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Flow of diffeomorphisms induced by a geometric multiplicative functional

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Abstract. In this paper it is shown that the unique multiplicative functional solution to a differential equation driven by a geometric multiplicative functional consitutes a flow of local diffeomorphisms. In the case where the driving geometric multiplicative functional is generated by a Brownian motion, the result in particular presents an answer to an open problem proposed in Ikeda and Watanabe [4].

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0. Introduction

The study of solution flows of deterministic dynamical systems is a well developed topic. Given N vector fields $A_1, \ldots, A_N$ on the euclidean space $\mathbb{R}^d$, we consider a differential equation:

$$dY_t = \sum_{i=1}^{N} A_i(Y_t)\, dX_t^i; \quad Y_0 = x \ , \tag{1}$$

where $X_t = (X_t^i)$ is a continuous path in $\mathbb{R}^N$ as a driving force. If the path $t \to X_t$ is smooth and each $A_i \in C_b^\infty(\mathbb{R}^d)$, then Eq.(1) possesses a unique solution, denoted by $F_t(X, x)$. We call the map $X \to F_\cdot(X, \cdot)$ the Itô map associated to vector fields $A_1, \ldots, A_N$, although the name usually has been used only for semimartingale paths.

If we fix a smooth path X and regard the unique solution $F_t(X, x)$ as a map

$$F_t(X,\cdot)\colon \mathbb{R}^d \to \mathbb{R}^d, \quad x \to F_t(X,x) \ , \tag{2}$$

then a fundamental theorem in the theory of ordinary differential equations asserts that the map $F_t(X,\cdot)$ is smooth, its tangent map $F_t(X,\cdot)*(x)$ is non-degenerate, and therefore $(F_t(X,\cdot))$ is a flow of diffeomorphisms. A similar result has been established for some stochastic dynamical systems. For example, if $t \to (X_t^i)$ is an N dimensional Brownian motion, then there is a solution flow $(F_t(X,x))$ to Eq.(1). Indeed such results have been established for more general stochastic processes, see Elworthy [2, 3], Ikeda and Watanabe [4], and Kunita [5, 6] for more details and more references.

The present paper aims to describe a new approach to constructing a (stochastic) flow of diffeomorphisms. In fact what we show is that the unique multiplicative functional solution to a differential equation driven by a geometric multiplicative functional constructed in Lyons [9] forms a flow of local diffeomorphisms. If the driving path X_t is a Brownian motion (or more generally, a continuous semimartingale), then we provide a precise version of the solution flow to a stochastic dynamical system. However our result can be applied to a more general rough path.

We next describe the setting of this paper. The concept of a multiplicative functional as a genuine rough path has been proposed in Lyons [7, 8, 9], and a calculus for multiplicative functionals has been established in [9].

Note that Itô map $F_t(X,x)$ obtained by solving Eq. (1) depends essentially on the interpretation of a differential dX we give to a path X. For example, if X is a Brownian motion, then Itô differential and Stratonovich differential lead to totally different Itô maps.

Let X be a continuous path in a vector space V. If we are going to define a kind of path integral of a 1-form α along the path X, $\int \alpha(X)\, dX$, it seems reasonable that one can also define iterated integrals $\mathbf{X}_{st}^k$ of X:

$$\mathbf{X}_{st}^k = \int_{s<t_1<\cdots<t_k<t} dX_{t_1} \otimes \cdots \otimes dX_{t_k} \ . \tag{3}$$

In fact it suffices to define $\mathbf{X}^k$, $k \le p$, where p is a constant relating to the roughness of the path X, the higher order iterated integrals are uniquely determined by $\mathbf{X}^k$, $k \le p$, and therefore one may define $\int \alpha(X) dX$ for any α which is smooth enough. For example, if X is a continuous and piece-wisely smooth path, then $\mathbf{X}_{st}^1 = X_t - X_s$, and consider $\mathbf{X}_{st}^k$ ($k \ge 2$) to be the conventional iterated integrals. That is, $\mathbf{X}_{st}^k$ is recursively defined by

$$\mathbf{X}_{st}^k = \lim_{m(D)\to 0} \sum_{l=1}^m \sum_{\substack{i+j=k \\ i,j\ge 1}} \mathbf{X}_{st_{l-1}}^i \otimes \mathbf{X}_{t_{l-1}t_l}^j \ , \tag{4}$$

where $D = \{s = t_0 \le \cdots \le t_m = t\}$, and we regard $\mathbf{X}_{st}^k$ as a tensor in the tensor product $V^{\otimes k}$. Therefore we only need to know $\mathbf{X}_{st}^1$, that is, the path $t \to X_t$ itself. Moreover, such choices of $\mathbf{X}_{st}^k$ lead to the following scaling control:

$$\sup_D \sum_l \left|\mathbf{X}_{t_{l-1}t_l}^k\right|^{\frac{1}{k}} < \infty, \quad k = 1, 2, \ldots, \tag{5}$$

where sup takes over all finite dissection D of $[s, t]$.

An advantage of using tensor form of $\mathbf{X}_{st}^k$ is that the basic property of any reasonable integral can be expressed via K.T.Chen formula (see [1]):

$$\mathbf{X}_{st} \otimes \mathbf{X}_{tu} = \mathbf{X}_{su}, \quad \forall\, 0 \le s \le t \le u \ , \tag{6}$$

where we set

$$\mathbf{X}_{st} = (1, \mathbf{X}_{st}^1, \ldots, \mathbf{X}_{st}^n), \quad 0 \le s \le t \ ,$$

and regard it as an element in the truncated tensor algebra $T^{(n)}(V)$:

$$T^{(n)}(V) = \sum_{k=0}^n \oplus V^{\otimes k}, \qquad V^{\otimes 0} = \mathbb{R} \ .$$

It is easily seen that if $\mathbf{X}_{st} = (1, \mathbf{X}_{st}^1, \ldots, \mathbf{X}_{st}^n)$ satisfies the analytic condition (5) and the algebraic relation (6), then $\mathbf{X}_{st}^2, \ldots, \mathbf{X}_{st}^n$ are uniquely determined by $\mathbf{X}_{st}^1$, that is, by the path $t \to X_t$ itself.

However, most rough paths we are interested rarely satisfy the analytic condition (5) even for $\mathbf{X}_{st}^1$. For example, almost all sample paths of a Brownian motion do not satisfy (5). On the other hand, almost all Brownian motion paths X are $\frac{1}{p}$-Hölder continuous for any $p > 2$ (but not for $p = 2$), and therefore

$$\sup_D \sum_l \left|\mathbf{X}_{t_{l-1}t_l}^1\right|^p < \infty \ .$$

Hence if X is a rough path, if we were able to define a kind of path integral, and therefore the iterated integrals $\mathbf{X}_{st}^k$ make sense, then by scaling property we expect that $\mathbf{X}_{st}^k$ satisfy a weaker analytic condition that

$$\sup_D \sum_l \left|\mathbf{X}_{t_{l-1}t_l}^k\right|^{\frac{p}{k}} < \infty \tag{7}$$

for some $p \ge 1$, even (5) does not hold.

It is shown in Lyons [9] that if $\mathbf{X}_{st} = (1, \mathbf{X}^1_{st}, \ldots, \mathbf{X}^n_{st})$ satisfies (6) and (7), then $\mathbf{X}^k_{st}$ $(k \geq [p]+1)$ are uniquely determined by $\mathbf{X}^1_{st}, \ldots, \mathbf{X}^{[p]}_{st}$. Moreover, given $\mathbf{X}^1_{st}$ (that is, a path $t \to X_t$), then there are many different choices of $\mathbf{X}^2_{st}, \ldots, \mathbf{X}^{[p]}_{st}$, such that $\mathbf{X}_{st} = (1, \mathbf{X}^1_{st}, \ldots, \mathbf{X}^{[p]}_{st})$ satisfies the algebraic relation (6) and analytic condition (7).

Following Lyons [9], a genuine rough path is a combination $(\mathbf{X}^1_{st}, \ldots, \mathbf{X}^{[p]}_{st})$ satisfying (6) and (7). We call such a combination a multiplicative functional with finite p-variation by an obvious reason (see section 1 below for a precise definition).

The paper is organised as following. In Sect. 1, we collect several results about multiplicative functionals and establish notations as well. Also we add several new results about multiplicative functionals. In Sect. 2 we show that the procedure of iterating the integrals of a 1-form gives the unique solution to a differential equation driven by a multiplicative functional. In the final Sect. 3, we prove that the unique multiplicative functional solution of a differential equation driven by a geometric multiplicative functional is smooth in initial date, and forms a flow of diffeomorphisms provided the vector field is smooth enough.

1. Integration

In this section we recall several basic facts about multiplicative functionals, and establish notations as well, for more details, see Lyons [9]. Given a $T > 0$, we will use I to denote the interval $[0, T]$, and $\triangle$ to denote the set $\{(s,t): 0 \leq s \leq t \leq T\}$.

A continuous function ω on $\triangle$ with values in $\mathbb{R}^+$ is called a control function if

$$\omega(s,t) + \omega(t,u) \leq \omega(s,u), \quad \forall\ (s,t),(t,u) \in \triangle\ ,$$

and ω is regular, that is, $\omega(s,s) = 0$ for all $s \in I$.

Given a real and separable Banach space V, we use $T^{(n)}(V)$ to denote the truncated tensor algebra over V of degree n:

$$T^{(n)}(V) = \sum_{k=0}^{n} \oplus V^{\otimes k}, \quad V^{\otimes 0} = \mathbb{R}\ .$$

Each $V^{\otimes k}$ $(k \leq n)$ is endowed with any but fixed compatible Banach tensor norm.

We say a map $\mathbf{X}: \triangle \to T^{(n)}(V)$ is of *finite p-variation* controlled by a regular control function ω if $\mathbf{X}_{st} = \left(1, \mathbf{X}^1_{st}, \ldots, \mathbf{X}^n_{st}\right)$, $\mathbf{X}^k_{st} \in V^{\otimes k}$, and

$$|\mathbf{X}_{st}^i| \le \frac{\omega(s,t)^{\frac{i}{p}}}{\beta\left(\frac{i}{p}\right)}, \quad \forall\ (s,t) \in \triangle, \quad i = 1, \ldots, n \ , \tag{8}$$

where β is a fixed positive constant depending only on p. For the precise value of β, see Lyons [9]. Such a function $\mathbf{X}$ is called an *almost multiplicative functional* with finite p-variation controlled by ω if in addition $n \ge [p]$ and

$$\left|(\mathbf{X}_{st} \otimes \mathbf{X}_{tu})^i - \mathbf{X}_{su}^i\right| \le K_1 \omega(s,t)^\theta, \quad \forall\ (s,t) \in \triangle, \quad i = 1, \ldots, [p] \tag{9}$$

for some constants K_1, $\theta > 1$.

A functional $\mathbf{X}: \triangle \to T^{(n)}(V)$ is called a *multiplicative functional* (of degree n) if $\mathbf{X}_{st} = (1, \mathbf{X}_{st}^1, \ldots, \mathbf{X}_{st}^n)$ and $\mathbf{X}$ satisfies K. T. Chen formula (see [1]),

$$\mathbf{X}_{st} \otimes \mathbf{X}_{tu} = \mathbf{X}_{su}, \quad \forall\ (s,t), (t,u) \in \triangle \ . \tag{10}$$

If $\mathbf{X}: \triangle \to T^{(n)}(V)$ is a multiplicative functional of degree n, and $\mathbf{X}_{st} = (1, \mathbf{X}_{st}^1, \ldots, \mathbf{X}_{st}^n)$, then we say $\mathbf{X}^1$ is the first level path, $\mathbf{X}^2$ is the second level path and etc. In this case, we use either X_t or X_t^1 to denote $\mathbf{X}_{0t}^1$, unless otherwise specified.

A calculus for multiplicative functionals has been established in Lyons [9]. Here we recall several results in [9] we need later.

