{=} \langle \xi_N \rangle_{N,f} \left(= \left\langle \frac{\varphi}{N} \right\rangle_{N,f} \right) \ , \tag{1.37}$$

and hence, as we shall see, the concentration properties of the random scaled interface ξ_N under $\mathbb{P}_{N,f}$, or, in other words, under a slowly varying external field $f \in \mathbb{C}_0^\infty(D)$:

Theorem 1.2. *Assume that ∂D is Lipschitz. Then, for every $f \in \mathbb{C}_0^\infty(D)$, the sequence of (interpolated) mean profiles $\{\tilde{u}_{N,f}\}$ converges strongly in $\mathbb{H}_0^1$ to the solution $u_{[f]}$ of the semi-linear elliptic problem (1.35) as $N \to \infty$:*

$$\lim_{N\to\infty} \|\tilde{u}_{N,f} - u_{[f]}\|_{1,2} = 0 \ . \tag{1.38}$$

The scaled random interface $\tilde{\xi}_N$ concentrates, under $\{\mathbb{P}_{N,f}\}$ around $u_{[f]}$ in the following sense:

$$\lim_{N\to\infty} \left\langle \|\tilde{\xi}_N - u_{[f]}\|_2^2 \right\rangle_{N,f} = 0 \ . \tag{1.39}$$

Furthermore, under $\mathbb{P}_{N,f}$ the gradient field $\eta(\cdot) = \nabla^N \xi_N(\cdot)$ locally relaxes to the Funaki–Spohn state $\mathbb{P}^{FS}_{\nabla u_{[f]}(\cdot)}$. More precisely, for every $g \in \mathbb{L}_2(D)$ and every bounded continuous local function F on the space of height differences (see Subsection 4.2 for the precise definitions)

$$\lim_{N\to\infty} \frac{1}{N^d} \sum_x^N g(x) \big\langle F(\theta_{-x}\eta) \big\rangle_{N,f} = \int_D g(x) \big\langle F(\eta) \big\rangle^{FS}_{\nabla u_{[f]}(x)} \, \mathrm{d}x \ , \tag{1.40}$$

where θ_x is the translation operator on the space of height differences ($\theta_x \eta(y) = \eta(y + x)$).

In a sense, (1.38) and (1.40) are the main results of this work, and their proof requires most of the techniques developed throughout the paper. The result (1.40) is a form of *weak local equilibrium*, in the hydrodynamic limit language [35], and it gives some information on the fluctuations of the interface $\tilde{\xi}_N$ around its mean value. It is a first step toward a Central Limit Theorem for $\tilde{\xi}_N$. For a Funaki–Spohn state $\mathbb{P}^{\mathrm{FS}}_v$, $v \in \mathbb{R}^d$ the CLT has been established (see [28] and [39]) and the limiting field is a continuum free field with covariance which depends on the tilt v. We believe that the fluctuations for $\mathbb{P}_{N,f}$ are determined by the spatially varying tilt $\nabla u_{[f]}(\cdot)$, however the extension of the quoted results to the case presented here seem rather challenging, due to the non constant tilt and to the presence of boundary conditions.

The concentration property (1.39) falls in the framework of large deviation results which follow more or less in a standard way once Theorem 1.1 is verified:

Theorem 1.3. *Assume that ∂D is Lipschitz. Then the family of random surfaces $\tilde{\xi}_N$ obeys a strong LDP on $\mathbb{L}_2(D)$ with speed N^d and rate function given by the integrated surface tension (1.16)*

$$\Sigma(u) \triangleq \begin{cases} \int_D \sigma(\nabla u(x))\mathrm{d}x & \text{if } u \in \mathbb{H}_0^1(D), \\ +\infty & \text{otherwise} \end{cases}, \tag{1.41}$$

that is, for every measurable $E \subset \mathbb{L}_2(D)$ we have that

$$\begin{aligned} -\inf_{u\in E^\circ} \Sigma(u) &\le \liminf_{N\to\infty} \frac{1}{N^d} \log\left[\mathbb{P}_N\left(\tilde{\xi}_N \in E\right)\right] \\ &\le \limsup_{N\to\infty} \frac{1}{N^d} \log\left[\mathbb{P}_N\left(\tilde{\xi}_N \in E\right)\right] \le -\inf_{u\in\overline{E}} \Sigma(u) \ , \end{aligned} \tag{1.42}$$

where E° and $\overline{E}$ are respectively interior and closure in $\mathbb{L}_2(D)$ of E.

The main application of the previous Theorem is the construction of a droplet with fixed volume over a hard wall: we interpret φ (and its rescaled version ξ_N) as the height of the phase separation surface of a droplet lying on a wall with a given quantity of liquid $v > 0$. Our result shows that the rescaled profile $\tilde{\xi}_N$ converges as $N \to \infty$ to the deterministic Wulff curve $u^{(v)}$, unique minimizer of the variational problem

$$\inf\{\Sigma(u) : u \in H_0^1(D), \int_D u(x)\,dx = v\} \ , \tag{1.43}$$

Note that $u^{(v)}$ solves the corresponding Euler equation

$$\mathrm{div}\big(\nabla\sigma(\nabla u^{(v)})\big) = -c_v \mathbf{1}_D \ , \tag{1.44}$$

where c_v is an appropriate constant.

More precisely, we give the following two definitions.

Definition. *The hard wall (or entropic repulsion) condition*. We restrict ourselves to non-negative configurations, that is we will consider the conditioned measure

$$\mathbb{P}_{N,+} \triangleq \mathbb{P}_N(\,\cdot\,|\Omega_N^+) \quad \text{where} \quad \Omega_N^+ = \left\{\varphi \in \mathbb{R}^{\mathbb{Z}_N^d} : \varphi(x) \ge 0 \text{ for every } x \in D_N\right\} \ . \tag{1.45}$$

The study of this measure, aimed at understanding the effect of a forbidden region on a random interface, is interesting in its own right and the arising phenomenon goes under the name of *entropic repulsion* (see e.g. [20] and references therein).

Definition. *The volume condition*. For given $v > 0$, let us introduce the event

$$A_N(v) = \{\varphi \in \mathbb{R}^{\mathbb{Z}_N^d} : \frac{1}{N^{d+1}}\sum^N \varphi(x) = \frac{1}{N^d}\sum^N \xi_N(x) \ge v\} \ . \tag{1.46}$$

We can obviously reformulate this event as

$$\{\varphi \in \mathbb{R}^{D_N} : \int_D \tilde{\xi}_N(x)\,dx \ge v\} \ , \tag{1.47}$$

introducing a negligible error.

Theorem 1.4. *Assume that ∂D is Lipschitz. Then for every $v > 0$ the scaled random interface $\tilde{\xi}_N$, under the measure $\mathbb{P}_{N,+}$ conditioned on $A_N(v)$, concentrates around $u^{(v)}$, unique minimizer of the variational problem (1.43), in the following sense: for every fixed $\delta > 0$*

$$\lim_{N\to\infty} \mathbb{P}_{N,+}\Big(\|\tilde{\xi}_N - u^{(v)}\|_2 > \delta \Big| A_N(v) \Big) = 0 \ . \tag{1.48}$$

Remark 1.5. Instead of the volume condition $A_N(v)$, one could also consider a volume shell of thickness $\epsilon > 0$:

$$A_{N,\epsilon}(v) = \left\{ \varphi \in \mathbb{R}^{\mathbb{Z}_N^d} : v - \epsilon/2 \le \frac{1}{N^d} \sum_x^N \xi_N(x) \le v + \epsilon/2 \right\} \ . \tag{1.49}$$

By letting first $N \to \infty$ and then $\epsilon \to 0$ yields the same concentration result. Moreover, as it will be clear from the proof, Theorem 1.4 holds also if we replace $\mathbb{P}_{N,+}$ with $\mathbb{P}_N$ in the statement.

1.4. Sketch of the proof

In order to identify the limit $\Lambda_D(h)$ in (1.33) we use the following decomposition of the finite volume log–moment generating functions:

$$\frac{1}{N^d} \log \Big\langle \exp\Big(\frac{1}{N} \sum^N h(x)\varphi(x) \Big) \Big\rangle_N = \int_0^1 \frac{1}{N^d} \sum^N h(x) \langle \xi_N(x) \rangle_{N,th} \, \mathrm{d}t \ . \tag{1.50}$$

This step immediately shifts our attention to the main assertion (1.38) of Theorem 1.2. In fact the identification formula (1.34) would follow from (1.50) even if only a weaker form of convergence is secured. In any case let us fix $f \in \mathbb{C}_0^\infty(D)$, and, in order to facilitate notations, let us use a shortcut notation $u_N \stackrel{\Delta}{=} u_{N,f} (= \langle \xi_N \rangle_{N,f})$.

Our analysis of u_N relies on the following simple observation:

$$\text{for every } x \in D_N, \qquad \Big\langle \frac{\partial}{\partial \varphi(x)} \mathrm{e}^{-\mathscr{H}_{N,f}(\varphi)} \Big\rangle_{\mathrm{m}_N} = 0 \ , \tag{1.51}$$

where, as before, m_N is the reference measure defined in (1.11). In fact, using B_N to denote the unit ball in $\mathbb{R}^{D_N}$,

$$B_N = \left\{ \varphi \in \mathbb{R}^{D_N} : |\varphi|_N^2 = \sum_x^N \varphi(x)^2 \le 1 \right\} \ , \tag{1.52}$$

by performing the integration with respect to $\varphi(x)$ we obtain

$$\begin{aligned} \left| \Big\langle \frac{\partial}{\partial \varphi(x)} \exp\{-\mathscr{H}_{N,f}\} \Big\rangle_{\mathrm{m}_N} \right| \le\ & 2a^{N-1} \mathrm{Vol}\,(B_{N-1}) \exp\Big\{ - \min_{|\varphi|_N = a} \mathscr{H}_{N,f}(\varphi) \Big\} \\ & + \int_{\varphi \in aB_N^c} \left| \frac{\partial}{\partial \varphi(x)} \exp\{-\mathscr{H}_{N,f}(\varphi)\} \right| \mathrm{m}_N(\mathrm{d}\varphi) \ . \end{aligned} \tag{1.53}$$

However, due to the quadratic lower bound (1.13) on V, for some $c = c(f, N)$ we have that

$$\min_{|\varphi|_N = a} \mathscr{H}_{N,f} \;\geq\; \frac{a^2}{c_V}\lambda^*_{D_N} - c_{N,f}a, \tag{1.54}$$

where $\lambda^*_{D_N} > 0$ is the leading eigenvalue of the discrete Dirichlet Laplacian on D_N with Dirichlet boundary conditions and $c_{N,f} = N^{-1+d/2}\|f\|_2$. Equation (1.51) then follows in the limit $a \to \infty$, by applying (1.54) after having developed the derivative in the last term in (1.53).

Performing the differentiation in (1.51), we see that at any point $x \in D_N$,

$$\mathrm{div}_N \left(\left\langle V'(\eta(x))\right\rangle_{N,f}\right) = -f(x) \ , \tag{1.55}$$

where η is the field of height differences defined in (1.28). Summing (1.55) by parts against test functions, we obtain that for any $j \in \mathbb{C}_0^\infty(D)$,

$$\frac{1}{N^d}\sum\nolimits^N \left(\left\langle V'(\eta(x))\right\rangle_{N,f}, \nabla^N j(x)\right) = \frac{1}{N^d}\sum\nolimits^N j(x)f(x) \ . \tag{1.56}$$

The latter equation lies in the heart of our approach, and the statement on convergence of u_N happens to be nothing but the statement on the local relaxation properties of the η-field under the slowly varying (on the microscopic scale of $\mathbb{Z}^d_N$) tilt f. We shall see that under $\mathbb{P}_{N,f}$ our η-field in an appropriate sense locally relaxes as $N \to \infty$ near $x \in D_N$ into the Funaki–Spohn state in the averaged direction $\langle\eta(x)\rangle_{N,f} = \nabla^N u_N(x)$. Much of Section 4 is devoted to the attempts to give a precise meaning to this assertion. Meanwhile formally substituting $\left\langle V'(\eta(x))\right\rangle_{N,f}$ by the expectation $\nabla\sigma(\nabla^N u_N(x))$ of η under the Funaki–Spohn state $\mathbb{P}^{\mathrm{FS}}_{\nabla^N u_N(x)}$ (see (1.31)), we obtain from (1.56),

$$\frac{1}{N^d}\sum\nolimits^N \left(\nabla\sigma(\nabla^N u_N(x)), \nabla^N j(x)\right) \cong \frac{1}{N^d}\sum\nolimits^N j(x)f(x) \ . \tag{1.57}$$

Various a-priori bounds on the sequence $\{u_N\}$ which we derive in Section 3 will enable us to conclude at this stage that this sequence is pre-compact, and, moreover, any limit point u of $\{\tilde{u}_N\}$ has to satisfy,

$$\int_D \left(\nabla\sigma(\nabla u(x)), \nabla j(x)\right)\mathrm{d}x = \int_D f(x)j(x)\mathrm{d}x \qquad \text{for every } j \in \mathbb{C}_0^\infty(D) \ , \tag{1.58}$$

and, by the uniqueness of the solution of (1.58), this implies that there is only one limit point, which is precisely $u_{[f]}$.

1.5. Guide to subsequent sections

In our approach we drew inspiration from two recent sources of results and ideas. The first one is the paper by Funaki and Spohn which we have already mentioned. Equally important for us are the representation techniques developed in the series of articles by Helffer and Sjöstrand (see e.g. [31]) as well as applications of these techniques to non-Gaussian fields on $\mathbb{Z}^d$ we consider here [39].

In Section 2 we develop a probabilistic counterpart of the Helffer–Sjöstrand representation. We feel that the results of Section 2 might be of independent interest, and, accordingly, try to present them in a closed form in the general setting of Gibbs states on finite graphs. The appropriate notation is set up in Subsection 2.1, where also the probabilistic representation formulas for the covariance (Proposition 2.2) and for the mean (Proposition 2.5) are derived for arbitrary tilting fields and boundary conditions.

Our probabilistic interpretation of Helffer–Sjöstrand ideas provides a useful intuition for studying the correlation structure of random fields with strictly convex potentials. We return to the $\mathbb{L}_2$ theory in Subsection 2.2: In Lemma 2.8 we prove a partial inverse to the Brascamp–Lieb inequality[1]. An important entropic upper bound on the $\mathbb{L}_2$ oscillation of the $\nabla\varphi$ field is formulated in Proposition 2.10. At last, the issue of exponential tightness of the φ field is briefly worked out in Lemma 2.11.

In Subsection 2.3 we adjust and specify the above mentioned results to the setting of the square lattice $\mathbb{Z}^d$.

Much of the weight of technical estimates falls on Section 3. The ubiquitous oscillation bound and the $\mathbb{L}_p$ estimate on the expectations of the $\nabla\varphi$ field under various tilted measures are proven in Subsections 3.1 and 3.3 respectively. Strict convexity of the surface tension σ has been established in Subsection 3.4, see [28, Appendix A] for an alternative proof, whereas the implications for the properties of the functional Σ have been discussed in Subsection 3.5.

The claim 1.38 of Theorem 1.2 follows by taking the limit $N \nearrow \infty$ in (1.56). A rigorous treatment of such a limit requires an analysis of local relaxation properties of the η-field along the lines of 1.40. This is constituted by two main steps: the proof of the local weak convergence to Funaki-Spohn states (Subsection 4.1) and the analysis of the related families of Young measures (Subsection 4.2). In both instances the crucial compactness conditions follows from the oscillation bound (3.1) and from the $\mathbb{L}_p$ estimate of Lemma 3.4, which are, in this respect, indispensable.

Finally the proofs of Theorem 1.3 and Theorem 1.4 are completed in Section 5.

2. The random walk representation

In this section we derive some useful estimates on the φ-field, by taking advantage of a representation of continuous spin systems in terms of the solution of a suitable elliptic problem. This representation has been introduced by B. Helffer and J. Sjöstrand [31]: we will reinterpret it in a probabilistic way, by using the well known fact that the solution of elliptic problems can be given via diffusion processes (see e.g. [24]). Since we are in a discrete setting, the diffusion process is a discrete random walk, but only in the case of quadratic potentials (i.e. in the Gaussian setting) the random walk is simple, that is the transition rates are independent of time: in the general non-Gaussian setting we obtain formulas involving a random walk *in time dependent random environment.*

[1] See also [12] for a reproduction of our proof and a clever application to the CLT for the Hopfield model.

We will first introduce the random walk version of the Helffer–Sjöstrand (H–S) representation. We will do this in a rather general setting, i.e. for a φ-field on a graph (Subsection 2.1). Then we will apply the representation to get some estimates (Subsection 2.2). Finally we will restrict ourselves to the d-dimensional square lattice and we will make the estimates explicit in this setting (Subsection 2.3).

We sum up the simplifications arising in the Gaussian case at the end of Subsection 2.1.

2.1. The H–S representation for fields on a graph

We start with a finite connected graph $\mathscr{G} = (E, \mathscr{E})$. We decompose the set of sites $E = I \cup B$ into its interior $I \neq \emptyset$ and its boundary $B \neq \emptyset$. $\mathscr{E}$ is the set of oriented bonds b, i.e. $b = (x, y)$, for some $x, y \in E$, and we write $-b = (y, x)$ for the reversed bond: we assume that if $b \in \mathscr{E}$ then $-b \in \mathscr{E}$. Moreover, given $b \in \mathscr{E}$, we denote by $x(b)$ and $y(b)$ the two entries of the bond: that is if $b = (x, y)$, then $x(b) = x$ and $y(b) = y$. Let $\varphi \in \Omega_E \equiv \mathbb{R}^E$ and $b = (x, y) \in \mathscr{E}$, then

$$\nabla\varphi(b) \equiv \varphi(y) - \varphi(x) \tag{2.1}$$

is the discrete gradient. Throughout this section we identify the set Ω_A, $A \subset E$, with the set of functions from A to $\mathbb{R}$.

We consider a family $V \equiv \{V_b\}_{b\in\mathscr{E}}$ of potentials with the following properties:

1. *smoothness*: $V_b \in C^{2,\delta}(\mathbb{R}; \mathbb{R}^+)$ for some $\delta \in (0, 1)$;
2. *symmetry*: for every $b \in \mathscr{E}$, V_b is an even function and $V_b = V_{-b}$.
3. *strict convexity*: there exist two constants, C_1 and C_2, with $0 < C_1 \le C_2 < \infty$, such that
$$C_1 \le V_b'' \le C_2, \qquad b \in \mathscr{E} \ . \tag{2.2}$$

The Hamiltonian $\mathscr{H}_{\mathscr{G}} : \Omega_E \to \mathbb{R}$ is defined by

$$\mathscr{H}_{\mathscr{G}}(\varphi) = \frac{1}{2}\sum_{b\in\mathscr{E}} V_b\left(\varphi(y(b)) - \varphi(x(b))\right) = \frac{1}{2}\sum_{b\in\mathscr{E}} V_b\left(\nabla\varphi(b)\right) \ . \tag{2.3}$$

Unless otherwise stated, an element $\varphi \in \Omega_I$ will be viewed also as element of $\Omega_E^0 = \{\varphi \in \Omega_E : \varphi(x) = 0 \text{ for } x \in B\}$. Next, let $\rho \in \Omega_E^0$ and set

$$\mathscr{H}_{\mathscr{G},\rho}(\varphi) = \mathscr{H}_{\mathscr{G}}(\varphi) - \langle\rho, \varphi\rangle \ , \tag{2.4}$$

where $\langle\rho, \varphi\rangle = \sum_{x\in E} \rho(x)\varphi(x)$. For a given boundary condition $\psi \in \Omega_B$, let us define the configuration $\varphi \wedge \psi \in \Omega_E$ which agrees with φ on I and with ψ on B, that is

$$(\varphi \wedge \psi)(x) = \begin{cases} \varphi(x), & \text{if } x \in I, \\ \psi(x), & \text{if } x \in B \ , \end{cases} \tag{2.5}$$

and write $\mathscr{H}^{\psi}_{\mathscr{G},\rho}(\varphi) = \mathscr{H}_{\mathscr{G},\rho}(\varphi \wedge \psi)$. Consider now the probability measure $\mathbb{P}^{\psi}_{I,\rho}$ on Ω_I defined by

$$\mathbb{P}^{\psi}_{I,\rho}(\mathrm{d}\varphi) = \frac{1}{\mathscr{Z}^{\psi}_{\mathscr{G},\rho}} \exp(-\mathscr{H}^{\psi}_{\mathscr{G},\rho}(\varphi))\, m_I\,(\mathrm{d}\varphi) \ , \tag{2.6}$$

where $m_I(\mathrm{d}\varphi) \equiv \prod_{x\in I} \mathrm{d}\varphi(x)$ and $\mathscr{Z}^{\psi}_{\mathscr{G},\rho} \in \mathbb{R}^+$ is the normalizing constant.

Notation. We denote by $\left\langle \cdot \right\rangle_{\mathbb{P}}$ and $\left\langle \cdot, \cdot \right\rangle_{\mathbb{P}}$ the expectation and the scalar product with respect to a measure $\mathbb{P}$. However we will often use simplified notations, like

$$\left\langle \cdot \right\rangle^{\psi}_{I,\rho} \quad \text{and} \quad \left\langle \cdot, \cdot \right\rangle^{\psi}_{I,\rho} \quad , \tag{2.7}$$

respectively for the expectation with respect to $\mathbb{P}^{\psi}_{\mathscr{G},\rho}$ and the scalar product in $\mathbb{L}_2(\mathbb{P}^{\psi}_{I,\rho})$. Moreover

$$\left\langle \cdot \right\rangle_{I,\rho}, \quad \left\langle \cdot \right\rangle^{\psi}_{I}, \quad \left\langle \cdot \right\rangle_{I} \quad , \tag{2.8}$$

are respectively the expectation with respect to

$$\mathbb{P}_{I,\rho}, \quad \mathbb{P}^{\psi}_{I}, \quad \mathbb{P}_{I} \quad , \tag{2.9}$$

which, in turn, are $\mathbb{P}^{\psi}_{I,\rho}$ with (respectively) $\psi \equiv 0$, $\rho \equiv 0$, both $\rho \equiv 0$ and $\psi \equiv 0$. Analogous short-cut notation for the scalar product.

In order to introduce the H–S representation let us introduce the second order elliptic operator

$$\begin{aligned} L^{\psi}_{I,\rho} &= e^{\mathscr{H}^{\psi}_{\mathscr{G},\rho}(\varphi)} \sum_{x\in I} \frac{\partial}{\partial\varphi(x)} \left[e^{-\mathscr{H}^{\psi}_{\mathscr{G},\rho}(\varphi)} \frac{\partial}{\partial\varphi(x)} \right] \\ &= \sum_{x\in I} \left(\frac{\partial^2}{\partial\varphi(x)^2} - \frac{\partial \mathscr{H}^{\psi}_{\mathscr{G},\rho}(\varphi)}{\partial\varphi(x)} \frac{\partial}{\partial\varphi(x)} \right) \quad , \end{aligned} \tag{2.10}$$

with domain

$$\begin{aligned} C^2_{\exp}(\Omega_I;\mathbb{R}) \equiv \; &\{F \in C^2(\Omega_I;\mathbb{R}) : \exists \varepsilon = \varepsilon(F) > 0 \text{ s.t.} \\ &\sup_{\varphi} |\partial F(x,\varphi)| \exp(-\varepsilon \sum_{x} |\varphi(x)|) < \infty\} \quad , \end{aligned} \tag{2.11}$$

where

$$\partial F(x,\varphi) \equiv \frac{\partial\, F(\varphi)}{\partial\varphi(x)}, \qquad x \in I \quad , \tag{2.12}$$

and $\partial F(x,\cdot) = 0$ at the boundary points $x \in B = E \setminus I$. Then a simple integration by parts shows that for all functions F, G in the domain of $L^{\psi}_{I,\rho}$

$$\left\langle F, \left(-L^{\psi}_{I,\rho}\right) G \right\rangle^{\psi}_{I,\rho} = \sum_{x\in I} \left\langle \partial F(x,\varphi), \partial G(x,\varphi) \right\rangle^{\psi}_{I,\rho} = \left\langle G, \left(-L^{\psi}_{I,\rho}\right) F \right\rangle^{\psi}_{I,\rho} \quad , \tag{2.13}$$

that is $L^{\psi}_{I,\rho}$ is symmetric with respect to $\mathbb{P}^{\psi}_{I,\rho}(\mathrm{d}\varphi)$. We can then [26] introduce the closed self-adjoint extension of $L^{\psi}_{I,\rho}$ in $\mathbb{L}_2(\mathbb{P}^{\psi}_{I,\rho})$, denoted again by $L^{\psi}_{I,\rho}$, which acts on a domain $\mathscr{D}^{\psi}_{I,\rho}$, and the corresponding Dirichlet form

$$\Gamma_{L^{\psi}_{I,\rho}}(F,G) = \left\langle \langle\langle \partial F, \partial G \rangle\rangle(\varphi) \right\rangle^{\psi}_{I,\rho} \quad , \tag{2.14}$$

where we have introduced the notation

$$\langle\langle f, g\rangle\rangle(\varphi) = \sum_{x\in I} f(x,\varphi)g(x,\varphi) \ , \tag{2.15}$$

for $f, g : I \times \Omega_E \to \mathbb{R}$. By the standard theory Dirichlet forms [26], interpreting ∂F and ∂G as weak derivatives, every term in (2.13) (and (2.14)) is well defined and the equalities in (2.13) hold for $F, G \in \mathscr{D}^{\psi}_{I,\rho}$.

