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Stochastic calculus with respect to free Brownian motion and analysis on Wigner space

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Abstract. We define stochastic integrals with respect to free Brownian motion, and show that they satisfy Burkholder-Gundy type inequalities in operator norm. We prove also a version of Itô's predictable representation theorem, as well as product form and functional form of Itô's formula. Finally we develop stochastic analysis on the free Fock space, in analogy with stochastic analysis on the Wiener space.

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Introduction

In this paper we develop a stochastic integration theory with respect to the free Brownian motion. A strong motivation for undertaking this work was provided by two sources. On one hand the stochastic quantization approach to Master Fields, as described in [D], requires the development of a stochastic calculus with respect to free Brownian motion, in order to be implemented in a mathematically rigourous way. On the other hand, the theory of free entropy developped by D. Voiculescu suggests the study of "free" Gibbs states, whose definition is analogous to the classical Gibbs states, but with free entropy replacing classical entropy, and as in the classical case, these free Gibbs states can be realized as invariant measures for some non-commutative diffusion processes driven by free Brownian motion.

Actually, these two sources are very much related, but we shall not say more about that subject in the present paper, and will come back to it in a subsequent work. Here we shall concentrate on the purely stochastic calculus aspects of the problems.

D. Voiculescu has shown (see [V2], [VDN]) that independent $N \times N$ random matrices give rise, in the large N limit, to free random variables. In this sense, free Brownian motion is the large N limit of Brownian motion with values in $N \times N$ hermitian matrices. Likewise, the stochastic calculus with respect to free Brownian motion that we are going to develop can be viewed as the large N limit of stochastic calculus with respect to $N \times N$ hermitian matrix valued Brownian motion, the processes to be integrated being matrix valued stochastic processes. Due to the non-commutativity of matrix algebras, a process to be integrated can be multiplied either to the right or to the left of the increments of the integrator process. One can even take two processes and multiply one of them on the right and the other on the left. The Itô formula for such matrix valued stochastic integrals is easy to obtain, from the Itô formula for the components. Indeed, normalizing the covariance of Brownian motion by

$$E\left[\frac{1}{N}\operatorname{tr}(X_t^2)\right] = t$$

Itô's formula takes the form

$$\begin{aligned}\left(\int_0^t A_s\,dX_sB_s\right)\left(\int_0^t C_s\,dX_sD_s\right) &= \int_0^t A_s\,dX_sB_s\left(\int_0^s C_u\,dX_uD_u\right)\\ &\quad+\int_0^t\left(\int_0^s A_u\,dX_uB_u\right)C_s\,dX_sD_s+\int_0^t A_s\frac{1}{N}\operatorname{tr}(B_sC_s)D_s\,ds\end{aligned}$$

for adapted matrix valued processes A, B, C and D. The Itô formula we are going to prove in section 4 is the large N limit of this formula. We shall however not use large matrix approximations for free Brownian motion, but rather take a direct approach starting from the definition of free Brownian motion.

Stochastic calculus with respect to free noise has already been considered before, namely in [KS], [S] and [F], inspired by the Hudson and Parthasarathy's quantum stochastic calculus [HP]. In the direct approach that we develop in this paper, we do not define stochastic integrals with respect to creation and annihilation processes, but only with respect to free Brownian motion, which does not require the use of free Fock space. This allows us to give a Burkholder-Gundy type inequality in operator norm, which means that the operator norm of a stochastic integral can be controlled by a suitable norm of its quadratic variation. It has been shown recently by Pisier and Xu [PX] that the classical Burholder-Gundy inequalities for martingales can be extended to a general non-commutative context. However as is well known these inequalities hold only in L^p for $1 < p < \infty$, and break down at $p = \infty$. So our result does not follow from this, it is intimately related with the freeness property, and with Haagerup's inequality on free groups. These inequalities allow us to derive a reasonable functional calculus version of the Itô formula, based on the product form which has been obtained in the previous works on free stochastic calculus.

It is well known that the classical Fock space associated with an infinite dimensional Hilbert space can be interpreted as the L^2 space of Wiener measure, and this gives rise to a rich analytical theory known as "analysis on Wiener space". Here we shall see that many results from Wiener space analysis have analogues when the Boson Fock space is replaced by the free (i.e. unsymmetrized) Fock space, and the free Brownian motion plays the role of the classical Brownian motion. Indeed, the free Fock space modelled on $L^2(\mathbb{R}_+)$ can be interpreted as giving chaos decomposition of the L^2 space of a free Brownian motion. Then we will define a free gradient operator, and its adjoint, which will play the role of a free Skorokhod integral, and we will have, among other results, free versions of the Itô Predictable Representation Theorem and the Bismut-Clarke-Ocone formula. Since the semicircular distribution, or "Wigner distribution" plays here the role of gaussian distribution in the classical theory, we have coined the name "analysis on Wigner space".

This paper is organized as follows. In section 1, we give some preliminary material on free probability theory and functional calculus. In section 2 we introduce simple bi-processes and define their stochastic integrals with respect to free Brownian motion. Section 3 is devoted to the Burkholder-Gundy inequality and extension of stochastic integrals to more general classes of bi-processes, and in section 4 we prove Itô's formula. Finally in section 5, inspired by Nualart's presentation of analysis on Wiener space in [N], we introduce the parallel theory of "analysis on Wigner space".

1. Preliminaries and notations

1.1. Free Brownian motion and martingales

We refer to [VDN] for the basic facts about free probability theory. Throughout the paper we shall denote by $(\mathscr{A}, \tau)$ a W^*-non-commutative probability space, namely $\mathscr{A}$ is a von Neumann algebra, and τ is a faithful normal tracial state on $\mathscr{A}$. We shall denote by $L^p(\mathscr{A},\tau)$ the non-commutative L^p spaces obtained by completion of $\mathscr{A}$ with respect to the norms $\|X\|_{L^p}=\tau[|X|^p]^{1/p}$, for $1\le p\le\infty$, where the L^∞ norm is just the algebra norm. We shall assume that $\mathscr{A}$ is filtered, so that there exists a family $(\mathscr{A}_t)_{t\in\mathbb{R}_+}$ of unital, weakly closed $*$-subalgebras of $\mathscr{A}$, such that $\mathscr{A}_s\subset\mathscr{A}_t$ for all s,t with $s\le t$. Further we shall assume that there exists an $(\mathscr{A}_t)_{t\in\mathbb{R}_+}$-free Brownian motion $(X_t)_{t\in\mathbb{R}_+}$ i.e. each X_t is a self adjoint element of $\mathscr{A}$ with semi-circular distribution of mean zero and variance t, one has $X_t\in\mathscr{A}_t$ for all $t\ge 0$, and for all s,t with $s\le t$, the element X_t-X_s is free with $\mathscr{A}_s$, and has semi-circular distribution of mean zero and variance $t-s$.

A concrete example of such a situtation is the following. Let $(\mathscr{B},\omega)$ be a non-commutative probability space, and consider, on the full Fock space

$$F(L^2(\mathbb{R}_+))=\Omega\mathbb{C}\oplus\bigoplus_{n=1}^{\infty}L^2(\mathbb{R}_+)^{\otimes n}$$

the creation and annihilation operators $l_t\equiv l_{1_{[0,t]}}$, $l_t^*\equiv l_{1_{[0,t]}}^*$ (see section 5 for definitions, we shall not use this before). Let $\mathscr{W}$ be the von Neumann algebra generated by the operators $X_t\equiv l_t+l_t^*$ for $t\ge 0$, with the state ρ induced by the pure state Ω. Take now $(\mathscr{A},\tau)=(\mathscr{B},\omega)*(\mathscr{W},\rho)$ the reduced free product, and let $\mathscr{A}_t$ be the weakly closed $*$-subalgebra of $\mathscr{A}$ generated by $\mathscr{B}\cup\{X_s;s\le t\}$, then $(\mathscr{A},\tau)$, $(\mathscr{A}_t)_{t\in\mathbb{R}_+}$ and $(X_t)_{t\in\mathbb{R}_+}$ satisfy the required properties.

Returning to the general situation, since the state τ is tracial, for any unital, weakly closed $*$-subalgebra $\mathscr{B}$ of $\mathscr{A}$, there exists a unique conditional expectation onto $\mathscr{B}$. Following a probabilistic tradition we shall denote by $\tau[.|\mathscr{B}]$ this conditional expectation. Recall that it extends to a contraction on all L^p spaces for $1\le p\le\infty$. A map $t\mapsto M_t$ from $[0,+\infty[$ to $L^p(\mathscr{A},\tau)$ will be called an L^p-martingale with respect to the filtration $(\mathscr{A}_t)_{t\in\mathbb{R}_+}$ if for every $s\le t$ one has $\tau[M_t|\mathscr{A}_s]=M_s$.

1.2. Functional calculus and differentiation

Let $\mathscr{B}$ be a unital C^*-algebra, then for any $X,Y\in\mathscr{B}$, one has Duhamel's formula

$$\mathrm{e}^X-\mathrm{e}^Y=\int_0^1\mathrm{e}^{\alpha X}(X-Y)\mathrm{e}^{(1-\alpha)Y}\,d\alpha$$

Let now f: $\mathbb{R}\to\mathbb{C}$ be a function such that $f(x)=\int_{\mathbb{R}}\mathrm{e}^{ixy}\mu(dy)$, where μ is a finite complex measure satisfying $\mathscr{I}_1(f)=\int_{\mathbb{R}}|y||\mu|(dy)<\infty$, then for any two hermitian elements in $\mathscr{B}$, one has

$$f(X) - f(Y) = \int_0^1 \int_{\mathbb{R}} \mathrm{i} y \mathrm{e}^{\mathrm{i}\alpha y X}(X - Y)\mathrm{e}^{\mathrm{i}(1-\alpha)yY}\mu(dy)\, d\alpha$$

where the integral is uniformly convergent. This implies that

$$\|f(X) - f(Y)\| \leq \|X - Y\| \mathscr{I}_1(f)$$

hence the function $X \mapsto f(X)$ is globally Lipschitz on $\mathscr{B}_h$, the space of hermitian elements of $\mathscr{B}$. Furthermore this map is also Fréchet differentiable at any point X, with differential

$$df(X).A = \int_0^1 \int_{\mathbb{R}} \mathrm{i} y \mathrm{e}^{\mathrm{i}\alpha y X} A \mathrm{e}^{\mathrm{i}(1-\alpha)yX}\mu(dy)\, d\alpha$$

We refer to Peller [P] for a discussion of conditions on the function f, ensuring that the map $X \mapsto f(X)$ is differentiable or globally Lipschitz. We shall be content with the preceding space of functions. If moreover one has $\mathscr{I}_2(f) = \int_{\mathbb{R}} |y|^2 |\mu|(dy) < \infty$ then f is twice differentiable, meaning that there exists a continuous bilinear map $d^2 f(X)$ on $\mathscr{B}_h$ with

$$\begin{aligned} d^2 f(X) \cdot (A_1, A_2) &= \lim_{\varepsilon_1, \varepsilon_2 \to 0} 1/\varepsilon_1 \varepsilon_2 (f(X + \varepsilon_1 A_1 + \varepsilon_2 A_2) \\ &\qquad - f(X + \varepsilon_1 A_1) - f(X + \varepsilon_2 A_2) + f(X)) \\ &= \lim_{\varepsilon \to 0} 1/\varepsilon (df(X + \varepsilon A_2) \cdot A_1 - df(X) \cdot A_1) \end{aligned}$$

where the limits are in operator norm. One has

$$\begin{aligned} d^2 f(X) \cdot (A_1, A_2) = &\iint_{\substack{\alpha,\beta \geq 0 \\ \alpha+\beta \leq 1}} \int_{\mathbb{R}} -y^2 \mathrm{e}^{\mathrm{i}\alpha y X} A_1 \mathrm{e}^{\mathrm{i}\beta y X} A_2 \mathrm{e}^{\mathrm{i}(1-\alpha-\beta)yX} \mu(dy)\, d\alpha\, d\beta \\ &+ \iint_{\substack{\alpha,\beta \geq 0 \\ \alpha+\beta \leq 1}} \int_{\mathbb{R}} -y^2 \mathrm{e}^{\mathrm{i}\alpha y X} A_2 \mathrm{e}^{\mathrm{i}\beta y X} A_1 \mathrm{e}^{\mathrm{i}(1-\alpha-\beta)yX} \mu(dy)\, d\alpha\, d\beta \end{aligned}$$

1.3. Piecewise constant maps

A map defined on $\mathbb{R}_+$ will be called piecewise constant if there exists a partition of $\mathbb{R}_+$ into finitely many intervals $[s, t[$ on which the function is constant.