Let $\mathbf{X}: \triangle \to T^{(n)}(V)$ be a multiplicative functional of degree n. Then

$$\mathbf{X}_{st}^k = \sum_{l=1}^m \left(\mathbf{X}_{t_{l-1}t_l}^k + \sum_{\substack{i+j=k \\ i,j\ge 1}} \mathbf{X}_{st_{l-1}}^i \otimes \mathbf{X}_{t_{l-1}t_l}^j\right), \quad k = 1, \ldots, n \ ,$$

for any finite dissection $D = \{s = t_0 \le t_1 \le \cdots \le t_m = t\}$ of $[s,t]$.

If in addition $\mathbf{X}$ is of finite p-variation and $n \ge [p]$, then it is shown in Lyons [9] that the following limits exist,

$$\mathbf{X}_{st}^k = \lim_{m(D)\to 0} \sum_{l=1}^m \sum_{\substack{i+j=k \\ i,j\ge 1}} \mathbf{X}_{st_{l-1}}^i \otimes \mathbf{X}_{t_{l-1}t_l}^j, \quad \forall\ (s,t) \in \triangle,\ k = [p]+1, \ldots,$$

$m(D) = \max_l (t_l - t_{l-1})$. Moreover

$$\tilde{\mathbf{X}}_{st} = (1, \mathbf{X}_{st}^1, \ldots, \mathbf{X}_{st}^n)$$

is the unique extension among those $\tilde{\mathbf{X}}$ which satisfies K.T.Chen formula and possesses finite p-variation for each $n > [p]$, and the extension $\mathbf{X} \to \tilde{\mathbf{X}}$ is continuous in the following sense: if $\mathbf{X}, \mathbf{Y}$ are two multiplicative functionals with finite p-variation controlled by ω, and

$$\left|\mathbf{X}^i_{st} - \mathbf{Y}^i_{st}\right| \le \varepsilon\omega(s,t)^{\frac{i}{p}}, \quad \forall\ (s,t) \in \triangle,\ i = 1, \ldots, [p] \tag{11}$$

for $\varepsilon \le \varepsilon_0$, where $\varepsilon_0 > 0$ is a fixed constant depending only on p, then the inequality (11) remains true for any i.

The following is a result in Lyons [9] which shows how to construct a multiplicative functional from an almost one.

Theorem 1 (Lyons [9]). *Let* $\mathbf{X}: \triangle \to T^{(n)}(V)$ *be an almost multiplicative functional with finite p-variation* (*so that* $n \ge [p]$). *Then there is a unique multiplicative functional* $\hat{\mathbf{X}}$ *with finite p-variation, such that*

$$\left|\hat{\mathbf{X}}^i_{st} - \mathbf{X}^i_{st}\right| \le K_2\omega(s,t)^{\theta}, \quad \forall\ (s,t) \in \triangle,\ i = 1, \ldots, [p]\ ,$$

for some control function ω, *constants* K_2, $\theta > 1$. *Moreover, the map* $\mathbf{X} \to \hat{\mathbf{X}}$ *is continuous in the following sense: if* $\mathbf{X}, \mathbf{Y}$ *are two almost multiplicative functionals with finite p-variation controlled by* ω *and if*

$$\left|\mathbf{X}^i_{st} - \mathbf{Y}^i_{st}\right| \le \varepsilon\omega(s,t), \quad \forall\ (s,t) \in \triangle, i = 1, \ldots, [p]\ ,$$

then

$$\left|\hat{\mathbf{X}}^i_{st} - \hat{\mathbf{Y}}^i_{st}\right| \le K_3K(\varepsilon)\omega(s,t)^{\frac{i}{p}}, \quad \forall\ (s,t) \in \triangle, \quad i = 1, \ldots, [p]\ ,$$

for some constants $K(\varepsilon)$ *depending only on* p, θ *and* $\varepsilon > 0$ *such that* $\lim_{\varepsilon \to 0} K(\varepsilon) = 0$, *and* K_3 *depending on* p, $\max\omega$, θ.

Indeed $\hat{\mathbf{X}}^k$ can be obtained by the following recursive definition:

$$\hat{\mathbf{X}}^k_{st} = \lim_{m(D)\to 0} \sum_l \left\{\mathbf{X}^k_{t_{l-1}t_l} + \sum_{\substack{i+j=k \\ i,j\ge 1}} \hat{\mathbf{X}}^i_{st_{l-1}} \otimes \hat{\mathbf{X}}^j_{t_{l-1}t_l}\right\}, \quad \forall\ (s,t) \in \triangle\ , \tag{12}$$

$k = 1, \ldots, [p]$.

Definition 1. 1) *A multiplicative functional* $\mathbf{X}: \triangle \to T^{(n)}(V)$ *is called a classical multiplicative functional, if* $t \to X_t = \mathbf{X}^1_{0t}$ *is continuous and piece-wisely smooth, and*

$$\mathbf{X}^k_{st} = \int_{s<t_1<\cdots<t_k<t} dX^1_{t_1} \otimes \cdots \otimes dX^1_{t_k}\ ,$$

where the right hand side is in the sense of the conventional integral.

2) *Let* $\mathbf{X}: \triangle \to T^{(n)}(V)$ *be a multiplicative functional with finite p-variation. We say it is geometric if there is a control function* ω *such that* $\mathbf{X}$ *is of finite p-variation controlled by* ω, *and for any* $\varepsilon > 0$ *there is a classical multiplicative functional* $\mathbf{X}(\varepsilon): \triangle \to T^{(n)}(V)$ *which is of finite p-variation controlled by* ω, *and*

$$\left|\mathbf{X}(\varepsilon)^i_{st} - \mathbf{X}^i_{st}\right| \le \varepsilon\omega(s,t)^{\frac{i}{p}}, \quad \forall (s,t) \in \triangle, \quad i = 1,\dots,[p] \ .$$

It is easily seen that if $\mathbf{X}$ is a geometric multiplicative functional, then the symmetric part of $\mathbf{X}^i_{st}$ is $\frac{1}{i!}(\mathbf{X}^1_{st})^{\otimes i}$ for each i.

Example 1 If W is a continuous semimartingale on a probability space $(\Omega,\ \mathrm{P})$, then we can prove that

$$\tilde{\mathbf{W}}(\sigma)_{st} = \left(1, \mathbf{W}(\sigma)^1_{st}, \mathbf{W}(\sigma)^2_{st}\right)$$

is a geometric multiplicative functional with finite p-variation for any $2 < p < 3$ and for almost all $\sigma \in \Omega$, where $\mathbf{W}^1_{st} = W_t - W_s$ and

$$\mathbf{W}^2_{st} = \int_{s<t_1<t_2<t} \circ dW_{t_1} \otimes \circ dW_{t_2} \ ,$$

where the right hand side is in the sense of Stratonovich's integral.

Let V, U be two real, separable Banach spaces. Then we will use $\mathrm{hom}(V, U)$ to denote the Banach space of all continuous linear operators from V to U endowed with the usual operator norm. A map $f: V \to U$ is called a $\mathrm{Lip}(r)$ map, if all j-th derivatives f^j up to degree $[r]$ exist and

$$f^j(x_t)(v) = \sum_{i+j\le[r]} f^{i+j}(x_s)(x^i_{st} \otimes v) + R_j(x_s, x_t)(v), \quad \forall\ v \in V^{\otimes j} \ ,$$

$$|f^j(x)| \le M, \quad |R_j(x,y)| \le M|x-y|^{r-j} \ , \tag{13}$$

for $j = 0, \dots, [r]$, and any smooth path $t \to x_t$, where M is a positive constant (called Lipschitz constant of f), and

$$x^k_{st} = \int_{s<t_1<\cdots<t_k<t} dx_{t_1} \otimes \cdots \otimes dx_{t_k}, \ \ k = 1, \dots,$$

where the right hand side is in the sense of the conventional iterated integrals.

Since $f^j \in \mathrm{hom}(V^{\otimes j}, U)$ is symmetric, so that we can replace x^i_{st} by its symmetric part $\frac{1}{i!}(x^1_{st})^{\otimes i}$ in (13). Hence we have

Proposition 1. *If* $f: V \to U$ *is a* $\mathrm{Lip}(r)$ *map, and* $\mathbf{X}: \triangle \to T^{(n)}(V)$ *is a geometric multiplicative functional of degree* n, $n \ge [r]$, *then*

$$f^j(X_t)(v) = \sum_{i+j\le[r]} f^{i+j}(X_s)(\mathbf{X}^i_{st} \otimes v) + R_j(X_s, X_t)(v), \quad \forall\ v \in V^{\otimes j} \ ,$$

$$\textit{for } j = 0, \dots, [r]. \tag{14}$$

In the sequel, for simplicity, we will assume that $2 \le p < 3$. Example 1 shows that our next discussion can be applied to almost all sample paths of a continuous semimartingale.

Proposition 2. *Let* $f\colon V \to U$ be a $\mathrm{Lip}(r)$ *map, and let* $r > 2$. *Then*

$$f(y) - f(x) - f^1(x)(y-x) = F(x,y)\left(\tfrac{1}{2}(y-x)^{\otimes 2}\right) ,$$

and

$$f^1(y) - f^1(x) = G(x,y)(y-x)$$

for any $x, y \in V$, *where*

$$F{:}V \oplus V \to \hom(V^{\otimes 2}, U),$$
$$G{:}V \oplus V \to \hom(V, \hom(V, U))$$

are two Lip$(r-2)$ *maps.*

A map $\alpha\colon V \to \hom(V, W)$ is called a W-valued 1-form on V. We make the following convention. Be the definition, α^i is a map from V to $\hom(V^{\otimes i}, \hom(V, W))$ which we identify as $\hom(V^{\otimes(i+1)}, W)$, and we regard α^i as a map (and use the same notation) which takes values in $\hom(V^{\otimes(i+1)}, W)$ by

$$\alpha^i(x)(\xi_1 \otimes \cdots \otimes \xi_{i+1}) = \alpha^i(x)(\xi_1)(\xi_2 \otimes \cdots \otimes \xi_{i+1}) .$$

Given a multiplicative functional with finite p-variation $\mathbf{X}\colon \triangle \to T^{(2)}(V)$, and a 1-form $\alpha\colon V \to \hom(V, W)$ which is of $\mathrm{Lip}(r)$, $r > 1$. Then following Lyons [9] we define a functional $\mathbf{Y}\colon \triangle \to T^{(2)}(W)$ by

$$\mathbf{Y}_{st} = \left(1, \mathbf{Y}^1_{st}, \mathbf{Y}^2_{st}\right) ,$$
$$\mathbf{Y}^1_{st} = \alpha(X_s)(\mathbf{X}^1_{st}) + \alpha^1(X_s)(\mathbf{X}^2_{st}) ,$$
$$\mathbf{Y}^2_{st} = \alpha(X_s) \otimes \alpha(X_s)(\mathbf{X}^2_{st}) .$$

It is shown in Lyons [9] that if $\frac{r+1}{p} > 1$, then $\mathbf{Y}\colon \triangle \to T^{(2)}(W)$ is an almost multiplicative functional with finite p-variation. The associated multiplicative functional $\hat{\mathbf{Y}}$ is called the integral of the 1-form α against the multiplicative functional $\mathbf{X}$, and denoted by $\int \alpha(X)\delta\mathbf{X}$. For simplicity we will use $\int_s^t \alpha(X)\delta\mathbf{X}$ to denote $(\int \alpha(X)\delta\mathbf{X})_{st}$ and $\int_s^t \alpha(X)\delta\mathbf{X}^i$ to denote the i-th component $(\int \alpha(X)\delta\mathbf{X})^i_{st}$, respectively.

Proposition 3 (Lyons [9]). *There is a* $K_M > 0$ *depending only on* M, r, p, *where* $\frac{r+1}{p} > 1$, *such that if* $\alpha\colon V \to \hom(V, W)$ *is a* $\mathrm{Lip}(r)$ *map with a Lipschitz constant* M, *if* $\mathbf{X}\colon \triangle \to T^{(2)}(V)$ *is a geometric multiplicative functional with finite p-variation controlled by* ω, *and if* $\omega \le 1$ *on* $\triangle$, *then*

$$\left|\int_s^t \alpha(X)\delta\mathbf{X}^i\right| \leq \frac{[K_M\omega(s,t)]^{\frac{i}{p}}}{\beta\left(\frac{i}{p}\right)!}, \quad \forall (s,t) \in \triangle, \quad i = 1,2 \ . \tag{15}$$

The following two propositions are crucial in our next development. In what follows, we will use K (with or without a lowerscript) to denote a constant which may be different from line to line.