The basic step for the H–S representation is the following observation: let $G \in C^2_{\exp}(\Omega_I; \mathbb{R})$ be of mean zero

$$\left\langle G\right\rangle^{\psi}_{I,\rho} = 0 \ , \tag{2.16}$$

and let $H \in \mathscr{D}^{\psi}_{I,\rho}$ satisfy

$$(-L^{\psi}_{I,\rho})\, H = G \qquad \text{with} \quad \left\langle H\right\rangle^{\psi}_{I,\rho} = 0 \ . \tag{2.17}$$

The first equality has to be interpreted in the weak sense: for every $F \in C^2_{\exp}$ $(= C^2_{\exp}(\Omega_I; \mathbb{R}))$,

$$\left\langle\langle\langle\partial F, \partial H\rangle\rangle(\varphi)\right\rangle^{\psi}_{I,\rho} = \langle F, G\rangle^{\psi}_{I,\rho} \ . \tag{2.18}$$

Let us first give a result of existence, uniqueness and regularity for (2.17). Recall the regularity assumptions on V listed at the beginning of this section:

Lemma 2.1. *There exists a unique solution H to (2.17). Moreover $H \in C^{3,\delta}\cap\mathscr{D}^{\psi}_{I,\rho}$ and therefore it is a classical solution.*

Proof. Let us first introduce the Hilbert space $\tilde{W}$, closure of

$$W_0 \equiv \left\{F \in C^{\infty}(\Omega_I; \mathbb{R}) \cap C^2_{\exp}(\Omega_I; \mathbb{R}) : \langle F\rangle^{\psi}_{I,\rho} = \langle 1, F\rangle^{\psi}_{I,\rho} = 0\right\} \ , \tag{2.19}$$

under the norm defined by

$$\|F\|^2_{\tilde{W}} = \langle F^2\rangle^{\psi}_{I,\rho} + \left\langle\langle\langle\partial F, \partial F\rangle\rangle\right\rangle^{\psi}_{I,\rho} \ . \tag{2.20}$$

The bilinear functional in (2.14), viewed as a map from $\tilde{W} \times \tilde{W} \longrightarrow \mathbb{R}$ is clearly bounded and its coercivity follows from the spectral gap of the operator $L^{\psi}_{I,\rho}$: there exists $\epsilon > 0$ such that

$$\left\langle\langle\langle\partial F, \partial F\rangle\rangle\right\rangle^{\psi}_{I,\rho} \geq \epsilon\langle F, F\rangle^{\psi}_{I,\rho} \ , \tag{2.21}$$

for every $F \in \tilde{W}$. This can be proven from classical results on Schrödinger operator using the unitary transformation $F \in \mathbb{L}_2(\mathbb{P}^{\psi}_{I,\rho}) \longrightarrow \exp(-\mathscr{H}^{\psi}_{\mathscr{G},\rho}/2)F \in \mathbb{L}_2(\mathrm{m}_I)$

which maps the operator $-L^{\psi}_{I,\rho}$ unitarily to the Schrödinger operator $-\Delta + V$ where $\Delta = \sum_{x\in I}(\partial^2/\partial\varphi(x)^2)$ and $V : \mathbb{R}^I \longrightarrow \mathbb{R}$ a multiplicative operator

$$V(\varphi) = \sum_{x\in I}\left[\frac{1}{4}\left(\frac{\partial \mathscr{H}^{\psi}_{\mathscr{G},\rho}(\varphi)}{\partial\varphi(x)}\right)^2 - \frac{1}{2}\frac{\partial^2 \mathscr{H}^{\psi}_{\mathscr{G},\rho}(\varphi)}{\partial\varphi(x)^2}\right] . \tag{2.22}$$

In particular the spectral gaps of $-L^{\psi}_{I,\rho}$ and $-\Delta + V$ agree. Since V is bounded below and, by (2.2), $\lim_{R\to\infty}\inf_{\sum_{x\in I}\varphi(x)^2\geq R} V(\varphi) = +\infty$ (cf. (1.54)), all the points in the spectrum of $-\Delta + V$ are isolated, cf. [44, Th.XIII.64 and Th.XIII.69]. Since, for $F \in \tilde{W}$, $\left\langle\langle\langle\partial F, \partial F\rangle\rangle\right\rangle^{\psi}_{I,\rho} = 0$ implies $F \equiv 0$ and since we are dealing with non-negative operators, (2.21) is proven.[2]

Observe now that the functional $\langle\cdot, G\rangle : \tilde{W} \longrightarrow \mathbb{R}$ is bounded. By the Lax–Milgram Theorem [30] we obtain the existence of a unique $H \in \tilde{W}$ which satisfies (2.18) for every $F \in \tilde{W}$. Since $\langle G\rangle^{\psi}_{I,\rho} = 0$, (2.18) holds without the restriction $\langle F\rangle^{\psi}_{I,\rho} = 0$ and the existence-uniqueness part of the Lemma is proven.

To prove the regularity of such a solution we refer to results which can be found in [30], to which we refer also for the Sobolev space notation. For notational convenience, in this proof we write $(\underline{\partial})_x = \partial/\partial\varphi(x)$ and

$$L \equiv L^{\psi}_{I,\rho} = \underline{\partial}^2 - \underline{c}\cdot\underline{\partial} \ , \tag{2.23}$$

with $(\underline{c}(\varphi))_x \equiv \partial\mathscr{H}^{\psi}_{\mathscr{G},\rho}(\varphi)/\partial\varphi(x)$ (cf. (2.10)). Since $V_b \in C^{2,\delta}$, $(\underline{c})_x \in C^{1,\delta}(\Omega_I)$. First of all note that the solution H we found is a weak solution in the sense of formulas (8.2), (8.3) and (8.4) of [30] in any open ball of Ω_I (equipped with the Euclidean norm). In particular $H \in W^{1,2}_{\mathrm{loc}}$. By [30, Cor.8.36], since both $\underline{c}$ and G are locally bounded, $H \in C^{1,\alpha}$ for every $\alpha \in (0,1)$ and by [30, Th.8.8], since both $\underline{c}$ and G are differentiable, $H \in W^{3,2}_{\mathrm{loc}}$. We can therefore differentiate once with respect to $\varphi(x)$, any $x \in I$, both sides of the equation in (2.17) and we obtain that $v : \Omega_I \to \mathbb{R}$ defined by $v = \partial H/\partial\varphi(x)$ solves weakly the equation

$$\underline{\partial}^2 v - \underline{c}\cdot\underline{\partial} v = -\frac{\partial G}{\partial\varphi(x)} + \frac{\partial\underline{c}}{\partial\varphi(x)}\cdot\underline{\partial} H \equiv \tilde{g} \ , \tag{2.24}$$

where $\tilde{g}$ is locally bounded. Therefore, again by [30, Cor.8.36], $v \in C^{1,\alpha}$ for every $\alpha \in (0,1)$. Therefore $H \in C^{2,\alpha}$ and H is a classical solution of (2.17). We can now use the Schauder interior regularity theorem [30, Th.6.17] to conclude that, since both $\underline{c}$ and G are $C^{1,\delta}$, $H \in C^{3,\delta}$. □

[2] A more explicit lower bound on the spectral gap can be obtained by using the Bakry–Emery criterion and the fact that we have an explicit lower bound on the Hessian of $\mathscr{H}^{\psi}_{\mathscr{G},\rho}$: in such a way it can be shown that in (2.21) ϵ can be chosen equal to $C_1\lambda^*_E$, where λ^*_E is the leading eigenvalue of discrete the Laplacian on E, with Dirichlet boundary conditions on B (see (2.79) below).

In view of (2.13) and (2.17), for $F \in C^2_{\exp}$ we have

$$\mathrm{cov}^{\psi}_{I,\rho}(F,G) = \left\langle F, (-L^{\psi}_{I,\rho})\, H \right\rangle^{\psi}_{I,\rho} = \left\langle \langle\langle \partial F, \partial H \rangle\rangle(\varphi) \right\rangle^{\psi}_{I,\rho} \quad . \tag{2.25}$$

At this stage, let us introduce the jump process generator $Q^{\psi,\varphi}_E$, acting on functions $j : E \to \mathbb{R}$,

$$Q^{\psi,\varphi}_E j(x) = \sum_{b\in\mathscr{E}:x(b)=x} a^{\psi}(b,\varphi)\nabla j(b) \quad , \tag{2.26}$$

where

$$a^{\psi}(b,\varphi) = a(b,\varphi\wedge\psi), \qquad \text{with} \quad a(b,\varphi) = V''_b(\nabla\varphi(b)) \quad . \tag{2.27}$$

Note that the jump rates are symmetric, that is

$$a(b,\varphi) = a(-b,\varphi) \quad , \tag{2.28}$$

for every $b \in \mathscr{E}$ and every $\varphi \in \Omega_I$. Then, using

$$\begin{aligned} &\frac{\partial}{\partial\varphi(x)}(-L^{\psi}_{I,\rho})\, H(\varphi) \\ &\quad = (-L^{\psi}_{I,\rho})\frac{\partial H(\varphi)}{\partial\varphi(x)} + \sum_{z\in I} \frac{\partial^2 \mathscr{H}^{\psi}_{\mathscr{G},\rho}(\varphi)}{\partial\varphi(x)\partial\varphi(z)} \frac{\partial H(\varphi)}{\partial\varphi(z)} \\ &\quad = (-L^{\psi}_{I,\rho})\partial H(x,\varphi) \\ &\qquad - \sum_{b\in\mathscr{E}:x(b)=x} V''_b\left((\nabla(\psi\wedge\varphi))(b)\right)\left[\partial H(y(b),\varphi) - \partial H(x(b),\varphi)\right] \quad , \end{aligned} \tag{2.29}$$

we see that $h \equiv \partial H$ satisfies the equation

$$(-L^{\psi}_{I,\rho})h(x,\varphi) + (-Q^{\psi,\varphi}_E)h(x,\varphi) = \partial G(x,\varphi), \qquad x\in I,\ \varphi\in\Omega_I \quad , \tag{2.30}$$

with 0-boundary conditions, i.e.

$$h(x,\varphi) = 0, \qquad x \in B \quad , \tag{2.31}$$

for all $\varphi \in \Omega_I$. Let us denote by $\mathscr{M}_E$ the space of functions $h : E \times \Omega_I \to \mathbb{R}$ and write $\mathscr{M}^0_E = \{f \in \mathscr{M}_E : f(x,\cdot) \equiv 0 \text{ if } x \in B\}$. For uniformity, every function $f : I \times \Omega_I \to \mathbb{R}$ will be implicitly extended to a function $f \in \mathscr{M}^0_E$, by setting $f(x,\cdot) \equiv 0$ for every x in the boundary B. If $u, w \in \mathscr{M}^0_E$ we have,

$$-\langle\langle u, Q^{\psi,\varphi}_E w \rangle\rangle(\varphi) = \frac{1}{2}\sum_{b\in\mathscr{E}} a^{\psi}(b,\varphi)\nabla u(b,\varphi)\nabla w(b,\varphi) \equiv \Gamma_{Q^{\psi,\varphi}_E}(u,w)(\varphi) \quad . \tag{2.32}$$

This implies that the operator

$$\mathscr{L}^{\psi}_{E,\rho} \equiv L^{\psi}_{I,\rho} + Q^{\psi,\varphi}_{E} \tag{2.33}$$

with domain

$$\hat{C}^2_{\exp}(E \times \Omega_I; \mathbb{R}) \equiv \left\{ f \in \mathscr{M}^0_E : f(x,\cdot) \in C^2_{\exp}(\Omega_I; \mathbb{R}) \text{ for each } x \in I \right\} \ , \tag{2.34}$$

is symmetric with respect to the measure[3] $\hat{\mathbb{P}}^{\psi}_{I,\rho} \equiv \epsilon_E \otimes \mathbb{P}^{\psi}_{I,\rho}$, with ϵ_E the counting measure on E. More precisely for $f, g \in \hat{C}^2_{\exp}(E \times \Omega_I; \mathbb{R})$, we have that

$$\begin{aligned}
&\left\langle \langle\langle f, (-\mathscr{L}^{\psi}_{E,\rho}) g\rangle\rangle(\varphi) \right\rangle^{\psi}_{I,\rho} \\
&= \int_{E\times\Omega_I} f(x,\varphi)\left(-\mathscr{L}^{\psi}_{E,\rho}\right) g(x,\varphi) \hat{\mathbb{P}}^{\psi}_{I,\rho}(dx, d\varphi) \\
&= \left\langle \langle\langle f, \left(-L^{\psi}_{I,\rho}\right) g\rangle\rangle(\varphi) \right\rangle^{\psi}_{I,\rho} + \frac{1}{2}\sum_{b\in\mathscr{E}} \left\langle a^{\psi}(b,\varphi)\nabla f(b,\varphi)\nabla g(b,\varphi) \right\rangle^{\psi}_{I,\rho} \\
&= \sum_{x\in I} \Gamma^{\psi}_{I,\rho}\left(f(x,\cdot), g(x,\cdot)\right) + \left\langle \Gamma_{Q^{\psi,\varphi}_E}(f,g)(\varphi) \right\rangle^{\psi}_{I,\rho} \\
&\equiv \Gamma_{\mathscr{L}^{\psi}_{E,\rho}}(f,g) \ ,
\end{aligned} \tag{2.35}$$

defines a Dirichlet form $\Gamma_{\mathscr{L}^{\psi}_{E,\rho}}$ on $\mathbb{L}_2(\epsilon_I \otimes \mathbb{P}^{\psi}_{I,\rho})$.

Following Freidlin [24, §2], let us give a process representation for $h(x,\varphi)$. Consider the joint Markov process $\{(X(t), \Phi(t))\}_{t\in\mathbb{R}^+}$ on $E \times \Omega^0_E$, (pre)generated by $\mathscr{L}^{\psi}_{E,\rho}$. Due to the fact that the generator is sum of two terms and that the first part of the generator acts only on the φ-variable, we can construct the process by constructing first the finite dimensional diffusion process $\{\Phi(t)\}_{t\in\mathbb{R}^+}$ generated by $L^{\psi}_{I,\rho}$ and then by constructing the random walk $\{X(t)\}_{t\in\mathbb{R}^+}$ which makes jumps from x to y at time t with time dependent rate $a^{\psi}((x,y), \Phi(t))$. We will be interested only up to the moment that the random walk hits B: therefore we set

$$\tau = \inf\{t \geq 0 : X(t) \notin I\} \ . \tag{2.36}$$

Let us denote by $\mathbf{P}^{\psi,\rho}_{x,\varphi}$ the law on $D([0,\infty); E) \times C^0([0,\infty); \Omega_I)$ (D is the Skorohod space of right continuous trajectories: the topology on $E \times \Omega_I$ is the product topology, while Ω_I is equipped with the uniform topology and E with the discrete one) of this process starting at $(x,\varphi) \in E \times \Omega_I$. We have the following representation of the solution of the elliptic problem (2.30) and (2.31)

$$h(x,\varphi) = \mathbf{E}^{\psi,\rho}_{x,\varphi}\left[\int_0^{\tau} \partial G(X(s), \Phi(s))\, \mathrm{d}s\right] \ . \tag{2.37}$$

We state this result in the following proposition.

---

[3] Beware that this is not a probability measure!

Proposition 2.2. *Representation of covariances. For every $F, G \in C^2_{exp}(\Omega_I; \mathbb{R})$ we have*

$$cov^{\psi}_{I,\rho}(F,G) = \sum_{x\in I} \left\langle \partial F(x,\varphi), \mathbf{E}^{\psi,\rho}_{x,\varphi}\left[\int_0^\tau \partial G(X(s),\Phi(s))\, ds\right]\right\rangle^{\psi}_{I,\rho} \ . \tag{2.38}$$

In the special case $F(\varphi) = \varphi(x)$ and $G(\varphi) = \varphi(y)$ for some $x, y \in I$, then we simply have

$$cov^{\psi}_{I,\rho}(\varphi(x),\varphi(y)) = \mathbf{E}^{\psi,\rho}_{x}\left[\int_0^\tau \mathbf{1}_y(X(s))\, ds\right] \ , \tag{2.39}$$

where

$$\mathbf{P}^{\psi,\rho}_{x} \equiv \int_{E\times\Omega_I} \mathbb{P}^{\psi}_{I,\rho}(d\varphi)\mathbf{P}^{\psi,\rho}_{x,\varphi}, \qquad x \in I \ . \tag{2.40}$$

Proof. It is simply based on the observation that, with respect to the natural filtration associated to $\{(X(t),\Phi(t))\}_{t\in\mathbb{R}^+}$,

$$M(t) = \partial H(X(t),\Phi(t)) - \partial H(x,\varphi) - \int_0^t \mathscr{L}^{\psi}_{E,\rho}\partial H(X(s),\Phi(s))\mathrm{d}s \ , \tag{2.41}$$

is a $\mathbf{P}^{\psi,\rho}_{x,\varphi}$-local martingale.

Now observe that $\tau < \infty$ $\mathbf{P}^{\psi,\rho}_{x,\varphi}$-almost surely, since the graph is finite and connected, and the jump rates are bounded away from zero and ∞. Moreover we will show below that

$$\sup_{t\in\mathbb{R}^+} \mathbf{E}^{\psi,\rho}_{x,\varphi}\left[(M(t\wedge\tau))^2\right] < \infty \tag{2.42}$$

for every x and $\mathbb{P}^{\psi}_{I,\rho}$-a.e. φ. Therefore, by the Optional Stopping Theorem [45] for u.i. martingales, the result is easily achieved by considering $t = \tau$ (2.41) and taking expectations, recalling that $\partial H(x,\cdot) \equiv 0$ for $x \in B$. We are therefore left with the proof of (2.42).

Let us set

$$\begin{aligned} R(\partial H)(x,\varphi) &= \mathscr{L}^{\psi}_{E,\rho}\left(\partial H\right)^2(x,\varphi) - 2\partial H \mathscr{L}^{\psi}_{E,\rho}\partial H(x,\varphi), \\ &= 2\sum_{y\in I}\left(\frac{\partial}{\partial\varphi(y)}\partial H(x,\varphi)\right)^2 + \sum_{b:x(b)=x} a^{\psi}(b,\varphi)\left(\nabla\partial H(b,\varphi)\right)^2 \ , \end{aligned} \tag{2.43}$$

and

$$R^*(\partial H)(\varphi) = \sum_{x\in I} R\left(\partial H\right)(x,\varphi) \ . \tag{2.44}$$

Observe that

$$\begin{aligned} \left\langle R^*(\partial H)\right\rangle^{\psi}_{I,\rho} &= \sum_{x\in I}\left\langle \partial H(x,\cdot)\partial G(x,\cdot)\right\rangle^{\psi}_{I,\rho} \\ &\le \left\langle\langle\langle \partial H,\partial H\rangle\rangle\right\rangle^{\psi\ \ 1/2}_{I,\rho}\left\langle\langle\langle \partial G,\partial G\rangle\rangle\right\rangle^{\psi\ \ 1/2}_{I,\rho} < \infty \ . \end{aligned} \tag{2.45}$$

By standard martingale theory and by dominating $R(\partial H)$ with $R^*(\partial H)$ we obtain

$$\begin{aligned} \mathbf{E}^{\psi,\rho}_{x,\varphi}\left[(M(t\wedge\tau))^2\right] &\leq \int_0^\infty \mathbf{E}^{\psi,\rho}_{x,\varphi}\left[R^*(\partial H)(\Phi(s))\mathbf{1}_{[0,\tau]}(s)\right]\mathrm{d}s \\ &= \int_0^\infty \mathbf{E}^{\psi,\rho}_{x,\varphi}\left[R^*(\partial H)(\Phi(s))\mathbf{E}^{\psi,\rho}_{x,\varphi}\left[\mathbf{1}_{[0,\tau]}(s)\middle|\sigma(\Phi(s))\right]\right]\mathrm{d}s \ . \end{aligned} \tag{2.46}$$

but since by (2.2) there exists $\varepsilon > 0$ such that uniformly in x, φ and $\Phi(s)$

$$\mathbf{P}^{\psi,\rho}_{x,\varphi}\left(\tau \geq s\middle|\sigma(\Phi(s))\right) \leq \varepsilon^{-1}\exp(-\varepsilon s) \ , \tag{2.47}$$

from (2.45) and (2.46) we conclude that

$$\left\langle \sup_{t\in\mathbb{R}^+}\mathbf{E}^{\psi,\rho}_{x,\varphi}\left[(M(t\wedge\tau))^2\right]\right\rangle^{\psi}_{I,\rho} \leq \varepsilon^{-2}\left\langle R^*(\partial H)\right\rangle^{\psi}_{I,\rho} < \infty \ . \tag{2.48}$$

This proves (2.42) and the proof of Proposition 2.2 is complete. □

Remark 2.3. Actually, the representation is more general, for example we can replace the linear self-potential $\langle\rho,\varphi\rangle$ by a nonlinear self-potential. In this case the Hamiltonian is

$$\mathscr{H}_{\mathscr{G},U}(\varphi) \equiv \mathscr{H}_{\mathscr{G}}(\varphi) - \sum_{x\in I} U_x(\varphi_x) \ , \tag{2.49}$$

and we assume that $U_x \in C^2(\mathbb{R};\mathbb{R})$ and that

$$U_x'' \leq 0 \ , \tag{2.50}$$

for every $x \in I$. For simplicity we will assume also that, for all $x \in I$, $U_x'' \geq -C_3$ for some constant $C_3 \in \mathbb{R}^+$. Then denoting by $L^{\psi}_{I,U}$ the corresponding diffusion generator, and proceeding as before (in particular $(-L^{\psi}_{I,U})H = G$ with G and H of mean zero) we see that $\partial H(x,\varphi)$ satisfies the equation

$$\begin{aligned} &(-L^{\psi}_{I,U})\partial H(x,\varphi) + (-Q^{\psi,\varphi}_{E})\partial H(x,\varphi) + U_x''(\varphi(x))\partial H(x,\varphi) \\ &\quad = \partial G(x,\varphi), \quad x \in I \ , \end{aligned} \tag{2.51}$$

with boundary conditions (2.31). This yields

$$\partial H(x,\varphi) = \mathbf{E}^{\psi,U}_{x,\varphi}\left[\int_0^\tau \exp\left(\int_0^s U''_{X(t)}(\Phi(t))\,\mathrm{d}t\right)\partial G(X(s),\Phi(s))\,ds\right] \ , \tag{2.52}$$

cf. [24]. In particular

$$\begin{aligned} \mathrm{cov}^{\psi}_{I,U}(F,G) \ &= \mathbf{E}^{\psi,U}\left[\partial F(X(0),\Phi(0))\int_0^\tau \exp\left(\int_0^s U''_{X(t)}(\Phi(t))\,\mathrm{d}t\right)\right. \\ &\qquad\left.\times\ \partial G(X(s),\Phi(s))\,\mathrm{d}s\right] \ , \end{aligned} \tag{2.53}$$

and

$$\mathrm{cov}^{\psi}_{I,U}(\varphi(x),\varphi(y)) = \mathbf{E}^{\psi,U}_{x}\left[\textstyle\int_0^\tau \exp\left(\int_0^s U''_{X(t)}(\Phi(t))\,\mathrm{d}t\right)\mathbf{1}_y(X(s))\,\mathrm{d}s\right] \ . \tag{2.54}$$

Remark 2.4. *On the FKG inequality.* Let $F, G \in C^2_{\exp}(\Omega_I;\mathbb{R})$ be such that $\partial F \geq 0, \partial G \geq 0$. Then $\partial H \geq 0$ and therefore $\mathrm{cov}^{\psi}_{I,U}(F,G) \geq 0$. Thus the measure $\mathbb{P}^{\psi}_{I,U}$ satisfies the FKG property.