1.4. Tensor products and bimodules

Let $(\mathscr{A}, \tau)$ be a non-commutative probability space. Together with $\mathscr{A}$ we shall also consider the opposite algebra $\mathscr{A}^{\mathrm{op}}$, with the trace τ^{op}, namely $\tau = \tau^{\text{op}}$ as a linear map on $\mathscr{A}$, but the notation is meant to stress the algebra structure we are using. The spaces $\mathscr{A}$ and $\mathscr{A} \otimes \mathscr{A}$ have natural $\mathscr{A} - \mathscr{A}$ bimodule structures given by multiplication on the right and on the left, namely $a \cdot u \cdot b = aub$ and $a \cdot (u \otimes v) \cdot b = au \otimes vb$, or equivalently they have a left $\mathscr{A} \otimes \mathscr{A}^{\text{op}}$-module structure (here we are considering the algebraic tensor product). We shall denote by $\sharp$ these actions, namely one has $(a \otimes b)\sharp u = aub$ and $(a \otimes b)\sharp(u \otimes v) = au \otimes vb$. Of course, the action of $\mathscr{A} \otimes \mathscr{A}^{\text{op}}$ on $\mathscr{A} \otimes \mathscr{A}$ corresponds to the multiplication on the left in the algebra $\mathscr{A} \otimes \mathscr{A}^{\text{op}}$.

The map $\tau \otimes \tau^{\text{op}}$ defines a tracial state on the $*$-algebra $\mathscr{A} \otimes \mathscr{A}^{\text{op}}$, and we shall denote by $L^p(\tau \otimes \tau^{\text{op}})$ the corresponding L^p-spaces, thus $L^\infty(\tau \otimes \tau^{\text{op}})$ is the von Neumann algebra tensor product of $\mathscr{A}$ and $\mathscr{A}^{\text{op}}$.

2. Stochastic integrals of simple biprocesses

In this section we shall investigate stochastic integrals with respect to free Brownian motion. The definition of such stochastic integrals will follow the classical procedure of first defining integrals of piecewise constant processes, and then after some norm estimates, extending to more general classes of processes. One peculiar feature, however, of non commutative integration is that, since the integrator is composed of operators which do not commute with the process to be integrated, we have the choice of multiplying the integrand with the increments of the integrator either on the left or on the right. In fact, we will even consider a more general kind of integration, where the integrand is multiplied both on the right and on the left of the integrator, and thus we will be lead to integrate what we call biprocesses. It turns out that this is a rather natural thing to do, as will be shown in section 5.3 below when we prove the Itô predictable representation Theorem (see 5.3.8). Let us start by defining biprocesses.

2.1. Biprocesses

Definition 2.1.1. *A simple biprocess is a piecewise constant map $t \mapsto U_t$ from $\mathbb{R}_+$ into the algebraic tensor product $\mathscr{A} \otimes \mathscr{A}^{\text{op}}$, such that $U_t = 0$ for t large enough.*

By definition of a simple biprocess there exists finitely many piecewise constant maps, $t \mapsto A_t^j$, $t \mapsto B_t^j, j = 1, \ldots, n$, with values in $\mathscr{A}$, such that $A_t^j = B_t^j = 0$ for t large enough, and $U_t = \sum_{j=1}^n A_t^j \otimes B_t^j$ for all $t \geq 0$.

Definition 2.1.2. *A simple biprocess is called adapted if one has* $U_t \in \mathscr{A}_t \otimes \mathscr{A}_t$ *for all* $t \geq 0$.

If a simple biprocess is adapted, it is clear that one can choose a decomposition $U_t = \sum_{j=1}^n A_t^j \otimes B_t^j$ as above in which A_t^j and B_t^j belong to $\mathscr{A}_t$ for all $t \geq 0$. The simple biprocesses form a complex vector space, which we shall endow with the norms

$$\|U\|_{\mathscr{B}_p} = \left(\int_0^\infty \|U_s\|^2_{L^p(\tau \otimes \tau^{\mathrm{op}})} \, ds \right)^{1/2}$$

The completion of the space of simple biprocesses for these norms will be denoted by $\mathscr{B}_p$. Note that $\mathscr{B}_2$ is the Hilbert space associated with the inner product

$$\langle U, V \rangle = \int_0^\infty \langle U_s, V_s \rangle \, ds$$

where the inner product $\langle U_s, V_s \rangle$ is in $L^2(\mathscr{A}, \tau) \otimes L^2(\mathscr{A}, \tau)$.

The closed subspaces of $\mathscr{B}_p$ generated by adapted simple processes will be called $\mathscr{B}_p^a$.

The space of adapted simple biprocesses has an antilinear involution, coming from the antilinear involution on $\mathscr{A} \otimes \mathscr{A}$

$$\left(\sum a^j \otimes b^j \right)^* = \sum (b^j)^* \otimes (a^j)^*$$

This involution can be extended isometrically to either one of the spaces $\mathscr{B}_p$, or $\mathscr{B}_p^a$.

2.2. *Stochastic integrals of adapted simple biprocesses*

Let U be a simple adapted biprocess, one can choose a decomposition $U = \sum_{j=1}^n A^j \otimes B^j$ such that there exists times $0 = t_0 \leq t_1 \leq \cdots \leq t_m$ with $A_t^j = A_{t_k}^j \in \mathscr{A}_{t_k}$, $B_t^j = B_{t_k}^j \in \mathscr{A}_{t_k}$ for $t \in [t_k, t_{k+1}[$, $A_t^j = B_t^j = 0$ for $t \geq t_m$ (in the sequel we shall always assume that the decompositions we choose satisfy such properties).

Definition 2.2.1. *Let U be simple adapted biprocess, with a decomposition as above, then the stochastic integral of U is the operator*

$$\int_0^\infty U_s \sharp \, dX_s \equiv \sum_{k=0}^{m-1} U_{t_k} \sharp (X_{t_{k+1}} - X_{t_k}) = \sum_{j=1}^{n} \sum_{k=0}^{m-1} A_{t_k}^j (X_{t_{k+1}} - X_{t_k}) B_{t_k}^j$$

This is clearly independent of the decomposition chosen.

Remark. The adjoint of the stochastic integral is again a stochastic integral, namely with the adjoint of a biprocess defined as in 2.1, one has

$$\left(\int_0^\infty U_s \sharp \, dX_s\right)^* = \int_0^\infty U_s^* \sharp \, dX_s$$

For a simple adapted biprocess U, and $s \le t$, we shall denote $U^{(s,t)}$ the stopped simple adapted biprocess given by $U_r^{(s,t)} = U_r$ for $s \le r < t$ and $U_r^{(s,t)} = 0$ for $r < s$ and $r \ge t$. Then we define $\int_s^t U_r \sharp \, dX_r = \int_0^\infty U_r^{(s,t)} \sharp \, dX_r$.

Proposition 2.2.2. *Let U be a simple adapted biprocess, then $t \mapsto \int_0^t U_s \sharp \, dX_s$ is a martingale.*

Proof. Let us prove it for a process of the form $U_t = A \otimes B 1_{[t_1,t_2[}(t)$ where $A, B \in \mathscr{A}_{t_1}$. Let $s \le t$ and $Y \in \mathscr{A}_s$, we have to check that $\tau[\int_s^t U_r \sharp \, dX_r Y] = 0$. One has $\int_s^t U_r \sharp \, dX_r = A(X_{(t \vee t_1) \wedge t_2} - X_{(s \vee t_1) \wedge t_2})B$. Since $X_{(t \vee t_1) \wedge t_2} - X_{(s \vee t_1) \wedge t_2}$ is centered, free with $\mathscr{A}_{s \vee t_1}$, and A, B and Y are in $\mathscr{A}_{s \vee t_1}$, we get the result. The general case follows since linear combinations of martingales are martingales. $\square$

The following result, which is a weak version of the Itô formula we shall prove in section 4, will be crucial for proving inequalities on stochastic integrals, see section 3 below.

Lemma 2.2.3. *Let $U^1, \ldots, U^r$ be simple adapted biprocesses with decompositions*

$$U^k = \sum_{j=1}^{n_k} A^{k,j} \otimes B^{k,j} \quad \textit{for} \quad k = 1, \ldots, r \quad \textit{and} \quad n_k \ge 1$$

and let $N_t^k = \int_0^t U_s^k \sharp \, dX_s$, then one has, for any $t \ge 0$,

$$\begin{aligned}
&\tau[N_t^1 N_t^2 \cdots N_t^r] \\
&= \int_0^t \sum_{1 \le k_1 < k_2 \le r} \sum_{\substack{1 \le j_1 \le n_{k_1} \\ 1 \le j_2 \le n_{k_2}}} \tau[N_s^1 \cdots N_s^{k_1 - 1} A_s^{k_1, j_1} B_s^{k_2, j_2} N_s^{k_2 + 1} \cdots N_s^r] \\
&\quad \times \tau[B_s^{k_1, j_1} N_s^{k_1 + 1} \cdots N_s^{k_2 - 1} A_s^{k_2, j_2}] \, ds
\end{aligned}$$

Proof. Since A and B are piecewise constant, one has for $s > 0$ small enough

$$N_{t+s}^k - N_t^k = \sum_{j=1}^{n_k} A_t^{k,j} (X_{t+s} - X_t) B_t^{k,j} \tag{1}$$

hence $\|N_{t+s}^k - N_t^k\| = O(\sqrt{s})$ as $s \to 0$. Let us compute for small $s > 0$

$$\begin{aligned}
&\tau[N_{t+s}^1 N_{t+s}^2 \cdots N_{t+s}^r] - \tau[N_t^1 N_t^2 \cdots N_t^r] \\
&\quad = \sum_{k=1}^{r} \tau[N_t^1 \cdots (N_{t+s}^k - N_t^k) \cdots N_t^r] \\
&\qquad + \sum_{1 \le k_1 < k_2 \le r} \tau[N_t^1 \cdots (N_{t+s}^{k_1} - N_t^{k_1}) \cdots (N_{t+s}^{k_2} - N_t^{k_2}) \cdots N_t^r] + O(s^{3/2})
\end{aligned}$$

Using (1), the adaptedness of A and B, and the freeness assumption, we get that the terms in the first sum of the right hand side are zero, and

$$\begin{aligned}
&\tau[N_t^1 \cdots (N_{t+s}^{k_1} - N_t^{k_1}) \cdots N_{t+s}^{k_2} - N_t^{k_2}) \cdots N_t^r] \\
&\quad = s \sum_{\substack{1 \le j_1 \le n_{k_1} \\ 1 \le j_2 \le n_{k_2}}} \tau[N_t^1 \cdots A_t^{k_1, j_1} B_t^{k_2, j_2} N_t^{k_2+1} \cdots N_t^r] \\
&\qquad \times \tau[B_t^{k_1, j_1} N_t^{k_1+1} \cdots N_t^{k_2-1} A_t^{k_2, j_2}]
\end{aligned}$$

Here we used the fact that if S is free with $\{X, Y\}$ and $\tau(S) = 0$ one has $\tau(XSYS) = \tau(S^2)\tau(X)\tau(Y)$. Hence we obtain

$$\begin{aligned}
&\frac{\partial}{\partial t} \tau[N_t^1 N_t^2 \cdots N_t^r] \\
&\quad = \sum_{1 \le k_1 < k_2 \le r} \sum_{\substack{1 \le j_1 \le n_{k_1} \\ 1 \le j_2 \le n_{k_2}}} \tau(N_t^1 \cdots A_t^{k_1, j_1} B_t^{k_2, j_2} N_t^{k_2+1} \cdots N_t^r] \\
&\qquad \times \tau[B_t^{k_1, j_1} N_t^{k_1+1} \cdots N_t^{k_2-1} A_t^{k_2, j_2}]
\end{aligned}$$

Integrating from 0 to t gives the result. $\square$

3. Extensions of stochastic integral

3.1. Itô isometry

We will now extend the stochastic integrals to square integrable processes in the space $\mathscr{B}_2^a$, using an isometry property for the stochastic integral.

Proposition 3.1.1. *For all adapted simple biprocesses U and V, one has*

$$\tau\left[\int_0^\infty U_s \sharp\, dX_s \left(\int_0^\infty V_s \sharp\, dX_s\right)^*\right] = \langle U, V \rangle$$

Proof. By bilinearity it is enough to prove it for processes $U_t = A1_{[t_0,t_1[}(t)$ and $V_t = C \otimes D1_{[t_2,t_3[}(t)$. The left hand side is then, denoting by λ the Lebesgue measure

$$\begin{aligned}\tau[A(X_{t_1} - X_{t_0})BD^*(X_{t_3} - X_{t_2})C^*] &= \lambda([t_0, t_1[\cap[t_2, t_3[)\tau[BD^*]\tau[AC^*] \\ &= \langle U, V\rangle\end{aligned}$$

as claimed. □

Corollary 3.1.2. *The map $U \mapsto \int_0^\infty U_s \sharp\, dX_s$ can be extended isometrically from $\mathscr{B}_2^a$ into $L^2(\mathscr{A}, \tau)$.*

3.2. Burkholder-Gundy inequality

We have seen that stochastic integrals of adapted simple biprocesses give rise to martingales. Using this fact, it is possible to use some general results of Pisier and Xu [PX], on non-commutative martingales, in order to control the L^p norms (for $1 < p < \infty$) of stochastic integrals in terms of their quadratic variation, and we shall do this in section 4.2 below. However, it is well known that such estimates break down when one tries to estimate the L^∞ norms. So it is quite remarkable that in our context the L^∞ norm of the stochastic integral is controlled by a suitable quadratic variation L^∞ norm of the integrand, namely the norm in the space $\mathscr{B}_\infty$. This will allow us to extend stochastic integrals in a continuous way to this space of biprocesses. More precisely we shall now prove the following free L^∞ version of the well known Burkholder-Gundy inequalities.