Proposition 4. *Let* $\mathbf{X}, \tilde{\mathbf{X}}: \triangle \to T^{(2)}(V)$ *be two geometric multiplicative functionals with finite p-variation controlled by* ω, *and let* $\alpha: V \to \hom(V,W)$ *be a* $\mathrm{Lip}(r)$ 1*-form*, $p < r \leq 3$. *If*

$$\left|\mathbf{X}_{st}^i - \tilde{\mathbf{X}}_{st}^i\right| \leq \varepsilon\omega(s,t)^{\frac{i}{p}}, \qquad \forall\ (s,t) \in \triangle, \ \ i = 1,2,$$

where $\varepsilon \leq 1$, *then*

$$\begin{aligned}\left|\left[(\mathbf{Z}_{st}\otimes\mathbf{Z}_{tu})^1 - \mathbf{Z}_{su}^1\right] - \left[(\tilde{\mathbf{Z}}_{st}\otimes\tilde{\mathbf{Z}}_{tu})^1 - \tilde{\mathbf{Z}}_{su}^1\right]\right| \\ \leq K\varepsilon^{r-2}\omega(s,u)^{\frac{r}{p}} \ ,\end{aligned} \tag{16}$$

and

$$\begin{aligned}\left|\left(\int_s^t \alpha(X)\delta\mathbf{X}^1 - \mathbf{Z}_{st}^1\right) - \left(\int_s^t \alpha(\tilde{X})\delta\tilde{\mathbf{X}}^1 - \tilde{\mathbf{Z}}_{st}^1\right)\right| \\ \leq K\varepsilon^{r-2}\omega(s,t)^{\frac{r}{p}}\end{aligned} \tag{17}$$

for $(s,t),(t,u) \in \triangle$, *where* K *is a constant depending only on* $\max\omega$, *p*, *r*, *and Lipschitz constant*, *and* $\mathbf{Z}_{st} = (1, \mathbf{Z}_{st}^1, \mathbf{Z}_{st}^2)$,

$$\begin{aligned}\mathbf{Z}_{st}^1 &= \alpha(X_s)(\mathbf{X}_{st}^1) + \alpha^1(X_s)(\mathbf{X}_{st}^2), \\ \mathbf{Z}_{st}^2 &= \alpha(X_s)\otimes\alpha(X_s)(\mathbf{X}_{st}^2) \ ,\end{aligned}$$

and similarly to $\tilde{\mathbf{Z}}$.

Proof. A simple calculation shows that

$$\begin{aligned}(\mathbf{Z}_{st}\otimes\mathbf{Z}_{tu})^1 - \mathbf{Z}_{su}^1 = &\left[\alpha(X_t) - \alpha(X_s) - \alpha^1(X_s)(\mathbf{X}_{st}^1)\right](\mathbf{X}_{tu}^1) \\ &+ \left[\alpha^1(X_t) - \alpha^1(X_s)\right](\mathbf{X}_{tu}^2) \ ,\end{aligned}$$

and a similar equality for $\tilde{\mathbf{Z}}$, so that

$$\begin{aligned}&\left[(\mathbf{Z}_{st}\otimes\mathbf{Z}_{tu})^1 - \mathbf{Z}_{su}^1\right] - \left[(\tilde{\mathbf{Z}}_{st}\otimes\tilde{\mathbf{Z}}_{su})^1 - \tilde{\mathbf{Z}}_{su}^1\right] \\ &\quad = (H - \tilde{H})(\mathbf{X}_{tu}^1) \\ &\qquad + \left[\alpha(\tilde{X}_t) - \alpha(\tilde{X}_s) - \alpha^1(\tilde{X}_s)(\tilde{\mathbf{X}}_{st}^1)\right](\mathbf{X}_{tu}^1 - \tilde{\mathbf{X}}_{tu}^1) \\ &\qquad + (E - \tilde{E})(\mathbf{X}_{tu}^2) + \left[\alpha^1(\tilde{X}_t) - \alpha^1(\tilde{X}_s)\right](\mathbf{X}_{tu}^2 - \tilde{\mathbf{X}}_{tu}^2) \ ,\end{aligned} \tag{18}$$

where for simplicity, we have used the following notations:

$$H = \alpha(X_t) - \alpha(X_s) - \alpha^1(X_s)(\mathbf{X}_{st}^1),$$
$$E = \alpha^1(X_t) - \alpha^1(X_s) \ ,$$

and similarly for $\tilde{H}$, and $\tilde{E}$. We first estimate the first term of the right hand side in (18). By Prop. 2 we let

$$\alpha(y) - \alpha(x) - \alpha^1(x)(y-x) = F(x,y)\left(\tfrac{1}{2}(y-x)^{\otimes 2}\right) \ ,$$

where $F: V \oplus V \to \hom(V^{\otimes 2}, \hom(V,W))$ is a Lip$(r-2)$ map. Since $\tilde{\mathbf{X}}$, $\mathbf{X}$ are geometric multiplicative functionals, so that

$$\begin{aligned} H &= F(X_s, X_t)(\mathbf{X}_{st}^2), \\ \tilde{H} &= F(\tilde{X}_s, \tilde{X}_t)(\tilde{\mathbf{X}}_{st}^2) \ . \end{aligned} \tag{19}$$

However,

$$F(X_s, X_t) - F(\tilde{X}_s, \tilde{X}_t) = R_F((\tilde{X}_s, \tilde{X}_t), (X_s, X_t)) \ , \tag{20}$$

where R_F denotes the remaining term of Lip$(r-2)$ function F. Similar notation is applied to other Lip functions. Hence

$$|F(X_s, X_t) - F(\tilde{X}_s, \tilde{X}_t)| \le K\left(\varepsilon\omega(s,t)^{\frac{1}{p}}\right)^{r-2} \ . \tag{21}$$

Note that

$$\begin{aligned} H - \tilde{H} = &\big(F(X_s, X_t) - F(\tilde{X}_s, \tilde{X}_t)\big)(\mathbf{X}_{st}^2) \\ &+ F(\tilde{X}_s, \tilde{X}_t)(\mathbf{X}_{st}^2 - \tilde{\mathbf{X}}_{st}^2) \ , \end{aligned}$$

so that

$$\begin{aligned} (H - \tilde{H})(\mathbf{X}_{tu}^1) = &\left[F(X_s, X_t) - F(\tilde{X}_s, \tilde{X}_t)\right](\mathbf{X}_{st}^2 \otimes \mathbf{X}_{tu}^1) \\ &+ F(\tilde{X}_s, \tilde{X}_t)\left((\mathbf{X}_{st}^2 - \tilde{\mathbf{X}}_{st}^2) \otimes \mathbf{X}_{tu}^1\right) \ . \end{aligned}$$

Using (21) we get that

$$\begin{aligned} \left|(H - \tilde{H})(\mathbf{X}_{tu}^1)\right| \le &K\varepsilon^{r-2}\omega(0,t)^{\frac{r-2}{p}}\omega(s,u)^{\frac{3}{p}} \\ &+ K\varepsilon\omega(s,u)^{\frac{3}{p}} \ . \end{aligned} \tag{22}$$

Since

$$\alpha(\tilde{X}_t) - \alpha(\tilde{X}_s) - \alpha^1(\tilde{X}_s)(\tilde{\mathbf{X}}_{st}^1) = R_\alpha(\tilde{X}_s, \tilde{X}_t) \ ,$$

where

$$|R_\alpha(x,y)| \le M|x-y|^{r-1} \ ,$$

so that

$$\left|\left[\alpha(\tilde{X}_t) - \alpha(\tilde{X}_s) - \alpha^1(\tilde{X}_s)(\tilde{\mathbf{X}}^1_{st})\right](\mathbf{X}^1_{tu} - \tilde{\mathbf{X}}^1_{tu})\right| \\ \leq K\omega(s,t)^{\frac{r-1}{p}}\varepsilon\omega(t,u)^{\frac{1}{p}} \ . \tag{23}$$

Now we estimate the third term in (18). Let

$$\alpha^1(y) - \alpha^1(x) = Q(x,y)(y-x) \ ,$$

where Q: $V \oplus V \to \hom(V, \hom(V^{\otimes 2}, W))$ is a Lip$(r-2)$ map, so that

$$\left|Q(X_s,X_t) - Q(\tilde{X}_s,\tilde{X}_t)\right| = \left|R_Q((\tilde{X}_s,\tilde{X}_t),(X_s,X_t))\right| \\ \leq K\left(\varepsilon\omega(0,t)^{\frac{1}{p}}\right)^{r-2} \ .$$

Hence we have

$$|E - \tilde{E}| \leq |(Q(X_s,X_t) - Q(\tilde{X}_s,\tilde{X}_t))(\mathbf{X}^1_{st})| \\ + |Q(\tilde{X}_s,\tilde{X}_t)(\mathbf{X}^1_{st} - \tilde{\mathbf{X}}^1_{st})| \\ \leq K\left(\varepsilon\omega(0,t)^{\frac{1}{p}}\right)^{r-2}\omega(s,t)^{\frac{1}{p}} \\ + K\varepsilon\omega(s,t)^{\frac{1}{p}} \ ,$$

so that

$$|(E - \tilde{E})(\mathbf{X}^2_{tu})| \leq K\varepsilon^{r-2}\omega(0,t)^{\frac{r-2}{p}}\omega(s,u)^{\frac{3}{p}} + K\varepsilon\omega(s,u)^{\frac{3}{p}} \ . \tag{24}$$

It is easily seen that

$$\left|\left[\alpha^1(\tilde{X}_t) - \alpha^1(\tilde{X}_s)\right](\mathbf{X}^2_{tu} - \tilde{\mathbf{X}}^2_{tu})\right| \leq K\varepsilon\omega(s,u)^{\frac{r}{p}} \ . \tag{25}$$

Combining (22)–(25), we finally get that

$$\left|\left[(\mathbf{Z}_{st}\otimes\mathbf{Z}_{tu})^1 - \mathbf{Z}^1_{su}\right] - \left[(\tilde{\mathbf{Z}}_{st}\otimes\tilde{\mathbf{Z}}_{su})^1 - \tilde{\mathbf{Z}}^1_{su}\right]\right| \\ \leq K\varepsilon^{r-2}\omega(0,t)^{\frac{r-2}{p}}\omega(s,u)^{\frac{3}{p}} + K\varepsilon\omega(s,u)^{\frac{3}{p}} \\ + K\varepsilon\omega(s,t)^{\frac{r}{p}} + K\varepsilon^{r-2}\omega(0,t)^{\frac{r-2}{p}}\omega(s,u)^{\frac{3}{p}} \\ + K\varepsilon\omega(s,u)^{\frac{3}{p}} + K\varepsilon\omega(s,u)^{\frac{r}{p}} \\ \leq K\varepsilon^{r-2}\omega(s,u)^{\frac{r}{p}} \ . \tag{26}$$

Thus we have proved (16). Now we prove (17). For simplicity, we denote by $\rho = \varepsilon^{r-2}$. By (12), we have

$$\int_s^t \alpha(X)\delta\mathbf{X}^1 = \lim_{m(D)\to 0}\mathbf{Z}^D_{st} \ ,$$

where we denote by

$$\mathbf{Z}_{st}^{D} = \sum_{l=1}^{m} \mathbf{Z}_{t_{l-1}t_l}^{1} \ ,$$

if $D = \{s = t_0 \leq t_1 \leq \cdots \leq t_m = t\}$ is a finite dissection of $[s, t]$. Similar notation used for $\tilde{\mathbf{X}}$. Let $t_l \in D$, and let $D' = D - \{t_l\}$. Then we have

$$\mathbf{Z}_{st}^{D} - \mathbf{Z}_{st}^{D'} = (\mathbf{Z}_{t_{l-1}t_l} \otimes \mathbf{Z}_{t_l t_{l+1}})^1 - \mathbf{Z}_{t_{l-1}t_{l+1}}^{1} \ .$$

Using the estimate (16), we obtain that

$$\left| \left(\mathbf{Z}_{st}^{D} - \mathbf{Z}_{st}^{D'} \right) - \left(\tilde{\mathbf{Z}}_{st}^{D} - \tilde{\mathbf{Z}}_{st}^{D'} \right) \right| \leq K\rho\omega(t_{l-1}, t_{l+1})^{\frac{r}{p}} \ .$$

Choosing $t_l \in D$ such that

$$\begin{aligned} \omega(t_{l-1}, t_{l+1}) \ &\leq \frac{1}{m-2}\omega(s,t), \quad \text{if } m > 3 \ , \\ &= \omega(s,t), \quad \text{if } m = 3 \ , \end{aligned} \tag{27}$$

we have

$$\left| \left(\mathbf{Z}_{st}^{D} - \mathbf{Z}_{st}^{D'} \right) - \left(\tilde{\mathbf{Z}}_{st}^{D} - \tilde{\mathbf{Z}}_{st}^{D'} \right) \right| \leq K\rho \left(\frac{1}{m-2} \right)^{\frac{r}{p}} \omega(s,t)^{\frac{r}{p}} \ .$$

Repeating the same procedure and using the fact that $\frac{r}{p} > 1$, we finally get that

$$\left| (\mathbf{Z}_{st}^{D} - \mathbf{Z}_{st}^{1}) - (\tilde{\mathbf{Z}}_{st}^{D} - \tilde{\mathbf{Z}}_{st}^{1}) \right| \leq K\rho\omega(s,t)^{\frac{r}{p}} \ . \tag{28}$$

Letting $m(D) \to 0$ to get (17).