The main result of this section is a representation for the mean (and exponential expectations). As it will be clear from below, dealing with the change in the mean of the field due to the presence of an external magnetic field (or chemical potential) ρ will essentially reduce, via a differentiation trick, to the previous covariance representation. More delicate is the situation in the case of non zero boundary conditions. We give the following

Proposition 2.5. *Representation of the mean. Let $\psi \in \Omega_B$ and $\rho \in \Omega^0_E$ be given, then*

$$\Big\langle \varphi(x) \Big\rangle^{\psi}_{I,\rho} = \Big\langle \varphi(x) \Big\rangle^{\psi}_{I} + \int_0^1 \mathrm{cov}^{\psi}_{I,t\rho}\left(\varphi(x), \langle \rho, \varphi\rangle\right) \mathrm{d}t \quad , \tag{2.55}$$

where the first term in the right-hand side can be expressed as

$$\Big\langle \varphi(x) \Big\rangle^{\psi}_{I} = \int_0^1 \sum_{y\in B} \mathbf{P}^{t\psi}_x(X(\tau)=y)\psi(y)\,\mathrm{d}t = \int_0^1 \mathbf{E}^{t\psi}_x\left[\psi(X(\tau))\right] \mathrm{d}t \quad , \tag{2.56}$$

and

$$\mathrm{cov}^{\psi}_{I,t\rho}\left(\varphi(x), \langle\rho,\varphi\rangle\right) = \mathbf{E}^{\psi,t\rho}_x\left[\int_0^{\tau} \rho(X(s))\,\mathrm{d}s\right] \quad . \tag{2.57}$$

Proof. Let

$$h(t) \equiv \Big\langle \varphi(x) \Big\rangle^{\psi}_{I,t\rho} = \frac{\Big\langle \varphi(x)\exp\left(t\langle\rho,\varphi\rangle\right)\Big\rangle^{\psi}_{I}}{\Big\langle \exp\left(t\langle\rho,\varphi\rangle\right)\Big\rangle^{\psi}_{I}} \quad , \tag{2.58}$$

then

$$\begin{aligned} h'(t) &= \frac{\Big\langle \varphi(x)\langle\rho,\varphi\rangle\exp(t\langle\rho,\varphi\rangle)\Big\rangle^{\psi}_{I}}{\Big\langle \exp\left(t\langle\rho,\varphi\rangle\right)\Big\rangle^{\psi}_{I}} \\ &\quad - \frac{\Big\langle \varphi(x)\exp\left(t\langle\rho,\varphi\rangle\right)\Big\rangle^{\psi}_{I}}{\Big\langle \exp\left(t\langle\rho,\varphi\rangle\right)\Big\rangle^{\psi}_{I}} \frac{\Big\langle \langle\rho,\varphi\rangle\exp\left(t\langle\rho,\varphi\rangle\right)\Big\rangle^{\psi}_{I}}{\Big\langle \exp\left(t\langle\rho,\varphi\rangle\right)\Big\rangle^{\psi}_{I}} \\ &= \mathrm{cov}^{\psi}_{I,t\rho}\left(\varphi(x), \langle\rho,\varphi\rangle\right) \quad , \end{aligned} \tag{2.59}$$

which is given by (2.39). Therefore the second term in the right-hand side of (2.55) (and (2.57)) is justified. In order to take care of $\Big\langle \varphi(x) \Big\rangle^{\psi}_{I}$, let

$$f(t) = \Big\langle \varphi(x) \Big\rangle^{t\psi}_{I} = \frac{\Big\langle \varphi(x)\exp\big(-\mathscr{H}_{\mathscr{G}}(\varphi\wedge t\psi)\big)\Big\rangle_{m_I}}{\Big\langle \exp(-\mathscr{H}_{\mathscr{G}}(\varphi\wedge t\psi))\Big\rangle_{m_I}} \quad , \tag{2.60}$$

then by symmetry $f(0)=0$ and, as above,

$$f'(t) = -\mathrm{cov}_I^{t\psi}\left(\varphi(x), \frac{\mathrm{d}}{\mathrm{d}t}\mathscr{H}_{\mathscr{G}}(\varphi\wedge t\psi)\right) = \mathrm{cov}_I^{t\psi}\left(\varphi(x), G^{t\psi}(\varphi)\right) \quad , \tag{2.61}$$

where

$$G^{t\psi}(\varphi) = -\frac{\mathrm{d}}{\mathrm{d}t}\mathscr{H}_{\mathscr{G}}(\varphi\wedge t\psi) = -\sum_{y\in B}\frac{\partial\mathscr{H}_{\mathscr{G}}(\tilde{\varphi})}{\partial\tilde{\varphi}(y)}\bigg|_{\tilde{\varphi}=\varphi\wedge t\psi}\psi(y) \ . \tag{2.62}$$

By (2.25)

$$\mathrm{cov}_I^{t\psi}\left(\varphi(x), G^{t\psi}(\varphi)\right) = \left\langle h(x,\varphi)\right\rangle_I^{t\psi} \quad , \tag{2.63}$$

where $h(x,\varphi)$ solves

$$(-L_I^{t\psi})h(x,\varphi) + \sum_{z\in I}\frac{\partial^2\mathscr{H}_{\mathscr{G}}}{\partial\varphi(z)\partial\varphi(x)}(\varphi\wedge t\psi)h(z,\varphi) = \frac{\partial G^{t\psi}(\varphi)}{\partial\varphi(x)}, \qquad x\in I \quad , \tag{2.64}$$

with boundary condition $h(y,\cdot)=0,\ y\in B$. Let

$$w(y,\varphi) = \begin{cases} h(y,\varphi), & y\in I \\ \psi(y), & y\in B \quad , \end{cases} \tag{2.65}$$

then since

$$\begin{aligned}\sum_{z\in I}\frac{\partial^2\mathscr{H}_{\mathscr{G}}(\varphi\wedge t\psi)}{\partial\varphi(z)\partial\varphi(x)}h(z,\varphi) - \frac{\partial G^{t\psi}(\varphi)}{\partial\varphi(x)} &= \sum_{z\in E}\frac{\partial^2\mathscr{H}_{\mathscr{G}}(\tilde{\varphi})}{\partial\tilde{\varphi}(z)\partial\tilde{\varphi}(x)}\bigg|_{\tilde{\varphi}=\varphi\wedge t\psi} w(z,\varphi) \\ &= (-Q_E^{t\psi,\varphi})w(x,\varphi) \quad ,\end{aligned} \tag{2.66}$$

we see that $w(x,\varphi)$ solves

$$\mathscr{L}_E^{t\psi,\varphi}w(x,\varphi) = 0, \qquad x\in I \quad , \tag{2.67}$$

with boundary condition $w(x,\cdot)=\psi(x)$, $x\in B$. Now the result (2.56) follows as before from the argument in [24, §2],

$$w(x,\varphi) = \sum_{y\in B}\psi(y)\mathbf{P}_{x,\varphi}^{t\psi}\left[X(\tau)=y\right] = \mathbf{E}_{x,\varphi}^{t\psi}\left[\psi\left(X(\tau)\right)\right] \quad , \tag{2.68}$$

and the proof is complete. □

Remark 2.6. *Of course nothing prevents us to replace $\varphi(x)$ with $F(\varphi)\in C^2_{\mathrm{exp}}$, in this case we get*

$$\left\langle F(\varphi)\right\rangle_{I,\rho}^{\psi} = \left\langle F(\varphi)\right\rangle_I^{\psi} + \int_0^1 cov_{I,t\rho}^{\psi}\left(F(\varphi), \langle\rho,\varphi\rangle\right)\,\mathrm{d}t \quad , \tag{2.69}$$

where the first term in the right-hand side can be expressed as

$$\left\langle F(\varphi)\right\rangle_I^{\psi} = \left\langle F(\varphi)\right\rangle_I^{0} + \int_0^1 \mathbf{E}^{t\psi}\left[\partial F(X(0),\Phi(0))\psi(X(\tau))\right]\,\mathrm{d}t \ . \tag{2.70}$$

Also, interchanging the order of differentiation yields a similar formula:

$$\left\langle F(\varphi)\right\rangle_{I,\rho}^{\psi} = \left\langle F(\varphi)\right\rangle_{I}^{0} + \int_0^1 \mathbf{E}^{t\psi,t\rho}\left[\partial F(X(0),\Phi(0))\left(\int_0^{\tau}\rho(X(s))\,ds + \psi\,(X\,(\tau))\right)\right]\mathrm{d}t \quad . \tag{2.71}$$

In particular it follows from the argument, that $(\psi,\rho) : \Omega_B \times \Omega_E^0 \longrightarrow \left\langle F(\varphi)\right\rangle_{I,\rho}^{\psi}$ *is monotone in* ψ *and* ρ *for any monotone* $F \in C^2_{\exp}$.

With the following Corollary we give the tool to compute the moment generating function with general ψ and ρ.

Corollary 2.7. *The exponential moment. Let* $\nu, \rho \in \Omega_E^0$*, then*

$$\begin{aligned} &\log\left\langle \exp\left(\langle\nu,\varphi\rangle - \left\langle\langle\nu,\varphi\rangle\right\rangle_{I,\rho}^{\psi}\right)\right\rangle_{I,\rho}^{\psi} \\ &\quad = \int_0^1\int_0^t \mathrm{var}_{I,\rho+s\nu}^{\psi}\left(\langle\nu,\varphi\rangle\right)\,\mathrm{d}s\,\mathrm{d}t \\ &\quad = \int_0^1\int_0^t \sum_{x\in I}\nu(x)\mathbf{E}_x^{\psi,\rho+s\nu}\left[\int_0^{\tau}\nu(X(u))\,\mathrm{d}u\right]\mathrm{d}s\,\mathrm{d}t \quad . \end{aligned} \tag{2.72}$$

Proof. It follows again from the differentiation trick. Set

$$f(t) = \log\left\langle \exp\left\{t\left[\langle\nu,\varphi\rangle - \left\langle\langle\nu,\varphi\rangle\right\rangle_{I,\rho}^{\psi}\right]\right\}\right\rangle_{I,\rho}^{\psi} \quad , \tag{2.73}$$

then $f(0) = 0$ and

$$f'(t) = \left\langle \left\{\langle\nu,\varphi\rangle - \left\langle\langle\nu,\varphi\rangle\right\rangle_{I,\rho}^{\psi}\right\}\right\rangle_{I,\rho+t\nu}^{\psi} \quad . \tag{2.74}$$

Thus $f'(0) = 0$ and

$$f''(t) = \mathrm{var}_{I,\rho+t\nu}^{\psi}\left(\langle\nu,\varphi\rangle\right) \quad . \tag{2.75}$$

Integrating and using (2.39) yields the result. □

2.1.1. The homogeneous Gaussian case

We single out the very particular case of quadratic potentials, independent of the particular bond, by putting a superscript $*$. Therefore

$$V_b^*\left(\nabla\varphi(b)\right) = \frac{1}{2}\left(\nabla\varphi(b)\right)^2, \qquad b \in \mathscr{E} \quad , \tag{2.76}$$

and we write $\mathscr{H}_{\mathscr{G}}^*(\varphi)$ for the harmonic Hamiltonian, that is $\sum_{b\in\mathscr{E}}(\nabla\varphi(b))^2$, and $\mathbb{P}_{I,\rho}^*$ for the corresponding measure. The Gaussian measure will be used as a comparison tool and we will mostly use it only with $\psi \equiv 0$ and $\rho \equiv 0$: as before, in this case we will omit ψ and ρ in the notation. Note that $\mathbb{P}_I^*$ is a Gaussian measure on Ω_I and therefore it is completely characterized by its mean (in this case 0) and its covariances. Let us observe that, in the Gaussian case, Φ and X, in this case denoted by X^*, evolve independently. In fact $V_b^{*\prime\prime} \equiv 1$ for all $b \in \mathscr{E}$, and therefore (2.39) reduces to computing the time spent in y for the simple random walk starting at x (or vice versa). In particular for $G(\varphi) = \langle \nu, \varphi \rangle$ a linear function, the equation (2.30) reduces to an equation independent of φ:

$$(-Q^*)h^*(x) = \sum_{b \in \mathscr{E}: x(b)=x} \nabla_b h^* = \nu(x) \tag{2.77}$$

with solution $h^*(x) = \mathbf{E}_x[\int_0^\tau \nu(X^*(s))\, ds]$, and Q^* is just the Laplacian on the graph. As it is well known, in the Gaussian case we can express also the mean of the field in terms of a simple random walk. For example, in the case of boundary conditions $\psi \in \Omega_B$ and $\rho \equiv 0$ and denoting by $\mathbf{P}_x$ the law of the simple symmetric random walk starting from $x \in I$, we have (see formula (2.56))

$$\Big\langle \varphi(x) \Big\rangle_{\mathbb{P}_I^{*,\psi}} = \mathbf{E}_x\left[\psi\left(X_\tau\right)\right] \quad , \tag{2.78}$$

for every $x \in I$ (see [27, Ch. 13]) for the case in which V is still quadratic, but bond dependent, and $\rho \neq 0$). Every expression with a superscript $*$ refers to the homogeneous Gaussian case: in particular $\mathscr{L}^* = L^* + Q^*$. In general the Gaussian computations may cast some light in understanding the general (non-Gaussian) behavior. As already remarked before, this is not the only advantage: it is in fact possible to get some inequalities on non-Gaussian quantities with respect to Gaussian quantities. This is the subject of the next subsection.

2.2. Inequalities and estimates for φ-fields on a graph

For simplicity, in this subsection we will only deal with 0-boundary conditions, i.e. $\psi_x = 0$, $x \in B$.

A general inequality for log-concave measures, due to H. Brascamp and E. Lieb [13] and applied to the massless field case for the first time in [14], says that any moment, including the exponential one, of $\langle \nu, \varphi \rangle - \Big\langle \nu, \varphi \Big\rangle_{I,\rho}^{\psi}$ is bounded above by the corresponding moment for the field with Hamiltonian $\mathscr{H}^*(\varphi)$ equals to $\frac{1}{2}\sum_b C_1 V^*(\nabla\varphi(b))$, where C_1 is the lower bound on V_b''. We start by reviewing this result in our set up and by giving a (partial) reverse inequality, i.e. we will find a lower bound on variances in terms of the corresponding Gaussian expectation, with potential $C_2 V^*$, where C_2 is the upper bound on V_b''.

Let λ_E^* be the principal eigenvalue of the random walk on the graph with Dirichlet boundary conditions:

$$\lambda_E^* = \inf\left\{ \Gamma_{Q^*}(u,u) = \frac{1}{2}\sum_{b \in \mathscr{E}} (\nabla u(b))^2 : u \in \Omega_E^0 \text{ with } \|u\| = \langle u, u \rangle^{1/2} = 1 \right\} \quad . \tag{2.79}$$

Note that $\lambda_E^* > 0$ since $B \neq \emptyset$ and the graph is finite and connected. Next, for $F \subseteq I$ let

$$\lambda_{F,E}^* = \inf\left\{\Gamma_{Q^*}(u,u) = \frac{1}{2}\sum_{b\in\mathscr{E}}(\nabla u(b))^2 : u \in \Omega_E^0 \text{ with } \sum_{x\in F}(u(x))^2 = 1\right\} \ . \tag{2.80}$$

Note that $\lambda_{I,E}^* = \lambda_E^*$ and $\lambda_{F,E}^* \geq \lambda_E^*$. For given $\alpha : \mathscr{E} \longrightarrow \mathbb{R}$ define the divergence $\operatorname{div}(\alpha) \in \Omega_E^0$

$$\operatorname{div}(\alpha)(x) = \begin{cases} \sum\limits_{y\in E}\left[\alpha\big((y,x)\big) - \alpha\big((x,y)\big)\right], & x \in I \\ 0, & x \in B \ . \end{cases} \tag{2.81}$$

Note that the divergence is characterized by the following summation by parts formula: for all $\varphi \in \Omega_E^0$,

$$\langle\alpha, \nabla\varphi\rangle = \sum_{b\in\mathscr{E}}\alpha(b)\nabla\varphi(b) = \sum_{x\in I}\operatorname{div}(\alpha)(x)\varphi(x) \ . \tag{2.82}$$

where we have extended the notation $\langle\cdot,\cdot\rangle$ to denote also the scalar product in $\mathbb{L}_2(\mathscr{E};\mathbb{R})$.

2.2.1. Brascamp–Lieb and reverse Brascamp–Lieb inequalities

Lemma 2.8. *If $\nu \in \Omega_F^0$, where $F \subseteq E$ and $\Omega_F^0 = \{\nu \in \Omega_E : \nu(x) = 0$ for $x \in E\backslash F\}$, then*

$$\operatorname{var}_{I,\rho}(\langle\nu,\varphi\rangle) \leq \frac{1}{C_1}\operatorname{var}_I^*(\langle\nu,\varphi\rangle) \leq \frac{1}{C_1\lambda_{F,E}^*}\|\nu\|^2 \ . \tag{2.83}$$

and

$$\operatorname{var}_{I,\rho}(\langle\nu,\varphi\rangle) \geq \frac{1}{C_2}\operatorname{var}_I^*(\langle\nu,\varphi\rangle) \ . \tag{2.84}$$

Moreover, for each $\alpha \in \Omega_{\mathscr{E}} \equiv \mathbb{R}^{\mathscr{E}}$, we have

$$\frac{1}{C_2}\operatorname{var}_I^*(\langle\alpha,\nabla\varphi\rangle) \leq \operatorname{var}_{I,\rho}(\langle\alpha,\nabla\varphi\rangle) \leq \frac{1}{C_1}\operatorname{var}_I^*(\langle\alpha,\nabla\varphi\rangle) \leq \frac{1}{C_1}\|\alpha\|^2 \ . \tag{2.85}$$

Proof. Take $G \in C^2_{\exp}$, then since $\hat{C}^2_{\exp}$ is a dense set of the $\mathbb{L}_2$ domain of $(-\mathscr{L}_{E,\rho})$, we get by simple $\mathbb{L}_2$ calculus

$$\begin{aligned}\operatorname{var}_{I,\rho}(G) &= \Big\langle\langle\langle\partial G, \left(-\mathscr{L}_{E,\rho}\right)^{-1}\partial G\rangle\rangle(\varphi)\Big\rangle_{I,\rho} \\ &= \sup_{f\in\hat{C}^2_{\exp}}\left\{2\Big\langle\langle\langle\partial G, f\rangle\rangle(\varphi)\Big\rangle_{I,\rho} - \Gamma_{\mathscr{L}_{E,\rho}}(f,f)\right\} \ .\end{aligned} \tag{2.86}$$

Next, in view of $V'' \geq C_1$, we have

$$\Gamma_{Q_E^\varphi}(f,f)(\varphi) \geq C_1\Gamma_{Q^*}(f,f)(\varphi) \ . \tag{2.87}$$

Thus, by Jensen's inequality,

$$\Gamma_{\mathscr{L}_{E,\rho}}(f,f) \geq \sum_{x\in I} \Gamma_{L_{I,\rho}}(f(x,\,\cdot\,), f(x,\,\cdot\,)) + C_1 \Big\langle \Gamma_{Q^*}(f,f)(\varphi) \Big\rangle_{I,\rho} \\ \geq C_1 \Gamma_{Q^*}\left(f_{I,\rho}, f_{I,\rho}\right) \quad , \tag{2.88}$$

where $f_{I,\rho}(x) = \Big\langle f(x,\varphi) \Big\rangle_{I,\rho}$. In view of (2.88), with $\partial G(x,\cdot) \equiv \nu(x)$, and (2.86), we get

$$\begin{aligned} \mathrm{var}_{I,\rho}(\langle \nu, \varphi \rangle) &= \sup_{f \in \hat{C}^2_{\exp}} \left\{ 2 \Big\langle \langle \langle \nu, f \rangle \rangle(\varphi) \Big\rangle_{I,\rho} - \Gamma_{\mathscr{L}_{E,\rho}}(f,f) \right\} \\ &\leq \sup_{f \in \hat{C}^2_{\exp}} \left\{ 2\langle \nu, f_{I,\rho} \rangle - C_1 \Gamma_{Q^*}\left(f_{I,\rho}, f_{I,\rho}\right) \right\} \\ &\leq \frac{1}{C_1} \mathrm{var}^*_I(\langle \nu, \varphi \rangle) \quad , \end{aligned} \tag{2.89}$$

which proves (2.83), since, for any ν with support in F, by the definition of the Gaussian measure and Cauchy–Schwarz

$$\mathrm{var}^*_I(\langle \nu, \varphi \rangle) = \sup_{u \in \Omega^0_I} \frac{\langle u, \nu \rangle^2}{\Gamma_{Q^*}(u,u)} \leq \frac{\langle \nu, \nu \rangle}{\lambda^*_{F,E}} \quad . \tag{2.90}$$

Finally, again using (2.86), restricting ourselves to $f(x,\varphi) = \beta(x)$ independent of φ, we get

$$\mathrm{var}_{I,\rho}(\langle \nu, \varphi \rangle) \geq \sup_{\beta \in \Omega^0_I} \left\{ 2\langle \nu, \beta \rangle - \Gamma_{\mathscr{L}_{E,\rho}}(\beta,\beta) \right\} \quad . \tag{2.91}$$

Moreover, since $V'' \leq C_2$ we have

$$\Gamma_{\mathscr{L}_{E,\rho}}(\beta,\beta) = \sum_{b \in \mathscr{E}} \Big\langle a(b,\varphi) \Big\rangle_{I,\rho} (\nabla\beta(b))^2 \leq C_2 \Gamma_{Q^*}(\beta,\beta) \quad , \tag{2.92}$$

which implies (2.84). The inequality (2.85) trivially follows from (2.82):

$$\begin{aligned} \mathrm{var}_{I,\rho}(\langle \alpha, \nabla\varphi \rangle) &= \mathrm{var}_{I,\rho}(\langle \mathrm{div}(\alpha), \varphi \rangle) \\ &\leq \frac{1}{C_1} \mathrm{var}^*_I(\langle \mathrm{div}(\alpha), \varphi \rangle) \leq \frac{1}{C_1} \|\alpha\|^2 \quad , \end{aligned} \tag{2.93}$$

since, by Cauchy–Schwarz

$$\mathrm{var}^*_I(\langle \mathrm{div}(\alpha), \varphi \rangle) = \sup_{u \in \Omega^0_I} \frac{\langle \mathrm{div}(\alpha), u \rangle^2}{\Gamma_{Q^*}(u,u)} = \sup_{u \in \Omega^0_I} \frac{\langle \alpha, \nabla u \rangle^2}{\langle \nabla u, \nabla u \rangle} \leq \|\alpha\|^2 \quad . \tag{2.94}$$

□

We now pass to exponential estimates.

Lemma 2.9. *For each $\nu \in \Omega_F^0$ and $\alpha \in \Omega_{\mathscr{E}}$*

$$\frac{\mathrm{var}_I^*\left(\langle \nu, \varphi\rangle\right)}{2C_2} \le \log\left\langle \exp\left(\langle \nu, \varphi\rangle - \left\langle\langle \nu, \varphi\rangle\right\rangle_{I,\rho}\right)\right\rangle_{I,\rho} \le \frac{var_I^*\left(\langle \nu, \varphi\rangle\right)}{2C_1} \le \frac{1}{2C_1\lambda^*_{F,E}}\|\nu\|^2 \ . \tag{2.95}$$

and

$$\frac{\mathrm{var}_I^*\left(\langle \alpha, \nabla\varphi\rangle\right)}{2C_2} \le \log\left\langle \exp\left(\langle \alpha, \nabla\varphi\rangle - \left\langle\langle \alpha, \nabla\varphi\rangle\right\rangle_{I,\rho}\right)\right\rangle_{I,\rho} \le \frac{var_I^*\left(\langle \alpha, \nabla\varphi\rangle\right)}{2C_1} \le \frac{1}{2C_1}\|\alpha\|^2 \ . \tag{2.96}$$

In particular, for each $T > 0$, we have

$$\begin{aligned}
\mathbb{P}_{I,\rho}\left(\left(\langle \nu, \varphi\rangle - \left\langle\langle \nu, \varphi\rangle\right\rangle_{I,\rho}\right) > T\right) &\le \exp\left(-\frac{C_1T^2}{2var_I^*(\langle \nu, \varphi\rangle)}\right)\\
&\le \exp\left(-\frac{C_1\lambda^*_{F,E}T^2}{2\|\nu\|^2}\right)\\
\mathbb{P}_{I,\rho}\left(\left(\langle \alpha, \nabla\varphi\rangle - \left\langle\langle \alpha, \nabla\varphi\rangle\right\rangle_{I,\rho}\right) > T\right) &\le \exp\left(-\frac{C_1T^2}{2var_I^*(\langle \alpha, \nabla\varphi\rangle)}\right)\\
&\le \exp\left(-\frac{C_1T^2}{2\|\alpha\|^2}\right) \ .
\end{aligned} \tag{2.97}$$

and

$$\mathbb{P}_{I,\rho}\left(\left(\langle \nu, \varphi\rangle - \left\langle\langle \nu, \varphi\rangle\right\rangle_{I,\rho}\right) > T\right) \ge \frac{\exp(-2\log 2C_2 - 1)}{2} \times \exp\left(-\frac{2C_2^2}{C_1}\frac{T^2}{var_I^*(\langle \nu, \varphi\rangle)}\right) \ . \tag{2.98}$$

Proof. The inequalities in (2.95) and (2.96) are immediate consequences of Corollary 2.7 and Lemma 2.8. The upper bounds (2.97) are just an easy consequence of Chebychev inequality. In order to prove the lower bound (2.98), using the rescaling $T' = T/\sqrt{\mathrm{var}_I^*(\langle \nu, \varphi\rangle)}$ we may assume that $\mathrm{var}_I^*(\langle \nu, \varphi\rangle) = 1$. Let $u(x) = \left\langle\varphi(x)\right\rangle_{I,\rho}$ and, for simplicity, let us write $\mathbb{P}' = \mathbb{P}_{I,\rho} \circ T_u^{-1}$, where $T_u\varphi(x) = \varphi(x) - u(x)$, $x \in I$. Set $\mathbb{P}'_t(\mathrm{d}\varphi) = \frac{\exp(t\langle \nu,\varphi\rangle)}{Z'(t)}\mathbb{P}'(\mathrm{d}\varphi)$. Using an entropy inequality [21, Lemma 5.4.21], we have

$$\log\frac{\mathbb{P}'(\langle \nu, \varphi\rangle > T)}{\mathbb{P}'_t(\langle \nu, \varphi\rangle > T)} \ge -\frac{\mathbf{H}(\mathbb{P}'\big|\mathbb{P}'_t) + e^{-1}}{\mathbb{P}'_t(\langle \nu, \varphi\rangle > T)} \ , \tag{2.99}$$

where for two probability measures P and Q, defined on the same measurable space, the *relative entropy of Q with respect to P* is defined as

$$\mathbf{H}(Q|P) = \begin{cases} \int \left(\log \frac{\mathrm{d}Q}{\mathrm{d}P}\right) \mathrm{d}Q & Q \ll P \\ \infty & Q \not\ll P \end{cases} . \tag{2.100}$$

Therefore

$$\mathbf{H}(\mathbb{P}'_t \big| \mathbb{P}') = \mathbb{E}'_t \left[\log \frac{d\mathbb{P}'_t}{d\mathbb{P}'}\right] = t\mathbb{E}'_t[\langle \nu, \varphi\rangle] - \log Z'(t) \ , \tag{2.101}$$

and since

$$\frac{\mathrm{d}}{\mathrm{d}t} \log Z'(t) = \mathbb{E}'_t[\langle \nu, \varphi\rangle], \quad \frac{\mathrm{d}}{\mathrm{d}t}\mathbb{E}'_t[\langle \nu, \varphi\rangle] = \mathrm{var}'_t(\langle \nu, \varphi\rangle) \ , \tag{2.102}$$

we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{H}(\mathbb{P}'_t \big| \mathbb{P}') = t\frac{\mathrm{d}}{\mathrm{d}t}\mathbb{E}'_t[\langle \nu, \varphi\rangle] + \mathbb{E}'_t[\langle \nu, \varphi\rangle] - \frac{\mathrm{d}}{\mathrm{d}t} \log Z'(t) = t\mathrm{var}'_t(\langle \nu, \varphi\rangle) \ . \tag{2.103}$$

Thus, in view of the Brascamp–Lieb inequality (second inequality in (2.83))

$$\mathbf{H}(\mathbb{P}'_t \big| \mathbb{P}') \le \frac{t^2}{2C_1} \ . \tag{2.104}$$

On the other hand, using the reversed Brascamp–Lieb inequality (first inequality in (2.83))

$$-\log \frac{Z'(t-\tau)}{Z'(t)} = \int_{t-\tau}^{t} s \ \mathrm{var}'_s(\langle \nu, \varphi\rangle)\, ds \ge \frac{1}{2C_2}\left(2t\tau - \tau^2\right) \ . \tag{2.105}$$

In particular, for $t \ge C_2 T$, this implies

$$\log \mathbb{P}'_t(\langle \nu, \varphi\rangle \le T) \le -\sup_{\tau \ge 0}\{-T\tau - \log \frac{Z'(t-\tau)}{Z'(t)}\} = -\frac{1}{2C_2}\left(t - C_2 T\right)^2 \ . \tag{2.106}$$

Choose $t = C_2 T + \sqrt{2\log 2C_2}$, then $t^2 \le 2C_2^2 T^2 + 2\log 2C_2$, and $\mathbb{P}'_t(\langle \nu, \varphi\rangle > T) \ge 1 - e^{-\log 2} = 1/2$, which implies (2.98) by (2.99). □

2.2.2. Entropy estimates

Let us consider two probability measures P and Q on Ω_F. The following proposition shows that we can control the difference of the expectations of φ with respect to P and Q with the relative entropy.