Theorem 3.2.1. *For any simple adapted process U, one has*

$$\left\| \int_0^\infty U_s \sharp\, dX_s \right\|_{L^\infty(\tau)} \leq 2\sqrt{2}\|U\|_{\mathscr{B}_\infty} .$$

We do not know which natural norm to put on the space of adapted biprocesses in order to have an equivalence of norms, but see nevertheless the end of section 4.2.

Proof. Choose a decomposition $U = \sum_j A^j \otimes B^j$. Let us denote $M_t = \int_0^t U_s \sharp\, dX_s$, we apply Lemma 2.2.3 to $N^k = M$ for k odd and $N^k = M^*$ for k even,

$$
\begin{aligned}
&\tau[|M_t|^{2m}] = \tau[(M_t M_t^*)^m] \\
&= \sum_{1\le k\le m} \sum_{1\le j_1, j_2\le n} \int_0^t \tau[(M_s M_s^*)^{k-1} A_s^{j_1} (A_s^{j_2})^* (M_s M_s^*)^{m-k}] \tau[B_s^{j_1} (B_s^{j_2})^*] \, ds \\
&\quad + \sum_{1\le k_1 < k_2 \le m} \sum_{1\le j_1, j_2\le n} \int_0^t \tau[(M_s M_s^*)^{k_1-1} A_s^{j_1} B_s^{j_2} M_s^* (M_s M_s^*)^{m-k_2}] \\
&\quad \times \tau[B_s^{j_1} M_s^* (M_s M_s^*)^{k_2-k_1-1} A_s^{j_2}] \, ds \\
&\quad + \sum_{1\le k_1 < k_2 \le m} \sum_{1\le j_1, j_2\le n} \int_0^t \tau[(M_s M_s^*)^{k_1-1} A_s^{j_1} (A_s^{j_2})^* (M_s M_s^*)^{m-k_2}] \\
&\quad \times \tau[B_s^{j_1} M_s^* (M_s M_s^*)^{k_2-k_1-1} M_s (B_s^{j_2})^*] \, ds \\
&\quad + \sum_{1\le k_1 < k_2 \le m} \sum_{1\le j_1, j_2\le n} \int_0^t \tau[(M_s M_s^*)^{k_1-1} M_s (B_s^{j_1})^* (A_s^{j_2})^* (M_s M_s^*)^{m-k_2}] \\
&\quad \times \tau[(A_s^{j_1})^* (M_s M_s^*)^{k_2-k_1-1} M_s (B_s^{j_2})^*] \, ds \\
&\quad + \sum_{1\le k_1 < k_2 \le m} \sum_{1\le j_1, j_2\le n} \int_0^t \tau[(M_s M_s^*)^{k_1-1} M_s (B_s^{j_1})^* B_s^{j_2} M_s^* (M_s M_s^*)^{m-k_2}] \\
&\quad \times \tau[(A_s^{j_1})^* (M_s M_s^*)^{k_2-k_1-1} (B_s^{j_2})^*] \, ds
\end{aligned}
$$

Applying Hölder's inequality for the trace $\tau \otimes \tau^{\mathrm{op}}$, one has e.g.

$$
\begin{aligned}
&\Big| \sum_{1\le j_1, j_2\le n} \tau[(M_s M_s^*)^{k_1-1} A_s^{j_1} B_s^{j_2} M_s^* (M_s M_s^*)^{m-k_2}] \tau[B_s^{j_1} M_s^* (M_s M_s^*)^{k_2-k_1-1} A_s^{j_2}] \Big| \\
&\quad = \Big| \tau \otimes \tau^{op}[(M_s^* (M_s M_s^*)^{m-k_2+k_1-1} \otimes 1) \sharp \Big(\sum_{1\le j_1\le n} A_s^{j_1} \otimes B_s^{j_1} \Big) \\
&\qquad \sharp \Big(\sum_{1\le j_2\le n} B_s^{j_2} \otimes A_s^{j_2} \Big) \sharp (1 \otimes M_s^* (M_s M_s^*)^{k_2-k_1-1})] \Big| \\
&\quad \le \tau[|M_s|^{2m-2k_2+2k_1-1}] \tau[|M_s|^{2k_2-2k_1-1}] \\
&\qquad \times \Big\| \sum_{1\le j\le n} A_s^j \otimes B_s^j \Big\|_{L^\infty(\tau\otimes\tau^{\mathrm{op}})} \Big\| \sum_{1\le j\le n} B_s^j \otimes A_s^j \Big\|_{L^\infty(\tau\otimes\tau^{\mathrm{op}})}
\end{aligned}
$$

This gives us an upper bound on the second term in the right hand side of the above equality. We can treat the other terms in a similar way, and after regrouping, since $\| \sum_{1\le j\le n} A_s^j \otimes B_s^j \|_{L^\infty(\tau\otimes\tau^{\mathrm{op}})} = \| \sum_{1\le j\le n} B_s^j \otimes A_s^j \|_{L^\infty(\tau\otimes\tau^{\mathrm{op}})}$, we get

$$\tau[|M_t|^{2m}] \leq m \sum_{k=0}^{2m-2} \int_0^t \tau[|M_s|^k]\tau[|M_s|^{2m-2-k}] \|U_s\|^2_{L^\infty(\tau\otimes\tau^{\mathrm{op}})}\, ds$$

For any $X \in \mathscr{A}$, and nonnegative integers p, q, one has

$$\tau(|X|^{2p+1})\tau(|X|^{2q+1}) \leq (1/2)(\tau(|X|^{2p})\tau(|X|^{2q+2}) + \tau(|X|^{2p+2})\tau(|X|^{2q}))$$

Indeed, let μ be the image by τ of the spectral measure of $|X|$, one has

$$\begin{aligned}\tau(|X|^{2p+1})\tau(|X|^{2q+1}) &= \int_{\mathbb{R}_+}\int_{\mathbb{R}_+} x^{2p+1}y^{2q+1}\mu(dx)\mu(dy)\\ &\leq \int_{\mathbb{R}_+}\int_{\mathbb{R}_+} (1/2)x^{2p}y^{2q}(x^2+y^2)\mu(dx)\mu(dy)\\ &= (1/2)(\tau(|X|^{2p})\tau(|X|^{2q+2}) + \tau(|X|^{2p+2})\tau(|X|^{2q}))\end{aligned}$$

Using this inequality we can get rid of the terms of odd degree in the inequality and this yields

$$\tau[|M_t|^{2m}] \leq 2m \sum_{k=0}^{m-1} \int_0^t \tau[|M_s|^{2k}]\tau[|M_s|^{2m-2-2k}] \|U_s\|^2_{L^\infty(\tau\otimes\tau^{\mathrm{op}})}\, ds$$

The Catalan numbers $C_n = \frac{1}{n+1}\frac{(2n)!}{n!n!}$ satisfy the recursion relations

$$C_n = \sum_{k=0}^{n-1} C_k C_{n-k-1}$$

From this we infer, by induction on m, that

$$\tau[|M_t|^{2m}] \leq C_m\left(2\int_0^t \|U_s\|^2_{L^\infty(\tau\otimes\tau^{\mathrm{op}})}\, ds\right)^m$$

Since $(C_m)^{1/2m} \to 2$ as $m \to \infty$, one has

$$\|M_t\| = \lim_{m\to\infty} \tau[|M_t|^{2m}]^{1/2m} \leq 2\sqrt{2}\left(\int_0^t \|U_s\|^2_{L^\infty(\tau\otimes\tau^{\mathrm{op}})}\, ds\right)^{1/2}$$

We can also put $t = \infty$ in this inequality. □

Corollary 3.2.2. *The stochastic integral map* $U \mapsto \int_0^\infty U_s \sharp\, dX_s$ *can be extended continuously to the space* $\mathscr{B}^a_\infty$.

Once again we can extend the martingale property

Proposition 3.2.3. *Let* $U \in \mathscr{B}^a_\infty$, *then* $t \mapsto \int_0^t U_s \sharp\, dX_s$ *is an* L^∞ *martingale.*

Observe that the map $t \mapsto \int_0^t U_s \sharp\, dX_s$ is continuous in the L^∞ norm.

Remark. The following example shows that the inequality in reverse order does not hold. Let $T_n(x)$ be the n^{th} Tchebycheff polynomial, $T_n(2\cos\theta) = \frac{\sin(n+1)\theta}{\sin\theta}$, then one has

$$T_n(X_1) = \int_0^1 t^{\frac{n-1}{2}} \left(\sum_{k=0}^{n-1} T_k(t^{-1/2}X_t) \otimes T_{n-k-1}(t^{-1/2}X_t) \right) \sharp\, dX_t$$

This follows from section 5.1 and the Bismut-Clarke-Ocone formula (Proposition 5.3.12), but see also [B1]. The norm of the left hand side is clearly equal to $n+1$. But the $\mathscr{B}_\infty$-norm of the integrated process on the right hand side is

$$\left(\int_0^1 t^{n-1}\, dt \operatorname{ess\,sup}_{-2\le x,y\le 2} \Big| \sum_{k=0}^{n-1} T_k(x) T_{n-k-1}(y) \Big|^2 \right)^{1/2}$$
$$= \frac{1}{\sqrt{n}} \sum_{k=0}^{n-1} (k+1)(n-k)$$

since the maximum is assumed for $x = y = 0$. This is of the order $n^{5/2}$ when $n \to \infty$.

4. Itô's formula

4.1. Product form

We shall now state the analogue for our stochastic integrals of the classical Itô formula. Such a formula was proved in [KS] for stochastic integrals with respect to annihilation and creation processes on the free Fock space. We start with the product form, before going to the functional calculus form of the formula. First we need some preliminaries.

Definition 4.1.1. *Let* $X = \sum_i x_i \otimes x_i'$ and $Y = \sum_j y_j \otimes y_j'$, *be elements in the algebraic tensor product* $\mathscr{A} \otimes \mathscr{A}$, *define their bracket as*

$$\langle\langle X, Y \rangle\rangle = \sum_{i,j} x_i y_j' \tau[x_i' y_j] \in \mathscr{A} \quad .$$

In more intrinsic notations, denoting $\tilde{Y} = \sum_j y_j' \otimes y_j$ one has

$$\langle\langle X, Y \rangle\rangle = I_{\mathscr{A}} \otimes \tau^{\text{op}}(X \sharp \tilde{Y})$$

clearly,

$$\begin{aligned} \|\langle\langle X, Y \rangle\rangle\|_{L^\infty(\tau)} &= \sup_{\tau(|a|)\le 1} \tau(a \langle\langle X, Y \rangle\rangle) \\ &= \sup_{\tau(|a|)\le 1} \tau \otimes \tau^{\text{op}}((a \otimes 1) \sharp X \sharp \tilde{Y}) \end{aligned}$$

$$\leq \|X\|_{L^\infty(\tau\otimes\tau^{op})}\|\tilde{Y}\|_{L^\infty(\tau\otimes\tau^{\mathrm{op}})} = \|X\|_{L^\infty(\tau\otimes\tau^{\mathrm{op}})}\|Y\|_{L^\infty(\tau\otimes\tau^{\mathrm{op}})}$$

Let U and V be simple adapted biprocesses, then one has, by Cauchy-Schwarz inequality,

$$\left\|\int_0^\infty \langle\langle U_s, V_s\rangle\rangle ds\right\| \leq \|U\|_{\mathscr{B}_\infty}\|V\|_{\mathscr{B}_\infty}$$

hence the bracket $(U,V)\mapsto \int_0^\infty \langle\langle U_s, V_s\rangle\rangle\, ds$ can be extended continuously to the space $\mathscr{B}^a_\infty$ with values in $\mathscr{A}$. Let U and V be biprocesses in $\mathscr{B}^a_\infty$ then, by Theorem 3.2.1, $s\mapsto \int_0^s U_r\sharp dX_r$ is a continuous and bounded map with values in $\mathscr{A}$, furthermore,

$$\left\|\left(\int_0^s U_r\sharp\, dX_r \otimes 1\right)V_s\right\|_{L^\infty(\tau\otimes\tau^{\mathrm{op}})} \leq 2\sqrt{2}\|U\|_{\mathscr{B}_\infty}\|V_s\|_{L^\infty(\tau\otimes\tau^{\mathrm{op}})}$$

hence $s\mapsto (\int_0^s U_r\sharp\, dX_r \otimes 1)V_s$ is an element in $\mathscr{B}^a_\infty$, and one has

$$\left\|\int_0^\infty \left(\int_0^t U_s\sharp\, dX_s \otimes 1\right)V_t\sharp\, dX_t\right\| \leq 8\|U\|_{\mathscr{B}_\infty}\|V\|_{\mathscr{B}_\infty}$$

With these remarks at hand, we can now state Itô's formula.