Proposition 5. *Keep the same assumptions and notation as in Prop.* 4. *Then*

$$\begin{aligned} &\left| \left[(\mathbf{K}_{st} \otimes \mathbf{K}_{tu})^2 - \mathbf{K}_{su}^2 \right] - \left[\left(\tilde{\mathbf{K}}_{st} \otimes \tilde{\mathbf{K}}_{tu} \right)^2 - \tilde{\mathbf{K}}_{su}^2 \right] \right| \\ &\quad \leq K\varepsilon^{r-2}\omega(s,u)^{\frac{r}{p}} \ , \end{aligned} \tag{29}$$

and

$$\begin{aligned} &\left| \left[\int_s^t \alpha(X)\delta\mathbf{X}^2 - \mathbf{K}_{st}^2 \right] - \left[\int_s^t \alpha(\tilde{X})\delta\tilde{\mathbf{X}}^2 - \tilde{\mathbf{K}}_{st}^2 \right] \right| \\ &\quad \leq K\varepsilon^{r-2}\omega(s,t)^{\frac{r}{p}} \ , \end{aligned} \tag{30}$$

where

$$\mathbf{K}_{st} = (1, \hat{\mathbf{Z}}_{st}^1, \mathbf{Z}_{st}^2), \quad \hat{\mathbf{Z}}_{st}^1 = \int_s^t \alpha(X)\delta\mathbf{X}^1 \ ,$$

and similarly to $\tilde{\mathbf{K}}$.

Proof. It is easily seen that

$$\begin{aligned}(\mathbf{K}_{st} \otimes \mathbf{K}_{tu})^2 - \mathbf{K}^2_{su} &= \mathbf{Z}^2_{st} + \mathbf{Z}^2_{tu} - \mathbf{Z}^2_{su} + \hat{\mathbf{Z}}^1_{st} \otimes \hat{\mathbf{Z}}^1_{tu} \\ &= [\alpha(X_t) \otimes \alpha(X_t) - \alpha(X_s) \otimes \alpha(X_s)](\mathbf{X}^2_{tu}) \\ &\quad + \hat{\mathbf{Z}}^1_{st} \otimes \hat{\mathbf{Z}}^1_{tu} - \alpha(X_s) \otimes \alpha(X_s)(\mathbf{X}^1_{st} \otimes \mathbf{X}^1_{tu}) \ .\end{aligned}$$

However

$$\begin{aligned}&\hat{\mathbf{Z}}^1_{st} \otimes \hat{\mathbf{Z}}^1_{tu} - \alpha(X_s) \otimes \alpha(X_s)(\mathbf{X}^1_{st} \otimes \mathbf{X}^1_{tu}) \\ &\quad = (\hat{\mathbf{Z}}^1_{st} - \mathbf{Z}^1_{st}) \otimes \hat{\mathbf{Z}}^1_{tu} + \alpha(X_s)(\mathbf{X}^1_{st}) \otimes (\hat{\mathbf{Z}}^1_{tu} - \mathbf{Z}^1_{tu}) \\ &\quad\quad + \alpha^1(X_s)(\mathbf{X}^2_{st}) \otimes \hat{\mathbf{Z}}^1_{tu} \\ &\quad\quad + \alpha(X_s)(\mathbf{X}^1_{st})\alpha^1(X_t)(\mathbf{X}^2_{tu}) \\ &\quad\quad + \alpha(X_s)(\mathbf{X}^1_{st}) \otimes H(\mathbf{X}^1_{tu}) \\ &\quad\quad + \alpha(X_s)(\mathbf{X}^1_{st}) \otimes \alpha^1(X_s)(\mathbf{X}^1_{st} \ \otimes \ \mathbf{X}^1_{tu}) \ ,\end{aligned}$$

and similarly for $\tilde{\mathbf{X}}$, where

$$H = \alpha(X_t) - \alpha(X_s) - \alpha^1(X_s)(\mathbf{X}^1_{st}) \ .$$

Hence we have

$$\left|\left[(\mathbf{K}_{st} \otimes \mathbf{K}_{tu})^2 - \mathbf{K}^2_{su}\right] - \left[(\tilde{\mathbf{K}}_{st} \otimes \tilde{\mathbf{K}}_{tu})^2 - \tilde{\mathbf{K}}^2_{su}\right]\right| \leq K\rho\omega(s,u)^{\frac{r}{p}} \ .$$

By (14), we have

$$\int_s^t \alpha(X)\delta\mathbf{X}^2 = \lim_{m(D)\to 0} \mathbf{Y}^D_{st},$$

$$\mathbf{Y}^D_{st} = \sum_l^m \mathbf{Z}^2_{t_{l-1}t_l} + \hat{\mathbf{Z}}^1_{st_{l-1}} \otimes \hat{\mathbf{Z}}^1_{t_{l-1}t_l} \ .$$

However it is clear that

$$\mathbf{Y}^D_{st} - \mathbf{Y}^{D'}_{st} = (\mathbf{K}_{t_{l-1}t_l} \otimes \mathbf{K}_{t_l t_{l+1}})^2 - \mathbf{K}^2_{t_{l-1}t_{l+1}} \ ,$$

so that a similar argument as in the proof of Prop. 4 leads to the inequality (30).

Corollary 1. *Let* $\alpha: V \to \mathrm{hom}(V, W)$ *be a* Lip(r) 1*-form*, $p < r \leq 3$, *and let* $\mathbf{X}, \tilde{\mathbf{X}}: \triangle \to T^{(2)}(V)$ *be two geometric multiplicative functionals with finite p-variation controlled by* ω. *If*

$$\left|\mathbf{X}^i_{st} - \tilde{\mathbf{X}}^i_{st}\right| \leq \varepsilon\omega(s,t)^{\frac{i}{p}}, \ \ \forall \ (s,t) \in \triangle, \ i = 1, 2 \ ,$$

$\varepsilon \leq 1$, *then*

$$\left|\int_s^t \alpha(X)\delta\mathbf{X}^i - \int_s^t \alpha(\tilde{X})\delta\tilde{\mathbf{X}}^i\right| \leq K\varepsilon^{r-2}\omega(s,t)^{\frac{i}{p}}, \quad \forall\ (s,t) \in \triangle, \quad i = 1, 2\ , \tag{31}$$

where K *is a constant depending only on* $p, r, \max\omega$ *and the Lipschitz constant of* α. *In particular, if* α *is a* Lip(r) 1*-form*, $r \geq 3$, *then*

$$\left|\int_s^t \alpha(X)\delta\mathbf{X}^i - \int_s^t \alpha(\tilde{X})\delta\tilde{\mathbf{X}}^i\right| \leq K\varepsilon\omega(s,t)^{\frac{i}{p}}, \quad \forall(s,t) \in \triangle, i = 1, 2\ , \tag{32}$$

for some constant K *depending only on* $p, r, \max\omega$ *and Lipschitz constant* M *of* α.

Proof. It is easily seen that

$$\left|\mathbf{Z}^1_{st} - \tilde{\mathbf{Z}}^1_{st}\right| \leq K\left(\varepsilon\omega(0,s)^{\frac{1}{p}}\right)^{r-2}\omega(s,t)^{\frac{1}{p}}\ ,$$

so that by (17), we have

$$\left|\int_s^t \alpha(X)\delta\mathbf{X}^1 - \int_s^t \alpha(\tilde{X})\delta\tilde{\mathbf{X}}^1\right| \leq K\varepsilon^{r-2}\omega(s,t)^{\frac{1}{p}}\ .$$

Using above estimate, we have

$$\left|\mathbf{K}^2_{st} - \tilde{\mathbf{K}}^2_{st}\right| \leq K\left(\varepsilon\omega(0,s)^{\frac{1}{p}}\right)^{r-2}\omega(s,t)^{\frac{2}{p}} + K\varepsilon^{r-2}\omega(s,t)^{\frac{2}{p}}\ .$$

By (30) we deduce that

$$\left|\int_s^t \alpha(X)\delta\mathbf{X}^2 - \int_s^t \alpha(\tilde{X})\delta\tilde{\mathbf{X}}^2\right| \leq K\varepsilon^{r-2}\omega(s,t)^{\frac{2}{p}}\ .$$

2. Differential equations

Suppose that $t \to x_t$ is a smooth path in V, and $f\colon W \to \hom(V, W)$ is a V-valued vector field on W. We consider the following differential equation,

$$dy_t = f(y_t)\,dx_t, \qquad y_0 = z\ . \tag{33}$$

By setting $y_{st} = y_t - y_s$ and $x_{st} = x_t - x_s$, (33) can be written

$$y_{st} = \int_s^t f(z + y_{0u})\, dx_{0u},\ y_t = z + y_{0u} \ . \tag{34}$$

Note that

$$x_{st} = \int_s^t dx_{0u} \ . \tag{35}$$

Combining (34) and (35), and letting $Z_{st} = (x_{st}, y_{st})$, we can rewrite (33) to be

$$Z_{st} = \int_s^t \alpha_f^z(Z_{0u}) dZ_{0u} \ , \tag{36}$$

where $\alpha_f^z : V \oplus W \to \hom(V \oplus W, V \oplus W)$ is a 1-form on $V \oplus W$ defined by

$$\alpha_f^z(x, y)(\xi, \eta) = (\xi, f(z + y)(\xi)), \quad \forall (x, y), (\xi, \eta) \in V \oplus W \ . \tag{37}$$

Note (36) has a meaning for a multiplicative functional of any degree.

Definition 2. *Let* $f : W \to \hom(V, W)$ *be a* Lip(r) *vector field*, $r > 1$, *and let* $\mathbf{X} : \triangle \to T^{(2)}(V)$ *be a multiplicative functional with finite p-variation. Then a multiplicative functional with finite p-variation* $\mathbf{Z} : \triangle \to T^{(2)}\ (V \oplus W)$ *is called a solution of the differential equation*

$$\delta \mathbf{Y} = f(Y) \delta \mathbf{X}, \qquad Y_0 = z \ , \tag{38}$$

if $\pi_V(\mathbf{Z}) = \mathbf{X}$, *and*

$$\mathbf{Z} = \int \alpha_f^z(Z) \delta \mathbf{Z} \ , \tag{39}$$

where $\alpha_f^z : V \oplus W \to \hom(V \oplus W, V \oplus W)$,

$$\alpha_f^z(x, y)(\xi, \eta) = (\xi, f(z + x)(\xi)), \quad \forall (x, y), (\xi, \eta) \in V \oplus W \ , \tag{40}$$

and π_V *(resp.* π_W*) is the lift map over* $T^{(2)}(V \oplus W)$ *of the natural projection* $V \oplus W \to V$ *(resp.* $V \oplus W \to W$*). In this case* $\mathbf{Y} = \pi_W(\mathbf{Z})$ *is called a multiplicative functional solution of Eq.* (38).

