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On weak mixing in lattice models

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Summary. For lattice models on $\mathbb{Z}^d$, weak mixing is the property that the influence of the boundary condition on a finite decays exponentially with distance from that region. For a wide class of models on $\mathbb{Z}^2$, including all finite range models, we show that weak mixing is a consequence of Gibbs uniqueness, exponential decay of an appropriate form of connectivity, and a natural coupling property. In particular, on $\mathbb{Z}^2$, the Fortuin-Kasteleyn random cluster model is weak mixing whenever uniqueness holds and the connectivity decays exponentially, and the q-state Potts model above the critical temperature is weak mixing whenever correlations decay exponentially, a hypothesis satisfied if q is sufficiently large. Ratio weak mixing is the property that uniformly over events A and B occurring on subsets Λ and Γ, respectively, of the lattice, $|P(A \cap B)/P(A)P(B) - 1|$ decreases exponentially in the distance between Λ and Γ. We show that under mild hypotheses, for example finite range, weak mixing implies ratio weak mixing.

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1. Introduction

For lattice models of all types, a fundamental question is as follows: how do events in one region of the lattice influence the probabilities for events in another distant region? When these "regions" are single points, the influence is quantified by the two-point function – the

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covariance in spin systems, and the connectivity in percolation models. Even when this two-point function has good behavior – exponential decay as a function of separation distance – it is not at all clear that for large regions similar exponential decay must hold; one cannot simply sum the two-point function over all pairs of sites, one from each region, to measure the influence of one region on the other. This motivates the study of mixing – the quantification of dependence between distant regions.

We will focus here on weak mixing, defined as follows for lattice models. Let the single-spin space be a finite or countably infinite set S, and let σ_x denote the spin at site x. Let $\mathrm{Var}(\cdot,\cdot)$ denote total variation distance between measures and let $|\cdot|$ denote the ℓ^1 norm. Let P be a measure on $S^{\mathbb{Z}^d}$ and let $P_{\Lambda,\eta}$ denote the distribution of the configuration $\sigma_\Lambda = \{\sigma_x, x \in \Lambda\}$ under P given the boundary condition $\eta \in S^{\Lambda^C}$. Following [24] we say that P has the *weak mixing* property if for some $C, \lambda > 0$, for all finite sets Δ, Λ with $\Delta \subset \Lambda$,

$$\begin{aligned} &\sup\{\mathrm{Var}(P_{\Lambda,\eta}(\sigma_\Delta \in \cdot), P_{\Lambda,\eta'}(\sigma_\Delta \in \cdot)) : \eta, \eta' \in S^{\Lambda^C}\} \\ &\quad \leq C \sum_{x \in \Delta, y \in \Lambda^C} \exp(-\lambda|x-y|) \ . \end{aligned} \tag{1.1}$$

Roughly, the influence of the boundary condition on a finite region decays exponentially with distance from that region. Of course this implies that P is the unique distribution with conditional probabilities $\{P_{\Lambda,\eta} : \Lambda \subset \mathbb{Z}^d \text{ finite}, \eta \in S^{\Lambda^C}\}$. Weak mixing has been given various names in the literature- it is the "Very Strong Decay Property" in [12], and it is a special form of the ψ-mixing of [14].

A completely analogous definition can be made for percolation models, with Λ, Δ replaced by sets of nearest-neighbor bonds and distance between bonds measured by ℓ^1 distance $|\cdot|$ between their midpoints.

For the Ising model a seemingly different definition of weak mixing appears in [26]-only $\Delta = \{0\}$ is required in (1.1). But in the Ising case this is readily shown to be equivalent to the above definition (see [21], proof of Theorem 2, 1st Step.)

Let $d(A,B) := \min\{|x-y| : x \in A, y \in B\}$ and write $d(x,A)$ for $d(\{x\},A)$.