Theorem 4.1.2. (Itô's formula). *Let U, V be in $\mathscr{B}^a_\infty$, then one has*

$$\begin{aligned}\left(\int_0^\infty U_t\sharp\, dX_t\right)\left(\int_0^\infty V_t\sharp\, dX_t\right) &= \int_0^\infty \left(\int_0^t U_s\sharp\, dX_s \otimes 1\right)V_t\sharp\, dX_t \\ &+ \int_0^\infty U_t\left(1\otimes \int_0^t V_s\sharp dX_s\right)\sharp\, dX_t \\ &+ \int_0^\infty \langle\langle U_s, V_s\rangle\rangle\, ds \ .\end{aligned}$$

Proof. Owing to the remarks before the statement of the proposition, we need only prove the formula for U and V simple adapted biprocesses, since then both sides extend by continuity to biprocesses in $\mathscr{B}^a_\infty$. By bilinearity, it is also enough to prove it for biprocesses of the form $U_s = A\otimes B1_{[t_0,t_1[}(s)$ and $V_s = C\otimes D1_{[t_2,t_3[}(s)$ with $A, B\in\mathscr{A}_{t_0}$ and $C, D\in\mathscr{A}_{t_2}$. In this case one has $\int_0^\infty U_s\sharp\, dX_s = A(X_{t_1} - X_{t_0})B$ and $\int_0^\infty V_s\sharp\, dX_s = C(X_{t_3} - X_{t_2})D$. By bilinearity again we can assume that either $[t_0, t_1[= [t_2, t_3[$, or these intervals are disjoint. The second case is easier and will be left to the reader. We shall only do the first case. There is no loss in generality in assuming that $t_0 = 0$ and $t_1 = t$, and that $X_0 = 0$, then it follows that

$$\left(\int_0^\infty U_t \sharp\, dX_t\right)\left(\int_0^\infty V_t \sharp\, dX_t\right) = \left(\sum_{k=0}^{n-1} A(X_{\frac{k+1}{n}t} - X_{\frac{k}{n}t})B\right) \times \left(\sum_{k=0}^{n-1} C(X_{\frac{k+1}{n}t} - X_{\frac{k}{n}t})D\right)$$

$$= \sum_{k=0}^{n-1} AX_{\frac{k}{n}t} BC(X_{\frac{k+1}{n}t} - X_{\frac{k}{n}t})D + \sum_{k=0}^{n-1} A(X_{\frac{k+1}{n}t} - X_{\frac{k}{n}t})BCX_{\frac{k}{n}t}D + \sum_{k=0}^{n-1} A(X_{\frac{k+1}{n}t} - X_{\frac{k}{n}t})BC(X_{\frac{k+1}{n}t} - X_{\frac{k}{n}t})D$$

Observe that $\|X_s - X_{\frac{k}{n}t}\| \le 2\sqrt{t/n}$ for $s \in [\frac{k}{n}t, \frac{k+1}{n}t[$. This implies that the simple biprocesses $s \mapsto \xi_s^{(n)} = \sum_{k=0}^{n-1} AX_{\frac{k}{n}t} BC \otimes D1_{[\frac{k}{n}t, \frac{k+1}{n}t[}(s)$ converge in $\mathscr{B}_\infty^a$, as $n \to \infty$, towards $s \mapsto AX_{s\wedge t}BC \otimes D = (\int_0^s U_r \sharp dX_r \otimes 1)V_s$. Thus we have

$$\sum_{k=0}^{n-1} AX_{\frac{k}{n}t} BC\left(X_{\frac{k+1}{n}t} - X_{\frac{k}{n}t}\right)D = \int_0^\infty \xi_s^{(n)} \sharp\, dX_s \to \int_0^\infty \left(\int_0^s U_r \sharp\, dX_r \otimes 1\right) V_s \sharp\, dX_s$$

as $n \to \infty$ and similarly

$$\sum_{k=0}^{n-1} A\left(X_{\frac{k+1}{n}t} - X_{\frac{k}{n}t}\right)BCX_{\frac{k}{n}t}D \to \int_0^\infty U_s\left(1 \otimes \int_0^s V_r \sharp\, dX_r\right) \sharp\, dX_s$$

Let now $Z \in \mathscr{A}_0$ be an arbitrary element, then the elements

$$\left(X_{\frac{k+1}{n}t} - X_{\frac{k}{n}t}\right)Z\left(X_{\frac{k+1}{n}t} - X_{\frac{k}{n}t}\right) \quad \text{for} \quad k = 0, \dots, n-1$$

form a free family (see [NS], application 1.10).

By a result of Voiculescu (see e.g. [V1]), if $X_1, \dots, X_n$ are free random variables, with $\tau(X_j) = 0$, then $\|X_1 + \dots + X_n\| \le \sup_j \|X_j\| + (\sum_{j=1}^n \tau(|X_j|^2))^{1/2}$, thus if $\tau[Z] = 0$ we have

$$\left\| \sum_{k=0}^{n-1} \left(X_{\frac{k+1}{n}t} - X_{\frac{k}{n}t}\right)Z\left(X_{\frac{k+1}{n}t} - X_{\frac{k}{n}t}\right) \right\| \le 4t/n\|Z\| + t/\sqrt{n}\|Z\| \to 0$$

as $n \to \infty$. On the other hand, using the same estimate,

$$\left\| \sum_{k=0}^{n-1} \left(X_{\frac{k+1}{n}t} - X_{\frac{k}{n}t} \right)^2 - t \right\| \leq 3t/n + t/\sqrt{n} \to 0$$

So finally, we get

$$\begin{aligned} \lim_{n\to\infty} \sum_{k=0}^{n-1} A\left(X_{\frac{k+1}{n}t} - X_{\frac{k}{n}t}\right) BC \left(X_{\frac{k+1}{n}t} - X_{\frac{k}{n}t}\right) D &= tA\tau[BC]D \\ &= \int_0^\infty \langle\langle U_s, V_s \rangle\rangle \, ds \end{aligned}$$

This ends the proof. □

We can also extend Itô's formula to ordinary integrals, indeed, let $t \mapsto K_t$ be a weakly measurable map into $\mathscr{A}$, such that $K_t \in \mathscr{A}_t$ for all $s \geq 0$ and $\int_0^\infty \|K_t\| \, dt < \infty$ then

$$\begin{aligned} \int_0^\infty K_t \, dt \int_0^\infty U_t \sharp dX_t &= \int_0^\infty K_t \left(\int_0^t U_s \sharp \, dX_s \right) dt \\ &\quad + \int_0^\infty \left(\int_0^t K_s \, ds \otimes 1 \right) U_t \sharp \, dX_t \end{aligned}$$

which we leave to the reader to verify.

4.2. L^p estimates

Observe that from the proof of Theorem 4.1.2, we have for all adapted biprocesses $U \in \mathscr{B}^a_\infty$, with $M_t = \int_0^t U_s \sharp dX_s$,

$$\lim_{n\to\infty} \sum_{k=0}^{\infty} \left(M_{\frac{k+1}{n}} - M_{\frac{k}{n}} \right) \left(M^*_{\frac{k+1}{n}} - M^*_{\frac{k}{n}} \right) = \int_0^\infty \langle\langle U_s, U_s^* \rangle\rangle \, ds$$

where the limit holds in $L^\infty(\tau)$. Since the process $(M_{\frac{k}{n}})_{k=0,1,\ldots}$ is a discrete time martingale, we can apply Pisier and Xu's generalization of Burkholder-Gundy inequality and obtain, for all $p \in [2, \infty[$, setting $d_k = M_{\frac{k+1}{n}} - M_{\frac{k}{n}}$

$$\begin{aligned} c_p \sup &\left(\left\| \sum_{k=0}^{\infty} d_k d_k^* \right\|^{1/2}_{L^{p/2}(\tau)}, \left\| \sum_{k=0}^{\infty} d_k^* d_k \right\|^{1/2}_{L^{p/2}(\tau)} \right) \\ &\leq \left\| \int_0^\infty U_s \sharp \, dX_s \right\|_{L^p(\tau)} \end{aligned}$$

$$\leq C_p \sup\left(\left\| \sum_{k=0}^{\infty} d_k d_k^* \right\|_{L^{p/2}(\tau)}^{1/2}, \left\| \sum_{k=0}^{\infty} d_k^* d_k \right\|_{L^{p/2}(\tau)}^{1/2} \right)$$

for some universal constants c_p, C_p. If we let $n \to \infty$ we obtain

$$c_p \sup\left(\left\| \int_0^\infty \langle\langle U_s, U_s^* \rangle\rangle\, ds \right\|_{L^{p/2}(\tau)}^{1/2}, \left\| \int_0^\infty \langle\langle U_s^*, U_s \rangle\rangle ds \right\|_{L^{p/2}(\tau)}^{1/2} \right)$$
$$\leq \left\| \int_0^\infty U_s \sharp\, dX_s \right\|_{L^p(\tau)}$$
$$\leq C_p \sup\left(\left\| \int_0^\infty \langle\langle U_s, U_s^* \rangle\rangle ds \right\|_{L^{p/2}(\tau)}^{1/2}, \left\| \int_0^\infty \langle\langle U_s^*, U_s \rangle\rangle ds \right\|_{L^{p/2}(\tau)}^{1/2} \right)$$

The constants given by Pisier and Xu's proof do diverge as $p \to \infty$, but since we have in our special case the estimations of part 3.2, it is tempting to think that the above inequality holds for $p = \infty$, with some constants $0 < c_\infty < C_\infty < \infty$, however, we have not been able to settle this question.

4.3. Functional calculus form

We shall now consider an integral of the form

$$M_t = M_0 + \int_0^t K_s\, ds + \int_0^t U_s \sharp\, dX_s$$

where $M_0 \in \mathscr{A}_0$, $s \mapsto K_s$ is weakly measurable, $K_s \in \mathscr{A}_s$ for all $s \geq 0$ and $\int_0^\infty \| K_s \| ds < \infty$, and $U \in \mathscr{B}_\infty^a$. Note that under these hypotheses there is a constant Φ such that $\|M_t\| < \Phi$ for all $t \geq 0$.

Definition 4.3.1. Let $\partial \colon \mathbb{C}[X] \to \mathbb{C}[X] \otimes \mathbb{C}[X]$ *be the canonical derivation, namely on monomials it is given by*

$$\partial X^n = \sum_{k=0}^{n-1} X^k \otimes X^{n-k-1}$$

and let $\partial^2 \colon \mathbb{C}[X] \to \mathbb{C}[X] \otimes \mathbb{C}[X] \otimes \mathbb{C}[X]$ *be the second derivative*

$$\partial^2 X^n = 2 \sum_{\substack{k,l\geq 0\\ k+l\leq n-2}} X^k \otimes X^l \otimes X^{n-k-l-2} \ .$$

Let $m\colon \mathscr{A} \otimes \mathscr{A} \to \mathscr{A}$ be the multiplication map, we consider the contraction $\eta = m \circ (I_{\mathscr{A}} \otimes \tau \otimes I_{\mathscr{A}})\colon \mathscr{A} \otimes \mathscr{A} \otimes \mathscr{A} \to \mathscr{A}$. Let $\sharp$ denote the product in $\mathscr{A} \otimes \mathscr{A}^{\mathrm{op}} \otimes \mathscr{A}$. We define, for $X \in \mathscr{A}$ and $U \in \mathscr{A} \otimes \mathscr{A}$,

$$\Delta_U P(X) = \eta((1 \otimes U)\sharp \partial^2 P(X) \sharp (U \otimes 1))$$

A more concrete formula can be obtained if we choose a decomposition $U = \sum_j A_j \otimes B_j$, and $P(X) = X^n$, namely

$$\Delta_U X^n = 2 \sum_{j_1,j_2} \sum_{\substack{k,l\geq 0\\ k+l\leq n-2}} X^k A_{j_1} B_{j_2} X^{n-k-l-2} \tau(B_{j_1} X^l A_{j_2})$$

One can check, as we did for the bracket $\langle\langle .,.\rangle\rangle$, that for all $X \in \mathscr{A}$, and polynomials $P \in \mathbb{C}[X]$, the map $U \mapsto \Delta_U P(X))$ extends by continuity to $U \in L^\infty(\tau \otimes \tau^{\mathrm{op}})$, with values in $L^\infty(\tau)$ (as the quadratic map associated with a bilinear continuous map).