The following proposition is obvious.

Proposition 6. *If* $\mathbf{Z} : \triangle \to T^{(2)}(V \oplus W)$ *is a solution of Eq.* (38), *then* $\mathbf{Y} = \pi_W(\mathbf{Z}) : \triangle \to T^{(2)}(W)$ *is a multiplicative functional with finite p-variation.*

The following theorem has been established in Lyons [9].

Theorem 2. *Let* $\mathbf{X} : \triangle \to T^{(2)}(V)$ *be a geometric multiplicative functional with finite p-variation, and let* $f : W \to \hom(V, W)$ *be a* Lip(r) *vector field*, $\frac{r}{p} > 1$. *Then for any* $z \in W$, *there is a unique solution* $\mathbf{Z} : \triangle \to T^{(2)}(V \oplus W)$ *to the differential equation*

$$\delta \mathbf{Y} = f(Y)\delta \mathbf{X}, \quad Y_0 = z \ . \tag{41}$$

Moreover, if $\mathbf{X}$ *is of finite p-variation controlled by* ω, *then* $\mathbf{Z}$ *is of finite p-variation controlled by* $K\omega$, *where K is a constant depending only on p, r,* $\max \omega$ *and Lipschitz constant of* f.

In this case, we call $\mathbf{Y} = \pi_W(\mathbf{Z})$ the unique multiplicative functional solution of Eq.(38), and denote it by $F(\mathbf{X}, z)$. The map $\mathbf{X} \to F(\mathbf{X}, z)$ for any fixed $z \in W$ is called the Itô map associated with the vector field f.

Furthermore, the Itô map $\mathbf{X} \to F(\mathbf{X}, z)$ is continuous. More precisely, we have

Theorem 3 (Lyons [9]). *Let* $f: W \to \hom(V, W)$ *be a Lip*(r) *vector field,* $\frac{r}{p} > 1$. *If* $\mathbf{X}, \tilde{\mathbf{X}}: \triangle \to T^{(2)}(V)$ *are two multiplicative functionals with finite p-variation controlled by* ω, *and if*

$$\left| \mathbf{X}^i_{st} - \tilde{\mathbf{X}}^i_{st} \right| \leq \varepsilon \omega(s,t)^{\frac{i}{p}}, \quad \forall \ (s,t) \in \triangle, \ \ i = 1, 2 \ ,$$

then

$$\left| F(\mathbf{X}, z)^i_{st} - F(\tilde{\mathbf{X}}, z)^i_{st} \right| \leq K K(\varepsilon) \omega(s,t)^{\frac{i}{p}}, \quad \forall \ (s,t) \in \triangle, \ \ i = 1, 2 \ .$$

Where $K(\varepsilon)$ *depending only on p, r,* $\max \omega$ *and the Lipschitz constant M, and* $\lim_{\varepsilon \to 0} \omega(\varepsilon) = 0$.

The first goal of this paper is to show that if f is a Lip(r) map, $r \geq 3$, then the sequence of multiplicative functionals obtained by iterating the integrals of 1-form α^z_f against a geometric multiplicative functional $\mathbf{X}$ converges to the unique solution $\mathbf{Z}$ of Eq. (38). More precisely, define a sequence of multiplicative functionals $\mathbf{Z}(n): \triangle \to T^{(2)}(V \oplus W)$ recursively by

$$\mathbf{Z}(n+1) = \int \alpha^z_f(Z(n)) \delta \mathbf{Z}(n), \quad n = 0, 1, \ldots \ , \tag{42}$$

and

$$\begin{aligned} \mathbf{Z}(0)_{st} &= \left(1, \mathbf{Z}(0)^1_{st}, \mathbf{Z}(0)^2_{st}\right), \\ \mathbf{Z}(0)^1_{st} &= (\mathbf{X}^1_{st}, 0), \quad \mathbf{Z}(0)^2_{st} = (\mathbf{X}^2_{st}, 0, 0, 0) \ , \end{aligned} \tag{43}$$

then we shall prove that $\mathbf{Z}(n)$ converges to the unique solution $\mathbf{Z}$ of Eq. (38).

In order to simplify our notations, we will suppress the uperscript z in α^z_f, and simply denote α^z_f by α_f (as z will be fixed as an initial value).

Define a sequence of almost multiplicative functionals $\tilde{\mathbf{Z}}(n)$ as follows,

$$\begin{aligned}&\tilde{\mathbf{Z}}(n+1)_{st} = (1, \tilde{\mathbf{Z}}(n+1)^1_{st}, \tilde{\mathbf{Z}}(n+1)^2_{st}),\\&\tilde{\mathbf{Z}}(n+1)^1_{st} = \alpha_f(Z(n)_s)(\mathbf{Z}(n)^1_{st}) + \alpha^1_f(Z(n)_s)(\mathbf{Z}(n)^2_{st}),\\&\tilde{\mathbf{Z}}(n+1)^2_{st} = \alpha_f(Z(n)_s) \otimes \alpha_f(Z(n)_s)(\mathbf{Z}(n)^2_{st})\ .\end{aligned} \tag{44}$$

To estimate $\mathbf{Z}(n)$ and $\tilde{\mathbf{Z}}(n)$, we need a decomposition of $T^{(2)}(V \oplus W)$. Let

$$\begin{aligned}H_1 &= \{(v,0) \in V \oplus W : v \in V\},\\H_2 &= \{(0,w) \in V \oplus W : w \in W\}\ .\end{aligned}$$

Then we have

$$V \oplus W = H_1 \oplus H_2, \quad H_1 \cong V, \quad H_2 \cong W \tag{45}$$

by obvious identification. Now consider $(V \oplus W)^{\otimes 2}$. It is obvious that

$$\begin{aligned}(u,v)\otimes(\xi,\eta) &= [(u,0)+(0,v)]\otimes[(\xi,0)+(0,\eta)]\\&= (u,0)\otimes(\xi,0) + (0,v)\otimes(\xi,0)\\&\quad + (u,0)\otimes(0,\eta) + (0,v)\otimes(0,\eta)\ .\end{aligned}$$

Let

$$\begin{aligned}H_{20} &= \operatorname{span}\{(u,0)\otimes(\xi,0) : u,\xi \in V\} \cong V^{\otimes 2}\\H^1_{11} &= \operatorname{span}\{(u,0)\otimes(0,\eta) : u \in V, \eta \in W\} \cong V \otimes W,\\H^2_{11} &= \operatorname{span}\{(0,v)\otimes(\xi,0) : \xi \in V, v \in W\} \cong W \otimes V,\\H_{02} &= \operatorname{span}\{(0,v)\otimes(0,\eta) : v,\eta \in W\} \cong W^{\otimes 2}\ .\end{aligned}$$

Then $(V \oplus W)^{\otimes 2}$ has a direct sum decomposition

$$(V \oplus W)^{\otimes 2} = H_{20} \oplus H^1_{11} \oplus H^2_{11} \oplus H_{02}\ . \tag{46}$$

Under this decomposition, if $\mathbf{K} : \triangle \to T^{(2)}(V \oplus W)$ is a functional, $\mathbf{X} = \pi_V(\mathbf{K})$ and $\mathbf{Y} = \pi_W(\mathbf{K})$, then we write

$$\begin{aligned}\mathbf{K}^1_{st} &= (\mathbf{X}^1_{st}, \mathbf{Y}^1_{st}),\\\mathbf{K}^2_{st} &= (\mathbf{X}^2_{st}, \mathbf{K}^{10}_{st}, \mathbf{K}^{01}_{st}, \mathbf{Y}^2_{st})\ .\end{aligned} \tag{47}$$

By definition, it is easily seen that

$$\alpha_f(x,y) :\ H_2 \to (0,0), \quad \forall\ (x,y) \in V \oplus W\ ,$$

and

$$\begin{aligned}&\alpha^1_f : V \oplus W \to \hom((V \oplus W)^{\otimes 2}, V \oplus W)\ ,\\&\alpha^1_f(x,y)[(\xi,\eta)\otimes(u,v)] = (0, f^1(z+y)(\xi)(v)),\end{aligned} \tag{48}$$

for any $(x,y), (\xi,\eta), (u,v) \in V \oplus W$. In particular we have

$$\alpha_f^1(x,y): H_{20} \oplus H_{11}^2 \oplus H_{02} \to (0,0),$$
$$\alpha_f^1(x,y)[(u,0) \otimes (0,\eta)] \to (0, f^1(z+y)(u)(\eta)) \ . \tag{49}$$

Proposition 7. *For any* $\zeta \in (V \oplus W)^{\otimes 2}$, *we have*

$$\alpha_f(x,y) \otimes \alpha_f(x,y)(\zeta) = \alpha_f(x,y) \otimes \alpha_f(x,y)(\pi(\zeta)) \ , \tag{50}$$

and

$$\begin{aligned}&\left[\alpha_f(x,y) \otimes \alpha_f(x,y)\right](\zeta)\\ &\quad = (\pi(\zeta), 1 \otimes f(z+y)(\pi(\zeta)), f(z+y) \otimes 1(\pi_V(\zeta)),\\ &\qquad f(z+y) \otimes f(z+y)(\pi(\zeta))) \ ,\end{aligned} \tag{51}$$

where

$$1 \otimes f(z+y): V^{\otimes 2} \to V \otimes W, \ u \otimes v \to u \otimes f(z+y)(v),$$
$$f(z+y) \otimes 1: V^{\otimes 2} \to W \otimes V, \ u \otimes v \to f(z+y)(u) \otimes v,$$
$$f(z+y) \otimes f(z+y): V^{\otimes 2} \to W^{\otimes 2}, u \otimes v \to f(z+y)(u) \otimes f(z+y)(v) \ .$$

Proof. Let $(\xi,\eta) \otimes (u,v) \in (V \oplus W)^{\otimes 2}$. Then we have

$$\begin{aligned}&\left[\alpha_f(x,y) \otimes \alpha_f(x,y)\right][(\xi,\eta) \otimes (u,v)]\\ &\quad = (\xi, f(z+y)(\xi)) \otimes (u, f(z+y)(u))\\ &\quad = \left[\alpha_f(x,y) \otimes \alpha_f(x,y)\right]((\xi,0) \otimes (u,0)) \ ,\end{aligned}$$

and

$$\begin{aligned}&(\xi, f(z+y)(\xi)) \otimes (u, f(z+y)(u))\\ &\quad = [(\xi,0) + (0, f(z+y)(\xi)] \otimes [(u,0) + (0, f(z+y)(u)]\\ &\quad = (\xi,0) \otimes (u,0) + (0, f(z+y)(\xi)) \otimes (u,0)\\ &\qquad + (\xi,0) \otimes (0, f(z+y)(u)) + (0, f(z+y)(\xi)) \otimes (0, f(z+y)(u)) \ .\end{aligned}$$

All conclusions follow immediately.

Proposition 8. 1) *For any* n,

$$\pi_V(\mathbf{Z}(n)) = \mathbf{X}, \qquad \pi_V(\tilde{\mathbf{Z}}(n)) = \mathbf{X} \ . \tag{52}$$

2) *For any* $(s,t) \in \triangle$,

$$\tilde{\mathbf{Z}}(n+1)_{st}^2 = \alpha_f(Z(n)_s) \otimes \alpha_f(Z(n)_s)(\mathbf{X}_{st}^2, 0, 0, 0) \ . \tag{53}$$

3) *For any* n *and* $(s,t) \in \triangle$,

$$\alpha_f(x,y)\left(\mathbf{Z}(n+1)_{st}^1 - \mathbf{Z}(n)_{st}^1\right) = 0 \ . \tag{54}$$

The proof of Prop. 8 is an easy calculation.

It is easy to check that

$$\begin{aligned}\tilde{\mathbf{Z}}(1)^1_{st} &= (\mathbf{X}^1_{st}, f(z)(\mathbf{X}^1_{st})),\\ \tilde{\mathbf{Z}}(1)^2_{st} &= \left(\mathbf{X}^2_{st}, 1\otimes f(z)(\mathbf{X}^2_{st}), f(z)\otimes 1(\mathbf{X}^2_{st}), f(z)\otimes f(z)(\mathbf{X}^2_{st})\right) \ .\end{aligned} \tag{55}$$

To show that $\mathbf{Z}(n)$ are uniformly bounded, we need a scaling estimate. If $\varepsilon, \gamma \in \mathbb{R}$, then we use $\Gamma(\varepsilon, \gamma)$ to denote the second quantisation of the linear map $(u, v) \to (\varepsilon u, \gamma v)\colon V \oplus W \to V \oplus W$.