Proposition 2.10. *Let P be the restriction of $\mathbb{P}_{I,\rho}$ on Ω_F, where $F \subseteq I$. Then for any measure Q on Ω_F we have*

$$\sum_{x\in F} \left(\mathbb{E}_P\left[\varphi(x)\right] - \mathbb{E}_Q\left[\varphi(x)\right]\right)^2 \le \frac{2}{C_1 \lambda^*_{F,E}}\mathbf{H}(Q|P) \ , \tag{2.107}$$

$$\sum_{b\in\mathscr{F}} \left(\mathbb{E}_P\left[\nabla\varphi(b)\right] - \mathbb{E}_Q\left[\nabla\varphi(b)\right]\right)^2 \leq \frac{2}{C_1}\mathbf{H}(Q|P) \quad , \tag{2.108}$$

where $\mathscr{F} = \{b \in \mathscr{E} : x(b), y(b) \in F\}$. *As a consequence*

$$\begin{aligned} \left(\sum_{x\in F} \left\langle\varphi(x)\right\rangle_{I,\rho}^2\right)^{1/2} &\leq \frac{\|\rho\|}{C_1(\lambda_E^*)^{1/2}(\lambda_{F,E}^*)^{1/2}}, \\ \left(\sum_{b\in\mathscr{E}} \left\langle\nabla\varphi(b)\right\rangle_{I,\rho}^2\right)^{1/2} &\leq \frac{\|\rho\|}{C_1(\lambda_E^*)^{1/2}} \quad . \end{aligned} \tag{2.109}$$

Proof. First note that for any $\beta \in \Omega_F$,

$$\begin{aligned} &\sum_{x\in F} \beta(x)\left[\mathbb{E}_Q\left[\varphi(x)\right] - \mathbb{E}_P\left[\varphi(x)\right]\right] \\ &\qquad - \log \mathbb{E}_P\left[\exp\left(\sum_{x\in F}\beta(x)\left(\varphi(x) - \mathbb{E}_P[\varphi(x)]\right)\right)\right] \leq \mathbf{H}(Q|P) \quad , \end{aligned} \tag{2.110}$$

(see [21, 3.2.12]). Next, in view of Lemma 2.9, we have

$$\log \mathbb{E}_P\left[\exp\left(\sum_{x\in F}\beta(x)\left(\varphi(x) - \mathbb{E}_P[\varphi(x)]\right)\right)\right] \leq \frac{\|\beta\|^2}{2C_1\lambda_{F,E}^*} \quad . \tag{2.111}$$

Choosing $\beta(x) = C_1\lambda_{F,E}^*(\mathbb{E}_Q[\varphi(x)] - \mathbb{E}_P[\varphi(x)])$ yields the first equality. The second follows with the very same argument. As for the last two statements: choose $P = \mathbb{P}_{I,\rho}$ and $Q = \mathbb{P}_I$. Then

$$\frac{\mathrm{d}Q}{\mathrm{d}P} = \frac{1}{Z}\exp\left(-\langle\rho,\varphi\rangle\right) \quad , \tag{2.112}$$

where $Z = \mathbb{E}_P[\exp(-\langle\rho,\varphi\rangle)] = 1/\mathbb{E}_Q[\exp(\langle\rho,\varphi\rangle)]$. Thus, using again Lemma 2.9, we have

$$\begin{aligned} \mathbf{H}(Q|P) &= \mathbb{E}_Q\left[-\langle\rho,\varphi\rangle\right] - \log Z = -\log Z \\ &= \log \mathbb{E}_Q\left[\exp\left(\langle\rho,\varphi\rangle\right)\right] \leq \frac{\|\rho\|^2}{2C_1\lambda_E^*} \quad , \end{aligned} \tag{2.113}$$

which implies the result. □

2.2.3. Exponential tightness

Lemma 2.11. *Let* $\rho \in \Omega_E^0$, *then for each* $\epsilon < C_1/2$ *and* $F \subseteq I$ *we have*

$$\begin{aligned} \log\left\langle\exp\left(\epsilon\lambda_{F,E}^*\sum_{x\in F}(\varphi(x))^2\right)\right\rangle_{I,\rho} &\leq \log\left\langle\exp\left(\epsilon\sum_{b\in\mathscr{E}}(\nabla\varphi(b))^2\right)\right\rangle_{I,\rho} \\ &\leq c_1\frac{|I|}{2} + c_2\lambda_E^*\|\rho\|^2, \end{aligned} \tag{2.114}$$

where c_1 *and* c_2 *depend only on* C_1, C_2 *and* ϵ *(see (2.116) below for explicit expressions).*

Proof. This result is quite elementary. The first inequality follows immediately from the definition of $\lambda^*_{F,E}$. For the second one, using $C_1 \le V'' \le C_2$ we have

$$C_1 \mathscr{H}^*_{\mathscr{G}}(\varphi) \le \mathscr{H}_{\mathscr{G}}(\varphi) \le C_2 \mathscr{H}^*_{\mathscr{G}}(\varphi) \quad . \tag{2.115}$$

Thus

$$\begin{aligned}\left\langle \exp\left(\epsilon \sum_{b\in\mathscr{E}} (\nabla(\varphi))^2\right)\right\rangle_{I,\rho} &= \frac{\left\langle \exp\left(\epsilon \sum_{b\in\mathscr{E}} (\nabla\varphi(b))^2 - \mathscr{H}_{\mathscr{G},\rho}(\varphi)\right)\right\rangle_{m_I}}{\left\langle \exp\left(-\mathscr{H}_{\mathscr{G},\rho}(\varphi)\right)\right\rangle_{m_I}} \\ &\le \left(\frac{C_2}{C_1 - 2\epsilon}\right)^{|I|/2} \exp\left(\frac{C_2 - C_1 + 2\epsilon}{2C_2(C_1 - 2\epsilon)} \mathrm{var}^*_I(\langle \rho, \varphi\rangle)\right) \quad ,\end{aligned} \tag{2.116}$$

and $\mathrm{var}^*_I(\langle\rho,\varphi\rangle) \le \lambda^*_E \|\rho\|^2$. □

2.3. *H–S representation and estimates for φ-fields on the square lattice*

We are now going to focus on the framework of the introduction (Subsection 1.2). In this case the graph $\mathscr{G}$ depends on $N \in \mathbb{Z}^+$. More precisely

$$I = D_N \subset \mathbb{Z}^d_N, \qquad B = \partial^+ D_N \subset \mathbb{Z}^d_N \quad , \tag{2.117}$$

where ∂^+ denotes the external boundary of a subset of $\mathbb{Z}^d_N$ and $\mathscr{E}$ are the bonds inherited from the nearest-neighbor graph structure of $\mathbb{Z}^d$, that is

$$\mathscr{E} \equiv \{(x,y) : x, y \in E = B \cup I, |x-y| = 1/N \text{ and } x \text{ or } y \in I\} \quad . \tag{2.118}$$

We will restrict ourselves once again to the 0-boundary condition case, therefore the superscript ψ will never be present. Moreover we will use all the notations and short-cuts of Subsection 1.2, in particular the Hamiltonian $\mathscr{H}_{\mathscr{G}}$ (2.3) will be simply denoted by $\mathscr{H}_N$ (1.9), the measure $\mathbb{P}_{N,h}$, defined in (1.27), corresponds to $\mathbb{P}_{I,\rho}$ with $\rho = \frac{1}{N} h$ (defined in (2.6) and (2.9)) and we will omit h from the notation if $h \equiv 0$. Observe also that $C_1 = c_V^{-1}$ and $C_2 = c_V$ (compare (1.13) and (2.2)). Moreover also here we use the notation $\partial F(x,\varphi) = \partial F(\varphi)/\partial\varphi(x)$ and $\partial G(x,\varphi) = \partial G(\varphi)/\partial\varphi(x)$.

2.3.1. Some H–S formulas on $\mathbb{Z}^d_N$

Below we restate, in this particular context, some of the formulas introduced in Proposition 2.2, Proposition 2.5 and Corollary 2.7.

On the square lattice, we will denote the stochastic process behind the H–S representation by $\{(X_N(t), \Phi(t))\}_{t\in\mathbb{R}^+}$, corresponding to the (X,Φ)-process introduced in Subsection 2.1. If we pass from the bond notation to the site notation, the pregenerator of this process can be written as

$$\begin{aligned}
\left(\mathscr{L}_{N,h}g\right)(x,\varphi) &= \sum_{z}^{N} e^{\mathscr{H}_{N,h}(\varphi)} \frac{\partial}{\partial\varphi(z)}\left[e^{-\mathscr{H}_{N,h}(\varphi)}\frac{\partial g}{\partial\varphi(z)}(x,\varphi)\right] \\
&\quad - \frac{1}{N^2}\sum_{i=1}^{d} \nabla_i^{N*}\left[V''(\eta_i(x))\nabla_i^N g\right](x,\varphi) \\
&= \left(L_{N,h}g\right)(x,\varphi) + \left(Q_N g\right)(x,\varphi) \qquad (2.119)
\end{aligned}$$

where $g : D_N \times \mathbb{R}^{D_N} \to \mathbb{R}$ is a smooth function of φ, for all x. The factor $1/N^2$ in front of the jump term of $\mathscr{L}_{N,f}$ is due to the fact that ∇^N is rescaled by N. We denote by τ_N the exit time of $X_N(\cdot)$ from D_N.

There are three formulas that will be particularly relevant for our analysis: the representation of the covariances and the representation of tilted expectations and exponential expectations of linear functionals. For $F, G \in C^2_{exp}(\mathbb{R}^{D_N};\mathbb{R})$ we have

$$\begin{aligned}
\mathrm{cov}_{N,h}(F,G) &= \left\langle \langle\langle \partial F, \left(-\mathscr{L}_{N,h}\right)^{-1}\partial G\rangle\rangle(\varphi)\right\rangle_{N,h} \\
&= \left\langle \sum_{x}^{N} \partial F(x,\varphi)\mathbf{E}^h_{x,\varphi}\left(\int_0^{\tau_N}\partial G(X_N(t),\Phi(t))\right)\mathrm{d}t\right\rangle_{N,h} .
\end{aligned} \qquad (2.120)$$

Here by $-\mathscr{L}^{-1}_{N,h}\partial G$ we mean the solution h of the elliptic problem $-\mathscr{L}_{N,h}h = \partial G$ (see (2.51)). The tilted averages are given by

$$\begin{aligned}
\left\langle \varphi(x)\right\rangle_{N,h} &= \frac{1}{N}\int_0^1 \left\langle \langle\langle \mathbf{1}_{\{x\}}, \left(-\mathscr{L}_{N,th}\right)^{-1} h\rangle\rangle(\varphi)\right\rangle_{N,th}\mathrm{d}t \\
&= \frac{1}{N}\int_0^1 \left\langle \mathbf{E}^{th}_{x,\varphi}(h(X_N(t)))\right\rangle_{N,th}\mathrm{d}t \ , \qquad (2.121)
\end{aligned}$$

and the exponential expectation value of linear functionals of φ

$$\begin{aligned}
&\log\left\langle \exp\left(\langle h_1,\varphi\rangle - \left\langle\langle h_1,\varphi\rangle\right\rangle_{N,h}\right)\right\rangle_{N,h} \\
&\quad = \frac{1}{2}\int_0^1\int_0^t \left\langle \langle\langle h_1, \left(-\mathscr{L}_{N,h+sh_1}\right)^{-1} h_1\rangle\rangle\right\rangle_{N,h+sh_1}\mathrm{d}s\mathrm{d}t \ , \qquad (2.122)
\end{aligned}$$

where h_1 is a function supported in D.

2.3.2. The Brascamp–Lieb inequalities, the lattice case

If we set $\Delta_N = -\nabla^{N*}\nabla^N$ and we denote by $\Delta_N{}^{-1} : D_N \times D_N \to \mathbb{R}$ the Green function associated to Δ_N (with Dirichlet boundary conditions), a change of notation in Lemma 2.8 and Lemma 2.9 leads, for $h, h_1 \in C_0^\infty(D)$, to the following inequalities, uniform in N:

$$\frac{1}{c_V}\sum_{x,y}^{N}\nabla^N h_1(x)\Delta_N^{-1}(x,y)\nabla^N h_1(y) \le \mathrm{var}_{N,h}\left(\sum_x^N h_1(x)\eta_i(x)\right) \le c_V N^d \|h_1\|_2^2\ , \tag{2.123}$$

$$\frac{1}{N^d}\log\left\langle \exp\left\{\sum_x^N\left[h_1(x)\eta_i(x) - \left\langle h_1(x)\eta_i(x)\right\rangle_{N,h}\right]\right\}\right\rangle_{N,h} \le \frac{c_V}{2}\|h_1\|_2^2\ , \tag{2.124}$$

$$\frac{1}{N^d}\log\left\langle \exp\left\{\sum_x^N\left[h_1(x)\xi_N(x) - \left\langle h_1(x)\xi_N(x)\right\rangle_{N,h}\right]\right\}\right\rangle_{N,h} \le c\|h_1\|_2^2\ , \tag{2.125}$$

where c is a constant, independent of h and h_1, which can be easily estimated from the definition (2.79) of $\lambda^*_{D_N}$. Here we simply use the fact that, since D is bounded, there exists c' such that $\lambda_{D_N} \ge c'N^{-2}$. In the first term in (2.123) we recognize the variance of $\sum^N h_1\eta_i$ with respect to the Gaussian massless field.

2.3.3. Entropy estimate on the gradient and exponential tightness

Again by using the fact that $\lambda_{D_N} \ge c'N^{-2}$, from (2.109) we obtain the following important bulk estimate

$$\sqrt{\frac{1}{N^d}\sum_x^N\left\langle\eta(x)\right\rangle_{N,h}^2} = \left\|\nabla^N u_{N,h}\right\|_2 \le \frac{c_V}{c'}\|h\|_2\ . \tag{2.126}$$

For the next estimate we need some notation: let $D_N^0 = \{x \in D_N : \mathrm{dist}(x, D_N^{\complement}) \ge 2/N\}$ and denote by $\tilde{\mathbb{P}}_0$ the restriction of $\mathbb{P}_{N,f}$ to $\mathbb{R}^{D_N^0}$. Moreover let $\tilde{\mathbb{Q}}_0$ be the restriction of $\mathbb{P}_{N,f}\circ(\theta_e^N)^{-1}$, $|e| = 1$, to $\mathbb{R}^{D_N^0}$, with θ_e^N the translation operator: $\theta_e^N f(x) = f(x + e/N)$. Note that $\theta_e^N D_N^0 \subset D_N$. Recall now that f is compactly supported in D: therefore for N sufficiently large f is supported in D_N^0. By applying Proposition 2.10, formula (2.108), we obtain the control of the oscillations in our profile in terms of a relative entropy:

$$\sum_{x\in D_N^0}\left[\left\langle\eta\left(x+\frac{e}{N}\right)\right\rangle_{N,h} - \left\langle\eta(x)\right\rangle_{N,h}\right]^2 \le 2c_V\mathbf{H}\left(\tilde{\mathbb{P}}_0\middle|\tilde{\mathbb{Q}}_0\right)\ . \tag{2.127}$$

We conclude this section by giving the estimate that implies the exponential tightness for the sequence of processes we are looking at: from Lemma 2.11 we obtain that there exists $\epsilon_0 > 0$ such that for every $\epsilon \in (0,\epsilon_0)$ there exists C such that

$$\sup_{N\in\mathbb{N}}\frac{1}{N^d}\log\left\langle\exp\left\{\epsilon\sum_x^N\left[|\xi_N(x)|^2 + \left|\nabla^N\xi_N(x)\right|^2\right]\right\}\right\rangle_{N,h} \le C\ . \tag{2.128}$$

3. A priori bounds and technical estimates

3.1. The oscillation inequality: statement and strategy of the proof

Recall the notation $u_{N,f}(x) = \langle \varphi(x)/N \rangle_{N,f} = \langle \xi_N(x) \rangle_{N,f}$. Below $e \in \mathbb{Z}^d$, $|e| = 1$. We have the following result, which controls the oscillations of the gradient of $u_{N,f}$.

Lemma 3.1. *Let us assume that ∂D is Lipschitz. Then for every $f \in \mathbb{C}_0^\infty(D)$ one can choose a function $\psi : \mathbb{N} \mapsto \mathbb{R}^+$ with $\lim_{N\to\infty} \psi(N) = 0$, such that,*

$$\frac{1}{N^d} \sum_x^N \sum_{i=1}^d \left[\nabla_i^N u_{N,f} \left(x + \frac{e}{N} \right) - \nabla_i^N u_{N,f}(x) \right]^2 \leq \psi(N) \ , \tag{3.1}$$

In fact our proof gives a control over the rate of convergence in (3.1): the function $\psi(N)$ can be chosen $O((1/N)^\delta)$ for some $\delta > 0$ (cf. (3.7)).

The scheme of the proof is the following:

1. By (2.127) the term we have to estimate is bounded by the $\mathbb{L}_2$-norm of the gradient *near the boundary* plus a relative entropy:

$$\begin{aligned} &\sum_x^N \sum_{i=1}^d \left[\nabla_i^N u_{N,f} \left(x + \frac{e}{N} \right) - \nabla_i^N u_{N,f}(x) \right]^2 \\ &\quad \leq \sum_{x \in D_N \setminus D_N^0} \sum_{i=1}^d \left[\nabla_i^N u_{N,f} \left(x + \frac{e}{N} \right) - \nabla_i^N u_{N,f}(x) \right]^2 + 2d\, c_V \mathbf{H}\left(\tilde{\mathbb{P}}_0 \middle| \tilde{\mathbb{Q}}_0 \right) \\ &\quad = T_1 + T_2 \ . \end{aligned} \tag{3.2}$$

   Note that in T_1 the sum is only over points at a distance smaller than $2/N$ from the boundary.
2. With Lemma 3.2 we will show that the relative entropy term T_2 can be decomposed into a bulk term and a boundary term: the boundary term is essentially the same as T_1, in the sense it is the sum of the square on $\nabla^N u_{N,f}(x)$ with x within a distance k/N from the boundary of D_N: this time k will be equal to 3.
3. The bulk term is controlled by applying the Brascamp–Lieb inequality (2.123).
4. The $\mathbb{L}_2$-norm of the gradient *near the boundary* will be shown (Lemma 3.3) to be bounded by the $\mathbb{L}_p$-norm of the gradient in the bulk; $p > 2$, and the latter will be controlled by adapting the $\mathbb{L}_p$ theory of elliptic equations to our discrete setting (see Lemma 3.4) [4].

The proof Lemma 3.1 can be found in Subsection 3.2, where one can find also the preparatory results we just described, with the exception of the $\mathbb{L}_p$ estimates, which are delayed till Subsection 3.3.

[4] Actually, Stefan Müller showed us a simple proof of the control of the $\mathbb{L}_2$-norm of the gradient at boundary using the equation (3.34). His technique works for very general domains, cf. [38]. Here we rely on our $\mathbb{L}_p$ estimates, since they play a key role in our compactness argument.

3.2. From the entropy estimate to $\mathbb{L}_p$ estimates

We recall that the notation for the lemma we are going to state has been introduced right before formula (2.127). In particular recall that $D_N^0 = \{x \in D_N : \text{dist}(x, D_N^{\complement}) \geq 2/N\}$.

Lemma 3.2. *There exists $C > 0$ such that for every N*

$$\mathbf{H}\left(\tilde{\mathbb{P}}_0 \middle| \tilde{\mathbb{Q}}_0\right) + \mathbf{H}\left(\tilde{\mathbb{Q}}_0 \middle| \tilde{\mathbb{P}}_0\right) \leq C \left[N^{d-1} \|\nabla f\|_2 \|f\|_2 + |\partial^- D_N^0| + \sum_{x \in B_N} \left| \left\langle \eta(x) \right\rangle_{N,f} \right|^2 \right] , \tag{3.3}$$

where $B_N = \{x \in D_N : \text{dist}(x, D_N^{\complement}) \leq 3/N\}$.

We postpone the proof of Lemma 3.2.