Proposition 4.3.2. *For all complex polynomials in one variable one has*

$$P(M_t) = P(M_0) + \int_0^t (\partial P(M_s)\sharp\, U_s)\sharp\, dX_s + \int_0^t \partial P(M_s)\sharp K_s ds + 1/2 \int_0^t \Delta_{U_s} P(M_s)\, ds$$

Proof. The formula can be checked for monomials X^n, by induction on n, using Itô's product formula of Theorem 4.1.2. The general case follows by linearity. □

Observe that, with the obvious notation $dM_s = U_s \sharp\, dX_s + K_s\, ds$, we can rewrite the formula

$$P(M_t) = P(M_0) + \int_0^t \partial P(M_s)\sharp\, dM_s + 1/2 \int_0^t \Delta_{U_s} P(M_s)\, ds$$

We shall now further extend the functional form of Itô's formula, to functions of the class considered in section 1.2. A similar idea has been used by G. F. Vincent-Smith in order to state Itô's formula for Hudson-Parthasarathy integrals, see [VS].

For this we shall need the following extension of Duhamel's formula

Lemma 4.3.3. *Let $\mathscr{M}$ be a Banach algebra with unit, and $\mathscr{B}$ be a Banach space, then for every continuous multilinear map $\varphi\colon \mathscr{M}^k \to \mathscr{B}$, one has*

$$\sum_{n=k}^{\infty} \frac{z^n}{n!} \sum_{\substack{n_1,\cdots,n_k \in \mathbb{N} \\ n_1+\cdots+n_k=n-k}} \varphi(a_1^{n_1}, \ldots, a_k^{n_k})$$

$$= z^k \int \cdots \int_{\substack{\alpha_1,\ldots,\alpha_{k-1} \geq 0 \\ \alpha_1+\ldots+\alpha_{k-1} \leq 1}} \varphi(\mathrm{e}^{\alpha_1 z a_1}, \ldots, \mathrm{e}^{\alpha_{k-1} z a_{k-1}}, \mathrm{e}^{(1-\alpha_1\cdots-\alpha_{k-1}) z a_k})\, d\alpha_1 \cdots d\alpha_{k-1}$$

Proof. We can expand the right hand side in powers of z. Interchanging integration and summation is easily justified by uniform estimates, so we are left with the evaluation of the integral

$$\int \cdots \int_{\substack{\alpha_1,\ldots,\alpha_{k-1} \geq 0 \\ \alpha_1+\ldots+\alpha_{k-1} \leq 1}} \alpha_1^{n_1} \ldots \alpha_{k-1}^{n_{k-1}} (1 - \alpha_1 - \ldots - \alpha_{k-1})^{n_k} d\alpha_1 \ldots d\alpha_{k-1}$$

$$= \frac{n_1! \ldots n_k!}{(n_1 + \cdots + n_k + k)!}$$

which is a classical result. □

Let $f(x) = \int_{\mathbb{R}} \mathrm{e}^{ixy} \mu(dy)$ with $\mathscr{I}_2(f) < \infty$, then for any self-adjoint $X \in \mathscr{A}$ one can define

$$\partial f(X) = \int_0^1 \int_{\mathbb{R}} \mathrm{i}y (\mathrm{e}^{\mathrm{i}\alpha y X} \otimes \mathrm{e}^{\mathrm{i}(1-\alpha) y X}) \mu(dy)\, d\alpha$$

For $U \in \mathscr{A} \otimes \mathscr{A}$, let

$$\Delta_U f(X) = \int \int_{\substack{\alpha,\beta \geq 0 \\ \alpha+\beta \leq 1}} \int_{\mathbb{R}} -y^2 \eta((1 \otimes U) \sharp (\mathrm{e}^{\mathrm{i}\alpha y X} \otimes \mathrm{e}^{\mathrm{i}\beta y X} \otimes \mathrm{e}^{\mathrm{i}(1-\alpha-\beta) y X})$$
$$\sharp (U \otimes 1)) \mu(dy)\, d\alpha\, d\beta$$

The first integral converges in $L^\infty(\tau \otimes \tau^{\mathrm{op}})$ and the second in $L^\infty(\tau)$. One can check again that the map $U \mapsto \Delta_U f(X)$ extends continuously to $U \in L^\infty(\tau \otimes \tau^{\mathrm{op}})$. Note that any polynomial P coincides on the spectrum of a self-adjoint X with some function f with $\mathscr{I}_2(f) < \infty$, and that the formulas for $\partial f(X)$ and $\Delta_U f(X)$ define the same elements as $\partial P(X)$ and $\Delta_U P(X)$.

Proposition 4.3.4. *Suppose that $U_s = U_s^*$ and $K_s = K_s^*$ for all $s \geq 0$, and $M_0 = M_0^*$, so that $M_t = M_t^*$ for all $t \geq 0$. Then one has, for all functions f with $\mathscr{I}_2(f) < \infty$*

$$f(M_t) = f(M_0) + \int_0^t (\partial f(M_s) \sharp U_s) \sharp\, dX_s$$
$$+ \int_0^t \partial f(M_s) \sharp K_s\, ds + 1/2 \int_0^t \Delta_{U_s} f(M_s)\, ds$$

Proof. For $y \in \mathbb{R}$, expand $e^{iyM_t} = \sum_{n=0}^{\infty} \frac{(iy)^n}{n!} M_t^n$ and apply the Lemma 4.3.3 twice, as well as Itô's formula for M_t^n, to obtain the result when $\mu = \delta_y$. The result follows for arbitrary functions with $\mathscr{I}_2(f) < \infty$ by integration. All exchanges of summation and integration are easily justified using the Burkholder-Gundy inequalities. $\square$

Let us restate Proposition 4.3.4 in the special important case where $U_s = (1 \otimes 1)1_{[0,T]}(s)$, i.e. $M_t = M_0 + X_t + \int_0^t K_s\, ds$ for $t \leq T$. If we denote by v_s the distribution of the self-adjoint element M_s, namely v_s is characterized by

$$\int_{\mathbb{R}} h(y) v_s(dy) = \tau[h(M_s)]$$

for bounded Borel functions h, and we let $\Delta_s f$ be the function

$$\Delta_s f(x) = \frac{\partial}{\partial x} \int_{\mathbb{R}} \frac{f(x) - f(y)}{x - y} v_s(dy)$$

then we have

$$f(M_t) = f(M_0) + \int_0^t \partial f(M_s) \sharp\, dM_s + 1/2 \int_0^t \Delta_s f(M_s)\, ds$$

The Itô correction term $1/2 \int_0^t \Delta_s f(M_s)\, ds$ turns out to be nicely connected with free entropy. We shall say more about these topics somewhere else.

5. Analysis on Wigner space

In this section we shall develop the first elements of the natural analogues of many results known commonly under the name of "analysis on Wiener space" (see e.g. [J], [M], [N], [U]). We shall start with an abstract version which corresponds to considering a semi-circular system modelled on an abstract Hilbert space. Then we shall investigate in more details the case where the Hilbert space is $L^2(\mathbb{R}_+)$ and the semi-circular system is the free Brownian motion. Most of our presentation is inspired by, and follows quite closely, the book of Nualart [N].

5.1. Free Fock space

Let H be a real Hilbert space with complexification $H_{\mathbb{C}}$, and $F(H)$ the associated free Fock space.

$$F(H) = \bigoplus_{n=0}^{\infty} H_{\mathbb{C}}^{\otimes n}$$

where $H_{\mathbb{C}}^{\otimes 0}$ is by definition the one dimensional Hilbert space generated by a unit vector Ω.

For each $h \in H_{\mathbb{C}}$, we let l_h and l_h^* be the left annihilation and creation operators defined as

$$l_h \Omega = 0$$
$$l_h h_1 \otimes \cdots \otimes h_n = \langle h_1, h \rangle h_2 \otimes \cdots \otimes h_n$$
$$l_h^* h_1 \otimes \cdots \otimes h_n = h \otimes h_1 \otimes \cdots \otimes h_n$$

For each $h \in H_{\mathbb{C}}$, l_h and l_h^* are bounded operators and adjoint of each other on $F(H)$. For each $h \in H$ we let $X(h) = l_h + l_h^*$. Let $\mathscr{SC}(H)$ be the von Neumann algebra of operators on $F(H^{\mathbb{C}})$ generated by $X(h)_{h \in H}$, and let τ be the restriction to $\mathscr{SC}(H)$ of the pure state associated to the vector Ω, i.e. $\tau(T) = \langle T\Omega, \Omega \rangle$ for $T \in \mathscr{SC}(H)$. Then the state τ is a faithful normal trace on $\mathscr{SC}(H)$ and the operators $\{X(h); h \in H\}$ form a semi-circular system in the non-commutative probability space $(\mathscr{SC}(H), \tau)$, in the sense of Voiculescu, see e.g. [VDN].

Furthermore, let $(T_k)_{k=0}^{\infty}$ be the Tchebycheff polynomials, determined by $T_0(x) = 1$, $T_1(x) = x$, and by the recursion $(k \geq 1)$

$$x T_k(x) = T_{k+1}(x) + T_{k-1}(x),$$

and let $(e_j)_{j=1}^{\dim(H)}$ be an orthonormal basis of H. Then for any choice of integers $k_1, \ldots, k_n$ and $j_1, \ldots, j_n$ such that $j_1 \neq j_2 \neq j_3 \cdots \neq j_{n-1} \neq j_n$, one has

$$T_{k_1}(X(e_{j_1})) T_{k_2}(X(e_{j_2})) \cdots T_{k_n}(X(e_{j_n}))\Omega = e_{j_1}^{\otimes k_1} \otimes \cdots \otimes e_{j_n}^{\otimes k_n}$$

see e.g. [VDN].

The map $X \mapsto X\Omega$ extends to a unitary isomorphism from $L^2(\mathscr{SC}(H), \tau)$ to $F(H^{\mathbb{C}})$.

There exists a free analogue of the second quantization functor, for which the analogue of Nelson's hypercontractivity estimates hold. For this compare [B2].

We define now the free analogue of the classical gradient and divergence operator on Wiener space.

Definition 5.1.1. *We define the gradient operator*

$$\nabla : F(H) \to F(H) \otimes H \otimes F(H) \ ,$$

with domain the algebraic sum $\bigoplus_{n=0}^{\infty} H^{\otimes n}$, *by*

$$\nabla \Omega = 0$$

$$\nabla h_1 \otimes \cdots \otimes h_n = \sum_{j=1}^{n} (h_1 \otimes \cdots \otimes h_{j-1}) \otimes h_j \otimes (h_{j+1} \otimes \cdots \otimes h_n)$$

Definition 5.1.2. *We define the divergence operator*

$$\delta : F(H) \otimes H \otimes F(H) \to F(H),$$

with domain the algebraic sum $\bigoplus_{n,m\geq 0}(H^{\otimes n} \otimes H \otimes H^{\otimes m})$, by

$$\delta\big((h_1 \otimes \cdots \otimes h_{j-1}) \otimes h_j \otimes (h_{j+1} \otimes \cdots \otimes h_n)\big) = h_1 \otimes \cdots \otimes h_n$$

It is easy to check that $\langle \nabla u, v\rangle = \langle u, \delta v\rangle$ for all $u \in \bigoplus_{n=0}^{\infty} H^{\otimes n}$, and for all $v \in \bigoplus_{n,m\geq 0}(H^{\otimes n} \otimes H \otimes H^{\otimes m})$, so that ∇ and δ are closable and their closures are mutually adjoint. In the following we will denote by $D(\nabla)$ and $D(\delta)$ the domains of the closures of ∇ and δ, respectively.

In the classical approach, although the gradient operator on Wiener space is a purely Hilbertian object, determined in terms of the Fock space structure only, it is often defined using derivatives of Wiener functionals along directions in the Cameron Martin space. The fundamental property of the gradient operator which makes these two definitions coincide is that the gradient operator is a derivation with respect to the product structure on Fock space induced by the probabilistic representation as a space of Wiener functionals. It turns out that there is a similar property of the gradient operator on Wigner space with respect to the underlying semi-circular system. This derivation property of ∇ will be presented in the next section.

5.2. The gradient operator

In the following we will suppress in our notation the dependence on the Hilbert space H and just write $\mathcal{SC} = \mathcal{SC}(H)$.