Lemma 1. *If* $\mathbf{Z}\colon \triangle \to T^{(2)}(V \oplus W)$ *is a multiplicative functional, then so is* $\Gamma(\varepsilon, \gamma)\mathbf{Z}$.

Proposition 9. Let $\mathbf{Z}\colon \triangle \to T^{(2)}(V \oplus W)$ *be a multiplicative functional with finite p-variation, and let* $f\colon W \to \mathrm{hom}(V, W)$ *be a* $\mathrm{Lip}(r)$ *vector field,* $\frac{r-1}{p} > 1$. *Then for any* $\varepsilon \in \mathbb{R}$,

$$\Gamma(\varepsilon,\varepsilon)\left(\int \alpha_f(Z)\delta\mathbf{Z}\right) = \int \alpha_f(\Gamma(\varepsilon,1)Z)\delta\Gamma(\varepsilon,1)\mathbf{Z} \ . \tag{56}$$

Proof. By definition, the integral $\int \alpha_f(Z(\varepsilon))\delta\mathbf{Z}(\varepsilon)$ is the associated multiplicative functional of $\tilde{\mathbf{K}}(\varepsilon)$ (see Prop. 7), where

$$\begin{aligned}\tilde{\mathbf{K}}(\varepsilon)^1_{st} &= \alpha_f(Z_s(\varepsilon))(\mathbf{Z}(\varepsilon)^1_{st}) + \alpha^1_f(Z(\varepsilon)_s)(\mathbf{Z}(\varepsilon)^2_{st}),\\ &= (\varepsilon\mathbf{X}^1_{st}, \varepsilon[f(Y_s)(\mathbf{X}^1_{st}) + f^1(Y_s)(\mathbf{Z}(\varepsilon)^{10}_{st})]),\\ \tilde{\mathbf{K}}(\varepsilon)^2_{st} &= (\varepsilon^2\mathbf{X}^2_{st}, \varepsilon^2 1\otimes f(Y_s)(\mathbf{X}^2_{st}), \varepsilon^2 f(Y_s)\otimes 1(\mathbf{X}^2_{st}),\\ &\qquad \varepsilon^2 f(Y_s)\otimes f(Y_s)(\mathbf{X}^2_{st})) \ ,\end{aligned}$$

$\mathbf{X} = \pi_V(\mathbf{Z})$ and $\mathbf{Y} = \pi_W(\mathbf{Z})$, and for simplicity, we denote $\Gamma(\varepsilon, 1)\mathbf{Z}$ by $\mathbf{Z}(\varepsilon)$. Hence

$$\tilde{\mathbf{K}}(\varepsilon) = \Gamma(\varepsilon,\varepsilon)\tilde{\mathbf{K}} \ ,$$

where $\tilde{\mathbf{K}} = \tilde{\mathbf{K}}(1)$. However $\int \alpha_f(Z)\delta\mathbf{Z}$ is the associated multiplicative functional of $\tilde{\mathbf{K}}$, so that

$$\Gamma(\varepsilon,\varepsilon)\left(\int \alpha_f(Z)\delta\mathbf{Z}\right) = \int \alpha_f(\Gamma(\varepsilon,1)Z)\delta\Gamma(\varepsilon,1)\mathbf{Z} \ .$$

Lemma 2. *For any real numbers* $\gamma \neq 0$ *and* ε, *we have*

$$\Gamma(1,\gamma^{-1})\Gamma(\varepsilon,\gamma) = \Gamma(\varepsilon,1) \ . \tag{57}$$

Proposition 10 (Lyons [9]). *Let* $\mathbf{Z}\colon \triangle \to T^{(2)}(V \oplus W)$ *be a multiplicative functional with finite p-variation, and let* $\mathbf{X} = \pi_V(\mathbf{Z})$. *If*

$$|\mathbf{X}_{st}^i| \le \frac{\omega(s,t)^{\frac{i}{p}}}{\beta\left(\frac{i}{p}\right)!}, \quad |\mathbf{Z}_{st}^i| \le \frac{[K\omega(s,t)]^{\frac{i}{p}}}{\beta\left(\frac{i}{p}\right)!}, \ \forall\ (s,t) \in \triangle, i = 1,2 \ ,$$

then

$$\left|(\Gamma(1,\gamma)\mathbf{Z}_{st})^i\right| \le \frac{[K_\gamma\omega(s,t)]^{\frac{i}{p}}}{\beta\left(\frac{i}{p}\right)!}, \ \ \forall\ (s,t) \in \triangle, \ \ i = 1,2 \ , \tag{58}$$

where

$$K_\gamma = \max\left\{1, \gamma^{\frac{kp}{j}}K \colon 1 \le k \le j \le 2\right\} \ . \tag{59}$$

In particular if $\gamma < K^{-\frac{2}{p}}$, *then*

$$\left|(\Gamma(1,\gamma)\mathbf{Z}_{st})^i\right| \le \frac{\omega(s,t)^{\frac{i}{p}}}{\beta\left(\frac{i}{p}\right)!}, \ \ \forall\ (s,t) \in \triangle, \ i = 1,2 \ . \tag{60}$$

In the sequel we fix the following data:

1) The constant $K_M \ge 1$ is determined by Prop. 3, which depends only on $\max \omega, p$ and Lipschitz constant M of f.
2) Let $\gamma > 0$ be a constant such that $\gamma < K_M^{-\frac{2}{p}}$.
3) Let $\varepsilon = \gamma^{-1} > 1$.

Theorem 4. *Under above conditions, we have*

$$\left|(\Gamma(\varepsilon,1)\mathbf{Z}(n))_{st}^i\right| \le \frac{\tilde{\omega}(s,t)^{\frac{i}{p}}}{\beta\left(\frac{i}{p}\right)!}, \ \forall\ (s,t) \in \triangle, \ i = 1,2 \ , \tag{61}$$

for $\tilde{\omega} = \varepsilon^p\omega$ *on an interval* I, *such that* $\tilde{\omega} \le 1$ *on* I.

Proof. Note

$$\Gamma(\varepsilon,\varepsilon)\mathbf{Z}(0) = \Gamma(\varepsilon,1)\mathbf{Z}(0), \ \ \forall\ \varepsilon \in \mathbb{R} \ ,$$

and

$$\begin{aligned} |\Gamma(\varepsilon,1)\mathbf{Z}(0)_{st}^i| \ &\le \varepsilon^i \frac{\omega(s,t)^{\frac{i}{p}}}{\beta\left(\frac{i}{p}\right)!} \\ &\le \frac{[\varepsilon^p\omega(s,t)]^{\frac{i}{p}}}{\beta\left(\frac{i}{p}\right)!} \ . \end{aligned}$$

Let $\tilde{\omega} = \varepsilon^p\omega$ and choose I such that $\tilde{\omega} \le 1$, so that

$$\left|\Gamma(\varepsilon,1)\mathbf{Z}(0)_{st}^i\right| \le \frac{\tilde{\omega}(s,t)^{\frac{i}{p}}}{\beta\left(\frac{i}{p}\right)!}, \ \ \forall\ (s,t) \in \triangle, \ i = 1,2 \ .$$

Now we use induction. Assume that

$$\left|\Gamma(\varepsilon,1)\mathbf{Z}(n)^i_{st}\right| \le \frac{\tilde{\omega}(s,t)^{\frac{i}{p}}}{\beta\left(\frac{i}{p}\right)!}, \quad \forall\ (s,t)\in\triangle,\ i=1,2\ .$$

Let

$$\mathbf{K}(n+1) = \int \alpha_f(\Gamma(\varepsilon,1)Z(n))\delta\Gamma(\varepsilon,1)\mathbf{Z}(n)\ .$$

Then by Prop. 3, we have

$$\left|\mathbf{K}(n+1)^i_{st}\right| \le \frac{[K_M\tilde{\omega}(s,t)]^{\frac{i}{p}}}{\beta\left(\frac{i}{p}\right)!}, \quad i=1,2\ .$$

so that by Prop. 10, we have

$$\left|\Gamma(1,\gamma)\mathbf{K}(n+1)^i_{st}\right| \le \frac{\tilde{\omega}(s,t)^{\frac{i}{p}}}{\beta\left(\frac{i}{p}\right)!}, \quad \forall\ (s,t)\in\triangle,\ i=1,2\ .$$

However by Lemma 1,

$$\begin{aligned}\mathbf{K}(n+1) &= \Gamma(\varepsilon,\varepsilon)\mathbf{Z}(n+1)\\ &= \Gamma(\varepsilon,\gamma^{-1})\mathbf{Z}(n+1)\ ,\end{aligned}$$

so that

$$\Gamma(1,\gamma)\mathbf{K}(n+1) = \Gamma(\varepsilon,1)\mathbf{Z}(n+1)\ .$$

Therefore

$$\left|\Gamma(\varepsilon,1)\mathbf{Z}(n+1)^i_{st}\right| \le \frac{\tilde{\omega}(s,t)^{\frac{i}{p}}}{\beta\left(\frac{i}{p}\right)!}, \quad \forall\ (s,t)\in\triangle,\ i=1,2\ .$$

Corollary 2. *There are constants $T>0, K$ depending only on $p, r, \max\omega$ and Lipschitz constant M, such that $K\omega(0,T)\le 1$, and*

$$\left|\mathbf{Z}(n)^i_{st}\right| \le \frac{[K\omega(s,t)]^{\frac{i}{p}}}{\beta\left(\frac{i}{p}\right)!}, \quad \forall\ (s,t)\in\triangle,\ i=1,2\ , \tag{62}$$

where $I=[0,T]$, and $\triangle=\{(s,t): s\le t, s,t\in I\}$.

Now we can prove the following proposition.

Proposition 11. *If $f: W\to \mathrm{hom}(V,W)$ is a $\mathrm{Lip}(r)$ map, $r\ge 3$, then there is a positive constant T depending only on $\max\omega, p, r$ and Lipschitz constant M of f, such that*

$$\left|\mathbf{Z}(n+1)^i_{st} - \mathbf{Z}(n)^i_{st}\right| \le K_0\left(K\omega(0,T)^{\frac{1}{p}}\right)^n \omega(s,t)^{\frac{i}{p}}, \quad \forall\ (s,t) \in [0,T] \ , \tag{63}$$

for $n = 1, 2, \ldots$, *where* K_0, K *are constants depending only on* $p, r, \max\omega$ *and Lipschitz constant* M.

Proof. By Corollary 2, we can choose a $T > 0$ and a constant K_1 depending only on $p, r, \max\omega$ and M, such that $\mathbf{X}$, $\mathbf{Z}(n)$ are of finite p-variation controlled by $K_1\omega$ on $[0,T]$, and $K_1\omega(0,T) \le 1$ on this interval. Set

$$K_2 = 3\max\left\{\frac{M}{\beta\left(\frac{1}{p}\right)!}, \frac{M^2+2M}{\beta\left(\frac{2}{p}\right)!}\right\} \vee 1 \ ,$$

and $K_3 = K_2^p$. Let $K_4 = K$ be the constant appeared in (32). Now fix a positive constant T, such that

$$K_4K_3K_1\omega(0,T) < 1 \ .$$

By (43) and (55), we have

$$\begin{aligned}\left|\tilde{\mathbf{Z}}(1)^1_{st} - \mathbf{Z}(0)^1_{st}\right| &\le \left|f(z)(\mathbf{X}^1_{st})\right| \\ &\le M|\mathbf{X}^1_{st}| \\ &\le \frac{M}{\beta\left(\frac{1}{p}\right)!}\omega(s,t)^{\frac{1}{p}} \ ,\end{aligned}$$

and

$$\begin{aligned}\left|\tilde{\mathbf{Z}}(1)^2_{st} - \mathbf{Z}(0)^2_{st}\right| &\le (2M+M^2)|\mathbf{X}^2_{st}| \\ &\le \frac{2M+M^2}{\beta\left(\frac{2}{p}\right)!}\omega(s,t)^{\frac{2}{p}} \ ,\end{aligned}$$

so that

$$\left|\tilde{\mathbf{Z}}(1)^i_{st} - \mathbf{Z}(0)^i_{st}\right| \le (K_3\omega(s,t))^{\frac{i}{p}}, \quad \forall\ (s,t) \in \triangle, \ \ i = 1, 2 \ .$$

By Corollary 1, we have

$$\left|\mathbf{Z}(1)^i_{st} - \mathbf{Z}(0)^i_{st}\right| \le K_4(K_3\omega(s,t))^{\frac{i}{p}}, \quad \forall\ (s,t) \in \triangle, \ \ i = 1, 2 \ .$$

Now we use induction. Assume that

$$\left|\mathbf{Z}(n)^i_{st} - \mathbf{Z}(n-1)^i_{st}\right| \le h(n-1,T)\omega(s,t)^{\frac{i}{p}}, \quad \forall\ (s,t) \in \triangle, \ \ i = 1, 2 \ , \tag{64}$$

where $h(n,T) \hat{=} K_4\left(K_2\omega(0,T)^{\frac{1}{p}}\right)^n$.