Lemma 3.3. *Assume that ∂D is Lipschitz, then there exists $C < \infty$ such that for every $p \geq 1$*

$$\sum_{x \in B_N} \left| \nabla^N u_{N,f}(x) \right|^2 \leq C N^{d-1+1/p} \left\| \nabla^N u_{N,f} \right\|_{2p}^2 . \tag{3.4}$$

Proof. It follows directly from Hölder inequality

$$\begin{aligned} \sum_{x \in B_N} \left| \nabla^N u_{N,f}(x) \right|^2 &= \sum_x{}^N \left| \nabla^N u_{N,f}(x) \right|^2 \mathbf{1}_{B_N}(x) \\ &\leq \left(\sum_x{}^N \left| \nabla^N u_{N,f}(x) \right|^{2p} \right)^{1/p} |B_N|^{1-1/p} . \end{aligned} \tag{3.5}$$

Since ∂D is Lipschitz, we have $|B_N| = O(N^{d-1})$, which implies the result. □

Proof of Lemma 3.1. By using (2.127), Lemma 3.2 and Lemma 3.3 we, obtain that there exists a constant c_1 such that

$$\begin{aligned} &\sum{}^N \sum_{i=1}^d \left[\nabla_i^N u_{N,f} \left(x + \frac{e_i}{N} \right) - \nabla_i^N u_{N,f}(x) \right]^2 \\ &\leq c_1 \left[N^{d-1} + N^{(d-1+1/p)} \left\| \nabla^N u_{N,f} \right\|_{2p}^2 \right] . \end{aligned} \tag{3.6}$$

By Lemma 3.4 below, we can choose $p > 1$ such that $\|\nabla^N u_{N,f}\|_{2p}$ is bounded uniformly in N. Therefore

$$\sum{}^N \sum_{i=1}^d \left[\nabla_i^N u_{N,f} \left(x + \frac{e_i}{N} \right) - \nabla_i^N u_{N,f}(x) \right]^2 \leq c_2 N^{d-1+1/p} = o(N^d) , \tag{3.7}$$

and we are done. □

Proof of Lemma 3.2. Set $D_N^1 = \theta_e^N D_N^0$ and denote by $\tilde{\mathbb{P}}_1$ (respectively $\tilde{\mathbb{Q}}_1$) the restriction of $\mathbb{P}_{N,f}$ (respectively $\mathbb{P} \circ (\theta_e^N)^{-1}$) to D_N^1. Let $p_i(\varphi)$, $i = 0, 1$, be the density of $\tilde{\mathbb{P}}_i$ with respect to the Lebesgue measure:

$$p_i(\varphi) = \frac{\tilde{\mathbb{P}}_i(\mathrm{d}\varphi)}{\mathrm{m}_{D_N^i}(\mathrm{d}\varphi)} \quad , \tag{3.8}$$

where $\varphi \in \mathbb{R}^{D_N^i}$. Observe that, since

$$\frac{\tilde{\mathbb{Q}}_0(\mathrm{d}\varphi)}{\tilde{\mathbb{P}}_0(\mathrm{d}\varphi)} = \frac{p_1\left(\theta_{-e}^N\varphi\right)}{p_0\left(\varphi\right)} \quad , \tag{3.9}$$

we have the following expressions for the relative entropies:

$$\mathbf{H}\left(\tilde{\mathbb{P}}_0\middle|\tilde{\mathbb{Q}}_0\right) = \left\langle \log p_0(\varphi)\right\rangle_{N,f} - \left\langle \log p_1(\theta_{-e}^N\varphi)\right\rangle_{N,f} \quad , \tag{3.10}$$

$$\mathbf{H}\left(\tilde{\mathbb{Q}}_0\middle|\tilde{\mathbb{P}}_0\right) = \left\langle \log p_1(\varphi)\right\rangle_{N,f} - \left\langle \log p_0(\theta_{e}^N\varphi)\right\rangle_{N,f} \quad . \tag{3.11}$$

We now look for a convenient expression for the densities. Let $\overline{D_N^i} = D_N \backslash D_N^i$ and decompose the Hamiltonian (with $f = 0$) in the following way

$$\mathscr{H}_N(\varphi) = \mathscr{H}_i^\circ(\varphi) + \overline{\mathscr{H}}_i(\varphi) \quad , \tag{3.12}$$

where

$$\mathscr{H}_i^\circ(\varphi) = \sum_{x\sim y: x,y\in D_N^i} V(\varphi(x) - \varphi(y)) \quad , \tag{3.13}$$

is the Hamiltonian in D_N^i, with free boundary conditions. In (3.12) φ is an element of $\mathbb{R}^{D_N}$ (or $\mathbb{R}^{\mathbb{Z}_N^d}$, with zero boundary conditions outside D_N), but it is clear that $\mathscr{H}_i^\circ$ and $\overline{\mathscr{H}}_i(\varphi)$ do not depend on all the coordinates of φ. A simple computation leads to

$$p_i(\varphi) = \frac{\overline{Z_i}(\varphi)}{\mathscr{Z}_{N,f}} \exp\left\{-\mathscr{H}_i^\circ(\varphi) + \frac{1}{N}\sum_x^N \varphi(x) f(x)\right\} \quad , \tag{3.14}$$

where $\mathscr{Z}_{N,f}$ is the normalization in (1.27) and

$$\overline{Z_i}(\varphi) = \int \exp\left\{-\overline{\mathscr{H}}_i(\psi_{\tilde{\varphi},\varphi})\right\} \mathrm{m}_{\overline{D_N^i}}(\mathrm{d}\tilde{\varphi}) \quad , \tag{3.15}$$

with

$$\psi_{\tilde{\varphi},\varphi}(x) = \begin{cases} \tilde{\varphi}(x) & \text{if } x \in \overline{D_N^i} \\ \varphi(x) & \text{otherwise} \end{cases} . \tag{3.16}$$

Note that $\overline{Z_i}(\varphi)$ depends on φ only through $\varphi(x)$ with $x \in \partial^- D_N^i$. If we take into account that

$$\mathscr{H}_0^\circ(\varphi) = \mathscr{H}_1^\circ\left(\theta_e^N\varphi\right) \quad , \tag{3.17}$$

from (3.10), (3.11) and (3.14) we obtain that

$$
\begin{aligned}
\mathbf{H}\left(\tilde{\mathbb{P}}_0\middle|\tilde{\mathbb{Q}}_0\right) + \mathbf{H}\left(\tilde{\mathbb{Q}}_0\middle|\tilde{\mathbb{P}}_0\right) &= \left\langle \frac{1}{N}\sum_x^N \varphi(x) f(x) + \log \overline{Z_1}(\theta_{-e}^N\varphi) \right\rangle_{N,f} \\
&\quad - \left\langle \frac{1}{N}\sum_x^N \theta_{-e}^N\varphi(x) f(x) + \log \overline{Z_0}(\varphi) \right\rangle_{N,f} \\
&\quad + \left\langle \frac{1}{N}\sum_x^N \varphi(x) f(x) + \log \overline{Z_1}(\varphi) \right\rangle_{N,f} \\
&\quad - \left\langle \frac{1}{N}\sum_x^N \theta_{e}^N\varphi(x) f(x) + \log \overline{Z_0}(\theta_e^N\varphi) \right\rangle_{N,f} \qquad (3.18) \\
&= \left\langle \frac{1}{N}\sum_x^N \left[2\varphi(x) - \theta_e^N\varphi(x) - \theta_{-e}^N\varphi(x)\right] f(x) \right\rangle_{N,f} \\
&\quad + \left\langle \log\left(\frac{\overline{Z_0}(\varphi)}{\overline{Z_0}(\theta_e^N\varphi)}\right)\right\rangle_{N,f} + \left\langle \log\left(\frac{\overline{Z_1}(\varphi)}{\overline{Z_1}(\theta_{-e}^N\varphi)}\right)\right\rangle_{N,f} \qquad (3.19) \\
&\equiv J_1 + J_2 + J_3.
\end{aligned}
$$

By summation by parts and Cauchy–Schwarz we obtain

$$
\begin{aligned}
J_1 &\le \left| \frac{1}{N}\sum_x^N (\nabla_i^N f)(x) \left\langle \eta_i(x) \right\rangle_{N,f} \right| \\
&\le \frac{1}{N}\left(\sum_x^N \left[\nabla_i^N f(x)\right]^2\right)^{1/2} \left(\sum_x^N \left\langle \eta_i(x)\right\rangle_{N,f}^2\right)^{1/2}, \qquad (3.20)
\end{aligned}
$$

where i is such that $e = e_i$. By using (2.126) and the regularity of f one easily see that J_1 is bounded by $const.N^{d-1}\|f\|_2\|\nabla f\|_2$, which is compatible with the bound we claimed (cf. (3.3)), and therefore we are left with estimating J_2 and J_3.

We use once again the differentiation trick, to interpolate between φ and $\theta_e^N\varphi$:

$$
\varphi_t = \varphi + t\left[\theta_e\varphi - \varphi\right], \quad t \in [0,1]. \qquad (3.21)
$$

We have:

$$
\frac{\mathrm{d}}{\mathrm{d}t}\log \overline{Z_0}(\varphi_t) = \overline{\mathbb{E}}_t\left[-\frac{\mathrm{d}}{\mathrm{d}t}\overline{\mathscr{H}}_0(\psi_{\overline{\varphi},\varphi_t})\right], \qquad (3.22)
$$

where this time $\psi_{\overline{\varphi},\varphi_t}(x) \in \mathbb{R}^{\overline{D^0{}_N}\cup\partial^- D_N^0}$

$$
\psi_{\overline{\varphi},\varphi_t}(x) = \begin{cases} \varphi_t(x) & \text{if } x \in \partial^- D_N^0, \\ \overline{\varphi}(x) & \text{otherwise} \end{cases}, \qquad (3.23)
$$

and $\overline{\mathbb{P}}_t$, $\overline{\mathbb{E}}_t$ for the expectation, is the measure on $\mathbb{R}^{\overline{D_N^0}}$, with zero boundary conditions outside D_N and φ_t boundary conditions on $\partial^- D_N^0$. Note that we are using $\overline{\varphi}$ for the field distributed according to $\overline{\mathbb{P}}_t$. By taking another derivative we obtain

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2} \log \overline{Z_0}(\varphi_t) = \overline{\mathbb{E}}_t \left[-\frac{\mathrm{d}^2}{\mathrm{d}t^2} \overline{\mathscr{H}}_0(\psi_{\overline{\varphi},\varphi_t}) \right] + \mathrm{var}_{\overline{\mathbb{P}}_t} \left(\frac{\mathrm{d}}{\mathrm{d}t} \overline{\mathscr{H}}_0(\psi_{\overline{\varphi},\varphi_t}) \right) \quad , \tag{3.24}$$

and by integrating twice, observing that $\overline{\mathbb{P}}_0(\cdot) = \mathbb{P}_{N,f}(\cdot|\mathscr{F}_{\partial^- D_N^0})$, we obtain

$$\begin{aligned}
\log\left(\frac{\overline{Z_0}(\varphi)}{\overline{Z_0}(\theta_e^N \varphi)} \right) \le\ & \overline{\mathbb{E}}_0 \left[\frac{\mathrm{d}}{\mathrm{d}t} \overline{\mathscr{H}}_0(\psi_{\overline{\varphi},\varphi_t}) \big|_{t=0} \right] \\
& + \int_0^1 \int_0^s \left\{ \overline{\mathbb{E}}_t \left[\frac{\mathrm{d}^2}{\mathrm{d}t^2} \overline{\mathscr{H}}_0(\psi_{\overline{\varphi},\varphi_t}) \right] - \mathrm{var}_{\overline{\mathbb{P}}_t} \left(\frac{\mathrm{d}}{\mathrm{d}t} \overline{\mathscr{H}}_0(\psi_{\overline{\varphi},\varphi_t}) \right) \right\} \mathrm{d}t\,\mathrm{d}s \\
\le\ & \mathbb{E}_{N,f} \left[\frac{\mathrm{d}}{\mathrm{d}t} \overline{\mathscr{H}}_0(\psi_{\overline{\varphi},\varphi_t}) \big|_{t=0} \Big| \mathscr{F}_{\partial^- D_N^0} \right] (\varphi) \\
& + \int_0^1 \int_0^s \overline{\mathbb{E}}_t \left[\frac{\mathrm{d}^2}{\mathrm{d}t^2} \overline{\mathscr{H}}_0(\psi_{\overline{\varphi},\varphi_t}) \right] \mathrm{d}t\,\mathrm{d}s \quad .
\end{aligned} \tag{3.25}$$

Now, by using $V'' \le c_V$, we obtain that

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t} \overline{\mathscr{H}}_0(\psi_{\overline{\varphi},\varphi_t}) \big|_{t=0} &= \sum_{x \in \partial^- D_N^0} \ \sum_{y \in \overline{D_N^0} : |y-x|=1} V'(\varphi(x) - \overline{\varphi}(y))(\varphi(x+e) - \varphi(x)) \\
&\le c_V \sum_{x \sim y : x \in \partial^- D_N^0} \left[(\varphi(x) - \overline{\varphi}(y))^2 + (\varphi(x) - \varphi(y))^2 \right] \quad ,
\end{aligned} \tag{3.26}$$

and

$$\begin{aligned}
\frac{\mathrm{d}^2}{\mathrm{d}t^2} \overline{\mathscr{H}^*}_0(\psi_{\overline{\varphi},\varphi_t}) &= \sum_{x \sim y : x \in \partial^- D_N^0} V'' \big(\varphi(x) - \overline{\varphi}_t(y) \big) \left(\varphi(x+e) - \varphi(x) \right)^2 \\
&\le 2dc_V \sum_{x \in \partial^- D_N^0} (\varphi(x+e) - \varphi(x))^2 \quad .
\end{aligned} \tag{3.27}$$

Using (3.26) and (3.27), from (3.25), recalling that $\varphi(x) - \varphi(y) = \eta\big((y,x)\big)$ and that $\eta_i^*(x) = \eta(x - e_i) - \eta(x)$, we obtain that there exists a constant c_1 (depending only on C_2 and d) such that for every N

$$J_2 \le c_1 \sum_{x \in \partial^- D_N^0} \left\langle |\eta(x)|^2 + |\eta^*(x)|^2 \right\rangle_{N,f} \quad . \tag{3.28}$$

Of course the same type of estimates apply to J_3:

$$J_3 \le c_1 \sum_{x \sim y : x \in \partial^- D_N^1} \left\langle |\eta(x)|^2 + |\eta^*(x)|^2 \right\rangle_{N,f} \quad . \tag{3.29}$$

To conclude the proof is sufficient to observe that by the Brascamp–Lieb inequality (2.123) there exists c_2 such that for every N

$$\sum_{x\in\partial^- D_N^i} \left\langle |\eta(x)|^2 \right\rangle_{N,f} = \sum_{x\in\partial^- D_N^i} \left[\left| \left\langle \eta(x) \right\rangle_{N,f} \right|^2 + \mathrm{var}_{N,f}\,(\eta(x)) \right]$$
$$\leq c_2 |\partial^- D_N^i| + \sum_{x\in\partial^- D_N^i} \left\langle \eta(x) \right\rangle_{N,f}^2 \ , \tag{3.30}$$

and the proof is complete. □

3.3. $\mathbb{L}_p$ estimates on $\nabla^N u_{N,f}$

Our $\mathbb{L}_p$ estimate on $\nabla^N u_{N,f}$ is based on the analysis of the main equation (1.55);

$$\mathrm{div}_N \left(\left\langle V'(\eta(x)) \right\rangle_{N,f} \right) = -f(x) \ . \tag{3.31}$$

Introducing the $d \times d$ diagonal matrix;

$$A_N(x) = \mathrm{diag}\left(\left\langle \int_0^1 V''\big(\eta_i(x) - (1-t)\langle \eta_i(x)\rangle_{N,f}\big)\mathrm{d}t \right\rangle_{N,f} \right) \ , \tag{3.32}$$

and the $\mathbb{Z}_N^d$-field

$$a_N(x) = \left\langle V'\big(\eta(x) - \langle\eta(x)\rangle_{N,f}\big)\right\rangle_{N,f} \ , \tag{3.33}$$

we rewrite (3.31) as

$$\mathrm{div}_N \left(A_N(x) \nabla^N u_{N,f}(x) \right) = -\mathrm{div}_N\big(a_N(x)\big) - f(x) \tag{3.34}$$

Notice that by the choice of $f \in \mathbb{C}_0^\infty(D)$ and due to our assumptions on the potential V,

$$\frac{1}{c_V}\mathbb{I} \ \leq \ A_N \ \leq \ c_V \mathbb{I} \ . \tag{3.35}$$

Furthermore, by the Brascamp–Lieb inequality (2.123), $a_N \in \mathbb{L}_\infty$.

Lemma 3.4. *There exists $p = p(D, c_V) > 2$ and a constant $c = c(D) < \infty$, such that uniformly in N,*

$$\|u_{N,f}\|_{1,p} \ \leq \ c\big(\|a_N\|_p + \|f\|_p\big) \ . \tag{3.36}$$

The claim of the lemma follows by an adaptation to the current discrete setting of various facts from the $\mathbb{L}_p$ theory of elliptic PDE-s. Below we sketch the main steps and provide the corresponding references.

Remark 3.5. *If no further information on the regularity of A_N is available, then* (3.36) *might, in general, fail for large enough values of p (see* [36]).

Step 1 (Reduction to the Laplacian). The arguments of [36] imply that under the uniform ellipticity condition (3.35), the bound (3.36) follows once there exists $q > 2$, such that

$$\inf_N \lambda_N(D,q) \stackrel{\Delta}{=} \inf_N \inf_{\|u\|_{1,q}\geq 1} \sup_{\|v\|_{1,q'}\leq 1} \frac{1}{N^d}\sum_x^N \left(\nabla^N u(x), \nabla^N v(x)\right) > 0 \ , \quad (3.37)$$

where q' is the Hölder conjugate of q, and $\|\cdot\|_{1,q}$ and $\|\cdot\|_{1,q'}$ are the norms of $\mathbb{H}_0^{1,q}(D_N)$ and $\mathbb{H}_0^{1,q'}(D_N)$ respectively. Since the proof of the latter claim is greatly simplified in the discrete setting, we sketch it here for the sake of completeness:

Let us define

$$\mathscr{D}_N\left(u,v\right) = \frac{1}{N^d}\sum_x^N (A_N(x)\nabla^N u(x), \nabla^N v(x)) \ . \quad (3.38)$$

Suppose that for some $p > 2$,

$$\inf_{\|u\|_{1,p}\geq 1} \sup_{\|v\|_{1,p'}\leq 1} \mathscr{D}_N\left(u,v\right) > 0 \ . \quad (3.39)$$

Then, using the integral form of (3.34),

$$\mathscr{D}_N\left(u_N,v\right) = \frac{1}{N^d}\sum_x^N (a_N(x), \nabla^N v(x)) + \frac{1}{N^d}\sum_x^N v(x)f(x) \quad (3.40)$$

and taking supremum over v; $\|v\|_{1,p'} \leq 1$, in both hand sides above, one immediately recovers the $\mathbb{L}_p$ estimate (3.36) we are after.

On the other hand, writing,

$$A_N = c_V\mathbb{I} - \left(c_V\mathbb{I} - A_N\right) \quad (3.41)$$

we obtain that

$$\begin{aligned}
\sup_{\|v\|_{1,p'}\leq 1} \mathscr{D}_N\left(u,v\right) &\geq c_V \sup_{\|v\|_{1,p'}\leq 1} \frac{1}{N^d}\sum_x^N (\nabla^N u, \nabla^N v) \\
&\quad -\left(c_V - \frac{1}{c_V}\right) \sup_{\|v\|_{1,p'}\leq 1} \frac{1}{N^d}\sum_x^N (\nabla^N u, \nabla^N v) \\
&\geq c_V \left(\inf_{\|u\|_{1,p}\leq 1} \sup_{\|v\|_{1,p'}\leq 1} \frac{1}{N^d}\sum_x^N (\nabla^N u, \nabla^N v) - \left(1 - \frac{1}{c_V^2}\right)\right) \\
&\geq c_V \left(\inf_M \lambda_M(D,p) - \left(1 - \frac{1}{c_V^2}\right)\right) \ . \quad (3.42)
\end{aligned}$$

As in [36] it follows from the Riesz-Thorin interpolation theorem that for each M the function $\lambda_M(D,p)$ is log-concave in p with the maximum $\lambda_M(D,2) \equiv 1$. Consequently (3.37) implies that

$$\lim_{p\to 2} \lambda_N(D,p) = 1 \quad (3.43)$$

uniformly in N. Since $c_V < \infty$, it is then possible to choose some $p > 2$, such that

$$\inf_M \lambda_M(D, p) - \left(1 - \frac{1}{c_V^2}\right) > 0 \ , \tag{3.44}$$

which, by the last line in (3.42), implies the p-coercivity bound (3.39) and hence the conclusion of the lemma.

It remains, therefore, to study the properties of discrete Laplacians on D_N. An equivalent reformulation of (3.37), asserts that uniformly in N,

$$\|v_N\|_{1,q} \le c_1 \|b_N\|_{1,q} \ , \tag{3.45}$$

whenever v_N is the solution to the discrete Poisson equation

$$\begin{aligned} &\mathrm{div}_N\left(\nabla^N v_N\right) = -\mathrm{div}_N\left(b_N\right) \qquad \text{on } D_N \\ &v_N\Big|_{\mathbb{Z}_N^d \setminus D_N} = 0 \ . \end{aligned} \tag{3.46}$$

We conjecture that (3.45) is indeed true as soon as similar bounds hold for the solutions of the continuous Poisson equation on D, which, in view of the Calderon–Zygmund inequality and the results of [2], would provide natural requirements on the regularity of ∂D.

Since, however, we were not able to verify this seemingly obvious conjecture, our proof of (3.45) rely on a completely different approach to the $\mathbb{L}_q$-regularity: by (1.20) it would be enough to prove (3.45) for the continuous interpolations $\tilde{v}_N$ and $\tilde{b}_N$;

$$\|\tilde{v}_N\|_{1,q} \le c_1 \|\tilde{b}_N\|_{1,q} \ . \tag{3.47}$$

In order to prove (3.47) we shall refer to the inverse Hölder inequality techniques of [29], while the input for these techniques is to be provided by the structure of solutions to (3.46) via a discretized version of the Caccioppoli inequality.

Step 2 (Caccioppoli inequality). It happens to be convenient to split (3.46) into two problems in the following way: Let C be a large enough cube in $\mathbb{R}^d$ such that $0 \in D \subset C$. Set $C_N = C \cap \mathbb{Z}_N^d$, and let w_N be the solution to

$$\begin{aligned} &\mathrm{div}_N\left(\nabla^N w_N\right) = -\mathrm{div}_N\left(b_N\right) \qquad \text{on } 4C_N \\ &w_N\Big|_{\mathbb{Z}_N^d \setminus 4C_N} = 0 \ , \end{aligned} \tag{3.48}$$

where we extend $b_N \equiv 0$ outside D_N. Also let ρ_N to denote the solution to

$$\begin{aligned} &\mathrm{div}_N\left(\nabla^N \rho_N\right) = 0 \qquad \text{on } D_N \\ &\rho_N\Big|_{2C_N \setminus D_N} = -w_N \ . \end{aligned} \tag{3.49}$$

Then, of course, $v_N = w_N + \rho_N$ is the solution to (3.46).

Given $x \in \mathbb{R}^d$, we use $B_R(x) \subset \mathbb{R}^d$ to denote the cube of the side-length R centered at x, and we use $B_{R,N}(x)$ to denote its discretization $B_{R,N} = B_R \cap \mathbb{Z}_N^d$.

For each R and x there exists a smooth function $\eta_R \in \mathbb{C}_0^\infty(B_R)$, such that $\eta_R \equiv 1$ on $B_{R/2}$, $0 \le \eta_R \le 1$ and

$$\|\nabla \eta_R\|_\infty \le c_2(d)/R \ . \tag{3.50}$$

Let now x and R be such that the cube $B_R(x) \subset 2C$. Then for any choice of $\lambda \in \mathbb{R}$ and $\theta > 0$,

$$\begin{aligned}
&- \sum_{y \in B_{R,N}(x)} \Big(b_N, \nabla^N\big[\eta_R^2(w_N - \lambda)\big](y)\Big) \\
&\quad = - \sum_{y \in B_{R,N}} \sum_{i=1}^d b_{N,i} \Big(\eta_R(y + \frac{e_i}{N})\nabla_i^N\big[\eta_R(w_N - \lambda)\big](y) \\
&\qquad\qquad + \eta_R(w_N - \lambda)(y)\nabla_i^N \eta_R(y)\Big) \\
&\quad \le (1 + \frac{1}{\theta}) \sum_{y \in B_{R,N}(x)} |b_N|^2 + \theta \sum_{y \in B_{R,N}(x)} \Big(\nabla^N\big[\eta_R(w_N - \lambda)\big](y)\Big)^2 \\
&\qquad + \frac{c_2(d)}{R^2} \sum_{y \in B_{R,N}(x)} \big(w_N(y) - \lambda\big)^2 \ .
\end{aligned} \tag{3.51}$$

On the other hand, by (3.48),

$$\begin{aligned}
&- \sum_{y \in B_{R,N}(x)} \Big(b_N, \nabla^N\big[\eta_R^2(w_N - \lambda)\big](y)\Big) \\
&\quad = \sum_{y \in B_{R,N}(x)} \Big(\nabla^N w_N(y), \nabla^N\big[\eta_R^2(w_N - \lambda)\big](y)\Big) \ .
\end{aligned} \tag{3.52}$$

However,

$$\begin{aligned}
&\nabla_i^N w_N(y) \nabla_i^N \big[\eta_R^2(w_N - \lambda)\big](y) \\
&\quad = \big(\nabla_i^N\big[\eta_R(w_N - \lambda)\big](y)\big)^2 - \big(w_N(y) - \lambda\big)\big(w_N(y + \tfrac{e_i}{N}) - \lambda\big)\big(\nabla_i^N \eta_R(y)\big)^2 \ .
\end{aligned} \tag{3.53}$$

Choosing $\theta < 1$ in 3.51 and successively substituting the left hand side there by 3.52 and 3.53, we, using the bound (3.50) on $\nabla\eta_R$, infer that there exist constants $c_3 = c_3(d)$ and $c_4 = c_4(d)$ such that, for all R and N,

$$\sum_{y \in B_{R,N}(x)} \Big(\nabla^N\big[\eta_R(w_N - \lambda)\big](y)\Big)^2 \le c_3 \sum_{y \in B_{R,N}(x)} |b_N(y)|^2 + \frac{c_4}{R^2} \sum_{y \in B_{R,N}(x)} (w_N(y) - \lambda)^2 \ . \tag{3.54}$$

Since $\eta_R \equiv 1$ on $B_{R/2}(x)$ we finally obtain from (3.54):

$$\sum_{y \in B_{R/2,N}(x)} \Big(\nabla^N w_N(y)\Big)^2 \le c_3 \sum_{y \in B_{R,N}(x)} |b_N(y)|^2 + \frac{c_4}{R^2} \sum_{y \in B_{R,N}(x)} (w_N - \lambda)^2. \tag{3.55}$$

Step 3 (Reverse Hölder inequality). By (1.20) a similar inequality is valid for the continuous interpolations $\tilde{w}_N$ and $\tilde{b}_N$. Thus, (3.55) implies,

$$\int_{B_{R/2}(x)} \left|\nabla\tilde{w}_N(y)\right|^2 \mathrm{d}y \;\leq\; c_5 \int_{B_R(x)} \left|\tilde{b}_N(y)\right|^2 \mathrm{d}y + \frac{c_6}{R^2} \int_{B_R(x)} \left(\nabla\tilde{w}_N(y) - \lambda\right)^2 \mathrm{d}y \;\; . \tag{3.56}$$

Let us choose

$$\lambda = \lambda_R = \fint_{B_R} \tilde{w}_N(y)\mathrm{d}y \stackrel{\Delta}{=} \frac{1}{|B_R|} \int_{B_R(x)} \tilde{w}_N(y)\mathrm{d}y \;\; . \tag{3.57}$$

By the Poincaré-Sobolev embedding theorem,

$$\int_{B_R(x)} (\tilde{w}_N(y) - \lambda_R)^2 \mathrm{d}y \;\leq\; c_7 R^{d+\frac{2(q-d)}{q}} \left(\int_{B_R(x)} \left(\nabla\tilde{w}_N(y)\right)^q \mathrm{d}y\right)^{\frac{2}{q}} \;\; , \tag{3.58}$$

where $q = q(d) = 2d/(d+2)$ is the Sobolev exponent of 2. Consequently,

$$\frac{1}{R^2} \fint_{B_R(x)} \left(\tilde{w}_N(y) - \lambda_R\right)^2 \mathrm{d}y \;\leq\; c_7 \left(\fint_{B_R(x)} \left(\nabla\tilde{w}_N(y)\right)^q \mathrm{d}y\right)^{\frac{2}{q}} \;\; , \tag{3.59}$$

and we finally obtain:

$$\fint_{B_{R/2}(x)} \left(\nabla\tilde{w}_N(y)\right)^2 \mathrm{d}y \;\leq\; c_8 \fint_{B_R(x)} \left(\tilde{b}_N(y)\right)^2 \mathrm{d}y + c_9 \left(\fint_{B_R(x)} \left(\nabla\tilde{w}_N(y)\right)^q \mathrm{d}y\right)^{\frac{2}{q}} \;\; . \tag{3.60}$$

Therefore, $\tilde{w}_N$ satisfies the assumptions of [29, Prop. 1.1, page 122] and, since the constants c_8 and c_9 in (3.60) do not depend on N, we infer that there exists an exponent $\bar{q} = \bar{q}(d) > 2$ and a constant c_{10}, such that the $\mathbb{L}_{\bar{q}}(2C)$ norms of $\nabla\tilde{w}_N$ are bounded above by

$$\|\nabla\tilde{w}_N\|_{\bar{q}} \;\leq\; c_{10}\|\tilde{b}_N\|_{\bar{q}} \;\; . \tag{3.61}$$

The treatment of (3.49) follows a similar pattern (c.f. p.152 in [29]), with the only exception that this time one should employ a different form of the Poincaré-Sobolev inequality, as provided, for example by the Corollary 4.5.3 in [50], and which is secured by our assumption on the Lipschitz character of ∂D. One then eventually obtains that the $\mathbb{L}_{\bar{q}}(C)$ norm of $\nabla\tilde{\rho}_N$ is bounded above as

$$\|\nabla\tilde{\rho}_N\|_{\bar{q}} \;\leq\; c_{11}\|\nabla\tilde{w}_N\|_{\bar{q}} \;\; , \tag{3.62}$$

and (3.47) follows as well as the claim (3.36) of Lemma 3.1.