Up to now, we have looked upon ∇ as a mapping on the full Fock space. Since the latter can be identified with the L^2-space $L^2(\mathcal{SC})$, we can also consider the restriction of ∇ to the L^p-spaces for $p \geq 2$. Our particular emphasis will be on the case $p = \infty$, i.e. we can also view ∇ as acting on the operator algebra $\mathcal{SC}$, where we take as domain the image of the algebraic sum $\bigoplus_{n=0}^{\infty} H^{\otimes n}$ under the map $A \mapsto A\Omega$, i.e. the space

$$\mathcal{SC}_{\text{polynom}} := \text{ unital } *\text{-algebra generated by all } X(h), h \in H$$

which is dense in all the L^p spaces, for $p < \infty$ (it is only weakly dense in L^∞). We consider now $\mathcal{SC}_{\text{polynom}} \otimes H \otimes \mathcal{SC}_{\text{polynom}}$ and $\mathcal{SC}_{\text{polynom}}$

$\otimes\mathscr{SC}_{\text{polynom}}$ as a $\mathscr{SC}_{\text{polynom}}$-bimodule in a canonical way, namely by linear extension of

$$A_1 \cdot (B_1 \otimes h \otimes B_2) \cdot A_2 := (A_1 B_1) \otimes h \otimes (B_2 A_2)$$

and

$$A_1 \cdot (B_1 \otimes B_2) \cdot A_2 := (A_1 B_1) \otimes (B_2 A_2).$$

Then we can formulate the derivation property of ∇.

Proposition 5.2.1. 1) *The mapping*

$$\nabla : \mathscr{SC}_{\text{polynom}} \to \mathscr{SC}_{\text{polynom}} \otimes H \otimes \mathscr{SC}_{\text{polynom}}$$

is a derivation, i.e. we have

$$\nabla(AB) = A \cdot (\nabla B) + (\nabla A) \cdot B \qquad \text{for all}\, A, B \in \mathscr{SC}_{\text{polynom}}.$$

2) *In particular, we have for all $n \in \mathbb{N}$ and all $h_1, \ldots, h_n \in H$ the formula*

$$\nabla(X(h_1)\cdots X(h_n)) = \sum_{j=1}^{n} (X(h_1)\cdots X(h_{j-1})) \otimes h_j \otimes (X(h_{j+1})\cdots X(h_n)).$$

Proof. 1) Let (e_j) be an orthonormal basis of H. Since linear combinations of all $T_{k_1}(X(e_{j_1}))\cdots T_{k_n}(X(e_{j_n}))$ for $j_1 \neq j_2 \neq \cdots \neq j_n$ generate $\mathscr{SC}_{\text{polynom}}$, it suffices to prove the assertion for the product of two operators of the above form. Furthermore, it suffices to consider the case $A = X(e_i)$ for some i. One sees easily that this reduces further to the consideration of $\nabla(X(e_i)T_k(X(e_j)))$. If $i \neq j$ then the statement follows directly by the definition of ∇. For $i = j$, on the other side, one can calculate explicitely that one has with $T_k := T_k(X(e))$ for $\|e\| = 1$

$$\nabla(X(e)T_k) = X(e) \cdot \nabla(T_k) + \nabla(X(e)) \cdot T_k,$$

by using the recursion formula for the T_k.

2) Part 2 follows immediately from Part 1 by taking into account $\nabla(X(h)) = \nabla(T_1(X(h))) = 1 \otimes h \otimes 1$. □

We will now consider the question whether ∇ is closable as an operator on the L^p-spaces and what can be said about its domain. For this we will need the following technical lemma.

We will extend the scalar product on H in a canonical way also to a pairing of $\mathscr{SC} \otimes H \otimes \mathscr{SC}$ with H, which takes values in $\mathscr{SC} \otimes \mathscr{SC}$, by linear extension of

$$\langle A \otimes h_1 \otimes B, h_2 \rangle_H = A \otimes B \cdot \langle h_1, h_2 \rangle.$$

Lemma 5.2.2. 1) *For* $Y \in \mathscr{SC}_{\text{polynom}}$ *and* $h \in H$ *we have*

$$\tau \otimes \tau[\langle \nabla Y, h \rangle_H] = \tau[YX(h)].$$

2) *For* $Y, Y_1, Y_2 \in \mathscr{SC}_{\text{polynom}}$ *and* $h \in H$ *we have*

$$\begin{aligned}\tau \otimes \tau[Y_1 \cdot \langle \nabla Y, h \rangle_H \cdot Y_2] = \tau[Y_1 Y Y_2 X(h)] - \tau \otimes \tau[Y_1 Y \cdot \langle \nabla Y_2, h \rangle_H] \\ - \tau \otimes \tau[\langle \nabla Y_1, h \rangle_H \cdot Y Y_2]\end{aligned}$$

Proof. It suffices to consider Y of the form $Y = X(h_1) \cdots X(h_n)$. Then it is well-known that the expectation of $YX(h)$ can be written recursively in the following way

$$\begin{aligned}&\tau[X(h_1) \cdots X(h_n) X(h)] \\ &= \sum_{j=1}^{n} \langle h_j, h \rangle \tau[X(h_1) \cdots X(h_{j-1})] \tau[X(h_{j+1}) \cdots X(h_n)].\end{aligned}$$

By using the concrete form of ∇Y according to Proposition 5.2.1 this gives exactly the left-hand side of our assertion.
2) Just apply the first part to $Y_1 Y Y_2$ and use the derivation property of ∇. $\square$

From now on we will specialize to the case where $H = L^2(\mathbb{R}_+)$. In this case, the process $X_t = X(1_{[0,t]})$ is a free Brownian motion. For $Y \in \mathscr{SC}_{\text{polynom}}$ the gradient ∇Y can be considered as a function on $\mathbb{R}_+$, $t \mapsto \nabla_t Y$, with values in $\mathscr{SC} \otimes \mathscr{SC}$. Thus ∇Y is a biprocess.

We can consider ∇Y with respect to the $\mathscr{B}_p$ norms, yielding the following family of norms for elements $Y \in \mathscr{SC}_{\text{polynom}}$:

$$\|Y\|_{1,p} := \left(\|Y\|^p_{L^p(\tau)} + \|\nabla Y\|^p_{\mathscr{B}_p}\right)^{1/p}$$

and

$$\|Y\|_{1,\infty} := \max(\|Y\|, \|\nabla Y\|_{\mathscr{B}_\infty}).$$

Let us denote by $\mathbb{D}^p$ the closure of $\mathscr{SC}_{\text{polynom}}$ with respect to $\|\cdot\|_{1,p}$.

Note that the pairing between $\mathscr{SC} \otimes H \otimes \mathscr{SC}$ and H, which reads now

$$\langle U, h \rangle = \int_{\mathbb{R}_+} U_t \overline{h(t)}\, dt,$$

can, by the Cauchy-Schwarz inequality

$$\|\langle U, h \rangle\|_{L^\infty(\tau \otimes \tau^{\text{op}})} \leq \|U\|_{\mathscr{B}_\infty} \|h\|_{L^2(\mathbb{R}_+)},$$

be extended continuously to a pairing between $\mathscr{B}_\infty$ and $L^2(\mathbb{R}_+)$ with values in $L^\infty(\tau \otimes \tau^{\text{op}})$, where τ is the trace on $\mathscr{SC}$ (and in the same way for the L^p-spaces).

Proposition 5.2.3. *Let $1 \le p < \infty$, then the operator ∇ is closable as an operator from $L^p(\mathcal{SC})$ to $\mathcal{B}_p$ and the domain of its closure is $\mathbb{D}^p$.*

The case $p = \infty$ is slightly more subtle since the closure of $\mathcal{SC}_{\text{polynom}}$ for the L^∞ norm is not $L^\infty(\tau)$, but rather the C^*-algebra generated by $\mathcal{SC}_{\text{polynom}}$.

Proposition 5.2.4. *The operator ∇ is closable on $C^*(\mathcal{SC}_{\text{polynom}})$, and also on $L^\infty(\tau)$ for the weak topology.*

Proof of 5.2.3 and 5.2.4. Assume we have a sequence of elements $Y_n \in \mathcal{SC}_{\text{polynom}}$ such that $Y_n \to 0$ in L^p (resp. weakly), and $\nabla Y_n \to U$ in $\mathcal{B}_\infty$. We have to show that $U = 0$. It will suffice to show

$$\tau \otimes \tau[Z_1 \cdot \langle U, h\rangle \cdot Z_2] = 0$$
$$\text{for all } Z_1, Z_2 \in \mathcal{SC}_{\text{polynom}} \text{ and all } h \in L^2(\mathbb{R}_+).$$

By Lemma 5.2.2, we have

$$\begin{aligned}
\tau\otimes\tau&[Z_1 \cdot \langle U, h\rangle \cdot Z_2] \\
&= \lim_{n\to\infty} \tau \otimes \tau[Z_1 \cdot \langle \nabla Y_n, h\rangle_H \cdot Z_2] \\
&= \lim_{n\to\infty} (\tau[Z_1 Y_n Z_2 X(h)] - \tau \otimes \tau[Z_1 Y_n \cdot \langle \nabla Z_2, h\rangle_H] \\
&\quad - \tau \otimes \tau[\langle \nabla Z_1, h\rangle_H \cdot Y_n Z_2]) = 0
\end{aligned}$$

since $Y_n \to 0$ in L^p (resp. weakly). □

5.3. Multiple stochastic integrals and chaotic decomposition

We now investigate the case where $H = L^2(\mathbb{R}_+)$. In this case, the process $X_t = X(1_{[0,t]})$ is a free Brownian motion. We shall define stochastic integrals

$$\int f(t_1, \ldots, t_n)\, dX_{t_1} \ldots dX_{t_n}$$

for $f \in L^2(\mathbb{R}_+^n) \cong L^2(\mathbb{R}_+)^{\otimes n}$. This will give an explicit description of the isometry $F(H) \cong L^2(\mathcal{SC}(L^2(\mathbb{R}_+)))$ (chaotic decomposition). Using this isometry we can give more concrete formulas for the gradient and divergence operators.

Definition 5.3.1. *Let $D^n \subset \mathbb{R}_+^n$ be the collection of all diagonals, i.e.*

$$D^n := \{(t_1, \ldots, t_n) \in \mathbb{R}_+^n \mid t_i = t_j \textit{ for some} 1 \le i, j \le n \textit{ with } i \ne j\}.$$

For a characteristic function $f = 1_A$ with $A \subset \mathbb{R}_+^n$ of the form

$$A = [u_1, v_1] \times \cdots \times [u_n, v_n]$$

with $A \cap D^n = \emptyset$, we define the multiple stochastic integral

$$I(f) = \int f(t_1, \ldots, t_n)\, dX_{t_1} \cdots dX_{t_n}$$

by

$$I(f) := (X_{v_1} - X_{u_1}) \cdots (X_{v_n} - X_{u_n})$$

and extend this by linearity to simple functions of the form

$$f = \sum_{i=1}^{k} \alpha_i 1_{A_i} \ ,$$

where

$$A_i = [u_1^i, v_1^i] \times \cdots \times [u_n^i, v_n^i]$$

are disjoint n-dimensional rectangles as above which do not meet the diagonals.

A simple computation shows that

$$\langle I(f), I(g) \rangle_{L^2(\mathcal{SC})} = \langle f, g \rangle_{L^2(\mathbb{R}_+^n)}.$$

Since each $f \in L^2(\mathbb{R}_+^n)$ can be approximated in L^2-norm by functions of the above form we can extend the definition of $I(f) = \int f(t_1, \ldots, t_n)\, dX_{t_1} \cdots dX_{t_n}$ to all $f \in L^2(\mathbb{R}_+^n)$.

Let now $f = \oplus_{n=0}^{\infty} f_n \in \bigoplus_{n=0}^{\infty} L^2(\mathbb{R}_+^n) \cong F(L^2(\mathbb{R}_+))$. Then one easily sees that

$$\langle I(f_n), I(f_m) \rangle_{L^2(\mathcal{SC})} = 0 \ \text{ for } n \neq m,$$

and thus we have

$$\left\| \sum_{n=0}^{\infty} I(f_n) \right\|_{L^2(\mathcal{SC})}^2 = \sum_{n=0}^{\infty} \|f_n\|_{L^2(\mathbb{R}_+^n)}^2 = \|f\|_{F(L^2(\mathbb{R}_+))}^2 \ .$$

In this way we can assign to each $f = \oplus f_n \in F(L^2(\mathbb{R}_+))$ the multiple integral

$$I(f) := \sum_{n=0}^{\infty} I(f_n) = \sum_{n=0}^{\infty} \int f_n(t_1, \ldots, t_n)\ dX_{t_1} \cdots dX_{t_n} \in L^2(\mathcal{SC}).$$

Proposition 5.3.2. *The map*

$$\begin{aligned} I \colon F(L^2(\mathbb{R}_+)) &\to L^2(\mathcal{SC}) \\ f &\mapsto I(f) \end{aligned}$$

is determined by

$$I(f)\Omega = f \ ;$$

and the map I is an isomorphism.