Using Prop. 7 and Prop. 8, we get

$$\begin{aligned}\tilde{\mathbf{Z}}(n+1)^1_{st} - \tilde{\mathbf{Z}}(n)^1_{st} =& \left[\alpha_f\left(Z(n)^1_s\right) - \alpha_f\left(Z(n-1)^1_s\right)\right]\left(\mathbf{Z}(n)^1_{st}\right)\\ &+ \left[\alpha^1_f\left(Z(n)^1_s\right) - \alpha^1_f\left(Z(n-1)^1_s\right)\right]\left(\mathbf{Z}(n)^2_{st}\right)\\ &+ \alpha^1_f\left(Z(n-1)^1_s\right)\left(\mathbf{Z}(n)^2_{st} - \mathbf{Z}(n-1)^2_{st}\right),\end{aligned}$$

so that

$$\begin{aligned}\left|\tilde{\mathbf{Z}}(n+1)^1_{st} - \tilde{\mathbf{Z}}(n)^1_{st}\right| \le\ & M\left|Z(n)^1_s - Z(n-1)^1_s\right||\mathbf{Z}(n)^1_{st}|\\ &+ M\left|Z(n)^1_s - Z(n-1)^1_s\right|\left|\mathbf{Z}(n)^2_{st}\right|\\ &+ M\left|\mathbf{Z}(n)^2_{st} - Z(n-1)^2_{st}\right|\\ \le\ & \frac{M}{\beta\left(\frac{1}{p}\right)!}\left|Z(n)^1_s - Z(n-1)^1_s\right|\omega(s,t)^{\frac{1}{p}}\\ &+ \frac{M}{\beta\left(\frac{2}{p}\right)!}\left|Z(n)^1_s - Z(n-1)^1_s\right|\omega(s,t)^{\frac{2}{p}}\\ &+ M\left|\mathbf{Z}(n)^2_{st} - \mathbf{Z}(n-1)^2_{st}\right|\\ \le\ & \frac{M}{\beta\left(\frac{1}{p}\right)!}h(n-1,T)\omega(0,t)^{\frac{1}{p}}\omega(s,t)^{\frac{1}{p}}\\ &+ \frac{M}{\beta\left(\frac{2}{p}\right)!}h(n-1,T)\omega(0,t)^{\frac{1}{p}}\omega(s,t)^{\frac{2}{p}}\\ &+ \frac{M}{\beta\left(\frac{2}{p}\right)!}h(n-1,T)\omega(s,t)^{\frac{2}{p}}\\ \le\ & K_2 h(n-1,T)\omega(0,T)^{\frac{1}{p}}\omega(s,t)^{\frac{1}{p}}\ .\end{aligned}$$

Again using Prop. 7 and Prop. 8, we have

$$\begin{aligned}\tilde{\mathbf{Z}}(n+1)^2_{st} - \tilde{\mathbf{Z}}(n)^2_{st} =& \left[\alpha_f\left(Z(n)^1_s\right) \otimes \alpha_f\left(Z(n)^1_s\right)\right.\\ &\left. - \alpha_f\left(Z(n-1)^1_s\right) \otimes \alpha_f\left(Z(n-1)^1_s\right)\right]\left(\mathbf{X}^2_{st}, 0, 0, 0\right)\ ,\end{aligned}$$

which yields that

$$\begin{aligned}\left|\tilde{\mathbf{Z}}(n+1)^2_{st} - \tilde{\mathbf{Z}}(n)^2_{st}\right| &\le M^2\left|Z(n)^1_s - Z(n-1)^1_s\right||\mathbf{X}^2_{st}|\\ &\le \frac{M^2}{\beta\left(\frac{2}{p}\right)!}h(n-1,T)\omega(0,t)^{\frac{1}{p}}\omega(s,t)^{\frac{2}{p}}\\ &\le K_2 h(n-1,T)\omega(0,T)^{\frac{1}{p}}\omega(s,t)^{\frac{2}{p}}\ .\end{aligned}$$

Thus we have proved that

$$\left|\tilde{\mathbf{Z}}(n+1)^i_{st} - \tilde{\mathbf{Z}}(n)^i_{st}\right| \le K_2 h(n-1,T)\omega(0,T)^{\frac{1}{p}}\omega(s,t)^{\frac{i}{p}} \ ,$$

for any $(s,t) \in \triangle$, and $i = 1,2$. Using Corollary 1 to get

$$\left|\mathbf{Z}(n+1)^i_{st} - \mathbf{Z}(n)^i_{st}\right| \le K_4 K_2 h(n-1,T)\omega(0,T)^{\frac{1}{p}}\omega(s,t)^{\frac{i}{p}} \ ,$$

for any $(s,t) \in \triangle, i = 1,2$. Now (63) follows immediately. Therefore we have proved the proposition.

By Prop. 11, we know that there is a positive constant T depending only on $\max \omega, p, r$ and Lipschitz constant M of f, such that

$$\mathbf{Z}^i_{st} = \lim_{n\to\infty} \mathbf{Z}(n)^i_{st}, \quad \text{on } [0,T] \ ,$$

exists. It is easily seen that $\mathbf{Z}_{st} = (1, \mathbf{Z}^1_{st}, \mathbf{Z}^2_{st})$ is a geometric multiplicative functional with finite p-variation controlled by ω. Moreover we have

$$\mathbf{Z} = \int \alpha_f(Z)\delta\mathbf{Z} \ ,$$

i.e. $\mathbf{Z}$ is a solution of the differential equation

$$\delta\mathbf{Y} = f(Y)\delta\mathbf{X}, \qquad Y_0 = z \ . \tag{65}$$

It is routine to extend the solution $\mathbf{Z}$ to the original interval I by the uniform estimates obtained above.

Theorem 5. *Let* $\mathbf{X}, \hat{\mathbf{X}}: \triangle \to T^{(2)}(V)$ *be two multiplicative functionals with finite p-variation controlled by* ω, *and let* $f: W \to \mathrm{hom}(V,W)$ *be a* $\mathrm{Lip}(r)$ *vector field*, $r \ge 3$. *If*

$$\left|\mathbf{X}^i_{st} - \hat{\mathbf{X}}^i_{st}\right| \le \varepsilon\omega(s,t)^{\frac{i}{p}}, \quad \forall\ (s,t) \in \triangle, \ \ i = 1,2 \ ,$$

then

$$\left|\mathbf{Z}^i_{st} - \hat{\mathbf{Z}}^i_{st}\right| \le K\varepsilon\omega(s,t)^{\frac{i}{p}}, \quad \forall\ (s,t) \in \triangle, \ \ i = 1,2 \ , \tag{66}$$

where $\mathbf{Z}$ (*resp.* $\hat{\mathbf{Z}}$) *is the solution of Eq.* (65) *with driving multiplicative functional* $\mathbf{X}$ (*resp.* $\hat{\mathbf{X}}$), K *is a constant depending only on* $\max \omega, p, r$ *and Lipschitz constant* M of f.

Proof. We use notations as before, and use $\mathbf{K}(n)$ and $\tilde{\mathbf{K}}(n)$ to denote the corresponding sequences $\mathbf{Z}(n)$ and $\tilde{\mathbf{Z}}(n)$ obtained by replacing $\mathbf{X}$ by $\hat{\mathbf{X}}$, so that

$$\lim_{n\to\infty} \mathbf{K}(n)^i_{st} = \hat{\mathbf{Z}}, \quad \forall\ (s,t) \in \triangle, \ i = 1,2 \ .$$

We have

$$\left|\tilde{\mathbf{Z}}(n+1)^1_{st} - \tilde{\mathbf{K}}(n+1)^1_{st}\right| = \alpha_f\left(Z(n)^1_s\right)\left(\mathbf{Z}(n)^1_{st}\right) - \alpha_f\left(K(n)^1_s\right)\left(\mathbf{K}(n)^1_{st}\right)$$
$$+ \alpha^1_f\left(Z(n)^1_s\right)\left(\mathbf{Z}(n)^2_{st}\right) - \alpha_f\left(K(n)^1_s\right)\left(\mathbf{K}(n)^2_{st}\right)$$
$$= M\varepsilon\omega(0,s)^{\frac{1}{p}}\omega(s,t)^{\frac{1}{p}} + M\varepsilon\omega(s,t)^{\frac{1}{p}}$$
$$\leq 4M\varepsilon\omega(s,t)^{\frac{1}{p}} \ .$$

Similarly

$$\left|\tilde{\mathbf{Z}}(n+1)^2_{st} - \tilde{\mathbf{K}}(n+1)^2_{st}\right| \leq 2M^2\varepsilon\omega(s,t)^{\frac{2}{p}} \ ,$$

for any n. By Prop. 4 and Prop. 5 we deduce that

$$\left|\left(\mathbf{Z}(n+1)^i_{st} - \tilde{\mathbf{Z}}(n+1)^i_{st}\right) - \left(\mathbf{K}(n+1)^i_{st} - \tilde{\mathbf{K}}(n+1)^i_{st}\right)\right| \leq K\varepsilon\omega(s,t)^{\frac{3}{p}} \ ,$$

for any $(s,t) \in \triangle, i = 1,2$, so that

$$\left|\mathbf{Z}^i_{st} - \hat{\mathbf{Z}}^i_{st}\right| \leq K\varepsilon\omega(s,t)^{\frac{i}{p}}, \quad \forall (s,t) \in \triangle, \ i = 1,2 \ .$$

Remark. Theorem 5 is a slight improvement of Theorem 3 in the sense that the Itô map is in fact Lipschitz continuous in p-variation topology if the vector field is C^3_b.

3. Flow of diffeomorphisms

In this section we assume that $V = \mathbb{R}^N$ and $W = \mathbb{R}^d$. Let $\mathbf{X}\colon \triangle \to T^{(2)}(V)$ be a geometric multiplicative functional with finite p-variation, and let $f\colon W \to \mathrm{hom}(V,W)$ be a Lip(r) vector field, $r \geq 4$. Let $F(\mathbf{X},z)$ be the Itô map associated with the vector field f, i.e. $F(\mathbf{X},z)$ is the unique multiplicative functional solution of the differential equation

$$\delta\mathbf{Y} = f(Y)\delta\mathbf{X}, \qquad Y_0 = z \ . \tag{67}$$

Define

$$F_t(\mathbf{X},\cdot)\colon W \to W, \quad F_t(\mathbf{X},z) = z + F(\mathbf{X},z)^1_{0t} \ .$$

We also call $F_t(\mathbf{X},z)$ the Itô map associated with the vector field f.

For $\tau \in \mathbb{R}^+$, θ_τ denotes the natural shift, i.e. if $\mathbf{X}\colon \triangle \to T^{(2)}(V)$ is a functional, then $\theta_\tau\mathbf{X}\colon \triangle \to T^{(2)}(V)$,

$$(\theta_\tau\mathbf{X})^i_{st} = \mathbf{X}^i_{s+\tau,t+\tau}, \quad \forall\ (s,t) \in \triangle, \ i = 0,1,2 \ .$$

It is clear that if $\mathbf{X}$ is a geometric multiplicative functional, so is $\theta_\tau\mathbf{X}$.