3.4. Strict convexity of σ

Lemma 3.6. *For any Hamiltonian $\mathscr{H}$ with the $\mathbb{C}^2$, even interaction potential V satisfying (1.13), the surface tension σ is strictly convex. Moreover, for any $u, v \in \mathbb{R}^d$,*

$$\frac{c_V|u-v|^2}{2} \geq \sigma(v) - \sigma(u) - \big(\nabla\sigma(u), v-u\big) \geq \frac{|u-v|^2}{2c_V} \ . \tag{3.63}$$

Proof. In [25] the surface tension $\sigma(u)$ in the direction $(u,1) \in \mathbb{R}^{d+1}$ was identified as the limit

$$\sigma(u) = \lim_{M\to\infty} \sigma_M(u) \ , \tag{3.64}$$

where the *finite volume surface tension* σ_M is defined via the Gibbs state $\mathbb{P}_{M,u}$ on the finite graph $(\mathbb{T}^d_M, \mathscr{E}^d_M)$, where $\mathbb{T}^d_M$ is just the d-dimensional lattice torus $\mathbb{T}^d_M \stackrel{\Delta}{=} \mathbb{Z}^d \mathrm{mod}(M)$, and $\mathscr{E}^d_M$ is the corresponding set of all nearest neighbour oriented bonds. To fit the framework of Section 2 we use the decomposition $\mathbb{T}^d_M = I_M \cup B$, where the boundary B contains only one point, the origin ($B = \{0\}$), and the interiour I_M is, accordingly, given by $I_M = \mathbb{T}^d_M \setminus \{0\}$. Thus the reference measure $\mathrm{m}_{I_M} \stackrel{\Delta}{=} \mathrm{m}_M$ on $\mathbb{R}^{I_M}$ is given by

$$\mathrm{m}_M(\mathrm{d}\varphi) = \prod_{x\in\mathbb{T}^d_M\setminus\{0\}} \mathrm{d}\varphi(x) \ . \tag{3.65}$$

We impose the zero boundary condition on B; $\varphi(0) = 0$. Thus the field of bond differences $\eta_i(x) = \varphi(x+e_i) - \varphi(x)$ is defined on the whole of $\mathbb{T}^d_M$, where $x + e_i$ is understood this time as the appropriate shift on $\mathbb{T}^d_M$.

With this notations σ_M is defined as

$$\sigma_M(u) \stackrel{\Delta}{=} -\frac{1}{M^d} \log\Big\langle \exp\Big\{ -\sum_{x\in\mathbb{T}^d_M}\sum_{i=1}^d V(\eta_i(x)+u_i)\Big\}\Big\rangle_{\mathrm{m}_M} \ , \tag{3.66}$$

Furthermore, it has been proven in [25], that σ_M actually converges to σ in $\mathbb{C}^1_{\mathrm{loc}}\big(\mathbb{R}^d\big)$. Consequently, the strict convexity assertion (3.63) of the Lemma follows as soon as the similar statement is verified for σ_M for each M large enough. To this end let us investigate the Hessian of σ_M at u:

For any $\lambda \in \mathbb{R}^d$,

$$\begin{aligned}\Big(\mathrm{D}^2\sigma_M(u)\lambda, \lambda\Big) &= \sum_{i=1}^d \lambda_i^2 \langle V''(\eta_i(0)+u_i)\rangle_{M,u} \\ &\quad - \frac{1}{M^d}\mathrm{var}_{M,u}\left(\sum_{i=1}^d \sum_{x\in\mathbb{T}^d_M} \lambda_i V'(\eta_i(x)+u_i)\right) \ , \end{aligned} \tag{3.67}$$

where the subscript (M, u) corresponds to the expectation with respect to the Gibbs measure $\mathbb{P}_{M,u}$;

$$\begin{aligned}\mathbb{P}_{M,u}\big(\mathrm{d}\varphi\big) \;&=\; \frac{\mathrm{e}^{-\mathscr{H}_{M,u}(\varphi)}}{\mathscr{Z}_{M,u}}\mathrm{m}_M(\mathrm{d}\varphi)\\ &\stackrel{\Delta}{=}\; \frac{1}{\mathscr{Z}_{M,u}}\exp\Big\{-\sum_{x\in\mathbb{T}_M^d}\sum_{i=1}^d V(\eta_i(x)+u_i)\Big\}\mathrm{m}_M(\mathrm{d}\varphi)\ .\end{aligned} \tag{3.68}$$

Notice that we have used above the obvious fact that the distribution of the height differences $\{\eta(\cdot)\}$ under $\mathbb{P}_{M,u}$ is invariant with respect to the shifts on $\mathbb{T}_M^d$.

Our next step is to take advantage of the Helffer-Sjöstrand representation in order to develop the variance term on the right hand side of (3.67): For each $x \in I_M = \mathbb{T}_M^d \setminus \{0\}$ define

$$\xi_\lambda(x,\varphi) = \frac{\partial}{\partial\varphi(x)}\Big(\sum_{y\in\mathbb{T}_M^d}\sum_{i=1}^d \lambda_i V'(\eta_i(y)+u_i)\Big) = \sum_{i=1}^d \lambda_i \nabla_i^* V''(\eta_i(x)+u_i)\ . \tag{3.69}$$

As in Section 2 we use the notation

$$\langle\langle f, g\rangle\rangle(\varphi) \stackrel{\Delta}{=} \sum_{x\in I_M} f(x,\varphi)g(x,\varphi)\ . \tag{3.70}$$

Then,

$$\mathrm{var}_{M,u}\left(\sum_{i=1}^d\sum_{x\in\mathbb{T}_M^d}\lambda_i V'(\eta_i(x)+u_i)\right) = \Big\langle\langle\langle[-L_{M,u}-Q_{M,u}]^{-1}\xi_\lambda,\xi_\lambda\rangle\rangle(\varphi)\Big\rangle_{M,u}\ , \tag{3.71}$$

where the diffusion part of the operator $L_{M,u} + Q_{M,u}$ is given by

$$L_{M,u} = \sum_{x\in I_M}\mathrm{e}^{\mathscr{H}_{M,u}(\varphi)}\frac{\partial}{\partial\varphi(x)}\left(e^{-\mathscr{H}_{M,u}(\varphi)}\frac{\partial}{\partial\varphi(x)}\right)\ , \tag{3.72}$$

and $Q_{M,u}$ is the generator of a transient random walk on $\mathbb{T}_M^d \setminus \{0\}$ killed upon reaching the origin, that is

$$Q_{M,u}f(x,\varphi) = \sum_{i=1}^d \nabla_i^*\big(V''(\eta_i(x)+u_i)\nabla_i f(x,\varphi)\big)\ , \tag{3.73}$$

where, of course, we have used the convention

$$f(0,\cdot)\;\equiv\;0\ , \tag{3.74}$$

and the meaning of $[-L_{M,u} - Q_{M,u}]^{-1}\xi_\lambda$ is the same as in (2.120). On the other hand (again with the above convention (3.74) in mind),

$$\begin{aligned}&\Big\langle\langle\langle[-L_{M,u}-Q_{M,u}]^{-1}\xi_\lambda,\xi_\lambda\rangle\rangle(\varphi)\Big\rangle_{M,u}\\ &\quad=\;\sup_f\left\{2\Big\langle\langle\langle f,\xi_\lambda\rangle\rangle(\varphi)\Big\rangle_{M,u} - \Big\langle\langle\langle[-L_{M,u}-Q_{M,u}]f,f\rangle\rangle(\varphi)\Big\rangle_{M,u}\right\}\ .\end{aligned} \tag{3.75}$$

However, in view of the formula (3.69), we, after the summing by parts, obtain:

$$\Big\langle \langle\langle f, \xi_\lambda \rangle\rangle(\varphi) \Big\rangle_{M,u} = \sum_{x\in\mathbb{T}_M^d} \sum_{i=1}^d \lambda_i \big\langle \nabla_i f(x,\varphi) V''(\eta_i(x)+u_i) \big\rangle_{M,u} \ . \tag{3.76}$$

In a similar fashion,

$$\begin{aligned}\Big\langle \langle\langle [-L_{M,u} - Q_{M,u}] f, f \rangle\rangle(\varphi) \Big\rangle_{M,u} &= \sum_{x\in\mathbb{T}_M^d} \sum_{i=1}^d \big\langle V''(\eta_i(x)+u_i)(\nabla_i f(x,\varphi))^2 \big\rangle_{M,u} \\ &\quad + \sum_{x\in\mathbb{T}_M^d\setminus\{0\}} \big\langle \big[\frac{\partial f}{\partial\varphi(x)}\big]^2 \big\rangle_{M,u} \ .\end{aligned} \tag{3.77}$$

As a result we derive from (3.67) and (3.75) the following variational formula:

$$\begin{aligned}&(\mathrm{D}^2\sigma_M(u)\lambda,\lambda) \\ &\quad = \frac{1}{M^d} \inf_{f\in\hat{C}^2_{\exp}} \left\{ \sum_{x\in\mathbb{T}_M^d} \sum_{i=1}^d \big\langle V''(\eta_i+u_i)(\lambda_i - \nabla_i f)^2 \big\rangle_{M,u} \right. \\ &\qquad\qquad \left. + \sum_{x\in\mathbb{T}_M^d\setminus\{0\}} \big\langle \big[\frac{\partial f}{\partial\varphi(x)}\big]^2 \big\rangle_{M,u} \right\} \ .\end{aligned} \tag{3.78}$$

But $\sum_{x\in\mathbb{T}_M^d} \nabla_i f \equiv 0$, and we immediately infer from (1.13), that for any $u\in\mathbb{R}^d$

$$c_V \sum_{i=1}^d \lambda_i^2 \ \geq \ (\mathrm{D}^2\sigma_M(u)\lambda,\lambda) \ \geq \ \frac{1}{c_V}\sum_{i=1}^d \lambda_i^2 \ . \tag{3.79}$$

Since for any two vectors $u, v\in\mathbb{R}^d$,

$$\begin{aligned}&\sigma_M(u) - \sigma_M(v) - \big(\nabla\sigma_M(v), u-v\big) \\ &\quad = \int_0^1\int_0^t \Big(\mathrm{D}^2\sigma_M\big(v+s(u-v)\big)(u-v), u-v\Big)\,\mathrm{d}s\,\mathrm{d}t \ ,\end{aligned} \tag{3.80}$$

the proof of Lemma 3.6 is concluded. □

3.5. The functional Σ

For each $f\in\mathbb{L}_2(D)$ let us define $\Sigma_f : \mathbb{H}_0^1 \mapsto \mathbb{R}$ via

$$\Sigma_f(u) = \int_D \sigma\,(\nabla u(x))\,\mathrm{d}x - \int_D u(x)f(x)\mathrm{d}x \ . \tag{3.81}$$

The functional Σ_f is then everywhere finite on $\mathbb{H}_0^1\big(D\big)$ and, moreover, by Lemma 3.6 it is strictly convex continuous and coercive on the later space. It has, thereby, a unique minimizer which is precisely the variational solution $u_{[f]}$ to the Euler equation (1.35). Using this and the right hand side inequality in (3.79) we obtain

$$\begin{aligned}
&\Sigma_f\big(u_{[f]}+g\big)-\Sigma_f\big(u_{[f]}\big)\\
&=\lim_{M\to\infty}\int_0^1\int_0^t\int_D\Big(\mathrm{D}^2\sigma_M\big(\nabla u_{[f]}(x)+s\nabla g(x)\big)\nabla g,\nabla g\Big)\,\mathrm{d}x\,\mathrm{d}s\,\mathrm{d}t\\
&\geq\frac{1}{2c_V}\|g\|_{1,2}^2\ .
\end{aligned}\tag{3.82}$$

There are two simple consequences of (3.82). First of all, it immediately follows the minimum is stable:

Lemma 3.7. *Assume that*

$$\lim_{n\to\infty}\Sigma_f\big(u_n\big)=\min_u\Sigma_f\big(u\big)=\Sigma_f\big(u_{[f]}\big)\ .\tag{3.83}$$

Then,

$$\lim_{n\to\infty}u_n=u_{[f]}\tag{3.84}$$

strongly in $\mathbb{H}_0^1\big(D\big)$.

Secondly, the map $\Psi:\mathbb{L}_2(D)\mapsto\mathbb{H}_0^1\big(D\big)$ given by

$$\Psi\big(f\big)\stackrel{\Delta}{=}u_{[f]}\tag{3.85}$$

is Lipschitz. Indeed, for every couple $h,g\in\mathbb{L}_2(D)$;

$$\begin{aligned}
0&\leq\Sigma_h(u_{[g]})-\Sigma_h(u_{[h]})\\
&=\Sigma_g(u_{[g]})-\Sigma_g(u_{[h]})+\int_D(g(x)-h(x))\big(u_{[g]}(x)-u_{[h]}(x)\big)\\
&\leq\int_D(g(x)-h(x))\big(u_{[g]}(x)-u_{[h]}(x)\big)\ .
\end{aligned}\tag{3.86}$$

Proceeding as in 3.82, we, therefore, conclude;

$$\frac{1}{2c_V}\|u_{[g]}(x)-u_{[h]}\|_{1,2}^2\ \leq\ \|g-h\|_2\,\|u_{[g]}(x)-u_{[h]}\|_2\ ,\tag{3.87}$$

which, in view of the Poincaré inequality, yields the Lipschitz property of Ψ.

We are now in a position to prove that, still pending the proof of Theorem 1.2, two different identifications of Λ_D in (1.34) and, respectively, in (1.36) are consistent, that is:

Proposition 3.8. *For any* $f\in\mathbb{L}_2(D)$,

$$\Sigma_f\big(u_{[f]}\big)=-\int_0^1\int_D f(x)u_{[tf]}(x)\mathrm{d}x\,\mathrm{d}t\ .\tag{3.88}$$

Proof. Let $\mathscr{D}_n = (t_0, \ldots, t_n)$ be an increasing sequence partitioning $[0, 1]$, with $t_0 = 0$ and $t_n = 1$. Then, by using the fact that Ψ (cf. (3.85)) is Lipschitz, we obtain

$$\begin{aligned}
&\int_D \sigma\big(\nabla u_{[f]}(x)\big)\mathrm{d}x \\
&\quad = \sum_{k=1}^{n} \int_D \big(\sigma(\nabla u_{[t_k f]}) - \sigma(\nabla u_{[t_{k-1} f]})\big)\, \mathrm{d}x \\
&\quad = -\sum_{k=1}^{n} \int_D \big(\nabla\sigma(\nabla u_{[t_k f]}), \nabla(u_{[t_k f]} - u_{[t_{k-1} f]})\big)\, \mathrm{d}x + \mathrm{O}\Big(\|f\|_2^2 \sum_{k=1}^{n}(t_k - t_{k+1})^2\Big) \\
&\quad = \sum_{k=1}^{n} t_k \int_D f(x)(u_{[t_k f]} - u_{[t_{k-1} f]})(x)\mathrm{d}x + \mathrm{O}\Big(\|f\|_2^2 \sum_{k=1}^{n}(t_k - t_{k-1})^2\Big) \ .
\end{aligned} \tag{3.89}$$

The second summand in the last line above tends to zero with the mesh of $\mathscr{D}_n$, and, summing by parts, we obtain:

$$\begin{aligned}
\int_D \sigma\big(\nabla u_{[f]}(x)\big)\mathrm{d}x = &\int_D f(x)u_{[f]}(x)\mathrm{d}x \\
&- \sum_{k=1}^{n}(t_k - t_{k-1}) \int_D f(x)u_{[t_k f]}(x)\mathrm{d}x + \mathrm{o}(1) \ ,
\end{aligned} \tag{3.90}$$

which, in view of the continuity of the map (3.85), implies (3.88). □

Recall now that we have defined Σ in the whole of $\mathbb{L}_2(D)$ by setting $\Sigma(u) = \infty$, whenever $u \notin \mathbb{H}_0^1\big(D\big)$. We claim that;

$$\Sigma\big(u\big) = \sup_{h\in\mathbb{C}_0^\infty(D)} \left\{ \int_D h(x)u(x) + \Sigma_h\big(u_{[h]}\big) \right\} \ . \tag{3.91}$$

Actually there is almost nothing to prove: By the very definition

$$\Sigma_h\big(u_{[h]}\big) = -\Sigma^*\big(h\big) \ , \tag{3.92}$$

where Σ^* is the Fenchel–Young transform of Σ on $\mathbb{L}_2(D)$. Since Σ itself is obviously convex and lower-semicontinuous, $\Sigma^{**} \equiv \Sigma$, which would yield (3.91) with the supremum taken over the whole of $\mathbb{L}_2(D)$. By the virtue of (3.82), however, Σ^* is locally Lipschitz, thus any dense subspace of $\mathbb{L}_2(D)$, in particular $\mathbb{C}_0^\infty(D)$, suffices.

4. Convergence of average profiles

4.1. *Proof of Theorem 1.2*

To recall the notation: Given $f \in \mathbb{C}_0^\infty(D)$, we define the average profile u_N under $\mathbb{P}_{N,f}$ as

$$u_N(x) = \langle\xi_N(x)\rangle_{N,f} = \left\langle \frac{\varphi(x)}{N} \right\rangle_{N,f} ; \qquad x \in \mathbb{Z}_N^d \ , \tag{4.1}$$

and we denote by $\tilde{u}_N$ its polilinear interpolation cf. (1.17).

Lemma 4.1. *For every $j \in \mathbb{C}_0^\infty(D)$,*

$$\int_D \big(\nabla j(x), \nabla\sigma(\nabla\tilde{u}_N(x))\big)\mathrm{d}x - \int_D j(x) f(x)\mathrm{d}x \longrightarrow 0 \ , \tag{4.2}$$

and

$$\int_D \big(\nabla\tilde{u}_N(x), \nabla\sigma(\nabla\tilde{u}_N(x))\big)\mathrm{d}x - \int_D \tilde{u}_N(x) f(x)\mathrm{d}x \longrightarrow 0 \ . \tag{4.3}$$

Theorem 1.2 is a straightforward consequence of the above lemma: As it immediately follows from (4.2), (4.3) and convexity of σ (Lemma 3.6)

$$\begin{aligned}\liminf_{N\to\infty}\big(\Sigma_f(j) - \Sigma_f(\tilde{u}_N)\big) \ge \liminf_{N\to\infty}\bigg\{\int_D (\nabla\sigma(\nabla\tilde{u}_N(x)), \nabla j(x) - \nabla\tilde{u}_N(x))\,\mathrm{d}x \\ - \int_D f(x)\big(j(x) - \tilde{u}_N(x)\big)\mathrm{d}x\bigg\} = 0 \ , \end{aligned} \tag{4.4}$$

for every $j \in \mathbb{C}_0^\infty(D)$. Therefore,

$$\limsup_{N\to\infty} \Sigma_f\big(\tilde{u}_N\big) \ \le \inf_{j\in\mathbb{C}_0^\infty(D)} \Sigma_f\big(j\big) = \Sigma_f\big(u_{[f]}\big) \ , \tag{4.5}$$

which by the variational stability result of Lemma 3.7 implies the assertion (1.38) of Theorem 1.2.

The rest of the section is devoted to the proof of the weak convergence in Lemma 4.1. The input data is provided by the basic equation (1.56). Observe that $\mathbb{P}_{N,f}$ induces a measure on the space on height differences $\mathscr{X} = \{\eta \in (\mathbb{R}^{\mathbb{Z}^d})^d : \exists\varphi \in \mathbb{R}^{\mathbb{Z}^d}$ such that $\eta = \nabla\varphi\}$ and, with some abuse of notation we will write $\mathbb{P}_{N,f}(\mathrm{d}\eta)$. The relation to the local relaxation properties under $\mathbb{P}_{N,f}$ of the field η of bond differences enters the picture in the following fashion: in order to localize $\mathbb{P}_{N,f}$ near a point $x \in \mathbb{Z}_N^d$ define the shifted measure

$$\mathbb{P}^x_{N,f}(\mathrm{d}\eta) = \mathbb{P}_{N,f}\circ\theta_x(\mathrm{d}\eta) \ . \tag{4.6}$$

Consider now the following regularization of the family $\{\mathbb{P}^x_{N,f}\}$:

$$\mathbb{Q}_N\big(\mathrm{d}y, \mathrm{d}\eta\big) = \frac{1}{N^d}\sum_{x\in D_N}\delta_x(\mathrm{d}y)\mathbb{P}^x_{N,f}(\mathrm{d}\eta) \ . \tag{4.7}$$

Notice that for each $j \in \mathbb{C}_0^\infty(D)$,

$$\int_{\mathbb{R}^d}\int_{\mathscr{X}} \big(\nabla j(y), V'(\eta(0))\big)\mathbb{Q}_N(\mathrm{d}y, \mathrm{d}\eta) = \frac{1}{N^d}\sum\nolimits^N\Big(\nabla j(y), \big\langle V'(\eta(y))\big\rangle_{N,f}\Big) \ , \tag{4.8}$$

which is up to a o(1)-term exactly the left hand side of the master equation (1.56).

In Subsection 4.2 we show that $\mathbb{Q}_N$ relaxes to a certain integral mixture $\mathbb{Q}$ of Funaki–Spohn states (Lemma 4.3 below). The proof is inspired by the corresponding arguments in [25]. This result, however, is not sufficient for (4.2), since, in this way, we do not keep track of the running average $\nabla^N u_N$. The crucial connection between the representation (4.26) of $\mathbb{Q}$ and the limit properties of the family $\{\nabla^N u_N\}$ is established in Lemma 4.4 of Subsection 4.3 on the level of Young measures.

Similarly, in order to prove (4.3), we define the following sequence of measures:

$$\mathbb{V}_N\big(\mathrm{d}v, \mathrm{d}\eta\big) = \frac{1}{N^d}\sum_x^N \delta_{\nabla^N u_N(x)}(\mathrm{d}v)\mathbb{P}^x_{N,f}\big(\mathrm{d}\eta\big) \ . \tag{4.9}$$

Notice, that this time,

$$\begin{aligned}\int \big(v, V'(\eta)\big)\mathbb{V}_N(\mathrm{d}v, \mathrm{d}\eta) &= \frac{1}{N^d}\sum^N \big(\nabla^N u_N(x), \langle V'(\eta(x))\rangle_{N,f}\big) \\ &= \frac{1}{N^d}\sum^N u_N(x) f(x) \ . \end{aligned}\tag{4.10}$$

The relaxation properties of $\{\mathbb{V}_N\}$ are stated in the second part of Lemma 4.3, and they are related to the limit properties of the running average $\{\nabla^N u_N\}$, again via the notion of Young measures, in Subsection 4.3. We would like to stress that the $\mathbb{L}_p$ estimate of Lemma 3.4 is crucial for the proof of (4.3), since the method simply does not go through without such a uniform integrability condition.