Proof. One sees directly that the first statement is true for simple functions as used in the definition of $I(f)$, and thus it extends to all $f \in F(L^2(\mathbb{R}_+))$. As we argued before, in Section 5.1, by using Tchebycheff polynomials, this map has also to be onto, i.e. it is an isomorphism. □

This isomorphism yields the "chaos" decomposition of the space $L^2(\mathscr{S}\mathscr{C})$: Each element in $L^2(\mathscr{S}\mathscr{C})$ can be represented in a unique way as a multiple integral

$$I(f) = \sum_{n=0}^{\infty} \int f_n(t_1, \ldots, t_n)\, dX_{t_1} \cdots dX_{t_n}, \quad \text{with } f = \oplus f_n \in F(L^2(\mathbb{R}_+)).$$

The Itô-formula in this frame is now just a rule for expressing the product of multiple integrals as a sum of multiple integrals.

Notation. For functions $f \in L^2(\mathbb{R}_+^n)$ and $g \in L^2(\mathbb{R}_+^m)$ we denote by

$$f \overset{p}{\frown} g \in L^2(\mathbb{R}^{n+m-2p}) \quad \text{for } 0 \le p \le \min(n, m)$$

the functions given by

$$\begin{aligned}
&(f \overset{p}{\frown} g)(t_1, \ldots, t_{n+m-2p}) \\
&\quad := \int f(t_1, \ldots, t_{n-p}, s_p, \ldots, s_1) g(s_1, \ldots, s_p, t_{n-p+1}, \ldots, t_{n+m-2p}) \\
&\qquad \times ds_1 \ldots ds_p
\end{aligned}$$

In particular, for $p = 0$, we have $f \overset{0}{\frown} g = f \otimes g$.

Proposition 5.3.3. *For $f \in L^2(\mathbb{R}_+^n)$ and $g \in L^2(\mathbb{R}_+^m)$ we have*

$$I(f)I(g) = \sum_{p=0}^{\min(n,m)} I(f \overset{p}{\frown} g).$$

Proof. Just check for simple functions as in the definition of multiple integrals. □

We will now compare the L^∞-norm of a multiple integral with its L^2-norm. This result, which is a semi-circular version of Haagerup's inequality [H], is due to Bożejko [Boz], but we will reprove that statement here in a more combinatorial way-very much in the same spirit as our proof of the Burkholder-Gundy inequality.

Theorem 5.3.4. *For $f \in L^2(\mathbb{R}_+^n)$ we have*

$$\left\| \int f(t_1, \ldots, t_n)\, dX_{t_1} \cdots dX_{t_n} \right\| \leq (n+1)\|f\|_2 \ .$$

Proof. We use

$$\|I(f)\| = \lim_{m\to\infty} \left(\tau[(I(f)I(f)^*)^m]\right)^{1/2m} \ .$$

We note that for calculating the moment $\tau[(I(f)I(f)^*)^m]$ we have to pair all arguments of the appearing functions in such a way, that the pairing is non-crossing and that no arguments within each f are paired. Each such pairing contributes to the moment an nm-fold integral over $f^{\otimes 2m}$ where, according to the pairing, pairs of arguments of that function are identified and integrated; by an iterated application of Cauchy-Schwarz this integral can be estimated against $\|f\|_2^{2m}$ - independently of the pairing. Let us denote by d_m^n the number of non-crossing pairings of $2mn$ numbers which fulfill the constraint that we do not pair within any of the $2m$ sets $\{1, \ldots, n\}$,..., $\{(2m-1)n+1, \ldots, 2mn\}$. (For $n=1$ we have $d_m^1 = C_m$.) Then the constant we will get for our norm estimate is given by $\lim_{m\to\infty}(d_m^n)^{1/2m}$. We have not been able to calculate this number by combinatorial means, but we can recover this limit by looking at the above problem for a special choice of f. Namely, if we take $f = (1_{[0,1]})^{\otimes n}$, then $I(f)$ will be nothing but $T_n(X_1)$, where T_n is the n-th Tchebycheff polynomial and X_1 is the semi-circular variable of variance 1. But as a special case of the above reasoning we obtain that

$$\lim_{m\to\infty} (d_m^n)^{1/2m} = \|T_n(X_1)\| \ .$$

On the other side, this norm of $T_n(X_1)$ can be calculated as

$$\|T_n(X_1)\| = \sup_{|t|\leq 2} |T_n(t)| \ ,$$

which is equal to $n+1$ – as follows directly from the concrete representation

$$T_n(2\cos\theta) = \frac{\sin(n+1)\theta}{\sin\theta} \ . \qquad \square$$

In order to get a representation of square-integrable biprocesses in terms of multiple integrals we have to extend our notion of multiple integrals to “bi-multiple” integrals – essentially this means that we work with $I \otimes I$ instead of just I.

Notation. For a function $f \in L^2(\mathbb{R}_+; L^2(\mathbb{R}_+^n) \otimes L^2(\mathbb{R}_+^m))$,

$$t \mapsto f_t \in L^2(\mathbb{R}_+^n) \otimes L^2(\mathbb{R}_+^m) \cong L^2(\mathbb{R}_+^{n+m}),$$

we denote

$$(I \otimes I)(f) \in L^2(\mathbb{R}_+; L^2(\mathscr{SC}) \otimes L^2(\mathscr{SC})) = \mathscr{B}_2,$$

corresponding to the process

$$t \mapsto (I \otimes I)(f_t) \in L^2(\mathscr{SC}) \otimes L^2(\mathscr{SC}),$$

also by

$$(I \otimes I)(f_t) =: \int f_t(t_1, \dots, t_n, s_1, \dots, s_m)\, dX_{t_1} \cdots dX_{t_n} \otimes\, dX_{s_1} \cdots dX_{s_m}.$$

For clarity, we will also write sometimes $I^{(n)} \otimes I^{(m)}$ instead of $I \otimes I$. Because of the properties of I, it is clear that

$$\langle (I \otimes I)(f), (I \otimes I)(g) \rangle_{\mathscr{B}_2} = \langle f, g \rangle_{L^2(\mathbb{R}_+; L^2(\mathbb{R}_+^n) \otimes L^2(\mathbb{R}_+^m))},$$

and that for

$$f^{(n_i, m_i)} \in L^2(\mathbb{R}_+; L^2(\mathbb{R}_+^{n_i}) \otimes L^2(\mathbb{R}_+^{m_i})) \quad (i = 1, 2)$$

we have

$$\langle (I \otimes I)(f^{(n_1, m_1)}), (I \otimes I)(f^{(n_2, m_2)}) \rangle_{\mathscr{B}_2} = 0, \quad \text{ if } n_1 \neq n_2 \text{ or } m_1 \neq m_2.$$

Thus we obtain for

$$f = \oplus_{n,m=0}^{\infty} f^{(n,m)} \in L^2(\mathbb{R}_+; F(L^2(\mathbb{R}_+)) \otimes F(L^2(\mathbb{R}_+)))$$

with

$$f^{(n,m)} \in L^2(\mathbb{R}_+; L^2(\mathbb{R}_+^n) \otimes L^2(\mathbb{R}_+^m)) \ .$$

that

$$\left\| \sum_{n,m=0}^{\infty} (I \otimes I)(f^{(n,m)}) \right\|_{\mathscr{B}_2}^2 = \sum_{n,m=0}^{\infty} \|f^{(n,m)}\|^2 = \|f\| \ ,$$

and thus $I \otimes I$ gives an isometry

$$I \otimes I \colon L^2(\mathbb{R}_+; F(L^2(\mathbb{R}_+)) \otimes F(L^2(\mathbb{R}_+))) \to \mathscr{B}_2$$

$$f = \bigoplus_{n,m=0}^{\infty} f^{(n,m)} \mapsto (I \otimes I)(f) := \sum_{n,m=0}^{\infty} (I \otimes I)(f^{(n,m)}).$$

The following proposition is clear from the corresponding properties of I.

Proposition 5.3.5. *The map*

$$I \otimes I \colon L^2(\mathbb{R}_+; F(L^2(\mathbb{R}_+)) \otimes F(L^2(\mathbb{R}_+))) \to \mathscr{B}_2$$

is uniquely determined by

$$(I \otimes I)(f_t)(\Omega \otimes \Omega) = f_t;$$

and $I \otimes I$ is an isomorphism.

Definition 5.3.6. *We call a process*

$$f = \bigoplus_{n,m \geq 0} f^{(n,m)} \in L^2(\mathbb{R}_+; F(L^2(\mathbb{R}_+)) \otimes F(L^2(\mathbb{R}_+)))$$

adapted, if the following holds (almost surely): For all $n, m \in \mathbb{N}$ with $n + m \geq 1$ we have

$$f_t^{(n,m)}(t_1, \ldots, t_n, s_1, \ldots, s_m) = 0 \quad \textit{if} \max(t_1, \ldots, t_n, s_1, \ldots, s_m) > t.$$

The proof of the following proposition is straightforward.

Proposition 5.3.7. 1) *For a process $f \in L^2(\mathbb{R}_+; F(L^2(\mathbb{R}_+)) \otimes F(L^2(\mathbb{R}_+)))$ the following statements are equivalent:*

a) f is adapted.
b) $(I \otimes I)(f)$ is an adapted biprocess, i.e. $(I \otimes I)(f) \in \mathscr{B}_2^a$.
2) *Let $f = \oplus f^{(n,m)} \in L^2(\mathbb{R}_+; F(L^2(\mathbb{R}_+)) \otimes F(L^2(\mathbb{R}_+)))$ be adapted. Then we have*

$$\begin{aligned}
&\int (I \otimes I)(f_t) \sharp\, dX_t \\
&= \sum_{n,m=0}^{\infty} \int \left(\int f_t^{(n,m)}(t_1, \ldots, t_n, s_1, \ldots, s_m)\, dX_{t_1} \cdots dX_{t_n} \otimes\, dX_{s_1} \cdots dX_{s_m} \right) \sharp\, dX_t \\
&= \sum_{n,m=0}^{\infty} f_t^{(n,m)}(t_1, \ldots, t_n, s_1, \ldots, s_m)\, dX_{t_1} \cdots\, dX_{t_n} dX_t\, dX_{s_1} \cdots dX_{s_m}.
\end{aligned}$$

Theorem 5.3.8 (predictable representation theorem). *For every element $Y \in L^2(\mathscr{S}\mathscr{C})$, there exists a unique adapted $U \in \mathscr{B}_2^a$ such that*

$$Y = \tau[Y] + \int_0^\infty U_s \sharp\, dX_s$$

Proof. By the chaos decomposition, each $Y \in L^2(\mathscr{S}\mathscr{C})$ is of the form $Y = I(f)$ for some $f = \oplus f^{(n)} \in F(L^2(\mathbb{R}_+))$. It will be sufficient to consider the case $f = f^{(n)}$ for fixed n. Then we define processes $(k = 0, \ldots, n-1)$

$$t \mapsto f_t^{(k,n-k-1)} \in L^2(\mathbb{R}_+^k) \otimes L^2(\mathbb{R}_+^{n-k-1})$$

by

$$f_t^{(k,n-k-1)}(t_1,\ldots,t_k,s_1,\ldots,s_{n-k-1}) := \begin{cases} f(t_1,\ldots,t_k,t,s_1,\ldots,s_{n-k-1}), & \text{if } \max(t_1,\ldots t_k,s_1\ldots,s_{n-k-1}) \le t \\ 0, & \text{otherwise} \end{cases}$$

It is clear that all $f^{(k,n-k-1)}$ are adapted and that with

$$U_t := \sum_{k=0}^{n-1} (I^{(k)} \otimes I^{(n-k-1)})(f_t^{(k,n-k-1)})$$

we have

$$Y = I(f) = \int U_t \sharp\, dX_t.$$

Uniqueness follows from the isometry property. □

In terms of multiple integrals the gradient and the divergence operator can be written in a more concrete form. The statement is clear for simple functions and the general case follows by approximation.

Proposition 5.3.9. 1) *The gradient operator* ∇ *considered as a mapping*

$$\nabla : L^2(\mathcal{SC}) \to \mathcal{B}_2$$

is given by

$$\nabla_t\left(\int f(t_1,\ldots,t_n) dX_{t_1}\cdots dX_{t_n}\right) = \sum_{k=1}^{n} \int f(t_1,\ldots,t_{k-1},t,t_{k+1},\ldots,t_n)\, dX_{t_1}\cdots dX_{t_{k-1}} \otimes dX_{t_{k+1}}\cdots dX_{t_n}.$$

2) *The divergence operator* δ *considered as a mapping*

$$\delta : \mathcal{B}_2 \to L^2(\mathcal{SC})$$

is given by

$$\delta\left(\int f_t(t_1,\ldots,t_n,s_1,\ldots,s_m)\, dX_{t_1}\cdots dX_{t_n} \otimes dX_{s_1}\cdots dX_{s_m}\right) = \int f_t(t_1,\ldots,t_n,s_1,\ldots,s_m)\, dX_{t_1}\ldots dX_{t_n}\, dX_t\, dX_{s_1}\cdots dX_{s_m}.$$

By using the chaos decomposition for an element in $L^2(\mathcal{SC})$ it is easy to describe the domain $D(\nabla) = \mathbb{D}^2$ and the action of ∇ on that space. The proof is again straightforward.