Proposition 12. *If* $f: W \to \mathrm{hom}(V, W)$ *is a* $\mathrm{Lip}(r)$ *vector field*, $r \geq 3$, *and* $\mathbf{X}: \triangle \to T^{(2)}(V)$ *is a geometric multiplicative functional*, *then*

$$F_{t+s}(\mathbf{X}, z) = F_t(\theta_s \mathbf{X}, F_s(\mathbf{X}, z)), \quad \forall\ (s,t) \in \mathbb{R}^+ \ .$$

Proof. Note that $\mathbf{X} \to \theta_\tau \mathbf{X}$ is continuous, i.e. if

$$\left|\mathbf{X}^i_{st} - \tilde{\mathbf{X}}^i_{st}\right| \leq \varepsilon\omega(s,t)^{\frac{i}{p}}, \quad \forall\ (s,t) \in \triangle,\ i = 1, 2\ ,$$

then

$$\left|(\theta_\tau \mathbf{X})^i_{st} - (\theta_\tau \tilde{\mathbf{X}})^i_{st}\right| \leq \varepsilon\tilde{\omega}(s,t)^{\frac{i}{p}}, \ \ \forall\ (s,t) \in \triangle,\ i = 1, 2\ ,$$

where $\tilde{\omega}(s,t) = \omega(s+\tau, t+\tau)$. Now the conclusion follows from the uniqueness of Eq. (67) and Th. 5.

The goal of this section is to show that $(F_t(\mathbf{X}, z))$ is a flow of local diffeomorphisms.

Suppose $\mathbf{X}$ is a classical multiplicative functional, then $F_t(\mathbf{X}, z)$ is the unique solution of the ordinary differential equation,

$$dY_t = f(Y_t)\ dX^1_t, \quad Y_0 = z\ ,$$

so that $(F_t(\mathbf{X}, z))$ is a flow of diffeomorphisms, provided that f is smooth. If K_t denotes the differential $DF_t(\mathbf{X}, \cdot)(z)$, then $K_t \in \mathrm{hom}(W, W)$ and satisfies the following differential equation:

$$\begin{aligned} dF_t(\mathbf{X}, \cdot) &= f(F_t(\mathbf{X}, \cdot))\ dX_t, \\ dK_t &= \partial f(F_t(\mathbf{X}, \cdot)) \circ K_t\ dX_t, \\ (F_0(\mathbf{X}, \cdot), K_0) &= (z, \mathrm{id})\ . \end{aligned}$$

where $\partial f: W \to \mathrm{hom}(V, \mathrm{hom}(W, W))$,

$$\partial f(x)(\xi)(\eta) = \lim_{h\to 0} \frac{f(x+h\eta)(\xi) - f(x)(\xi)}{h}, \ \forall\ x \in W, \quad \xi \in V, \quad \eta \in W\ .$$

Moreover, K_t is invertible and K_t^{-1} satisfies the following differential equation:

$$\begin{aligned} dF_t(\mathbf{X}, \cdot) &= f(F_t(\mathbf{X}, \cdot))\ dX_t, \\ dK_t^{-1} &= \ -K_t^{-1} \circ \partial f(F_t(\mathbf{X}, \cdot))\ dX_t, \\ (F_0(X, \cdot), K_0^{-1}) &= (z,\ \mathrm{id}\ )\ . \end{aligned}$$

Let $H = W \oplus \mathrm{hom}(W, W)$, and let $\partial^1 f, \tilde{\partial}^1 f: H \to \mathrm{hom}(V, H)$ be two vector fields defined by

$$\begin{aligned} \partial^1 f(y,u)\xi\ &= (f(y)\xi, \partial f(y)(\xi) \circ u), \\ \tilde{\partial}^1 f(y,u)\xi\ &= (f(y)\xi, -u \circ \partial f(y)(\xi))\ , \end{aligned}$$

for all $(y,u) \in W \oplus \hom(W,W), \xi \in V$. Then we can rewrite above two equations as following:

$$dH_t = \partial^1 f(H_t)\delta X_t, \quad H_0 = (z, \mathrm{id}) \ ,$$

where $H_t = (F_t(\mathbf{X}, \cdot), K_t)$, and

$$d\tilde{H}_t = \tilde{\partial}^1 f(\tilde{H}_t)\delta X_t, \quad \tilde{H}_0 = (z, \mathrm{id}) \ ,$$

where $\tilde{H}_t = (F_t(\mathbf{X}, \cdot), K_t^{-1})$.

Theorem 6. *Let $\mathbf{X}\colon \triangle \to T^{2)}(V)$ be a geometric multiplicative functional with finite p-variation, and let $f\colon W \to \hom(V,W)$ be a $\mathrm{Lip}(r)$ vector field on W, $r \geq 4$. $F(\mathbf{X},z)$ denotes the Itô map associated with the vector field f and let $F_t(\mathbf{X},z) = F(\mathbf{X},z)^1_{0t} + z$. Then*

1) *The Itô map $F_t(\mathbf{X},\cdot)\colon W \to W$ is differentiable, and the differential $DF_t(\mathbf{X},\cdot)(z)$ is invertible.*
2) *Let $\mathbf{P}$, resp. $\mathbf{Q}$, be the multiplicative functional solution of the differential equation*:

$$\delta \mathbf{P} = \partial^1 f(P)\delta \mathbf{X}, \quad P_0 = (z, \ \mathrm{id}) \ , \tag{68}$$

resp.

$$\delta \mathbf{Q} = \tilde{\partial}^1 f(Q)\delta \mathbf{X}, \quad Q_0 = (z, \ \mathrm{id}) \ , \tag{69}$$

respectively. Then

$$\begin{aligned} DF_t(\mathbf{X},z) &= \mathrm{id} + \pi_{\hom(W,W)}(\mathbf{P})^1_{0t}, \\ DF_t(\mathbf{X},z)^{-1} &= \mathrm{id} + \pi_{\hom(W,W)}(\mathbf{Q})^1_{0t} \ . \end{aligned} \tag{70}$$

Proof. We prove the conclusions 1) and 2) together. Let $\mathbf{X}$ be of finite p-variation controlled by ω. Choosing a sequence of classical multiplicative functionals $\mathbf{X}(n)\colon \triangle \to T^{(2)}(V)$, such that $\mathbf{X}(n)$ are of finite p-variation controlled by ω, and

$$\left|\mathbf{X}(n)^i_{st} - \mathbf{X}^i_{st}\right| \leq \frac{1}{n}\omega(s,t)^{\frac{i}{p}}, \quad (s,t) \in \triangle, \ i = 1,2 \ ,$$

for $n = 1,2,\ldots$. By Th. 5, we have

$$\begin{aligned} \left|\mathbf{P}(n)^i_{st} - \mathbf{P}^i_{st}\right| &\leq \frac{1}{n}K\omega(s,t)^{\frac{i}{p}} \ , \\ \left|\mathbf{Q}(n)^i_{st} - \mathbf{Q}^i_{st}\right| &\leq \frac{1}{n}K\omega(s,t)^{\frac{i}{p}}, \quad \forall \ (s,t) \in \triangle, \ i = 1,2 \ , \end{aligned}$$

and

$$\left|\mathbf{Z}(n)^i_{st} - \mathbf{Z}^i_{st}\right| \leq \frac{1}{n}K\omega(s,t)^{\frac{i}{p}}, \quad \forall \ (s,t) \in \triangle, \ i = 1,2 \ ,$$

where K is a constant depending only on $p, \max\omega, r$ and Lipschitz constant M of f. For each n,

$$\mathbf{P}(n)^1_{0t} + (z, \mathrm{id}) = (F_t(\mathbf{X}(n), z), DF_t(\mathbf{X}(n), \cdot)(z)),$$
$$\mathbf{Q}(n)^1_{0t} + (z, \mathrm{id}) = \left(F_t(\mathbf{X}(n), z), DF_t(\mathbf{X}(n), \cdot)^{-1}(z)\right) ,$$

so that

$$|F_t(\mathbf{X}(n), z) - F_t(\mathbf{X}, z)| \le \frac{1}{n} K\omega(0, t)^{\frac{1}{p}} ,$$

and

$$\left| DF_t(\mathbf{X}(n), \cdot)(z) - \left(id + \pi_{\hom(W,W)}(\mathbf{P})^1_{0t} \right) \right| \le \frac{1}{n} K\omega(0, t)^{\frac{1}{p}} ,$$

which yields that $DF_t(\mathbf{X}, \cdot)(z)$ exists, and

$$DF_t(\mathbf{X}, \cdot)(z) = \mathrm{id} + \pi_{\hom(W,W)}(\mathbf{P})^1_{0t} .$$

Moreover,

$$\begin{aligned}
&(id + \pi_{\hom(W,W)}(\mathbf{P}(n))^1_{0t})(id + \pi_{\hom(W,W)}(\mathbf{Q}(n))^1_{0t}) \\
&\quad = DF_t(\mathbf{X}(n), z) DF_t(\mathbf{X}(n), z)^{-1} \\
&\quad = \mathrm{id}
\end{aligned}$$

for each n, so that

$$\begin{aligned}
&\left| \mathrm{id} - (\mathrm{id} + \pi_{\hom(W,W)}(\mathbf{P})^1_{0t})(\mathrm{id} + \pi_{\hom(W,W)}(\mathbf{Q})^1_{0t}) \right| \\
&\quad \le \frac{1}{n} K\omega(0, t)^{\frac{1}{p}}, \quad \forall\, n .
\end{aligned}$$

Letting $n \to \infty$ to get

$$\mathrm{id} + \pi_{\hom(W,W)}(\mathbf{Q})^1_{0t} = DF_t(\mathbf{X}, z)^{-1} .$$

Corollary 3. *Let* $\mathbf{X}, \tilde{\mathbf{X}} : \triangle \to T^{(2)}(V)$ *be two geometric multiplicative functionals with finite* p*-variation controlled by* ω*, and let* $f : W \to \hom(V, W)$ *be a* $\mathrm{Lip}(r)$ *vector field on* W, $r \ge 4$. *If*

$$\left| \mathbf{X}^i_{st} - \tilde{\mathbf{X}}^i_{st} \right| \le \varepsilon\omega(s, t)^{\frac{i}{p}}, \quad \forall\, (s, t) \in \triangle, \ i = 1, 2 ,$$

then

$$\begin{aligned}
&\left| F_t(\mathbf{X}, z) - F_t(\tilde{\mathbf{X}}, z) \right| \le \varepsilon K\omega(0, t)^{\frac{1}{p}}, \\
&\left| DF_t(\mathbf{X}, \cdot)(z) - DF_t(\tilde{\mathbf{X}}, \cdot)(z) \right| \le \varepsilon K\omega(0, t)^{\frac{1}{p}}, \quad \forall\, t \in I ,
\end{aligned}$$

and

$$\left| DF_t(\mathbf{X}, \cdot)(z)^{-1} - DF_t(\tilde{\mathbf{X}}, \cdot)(z)^{-1} \right| \leq \varepsilon K \omega(0, t)^{\frac{1}{p}}, \quad \forall\, t \in I \ ,$$

where K is a constant depending only $p, r, \max \omega$ and Lipschitz constant M of f.

In particular, if $\mathbf{X}$ is generated by a Brownian motion (see Example 1), then Corollary 3 gives an answer to an open problem proposed by Ikeda and Watanabe (see p.418 in [4]).

The following theorem follows from Th. 6 and Prop. 12 immediately.

Theorem 7. *Assume that $f: W \to \hom(V, W)$ is a C_b^∞ vector field, and that $\mathbf{X}: \triangle \to T^{(2)}(V)$ is a geometric multiplicative functional with finite p-variation. Let $F(\mathbf{X}, z)$ be the Itô map associated with the vector field f, and let $F_t(\mathbf{X}, z) = F(\mathbf{X}, z)^1_{0t} + z$. Then*

$$F_t(\mathbf{X}, \cdot): W \to W, \quad z \to F_t(\mathbf{X}, z)$$

is smooth, and $(F_t(\mathbf{X}, z))$ is a flow of local diffeomorphisms.

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