In what follows we shall frequently switch back and forth from discrete sums in terms of $\nabla^N u_N$ to continuous integrals in terms of the corresponding interpolations $\nabla\tilde{u}_N$. In all the cases the passage is secured by the following fact, which follows from our basic oscillation Lemma 3.1:

Assume that a function $\Phi : \mathbb{R}^d \mapsto \mathbb{R}$ satisfies

$$\text{for every } u, v \in \mathbb{R}^d \qquad |\Phi(u+v) - \Phi(u)| \ \le\ c_1|v|^2 + c_2|v||u| \ , \tag{4.11}$$

and $g \in \mathbb{C}_0(D)$. Then,

$$\frac{1}{N^d}\sum^N g(x)\Phi\big(\nabla^N u_N(x)\big) - \int_D g(y)\Phi\big(\nabla\tilde{u}_N(y)\big)\mathrm{d}y \ \longrightarrow\ 0 \ , \tag{4.12}$$

as $N \to \infty$. Actually the rate of convergence depends only on the upper bounds on the norms $\{\|\nabla^N u_N\|_2\}$, the modulus of continuity of g, the function $\psi(N)$, which controls the oscillations in Lemma 3.1, and the constants c_1 and c_2, as the following simple proposition shows:

Proposition 4.2. *Let $h : \mathbb{Z}^d_N \mapsto \mathbb{R}^d$ be a square integrable vector lattice field. Define*

$$\frac{1}{N^d}\sum^N \sum_{i=1}^d \big[h(x + \frac{e_i}{N}) - h(x)\big]^2 \stackrel{\Delta}{=} a \ . \tag{4.13}$$

Then for any function Φ satisfying (4.11) above and for any $g \in \mathbb{C}_0(D)$,

$$\begin{aligned}&\frac{1}{2}\left|\frac{1}{N^d}\sum^N g(x)\Phi\big(h(x)\big) - \int_D g(y)\Phi\big(\tilde{h}(y)\big)\mathrm{d}y\right| \\ &\quad\le\ \|g\|_{\sup}\big(c_1 a + c_2\sqrt{a}\|h\|_2\big) + \omega_g(\frac{1}{N})\|h\|_2^2 \ , \end{aligned}\tag{4.14}$$

where, as usual, we use $\tilde{h}$ to denote the interpolation (1.25) of h, and ω_g is the modulus of continuity of g.

The proof of (4.14) is a one line computation based on (4.11) and Jensen's inequality: For each $x \in \mathbb{Z}_N^d$ let $\mathrm{C}_N(x)$ to denote the elementary plaquette centered in x:

$$\mathrm{C}_N(x) \stackrel{\Delta}{=} \left\{ y \in \mathbb{R}^d \,:\, |y-x| \leq \frac{1}{2N} \right\} \quad . \tag{4.15}$$

Then,

$$\begin{aligned} &\Big| \frac{1}{N^d} \sum\nolimits^N g(x)\Phi\big(h(x)\big) - \int_D g(y)\Phi\big(\tilde{h}(y)\big)\mathrm{d}y \Big| \\ &\quad \leq \frac{1}{N^d} \sum\nolimits^N |g(x)| \int_{\mathrm{C}_N(x)} \Big| \Phi\big(h(x)\big) - \Phi\big(\tilde{h}(y)\big) \Big| \mathrm{d}y \\ &\qquad + \sum\nolimits^N \int_{\mathrm{C}_N(x)} \big|\Phi\big(\tilde{h}(y)\big)\big|\, |g(x) - g(y)|\, \mathrm{d}y \quad . \end{aligned} \tag{4.16}$$

By (4.11) the second term is bounded above by $\omega_g(\frac{1}{N})\|h\|_2^2$. Using (4.11) and the assumption (4.13) on the oscillation of h we, furthermore, bound the first term as

$$\begin{aligned} &\|g\|_{\sup} \sum\nolimits^N \left\{ c_1 \int_{\mathrm{C}_N(x)} |h(x) - \tilde{h}(y)|^2 \mathrm{d}y + c_2 |h(x)| \int_{\mathrm{C}_N(x)} |h(x) - \tilde{h}(y)| \mathrm{d}y \right\} \\ &\quad \leq \frac{\|g\|_{\sup}}{N^d} \sum\nolimits^N \sum_{\|y-x\| \leq 2} \Big\{ c_1 |h(x) - h(y)|^2 + c_2 |h(x)||h(x) - h(y)| \Big\} \\ &\quad \leq 2\|g\|_{\sup} \left(c_1 a + c_2 \|h\|_2 \sqrt{a} \right) \quad . \end{aligned} \tag{4.17}$$

4.2. Regularization and Funaki-Spohn states

It is an appropriate moment to recall the construction of [25] in more details: First of all, as far as distributions of height differences are considered, we are going to identify the lattices $\mathbb{Z}_N^d$ and $\mathbb{Z}^d$, so that it makes sense to talk about limit properties of the family of measures $\{\mathbb{P}_{N,f}^x\}$. To fix further notations let $\mathscr{B}^d$ to denote the set of all oriented bounds of $\mathbb{Z}^d$. The basic space $\mathscr{X}$ of height differences, $\mathscr{X} \subset \big(\mathbb{R}\big)^{\mathscr{B}^d}$, is characterized by the following plaquette condition: for any closed loop $b_1, \ldots, b_n$ of oriented bonds,

$$\sum_{i=1}^n \eta(b_i) = 0 \quad , \tag{4.18}$$

for every $\eta \in \mathscr{X}$. In particular, $\eta(b) = -\eta(\bar{b})$, whenever b and $\bar{b}$ are two different orientations of the same edge of $\mathbb{Z}^d$. Clearly $\mathscr{X}$ is a linear $\mathbb{Z}^d$-shift invariant subspace. Following [25] we also introduce a scale $\{\mathscr{X}_r\}_{r \in \mathbb{R}_+}$ of shift-invariant Hilbert subspaces of $\mathscr{X}$,

$$\mathscr{X}_r = \left\{ \eta \in \mathscr{X} \,:\, \|\eta\|_r^2 = \sum_{x \in \mathbb{Z}_N^d} e^{-rN|x|} |\eta(x)|^2 \, < \, \infty \right\} \tag{4.19}$$

Given a finite set $A \subset \mathbb{Z}^d$, let us define

$$A^* \triangleq \left\{ \langle x, y \rangle \in \mathscr{B}^d \, : \, x \in A \text{ or } y \in A \right\} \ . \tag{4.20}$$

For any configuration $\xi \in \mathscr{X}$ of height differences the finite volume Gibbs state $\mathbb{P}_{A^*,\xi}$ is defined on the affine space

$$\mathscr{X}_{A^*,\xi} = \left\{ \eta \in \mathbb{R}^{A^*} \, : \, \eta \wedge \xi \in \mathscr{X} \right\} \ , \tag{4.21}$$

where the symbol $\eta \wedge \xi$ stands for the composite configuration: $\eta \wedge \xi(b) = \eta(b)$, if $b \in A^*$, and $\eta \wedge \xi(b) = \xi(b)$ otherwise. There is a natural correspondence between $\mathscr{X}_{A^*,\xi}$ and $\mathbb{R}^A$: just *ground* a point x_0 outside A, and for each $x \in A$ define

$$\varphi(x) = \sum_{i=1}^{m} \eta \wedge \xi(b_i) \ , \tag{4.22}$$

along any finite chain $b_1, \dots, b_m$ of oriented bonds leading from x_0 to x. By (4.18) the value $\varphi(x)$ does not depend on the particular choice of the chain in (4.22). As a result, there is a natural uniform measure $m_{A^*,\xi}$ on $\mathscr{X}_{A^*,\xi}$. Set,

$$\mathbb{P}_{A^*,\xi}\big(\mathrm{d}\eta\big) = \frac{1}{\mathscr{Z}_{A^*,\xi}} \exp\left(-\frac{1}{2} \sum_{b \in A^*} V(\eta(b)) \right) m_{A^*,\xi}(\mathrm{d}\eta) \ . \tag{4.23}$$

By the definition [25] a probability measure $\mathbb{P}$ on $\mathscr{X}$ is an infinite volume Gibbs state for the interaction potential V, if for each finite $A \subset \mathbb{R}^d$,

$$\mathbb{P}\left(\cdot \middle| \mathscr{F}_{\mathscr{B}^d \setminus A^*} \right)(\xi) = \mathbb{P}_{A^*,\xi}(\,\cdot\,) \tag{4.24}$$

$\mathbb{P}$-a.s., where $\mathscr{F}_{\mathscr{B}^d \setminus A^*}$ is the σ-algebra generated by the height differences on the bonds from $\mathscr{B}^d \setminus A^*$.

At last let $\mathscr{S}_2(\mathscr{X})$ to denote the family of all shift-invariant Gibbs states $\mathbb{P}$ on $\mathscr{X}$ which in addition satisfy

$$\left\langle |\eta(0)|^2 \right\rangle_{\mathbb{P}} \ < \ \infty \ . \tag{4.25}$$

As we have already mentioned it had been proved in [25] that for each $u \in \mathbb{R}^d$ there is precisely one ergodic $\mathbb{P}_u^{\mathrm{FS}} \in \mathscr{S}_2(\mathscr{X})$ such that $\langle \eta(0) \rangle_{\mathbb{P}_u^{\mathrm{FS}}} = u$.

We are now in a position to state the results on the limit properties of the families of regularized measures $\{\mathbb{Q}_N\}$ and $\{\mathbb{V}_N\}$ which were defined in (4.7) and (4.9) respectively:

Lemma 4.3. *For each $r > 0$ both the family of measures $\{\mathbb{Q}_N\}$ and the family of measures $\{\mathbb{V}_N\}$ are tight on $\mathbb{R}^d \times \mathscr{X}_r$. Moreover, every limit point $\mathbb{Q}$ of $\{\mathbb{Q}_N\}$ enjoys the following representation in terms of Funaki-Spohn states:*

$$\mathbb{Q}\big(\mathrm{d}y, \mathrm{d}\eta\big) = \int_{\mathbb{R}^d} \nu_{\mathbb{Q}}\big(\mathrm{d}y, \mathrm{d}\lambda\big) \mathbb{P}_\lambda^{\mathrm{FS}}\big(\mathrm{d}\eta\big) \ , \tag{4.26}$$

where $\nu_{\mathbb{Q}}$ is a non-negative Radon measure on $\mathbb{R}^d \times \mathbb{R}^d$.

Similarly, each limit point $\mathbb{V}$ *of* $\{\mathbb{V}_N\}$ *has the following representation:*

$$\mathbb{V}\big(\mathrm{d}v, \mathrm{d}\eta\big) = \int_{\mathbb{R}^d} \nu_{\mathbb{V}}\big(\mathrm{d}v, \mathrm{d}\lambda\big)\mathbb{P}^{\lambda}_{\mathrm{FS}}\big(\mathrm{d}\eta\big) \ , \tag{4.27}$$

where $\nu_{\mathbb{V}}$ *is again a non-negative Radon measure on* $\mathbb{R}^d \times \mathbb{R}^d$.

Our proof essentially follows the scheme laid down in [25] (proof of Theorem 4.1 on p. 23 there) subject to the necessary adjustments to our case:

Step 1. For each $r > 0$ the family $\{\mathbb{Q}_N\}$ is tight on $\mathscr{X}_r \times \mathbb{R}^d$. Indeed, since D is bounded it is only the $\mathscr{X}_r$ component that we should take care of. To this end we simply use the fact that the embedding $\mathscr{X}_s \mapsto \mathscr{X}_r$ is compact for any pair $0 < s < r$, and that for every $b \in \mathscr{B}^d$,

$$\begin{aligned} \big\langle \|\eta(b)\|^2\big\rangle_{\mathbb{Q}_N} &\le \tfrac{1}{N^d}\sum_x^N \big\langle (|\eta(x)|^2\big\rangle_{N,f} \\ &= \tfrac{1}{N^d}\sum_x^N \big\langle \eta(x)\big\rangle_{N,f}^2 + \tfrac{1}{N^d}\sum_x^N \mathbb{V}\mathrm{ar}_{N,f}(\eta(x)) \\ &\le \|u_N\|^2_{1,2} + c_1 \ \le\ c_2 \ , \end{aligned} \tag{4.28}$$

where the last two inequalities follow respectively from th Brascamp–Lieb inequality (2.123) and the boundness of $\{u_N\}$ in $\mathbb{H}^1_0$. The proof of the tightness of the family $\{\mathbb{V}_N\}$ follows a similar track, except that now one needs an additional argument to ensure the tightness in the v-direction. This, however, readily follows from Chebychev's inequality based on the bound,

$$\big\langle |v|^2 + \|\eta\|_s^2\big\rangle_{\mathbb{V}_N} \ \le\ c_8\|u_N\|^2_{1,2} + c_9 \ . \tag{4.29}$$

For simplicity we proceed to consider only the case of $\{\mathbb{Q}_N\}$ measures. The proof for $\{\mathbb{V}_N\}$ measures is identical with the only difference that one should use the oscillation Lemma 3.1 to take care of the proof of the translation invariance on Step 3 below.

Because of the tightness we can assume (passing to a subsequence if necessary), that $\mathbb{Q}_N$ is convergent and $\mathbb{Q} = \lim \mathbb{Q}_N$. For each $g \in \mathbb{C}_0^\infty(D)$ define the sequence of measures $\{q_N\}$ on $\mathscr{X}_r$ via

$$q_N\big(\mathrm{d}\eta\big) = q_N[g]\big(\mathrm{d}\eta\big) = \int_{\mathbb{R}^d} \mathbb{Q}_N(\mathrm{d}x, \mathrm{d}\eta) g(x) \ . \tag{4.30}$$

Let

$$q = \lim q_N \ . \tag{4.31}$$

Step 2. For each $g \in \mathbb{C}_0(D)$ the (signed) measure $q\big(\mathrm{d}\eta\big) = q[g]\big(\mathrm{d}\eta\big)$ is Gibbs in the following sense: for every finite $A \subset \mathbb{Z}^d$ and for each $F \in \mathbb{C}_b\big(\mathscr{X}_r\big);\ F \in \mathscr{F}_{A^*}$;

$$\big\langle F\big\rangle_q = \big\langle \mathbb{E}_{A^*,\xi} F\big\rangle_q \ . \tag{4.32}$$

We deduce (4.32) as a consequence of the following observation:

$$\Big|\big\langle F\big\rangle_{N,f} - \big\langle \mathbb{E}_{A^*,\xi} F\big\rangle_{N,f}\Big| \ \le\ c_3(f)\|F\|_{\sup}|A|\sqrt{\delta_d(N)} \ , \tag{4.33}$$

where $\delta_d(N) = N^{-2}\log N$ if $d = 2$ and $\delta_d(N) = N^{-2}$ in the case of higher dimensions. Indeed, once (4.33) is verified, we readily obtain that

$$\begin{aligned}
&\left|\left\langle F\right\rangle_{q_N} - \left\langle \mathbb{E}_{A^*,\xi} F\right\rangle_{q_N}\right| \\
&\quad \leq \|g\|\text{sup}\left(c_4(f)\|F\|_{\text{sup}}|A|\sqrt{\delta_d(N)} + c_5(D)\|F\|\text{sup}\frac{\text{diam}(A)}{N}\right) \quad , \qquad (4.34)
\end{aligned}$$

and, since the function $\xi \mapsto \mathbb{E}_{A^*,\xi} F$ belongs to $\mathbb{C}_b(\mathscr{X}_r)$ as well, one can pass to the limit $N \to \infty$ in the above inequality, and (4.32) follows.

In order to check (4.33) notice, first of all, that

$$\left\langle F\right\rangle_{N,f} = \left\langle \mathbb{E}_{A^*,\xi} F\right\rangle_{N,f} \quad , \qquad (4.35)$$

whenever $f \equiv 0$ on A. Given then an arbitrary $f \in \mathbb{C}_0^\infty(D)$, let us simply kill it on A and define $f_A(x) \overset{\Delta}{=} \mathbb{I}_{A^c}(x)f(x)$. For every $\Phi \in \mathbb{C}_b(\mathscr{X}_r)$ we then obtain:

$$\left\langle \Phi\right\rangle_{N,f} = \frac{\left\langle \Phi \exp\left\{\frac{1}{N}\sum_{x\in A} f(x)\varphi(x)\right\}\right\rangle_{N,f_A}}{\left\langle \exp\left\{\frac{1}{N}\sum_{x\in A} f(x)\varphi(x)\right\}\right\rangle_{N,f_A}} \qquad (4.36)$$

In order to facilitate notations let us define

$$\epsilon_N = \frac{1}{N}\sum_{x\in A} f(x)(\varphi(x) - \langle\varphi(x)\rangle_{N,f_A}) \quad . \qquad (4.37)$$

Unfolding (4.36), we then compute (with an obvious abuse of notation):

$$\begin{aligned}
\left\langle \Phi\right\rangle_{N,f} &= \frac{\left\langle \Phi e^{\epsilon_N}\right\rangle_{N,f_A}}{\left\langle e^{\epsilon_N}\right\rangle_{N,f_A}} = \left\langle \Phi\right\rangle_{N,f_A+\epsilon_N} \\
&= \left\langle \Phi\right\rangle_{N,f_A} + \int_0^1 \text{cov}_{N,f_A+t\epsilon_N}\left(\Phi, \epsilon_N\right) \mathrm{d}t
\end{aligned} \qquad (4.38)$$

where

$$\begin{aligned}
\left|\int_0^1 \text{cov}_{N,f_A+t\epsilon_N}\left(\Phi, \epsilon_N\right)\mathrm{d}t\right| &\leq \|\Phi\|_{\text{sup}} \max_{t\in[0,1]} \sqrt{\text{var}_{N,f_A+t\epsilon_N}(\epsilon_N)} \\
&\leq c_6(f)\|\Phi\|_{\text{sup}}|A|\sqrt{\delta_d(N)} \quad . \qquad (4.39)
\end{aligned}$$

as a consequence of the Brascamp–Lieb inequality (2.123).

Since both F and $\mathbb{E}_{A^*,\xi} F$ belong to $\mathbb{C}_b\left(\mathscr{X}_r\right)$, the bound (4.33) follows now in a straightforward way:

$$\begin{aligned}
\left\langle F\right\rangle_{N,f} &= \left\langle F\right\rangle_{N,f_A} + \mathrm{O}\left(\|F\|_{\text{sup}}|A|\sqrt{\delta_d(N)}\right) \\
&= \left\langle \mathbb{E}_{A^*,\xi} F\right\rangle_{N,f_A} + \mathrm{O}\left(\|F\|_{\text{sup}}|A|\sqrt{\delta_d(N)}\right) \\
&= \left\langle \mathbb{E}_{A^*,\xi} F\right\rangle_{N,f} + \mathrm{O}\left(\|F\|_{\text{sup}}|A|\sqrt{\delta_d(N)}\right) \quad . \qquad (4.40)
\end{aligned}$$

Step 3. The measure q is translation invariant: Let $F \in \mathscr{F}_{A^*}$ be a local $\mathbb{C}_b(\mathscr{X}_r)$ function. Then

$$\left\langle F \circ \theta_y \right\rangle_{q_N} = \frac{1}{N^d} \sum_{x \in D_N} g(x) \mathbb{E}^x_{N,f} F \circ \theta_y = \frac{1}{N^d} \sum_{x \in D_N} g(x) \mathbb{E}^{x+y}_{N,f} F$$
$$= \left\langle F \right\rangle_{q_N} + \mathrm{O}\Big(\|F\|_{\sup} \frac{1}{N^d} \sum\nolimits^N |g(x+y) - g(x)| \Big) \ . \qquad (4.41)$$

Step 4. At this point we decompose g as

$$g(x) = g(x) \vee 0 - (-g(x) \vee 0) \ \ , \qquad (4.42)$$

and immediately conclude that the translation invariant Gibbs measure $q(\mathrm{d}\eta) = q[g](\mathrm{d}\eta)$ is subject to the Choquet decomposition with respect to the (extremal) Funaki–Spohn states:

$$q[g](\mathrm{d}\eta) = \int_{\mathbb{R}^d} \mathbb{P}^{\mathrm{FS}}_{\lambda}\big(\mathrm{d}\eta\big) \nu_{\mathbb{Q}}[g](\mathrm{d}\lambda) \ . \qquad (4.43)$$

It remains to prove, therefore, that there exists a non-negative Radon measure $\nu_{\mathbb{Q}}$ on $\mathbb{R}^d \times \mathscr{X}_r$, such that for every $g \in \mathbb{C}^\infty_0(D)$,

$$\nu_{\mathbb{Q}}[g]\,(\mathrm{d}\eta) = \int_{\mathbb{R}^d} \nu_{\mathbb{Q}}\big(\mathrm{d}x, \mathrm{d}\eta\big) g(x) \ . \qquad (4.44)$$

This, however, almost literally follows by the arguments in [25] (Step 3 on p. 24 there). □

4.3. *Young measures*

By (4.8) and the master equation (1.56) the first of the two crucial convergence statements (4.2) would follow, as soon as we show that for every $j \in \mathbb{C}^\infty_0(D)$,

$$\int_D \big(\nabla j(x), \nabla\sigma(\nabla \tilde{u}_N(x))\big)\mathrm{d}x - \int_{\mathbb{R}^d \times \mathscr{X}} \big(V'(\eta(0)), \nabla j(x)\big) \mathbb{Q}_N(\mathrm{d}x, \mathrm{d}\eta) \ \longrightarrow \ 0 \ . \qquad (4.45)$$

Notice that so far we do not require the individual convergence of each of the two terms above. Since by the assumption V' is sublinear, and in view of the tightness computation (4.28), it follows from Lemma 4.3 that

$$\lim_{N\to\infty} \int_{\mathbb{R}^d \times \mathscr{X}} \big(V'(\eta), \nabla j(x)\big)\mathbb{Q}_N(\mathrm{d}x, \mathrm{d}\eta) = \int_{D \times \mathbb{R}^d} \nu_{\mathbb{Q}}\big(\mathrm{d}x, \mathrm{d}v\big)\big(\nabla\sigma(v), \nabla j(x)\big) \ , \qquad (4.46)$$

along any convergent subsequence $\mathbb{Q}_N \to \mathbb{Q}$.

Similarly, the second crucial convergence statement (4.3) follows as soon as we show that

$$\int_D \big(\nabla \tilde{u}_N(x), \nabla\sigma(\nabla \tilde{u}_N(x))\big)\mathrm{d}x - \int_{\mathbb{R}^d \times \mathscr{X}} \big(V'(\eta), v\big)\mathbb{V}_N(\mathrm{d}v, \mathrm{d}\eta) \ \longrightarrow \ 0 \ . \qquad (4.47)$$

Since, by the $\mathbb{L}_p$ bound of Lemma 3.4, the family $\{\nabla\tilde{u}_N\}$ is uniformly integrable on in $\mathbb{L}_2$, we, in view of Lemma 4.3, obtain that

$$\lim_{N\to\infty}\int_{\mathbb{R}^d\times\mathscr{X}}\big(V'(\eta),v\big)\mathbb{V}_N(\mathrm{d}v,\mathrm{d}\eta)=\int_{\mathbb{R}^d\times\mathbb{R}^d}\nu_{\mathbb{V}}\big(\mathrm{d}v,\mathrm{d}\lambda\big)\big(v,\nabla\sigma(\lambda)\big)\ , \tag{4.48}$$

along any convergent subsequence $\mathbb{V}_N\to\mathbb{Q}$.

In order to describe possible limits of the first integrals in (4.45) and (4.47) we need to recall the notion of Young measures (see, for example, [5]):

Let Ω be a bounded open subset of $\mathbb{R}^d$. A sequence $f_N\in\mathbb{L}_1(\Omega,\mathbb{R}^d)$ is said to generate the family of (probability) Young measures $\{\mu_x\}_{x\in\Omega}$ if for every $\psi\in\mathbb{C}_0(\mathbb{R}^d)$,

$$\psi(f_N(x))\ \to\ \bar{\psi}(x)=\int_{\mathbb{R}^d}\psi(\lambda)\mu_x(\mathrm{d}\lambda) \tag{4.49}$$

weakly in $\mathbb{L}_1(\Omega,\mathbb{R})$ as $N\to\infty$.