Proposition 5.3.10. *Let $Y \in L^2(\mathscr{S}\mathscr{C})$ have chaos decomposition $Y = \sum_{n=0}^{\infty} I(f_n)$ with $f_n \in L^2(\mathbb{R}_+^n)$. Then*

$$\int_{\mathbb{R}_+} \|\nabla_t Y\|_2^2 dt = \sum_{n=0}^{\infty} n \|f_n\|^2_{L^2(\mathbb{R}_+^n)}$$

and Y belongs to $\mathbb{D}^2$ if and only if the latter sum converges to a finite value.

Note also that for functions $\varphi\colon \mathbb{R} \to \mathbb{R}$ with $\mathscr{I}_1(\varphi) < \infty$ we have a kind of chain rule for ∇: If $Y \in \mathbb{D}^\infty$, then

$$\nabla_t \varphi(Y) = \partial\varphi(Y) \sharp \nabla_t Y$$

with ∂ as defined in Sect. 4.3.

The concrete form of δ shows that the divergence operator can, as in the classical case, be seen as a kind of stochastic integral (the Skorohod integral). Indeed, by combining Propositions 5.3.7 and 5.3.9, we get that δ is a canonical generalization of the Itô-integral.

Proposition 5.3.11. *For an adapted biprocess $U \in \mathscr{B}_2^a$, one has $U \in D(\delta)$ and $\int_0^\infty U_s \sharp \, dX_s = \delta(U)$.*

This Proposition justifies to call δ the free Skorohod integral.

Proof. We write U in the form $U_t = (I \otimes I)(f_t)$ for an adapted $f = \oplus f^{(n,m)} \in L^2(\mathbb{R}_+; F(L^2(\mathbb{R}_+)) \otimes F(L^2(\mathbb{R}_+)))$. Then we have

$$\begin{aligned}
\int U_t \sharp \, dX_t &= \int (I \otimes I)(f_t) \sharp \, dX_t \\
&= \sum_{n,m=0}^{\infty} f_t^{(n,m)}(t_1, \ldots, t_n, s_1, \ldots, s_m) \, dX_{t_1} \cdots dX_{t_n} \, dX_t \, dX_{s_1} \cdots dX_{s_m} \\
&= \sum_{n,m=0}^{\infty} \delta\big((I^{(n)} \otimes I^{(m)})(f_t^{(n,m)})\big) \\
&= \delta\big((I \otimes I)(f_t)\big) \\
&= \delta(U) \ .
\end{aligned}$$

Since $\int U_t \sharp \, dX_t \in L^2(\mathscr{S}\mathscr{C})$, this equality also shows that U belongs to the domain of δ. $\square$

We also have a free analogue of the Bismut-Clark-Ocone formula. Denote by $\Gamma\colon \mathscr{B}_2 \to \mathscr{B}_2^a$ the orthogonal projection onto the space of square integrable adapted bi-processes.

Proposition 5.3.12 (free Bismut-Clark-Ocone formula). *For any $Y \in \mathbb{D}^2$, one has*

$$Y = \tau[Y] + \delta(\Gamma\nabla Y)$$

Proof. For $Y = I(f^{(n)})$ with $f^{(n)} \in L^2(\mathbb{R}_+^n)$ we obtain

$$\begin{aligned}
&\Gamma\nabla_t I(f^{(n)}) \\
&\quad = \sum_{k=1}^{n} \int_{t \geq \max\{t_i | i \neq k\}} f(t_1, \ldots, t_{k-1}, t, t_{k+1}, \ldots, t_n) \\
&\quad\quad \times dX_{t_1} \cdots dX_{t_{k-1}} \otimes dX_{t_{k+1}} \cdots dX_{t_n}
\end{aligned}$$

yielding

$$\begin{aligned}
\delta(\Gamma\nabla I(f^{(n)})) &= \sum_{k=1}^{n} \int_{t_k = \max(t_1, \ldots, t_n)} f(t_1, \ldots, t_n)\, dX_{t_1} \cdots dX_{t_n} \\
&= \int f(t_1, \ldots, t_n)\, dX_{t_1} \cdots dX_{t_n}. \qquad \square
\end{aligned}$$

The fact that stochastic integrals define martingales is again true for generalized stochastic integrals, namely one has the following.

Proposition 5.3.13 (martingale representation theorem). *A map $t \mapsto M_t$ from $\mathbb{R}_+$ to $L^2(\mathscr{S}\mathscr{C})$ is a martingale bounded in $L^2(\mathscr{S}\mathscr{C})$ if and only if there exists an adapted $U \in \mathscr{B}_2^a$ and a $M_0 \in \mathbb{C}$, such that $M_t = M_0 + \int_0^t U_s \sharp\, dX_s$ for all $t \geq 0$.*

Proof. By the predictable representation theorem, for all $T \geq 0$, there exists a $U^{(T)} \in \mathscr{B}_2^a$ such that $M_T = M_0 + \int U_s^{(T)} \sharp\, dX_s$. Since for $t \leq T$

$$M_t = \tau[M_T | \mathscr{A}_t] = M_0 + \int_0^t U_s^{(T)} \sharp\, dX_s,$$

we see that $U_t^{(T)}$ is for $t \leq T$ independent of T. Put $U := \lim_{T \to \infty} U^{(T)}$. Since

$$\begin{aligned}
\infty > \sup_{T \geq 0} \|M_T\|^2_{L^2(\mathscr{S}\mathscr{C})} &\geq \|M_t\|^2_{L^2(\mathscr{S}\mathscr{C})} \\
&= |M_0|^2 + \int_0^t \tau[\langle U_s, U_s \rangle] ds = \|U\|^2_{\mathscr{B}_2},
\end{aligned}$$

we have $\int_0^\infty \tau[\langle U_s, U_s \rangle]\, ds < \infty$ and thus $U \in \mathscr{B}_2$. Adaptedness of U is clear. This establishes the necessary condition, and the converse is easy. $\square$

5.4. The Skorohod integral

As seen before, the Skorohod integral

$$\delta\colon \mathscr{B}_2 \to L^2(\mathscr{S}\mathscr{C})$$

is the adjoint of the gradient operator, i.e. we have

$$\tau[\delta(U)Y] = \int_{\mathbb{R}_+} \tau\otimes\tau[U_t \nabla_t Y]\,dt \quad \text{for } U \in D(\delta) \text{ and } Y \in D(\nabla)\ .$$

More important than the natural domain $D(\delta)$ of the Skorohod integral will be a special subclass. To define this class we have to extend the action of

$$\nabla\colon L^2(\mathscr{S}\mathscr{C}) = L^2(\mathbb{R}^0_+; L^2(\mathscr{S}\mathscr{C})) \to \mathscr{B}_2 = L^2(\mathbb{R}_+; L^2(\mathscr{S}\mathscr{C})\otimes L^2(\mathscr{S}\mathscr{C}))$$

to

$$\begin{aligned}\nabla\colon \mathscr{B}_2 &= L^2(\mathbb{R}_+; L^2(\mathscr{S}\mathscr{C})\otimes L^2(\mathscr{S}\mathscr{C}))\\ &\to L^2(\mathbb{R}^2_+; L^2(\mathscr{S}\mathscr{C})\otimes L^2(\mathscr{S}\mathscr{C})\otimes L^2(\mathscr{S}\mathscr{C}))\end{aligned}$$

in the canonical way: For $U_t = A_t \otimes B_t$ we put

$$\nabla_s(A_t\otimes B_t) := (\nabla_s A_t)\otimes B_t + A_t \otimes (\nabla_s B_t).$$

The relevant subclass for our considerations is now given by the domain of this version of ∇.

Notation. We denote by $\mathbb{L}^2$ the class of biprocesses $U \in \mathscr{B}_2$ such that

$$\nabla U \in L^2(\mathbb{R}^2_+; L^2(\mathscr{S}\mathscr{C})\otimes L^2(\mathscr{S}\mathscr{C})\otimes L^2(\mathscr{S}\mathscr{C})).$$

$\mathbb{L}^2$ is a Hilbert space with the norm

$$\|U\|^2_{\mathbb{L}^2} := \|U\|^2_{\mathscr{B}_2} + \|\nabla U\|^2_{L^2(\mathbb{R}^2_+;L^2(\mathscr{S}\mathscr{C})\otimes L^2(\mathscr{S}\mathscr{C})\otimes L^2(\mathscr{S}\mathscr{C}))}.$$

On a formal level, ∇ and δ fulfill the Heisenberg commutation relations.

Proposition 5.4.1. *For $U \in D(\delta)$ with finite chaos expansion we have*

$$\nabla_t(\delta(U)) = U_t + \delta_s(\nabla_t U_s),$$

where the subscript at δ_s indicates that the Skorohod integration acts in the variable s.

Proof. It suffices to consider U of the form

$$U_t = (I^{(n)}\otimes I^{(m)})(f_t) = \int f_t(t_1,\dots,t_n,s_1,\dots,s_m)\,dX_{t_1}\cdots dX_{t_n}\otimes dX_{s_1}\cdots dX_{s_m}$$

with $f \in L^2(\mathbb{R}_+; L^2(\mathbb{R}^n_+)\otimes L^2(\mathbb{R}^m_+))$. Then we have

$$\nabla_t(\delta(U)) = \nabla_t\left(\int f_s(t_1,\ldots,t_n,s_1,\ldots,s_m)\,dX_{t_1}\cdots dX_{t_n}\,dX_s dX_{s_1}\cdots dX_{s_m}\right)$$

$$= U_t + \sum_{k=1}^{n}\int f_s(t_1,\ldots,t_{k-1},t,t_{k+1},\ldots,s_m)\,dX_{t_1}\cdots dX_{t_{k-1}}\otimes dX_{t_{k+1}}\cdots dX_{s_m}$$

$$+\sum_{k=1}^{m}\int f_s(t_1,\ldots,s_{k-1},t,s_{k+1},\ldots,s_m)\,dX_{t_1}\cdots dX_{s_{k-1}}\otimes dX_{s_{k+1}}\cdots dX_{s_m}$$

$$= U_t + \delta_s(\nabla_t U_s).$$

□

This commutativity relation between ∇ and δ yields the following relation for the covariance between two Skorohod integrals.

Proposition 5.4.2. 1) *We have* $\mathbb{L}^2 \subset D(\delta)$. 2) *Let* $U, V \in \mathbb{L}^2$. *Then we have*

$$\tau[\delta(U)\delta(V)] = \int_{\mathbb{R}_+}(\tau\otimes\tau)[U_tV_t]dt + \int_{\mathbb{R}_+^2}(\tau\otimes\tau\otimes\tau)[(\nabla_sU_t)(\nabla_tU_s)]\,ds\,dt.$$

Proof. We have for $U, V \in \mathscr{B}_2$ with finite chaos expansion

$$\begin{aligned}\tau[\delta(U)\delta(V)] &= \int_{\mathbb{R}_+}(\tau\otimes\tau)[U_t(\nabla_t(\delta(V))]\,dt\\ &= \int_{\mathbb{R}_+}(\tau\otimes\tau)[U_tV_t]\,dt + \int_{\mathbb{R}_+}(\tau\otimes\tau)[U_t\delta_s(\nabla_tV_s)]\,dt\\ &= \int_{\mathbb{R}_+}(\tau\otimes\tau)[U_tV_t]\,dt + \int_{\mathbb{R}_+^2}(\tau\otimes\tau\otimes\tau)[(\nabla_sU_t)(\nabla_tV_s)]\,dt\,ds.\end{aligned}$$

Put now $V = U^*$. The right hand side of the above formula extends continuously to all $U \in \mathbb{L}^2$. This implies that $\tau[\delta(U)\delta(U)^*]$ is finite, i.e. $U \in D(\delta)$, for all $U \in \mathbb{L}^2$. Hence we get the first part of the proposition. The second part follows by continuous extension of the above formula to all $U, V \in \mathbb{L}^2$. □

Note that Proposition 5.4.2 represents the generalization of the Ito isometry (Proposition 3.1.1) from the case of the Itô integral to the Skorohod integral. For adapted biprocesses the second term in the above formula vanishes.

It is conceivable that there should also exist analogues of the above formula for the other L^p-spaces. In view of our Burkholder-Gundy inequality one might suspect that we even have an estimate in operator norm for the Skorohod integral involving the gradient operator. In the classical case such estimates rely, for $p < \infty$, on so-called

Meyer's inequalities. It would be interesting to find an analogue of this for our case. Up to now, we could not prove such an estimate.

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