If $\{f_N\}$ is bounded in $\mathbb{L}_1(\Omega,\mathbb{R}^d)$, then it necessarily has a subsequence, which generates such a family. If, in addition, $\{f_N\}$ is bounded in $\mathbb{L}_p(\Omega,\mathbb{R}^d)$ (and generates a family of Young measures $\{\mu_x\}_{x\in\Omega}$), then the representation (4.49) is, in fact, valid for any $\psi\in\mathbb{C}(\mathbb{R})$, which does not grow too fast on infinity;

$$\lim_{t\to\infty}\frac{|\psi(t)|}{|t|^p}=0\ . \tag{4.50}$$

In particular, let us assume (possibly going to subsequence) that $\{\nabla\tilde{u}_N\}$ generates a family of Young measures $\{\mu_x\}_{x\in D}$. By the $\mathbb{L}_p$ estimate of Lemma 3.4 the condition (4.50) is satisfied for functions ψ of quadratic growth. Consequently, for every $j\in\mathbb{C}_0^\infty(D)$,

$$\lim_{N\to\infty}\int_D\big(\nabla j(x),\nabla\sigma(\nabla\tilde{u}_N(x))\big)\mathrm{d}x=\int_D\mathrm{d}x\int_{\mathbb{R}^d}\mu_x(\mathrm{d}v)\big(\nabla j(x),\nabla\sigma(v)\big)\ , \tag{4.51}$$

and

$$\lim_{N\to\infty}\int_D\big(\nabla\tilde{u}_N(x),\nabla\sigma(\nabla\tilde{u}_N(x))\big)\mathrm{d}x=\int_D\mathrm{d}x\int_{\mathbb{R}^d}\mu_x(\mathrm{d}v)\big(v,\nabla\sigma(v)\big)\ . \tag{4.52}$$

Lemma 4.4. *Assume (possibly after going to a subsequence) that both $\{\mathbb{Q}_N\}$ and $\{\mathbb{V}_N\}$ converge, and, furthermore, that $\{\nabla u_N\}$ generates the family $\{\mu_x\}$ of Young measures. Then,*

$$\nu_{\mathbb{Q}}\big(\mathrm{d}x,\mathrm{d}v\big)=\mu_x\big(\mathrm{d}v\big)\mathrm{d}x\ , \tag{4.53}$$

and,

$$\nu_{\mathbb{V}}\big(\mathrm{d}v,\mathrm{d}\lambda\big)=\int_D\mathrm{d}x\delta_\lambda\big(\mathrm{d}v\big)\mu_x\big(\mathrm{d}\lambda\big)=\int_D\mathrm{d}x\delta_v\big(\mathrm{d}\lambda\big)\mu_x\big(\mathrm{d}v\big)\ , \tag{4.54}$$

Both (4.45) and (4.47), and hence the claim of Theorem 1.2, are immediate consequence ofthe lemma above. Indeed, substituting (4.53) into (4.46) we obtain precisely the right hand side of (4.51). Similarly, the substitution of (4.54) into (4.48) gives the right hand side of (4.52).

Proof of Lemma 4.4 (4.53) follows if, for example, we show that for any $F \in \mathbb{C}_{\mathrm{b}}^{1}\big(\mathbb{R}^d\big)$ and $g \in \mathbb{C}_0^{\infty}(D)$;

$$\int_{D\times\mathbb{R}^d} \nu_{\mathbb{Q}}(\mathrm{d}x,\mathrm{d}\lambda)F(\lambda)g(x) = \int_D g(x)\mathrm{d}x \int_{\mathbb{R}^d} \mu_x(\mathrm{d}\lambda)F(\lambda) \ . \tag{4.55}$$

Due to the ergodicity of the Funaki–Spohn states,

$$F(\lambda) = \lim_{l\to\infty} \mathbb{E}_{\lambda}^{FS} F\big(\mathbb{A}\mathrm{v}_l\eta\big) \ , \tag{4.56}$$

for every $\lambda \in \mathbb{R}^d$, where the l-average $\mathbb{A}\mathrm{v}_l$ is defined via

$$\mathbb{A}\mathrm{v}_l\eta \triangleq \frac{1}{(2l+1)^d} \sum_{\|x\|\le l} \eta(x) \ . \tag{4.57}$$

Therefore it is possible to rewrite the left hand side of (4.55) as

$$\begin{aligned} &\lim_{l\to\infty} \int_{D\times\mathbb{R}^d} g(x)\nu_{\mathbb{Q}}(\mathrm{d}x,\mathrm{d}\lambda)\mathbb{E}_{\lambda}^{FS} F\big(\mathbb{A}\mathrm{v}_l\eta\big) \\ &\quad = \lim_{l\to\infty}\lim_{N\to\infty} \frac{1}{N^d}\sum\nolimits^N g(x)\left\langle F\big(\mathbb{A}\mathrm{v}_l\eta\big)\right\rangle_{N,f}^x \ . \end{aligned} \tag{4.58}$$

On the other hand,

$$\begin{aligned} &\left|\left\langle F\big(\mathbb{A}\mathrm{v}_l\eta\big)\right\rangle_{N,f}^x - F\big(\mathbb{A}\mathrm{v}_l\nabla^N u_N(x)\big)\right| \\ &\quad = \left|\left\langle F\big(\mathbb{A}\mathrm{v}_l\eta\big)\right\rangle_{N,f}^x - F\big(\langle\mathbb{A}\mathrm{v}_l\eta\rangle_{N,f}^x\big)\right| \\ &\quad \le \|F\|_{1,\infty}\mathrm{var}_{N,f}\left(\frac{1}{(2l+1)^d}\sum_{\|y-x\|\le l}\eta(y)\right)^{1/2} \ . \end{aligned} \tag{4.59}$$

By the Brascamp–Lieb inequality (2.123) the latter expression is bounded above by $c_1/\sqrt{l^d}$.

Finally, by the oscillation bound (3.1), for each l fixed,

$$\lim_{N\to\infty} \|\nabla\tilde{u}_N - \mathbb{A}\mathrm{v}_l\nabla\tilde{u}_N\|_2 = 0 \ , \tag{4.60}$$

which, in particular, implies that $\{\nabla\tilde{u}_N\}$ and $\{\mathbb{A}\mathrm{v}_l\nabla\tilde{u}_N\}$ generate the same family of Young measures $\{\mu_x\}$. Thus, in view of (4.14),

$$\begin{aligned} &\left|\frac{1}{N^d}\sum\nolimits^N g(x)\left\langle F\big(\mathbb{A}\mathrm{v}_l\eta\big)\right\rangle_{N,f}^x - \int_D g(x)\mathrm{d}x\int_{\mathbb{R}^d} F(\lambda)\mu_x(\mathrm{d}\lambda)\right| \\ &\quad \le \frac{c_1}{\sqrt{l^d}}\|F\|_{1,\infty}\|g\|_{\sup} + \mathrm{o}(1) \ , \end{aligned} \tag{4.61}$$

and (4.55) follows.

The proof of (4.54) follows the same pattern: for each $G \in \mathbb{C}^1_{\mathrm{b}}\big(\mathbb{R}^d \times \mathbb{R}^d\big)$,

$$\begin{aligned}\int_{\mathbb{R}^d\times\mathbb{R}^d} \nu_{\mathbb{V}}\big(\mathrm{d}v, \mathrm{d}\lambda\big) G(v,\lambda) &= \lim_{l\to\infty}\int_{\mathbb{R}^d\times\mathbb{R}^d} \nu_{\mathbb{V}}\big(\mathrm{d}v, \mathrm{d}\lambda\big)\big\langle G(v, \mathbb{A}\mathrm{v}_l\eta)\big\rangle_\lambda^{\mathrm{FS}} \\ &= \lim_{l\to\infty}\lim_{N\to\infty}\big\langle G(v, \mathbb{A}\mathrm{v}_l\eta)\big\rangle_{\mathbb{V}_N} \ . \end{aligned} \tag{4.62}$$

On the other hand,

$$\begin{aligned} &\Big|\big\langle G(v,\ \mathbb{A}\mathrm{v}_l\eta)\big\rangle_{\mathbb{V}_N} - \frac{1}{N^d}\sum\nolimits^N G(\nabla^N u_N, \nabla^N u_N)\Big| \\ &\quad\le \frac{1}{N^d}\|G\|_{1,\infty}\sum\nolimits^N \big\langle\big|\mathbb{A}\mathrm{v}_l\eta(x) - \langle\eta(x)\rangle_{N,f}\big|\big\rangle_{N,f} \\ &\quad\le \frac{1}{N^d}\|G\|_{1,\infty}\sum\nolimits^N\sqrt{\mathbb{V}\mathrm{ar}_{N,f}\big(\mathbb{A}\mathrm{v}_l\eta(x)\big)} \\ &\qquad + \frac{1}{(2l+1)^d}\sum\nolimits^N_{\|y-x\|\le l/N}\big|\nabla^N u_N(x) - \nabla^N u_N(y)\big| \\ &\quad\le \|G\|_{1,\infty}\left(\frac{c_2}{\sqrt{l^d}} + \frac{c_3}{\sqrt{\psi(N)}}\right) \ , \end{aligned} \tag{4.63}$$

where the first and the second terms in the last line above follow by the Brascamp–Lieb inequality (2.123) and by the oscillation bound (3.1) respectively.

As a result, by (4.12),

$$\begin{aligned}\int_{\mathbb{R}^d\times\mathbb{R}^d} \nu_{\mathbb{V}}\big(\mathrm{d}v\mathrm{d}\lambda\big) G(v,\lambda) &= \lim_{N\to\infty}\frac{1}{N^d}\sum\nolimits^N G(\nabla^N u_N, \nabla^N u_N) \\ &= \lim_{N\to\infty}\int_D G(\nabla u_N, \nabla u_N)\mathrm{d}x \\ &= \int_D \mathrm{d}x\int_{\mathbb{R}^d} G(\lambda,\lambda)\mu_x\big(\mathrm{d}\lambda\big) \ . \end{aligned} \tag{4.64}$$

Since (4.64) holds for every $G \in \mathbb{C}^1_{\mathrm{b}}\big(\mathbb{R}^d \times \mathbb{R}^d\big)$,

$$\nu_{\mathbb{V}}\big(\mathrm{d}v, \mathrm{d}\lambda\big) = \int_D \mathrm{d}x\, \delta_\lambda\big(\mathrm{d}v\big)\mu_x\big(\mathrm{d}\lambda\big) = \int_D \mathrm{d}x\delta_v\big(\mathrm{d}\lambda\big)\mu_x\big(\mathrm{d}v\big) \ , \tag{4.65}$$

as it has been claimed.

5. Large deviations and concentration results

Proof of Theorem 1.3. The proof of the upper bound:

$$\limsup_{N\to\infty}\frac{1}{N^d}\log\mathbb{P}_N(\tilde{\xi}_N \in E) \le -\inf_{\overline{E}}\Sigma \ , \tag{5.1}$$

where we recall that $\overline{E}$ denotes the closure in $\mathbb{L}_2(D)$ of E, follows from Theorem 1.1 and exponential tightness which is a simple consequence of (2.128), since closed balls with respect to the $\|\cdot\|_{1,2}$-norm are compact in $\mathbb{L}_2(D)$.

Because of the Lipschitz property of the map Ψ in (3.85) the large deviation lower bound

$$\liminf_{N\to\infty} \frac{1}{N^d} \log \mathbb{P}_N(\tilde{\xi}_N \in E) \geq -\inf_{E^\circ} \Sigma \ , \tag{5.2}$$

where E° denotes the $\mathbb{L}_2(D)$ interior of E, follows as soon as we show that

$$\liminf_{N\to\infty} \frac{1}{N^d} \log \mathbb{P}_N(\tilde{\xi}_N \in E) \geq -\Sigma(u_{[f]}) \ , \tag{5.3}$$

for any couple $(f, u_{[f]}) \in \mathbb{C}_0^\infty(D) \times E^\circ$. This, however, follows from Theorem (1.1), Theorem (1.2) and the usual change of measure argument: first note that by (1.38) and (1.36);

$$\begin{aligned}
\lim_{N\to\infty} \frac{1}{N^d} \mathbf{H}(\mathbb{P}_{N,f} \big| \mathbb{P}_N) &= \lim_{N\to\infty} \int_D u_{N,f}(x) f(x)\, \mathrm{d}x \\
&\quad - \lim_{N\to\infty} \frac{1}{N^d} \log \Big\langle \exp(\frac{1}{N} \sum^N \varphi(x) f(x)) \Big\rangle_N \\
&= \int_D u_{[f]}(x) f(x)\, \mathrm{d}x - \Lambda_D(f) = \Sigma(u_{[f]}) \ .
\end{aligned} \tag{5.4}$$

On the other hand, by 1.39, we know that

$$\lim_{N\to\infty} \mathbb{P}_{N,f}(\tilde{\xi}_N \in E) = 1 \tag{5.5}$$

Now the lower bound (5.3) is just a consequence of the standard entropy estimate:

$$\log \frac{\mathbb{P}_N(\tilde{\xi}_N \in E)}{\mathbb{P}_{N,f}(\tilde{\xi}_N \in E)} \geq -\frac{\mathbf{H}(\mathbb{P}_{N,f} \big| \mathbb{P}_N) + e^{-1}}{\mathbb{P}_{N,f}(\tilde{\xi}_N \in E)} \ , \tag{5.6}$$

cf. [21, Lemma 5.4.21]. □

Proof of Theorem 1.4. We first take care of the hard wall condition Ω_N^+ and prove that it can be neglected, more precisely, we claim that there exists a constant $C < \infty$ such that

$$\mathbb{P}_N(\Omega_N^+) \geq \begin{cases} \exp(-CN^{d-1} \log N), & d \geq 3 \\ \exp(-CN(\log N)^2) & d = 2 \ . \end{cases} \tag{5.7}$$

Actually, one can show with a more refined argument that the correct order is $\exp(-CN^{d-1})$, for $d \geq 2$, cf. [20], but for the sake of completeness we give a simple argument of the above estimate: for a given $a(N) > 0$, let $\mathbb{P}_N^{a(N)}$ denote the measure with boundary conditions $\varphi(x) = a(N), x \notin D_N$. Note that this corresponds to constant shift of the configurations $\varphi(x) + a(N), x \in D_N$. In particular, using Taylor formula, the symmetry of V

$$\int_{\mathbb{R}^{D_N}} \frac{\partial \mathscr{H}_N(\varphi)}{\partial \varphi_x} \mathbb{P}_N(\mathbf{d}\varphi) = 0, \qquad \text{for every } x \in D_N \ , \tag{5.8}$$

(1.7) and the fact that ∂D is Lipschitz we see that the relative entropy of $\mathbb{P}_N^{a(N)}$ with respect to $\mathbb{P}_N$ can be easily estimated by

$$\begin{aligned}\mathbf{H}(\mathbb{P}_N^{a(N)}\big|\mathbb{P}_N) &= \int_{\mathbb{R}^{D_N}} \big(\mathscr{H}_N(\varphi) - \mathscr{H}_N(\varphi + a(N))\big)\mathbb{P}_N^{a(N)}(\mathrm{d}\varphi) \\ &= \int_{\mathbb{R}^{D_N}} \big(\mathscr{H}_N(\varphi - a(N)) - \mathscr{H}_N(\varphi)\big)\mathbb{P}_N(\mathrm{d}\varphi) \\ &\le c_V C N^{d-1} a(N)^2 \ . \end{aligned} \tag{5.9}$$

On the other hand, by FKG we have that

$$\mathbb{P}_N^{a(N)}(\Omega_N^+) \ge \prod_{x\in D_N} \mathbb{P}_N^{a(N)}(\varphi(x)\ge 0) = \prod_{x\in D_N}\big(1 - \mathbb{P}_N(\varphi(x) < -a(N))\big) \ . \tag{5.10}$$

In view of the Brascamp–Lieb inequality (2.125) we have

$$\mathbb{P}_N(\varphi(x) < -a(N)) \le \begin{cases} \exp(-ca(N)^2), & d\ge 3 \\ \exp(-ca(N)^2/\log N), & d = 2 \ . \end{cases} \tag{5.11}$$

Thus choosing $a(N) = a\sqrt{\log N}$, for $d \ge 3$, or $a(N) = a\log N$, for $d = 2$, with $a > 0$ sufficiently large, yields $\mathbb{P}_N^{a(N)}(\Omega_N^+) \ge \frac{1}{2}$ and the above estimate follows by the entropy inequality [21, Lemma 5.4.21], as in (5.6).

Now the result follows from Theorem 1.3. Simply note that by FKG

$$\mathbb{P}_N\big(\|\tilde{\xi}_N - u^{(v)}\|_2 > \delta\big|\Omega_N^+\cap A_N(v)\big) \le \frac{\mathbb{P}_N\big(\{\|\tilde{\xi}_N - u^{(v)}\|_2 > \delta\}\cap A_N(v)\big)}{\mathbb{P}_N\big(\Omega_N^+\big)\mathbb{P}_N\big(A_N(v)\big)} \ . \tag{5.12}$$

Theorem 1.3 implies that, for all $\delta > 0$ fixed

$$\begin{aligned}&\limsup_{N\to\infty}\frac{1}{N^d}\log\mathbb{P}_N\big(\{\|\tilde{\xi}_N - u^{(v)}\|_2 > \delta\}\cap A_N(v)\big) \\ &\quad < -\inf\{\Sigma(u) : u\in\mathbb{H}_0^1(D), \int_D u(x)\,dx = v\} \ ,\end{aligned} \tag{5.13}$$

whereas in view of (5.7) estimate and Theorem 1.3

$$\begin{aligned}&\liminf_{N\to\infty}\frac{1}{N^d}\log\Big(\mathbb{P}_N\big(\Omega_N^+\big)\mathbb{P}_N\big(A_N(v)\big)\Big) \\ &\quad \ge -\inf\{\Sigma(u) : u\in\mathbb{H}_0^1(D), \int_D u(x)\,dx = v\} \ .\end{aligned} \tag{5.14}$$

□

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References

1. Abraham, D.B.: Surface structures and phase transitions: exact results, Phase Transitions and Critical Phenomena, **10**, Domb, C., Lebowitz, J.L. eds., Academic Press, London, 1–74 (1986)
2. Agmon, S., Douglis, A., Nirenberg, L.: Estimates near the boundary for solutions of elliptic partial differential equations satisfying general boundary conditions. II Comm. Pure Appl. Math. **17**, 35–92 (1964)
3. Alberti, G., Bellettini, G., Cassandro, M., Presutti, E.: Surface tension in Ising systems with Kac potentials, J. Stat. Phys. **82**, 743–796 (1996)
4. Alexander, K., Chayes, J.T., Chayes, L.: The Wulff construction and asymptotics of the finite cluster distribution for two-dimensional Bernoulli percolation, Comm. Math. Phys. **131**, 1–50 (1990)
5. Ball, J.M.: A version of the fundamental theorem for Young measures, in PDEs and continuum models of phase transitions (Nice, 1988), Springer-Verlag, Lecture Notes in Physics **344**, 207–215 (1989)
6. Ben Arous, G., Deuschel, J.-D.: The construction of the $d+1$-dimensional Gaussian droplet, Comm. Math. Phys. **179**, 467–488 (1996)
7. Benois, O., Bodineau, T., Buttà, P., Presutti, E.: On the validity of van der Waals theory of surface tension, Markov Proc. Rel. Fields **3**(2), 175–198 (1997)
8. Benois, O., Bodineau, T., Presutti, E.: Large deviations in the van der Waals limit, Stoch. Proc. Appl. **75**, 89–104 (1998)
9. Bodineau, T.: The Wulff construction in three and more dimensions, preprint, 1999
10. Bolthausen, E., Ioffe, D.: Harmonic crystal on the wall: a microscopic approach, Comm. Math. Phys. **187**, 523–566 (1997)
11. Borovkov, A.A.: Boundary-value problems for random walks and large deviations in function spaces, Th. Prob. Appl. **12**, 575–595 (1967)
12. Bovier, A., Gayrard, V.: An almost sure central limit theorem for the Hopfield model, Markov Proc. Rel. Fields **3**, 151–173 (1997)
13. Brascamp, H.J., Lieb, E.H.: On extensions of the Brunn–Minkowski and Prékopa–Leindler theorems, including inequalities for log concave functions, and with an application to the diffusion equation, J. Funct. Anal. **22**, 366–389 (1976)
14. Brascamp, H.J., Lieb, E.H., Lebowitz, J.L.: The statistical mechanics of anharmonic lattices, Bulletin of the international statistical institute, Proceedings of the 40th session, Warsaw 1975, Book 1: invited papers, 393–404 (1976)
15. Cerf, R.: Large deviations for three dimensional supercritical percolation, preprint (1998)
16. Cerf, R., Pisztora, A.: On the Wulff crystal in the Ising model, preprint (1999)
17. Cesi, F., Guadagni, G., Martinelli, F., Schonmann, R.H.: On the 2D stochastic Ising model in the phase coexistence region near the critical point, J. Stat. Phys. **85**, 55–102 (1996)
18. Chipot, M.: Variational Inequalities and Flows in Porous Media, Springer-Verlag (1984)
19. Comets, F.: Grandes deviations pour des champs de Gibbs sur $\mathbb{Z}^d$, C. R. Acad. Sci. Paris Ser. I Math. **303**, 511–513 (1986)
20. Deuschel, J.D., Giacomin, G.: Entropic repulsion for massless fields, Preprint 1999
21. Deuschel, J.D., Stroock, D.W.: Large Deviations, Academic Press, Vol. 137 in Pure and Applied Mathematics (1989)
22. Dobrushin, R.L., Kotecky, R., Shlosman, S.: Wulff Construction: a Global Shape from Local Interaction, AMS translation series **104** (1992)
23. Föllmer, H., Orey, S.: Large deviations for the empirical field of a Gibbs measure, Ann. Probab. **16**, 961–977 (1988)
24. Freidlin, M.I.: Markov Processes and Differential Equations: Asymptotics Problems, Birkhäuser, Basel, 1996
25. Funaki, T., Spohn, H.: Motion by mean curvature from the Ginsburg-Landau $\nabla\phi$ interface models, Comm. Math. Phys. **185**, 1–36 (1997)

26. Fukushima, M.: Dirichlet Forms and Markov Processes, North–Holland, Amsterdam 1980
27. Georgii, H.-O.: Gibbs Measures and Phase Transitions, de Gruyter ed., Studies in Mathematics **9** (1988)
28. Giacomin, G., Olla, S., Spohn, H.: Equilibrium fluctuations for $\nabla\varphi$ interface model, Preprint 1999
29. Giaquinta, M.: Multiple integrals in the calculus of variations and nonlinear elliptic systems, Annals of Mathematics Studies **105**, Princeton University Press, Princeton, NJ 1983
30. Gilbarg and Trudinger, Elliptic Partial Differential Equations, Springer-Verlag, 2nd edition 1982
31. Helffer, B., Sjöstrand, J.: On the correlation for Kac-like models in the convex case, J. Stat. Phys. **74**(1/2), 349–409 (1994)
32. Ioffe, D.: Large deviations for the 2D Ising model: a lower bound without cluster expansions, J. Stat. Phys., **74**, 411–432 (1994)
33. Ioffe, D.: Exact large deviation bound up to T_c for the Ising model in two dimensions, Prob. Th. Rel. Fields **102**, 313–330 (1995)
34. Ioffe, D., Schonmann, R.H.: Dobrushin-Kotecký-Shlosman theorem up to the critical temperature, Comm. Math. Phys. **199**, 117–167 (1998)
35. Kipnis, C., Landim, C.: Scaling limits of interacting particle systems, Springer-Verlag, Grundlehern der math. Wiss. **320** (1999)
36. Meyers, N.C.: An $\mathbb{L}_p$-estimate for the gradient of solutions of second order elliptic divergence equations, Ann. Sc. Norm. Sup. Pisa **17**, 189–206 (1963)
37. Messager, A., Miracle-Solé, S., Ruiz, J.: Convexity properties of surface tension and equilibrium crystals, J. Stat. Phys. **67**, 449–470 (1992)
38. Müller, S.: Private communication, 1999
39. Naddaf, A., Spencer, T.: On homogenization and scaling limit of some gradient perturbations of a massless free field, Comm. Math. Phys. **183**, 55–84 (1997)
40. Olla, S.: Large deviations for Gibbs random fields, Prob. Th. Rel. Fields **77**, 343–357 (1988)
41. Pisztora, A.: Surface order large deviations for Ising, Potts and percolation models, Prob. Th. Rel. Fields **104**, 427–466 (1996)
42. Pfister, Ch.-E.: Large deviations and phase separation in the two dimensional Ising model, Helv. Phys. Acta **64**, 953–1054 (1991)
43. Pfister, Ch.E., Velenik, Y.: Large deviations and continuum limit in the 2D Ising model, Prob. Th. Rel. Fields **109**, 435–506 (1997)
44. Reed, M., Simon, B.: Methods of modern mathematical physics IV: analysis of operators, Academic press inc. 1978
45. Rogers, L.C.G., Williams, D.: Diffusion, Markov processes, and martingales, 2d edition, Wiley Series in Probability and Mathematical Statistics, J. Wiley & Sons (1994)
46. Schonmann, R.H., Shlosman, S.: Complete analyticity for 2D Ising model completed, Comm. Math. Phys. **170**, 453–482 (1995)
47. Stroock, D., Varadhan, S.R.S.: Multidimensional Diffusion Processes, Springer-Verlag 1979
48. Winterbottom, W.L.: Equilibrium shape of a small particle in a contact with a foreign substrate, Acta Metal. **15**, 303–310 (1967)
49. Wulff, G.: Zur Frage der Geschwindigkeit des Wachstums und der Auflösung der Krystallflagen, Z. Kryst. Mineral. **34**, 449 (1901)
50. Ziemer, W.P.: Weakly differentiable functions. Sobolev spaces and functions of bounded variation, Graduate Texts in Mathematics **120**, Springer-Verlag, New York 1989