A related important property of measures on $S^{\mathbb{Z}^d}$ is *strong mixing for cubes*, also examined in [24] and [25]. Roughly, strong mixing for cubes says that when Λ is a cube, if two boundary conditions η, η' differ only at a single site z, then the influence of this difference decays exponentially in $d(z,\Delta)$, instead of exponentially in $d(\Lambda^C,\Delta)$ as in (1.1). This means roughly that the influence of the change at z cannot propagate along the boundary of Λ, whereas weak mixing only guarantees that the influence cannot propagate through the bulk of Λ. In three and higher dimensions, there are examples due to Shlosman [27] in which weak mixing holds but strong mixing for cubes fails, essentially because the influence of a single site does indeed propagate along the boundary. But in two dimensions, it was shown in [25] that for finite-range spin systems, weak mixing is actually equivalent to strong mixing for squares, as well as being equivalent to several other useful properties, such as exponential convergence to equilibrium of the associated Glauber dynamics, and the Dobrushin-Shlosman uniqueness condition of [12]. Unrestricted strong mixing (with Λ arbitrary) is not equivalent – it is a very strong property, not necessarily satisfied for common systems such as Ising models; see the example due to Schonmann described in ([24], p. 458–459.)

In arbitrary dimension, Dobrushin and Shlosman showed in [6] that weak mixing holds whenever their uniqueness condition is satisfied. For the Ising model in two dimensions, weak mixing is known to hold throughout the uniqueness region, other than at the critical point. For all supercritical temperatures and arbitrary external field, this was proved by Higuchi in [21] (in fact for arbitrary dimension.) For sufficiently low temperatures and arbitrary nonzero external field, weak mixing was proved in [24]. For sufficiently large fields at arbitrary temperature, weak mixing holds because the Dobrushin-Shlosman uniqueness condition is satisfied. For the remaining temperatures and external fields in the uniqueness region, weak mixing was proved in [26]. For the q-state Potts model in two dimensions with sufficiently large q, weak mixing above the critical temperature was proved in [28]. In this paper we will establish weak mixing for a very wide class of two-dimensional models, which includes the Ising model throughout the uniqueness region. For the Fortuin-Kastelyn random cluster model (abbreviated to "the FK model") in two dimensions, we will show that weak mixing holds everywhere in the uniqueness region where the connectivity or dual connectivity decays exponentially. For a Potts model in two dimensions above the critical temperature, we will show that weak mixing holds provided correlations decay exponentially, as is believed to always be the case. For general finite-range models we will show that weak mixing is a consequence of uniqueness, exponential decay of an appropriate form of connectivity, and a natural coupling property.

Let σ, η, etc. denote generic configurations in $S^{\mathbb{Z}^d}$. We say a function f on $S^{\mathbb{Z}^d}$ is *determined on* $A \subset \mathbb{Z}^d$ if $f(\sigma) = f(v)$ whenever $\sigma_x = v_x$ for all $x \in A$; an event A is determined on Λ if its indicator function δ_A

has this property. An event is *local* if it is determined on some finite Λ. Let $\mathfrak{F}_\Lambda$ denote the σ-field of all events determined on Λ. An event $A \in \mathfrak{F}_\Lambda$ may be viewed as a subset of either S^Λ or $S^{\mathbb{Z}^d}$; we will use these interpretations interchangeably, without serious risk of confusion.

Despite the exponential speed of the convergence given in (1.1), weak mixing does not give enough independence between distant events for certain applications. For example, for $m > 0$, consider the following events for a bond percolation model P on the square lattice having exponential decay of connectivity:

$$\begin{aligned} A_m &:= [(m,0) \leftrightarrow (m+m^2,0) \text{ by a path in the square} \\ &\qquad \{(x_1,x_2) : m \le x_1 \le m+m^2, -m^2/2 \le x_2 \le m^2/2\}] \\ B_m &:= [(-m,0) \leftrightarrow (-m-m^2,0) \text{ by a path in the square} \\ &\qquad \{(x_1,x_2) : -m-m^2 \le x_1 \le -m, -m^2/2 \le x_2 \le m^2/2\}] \ . \end{aligned}$$

Then $P(A_m)$ decays exponentially in m^2, and one would like to know that $P(A_m|B_m)$ also decays exponentially in m^2. The weak mixing property (1.1) can be re-expressed as

$$\sup\{|P(A|B) - P(A)| : A \in \mathfrak{F}_\Delta, B \in \mathfrak{F}_\Gamma\} \le C \sum_{x\in\Delta, y\in\Gamma} \exp(-\lambda|x-y|) \ ,$$

so weak mixing only ensures that $P(A_m|B_m)$ decays exponentially in m, which is much weaker. The difficulty may be avoided if, instead of knowing that differences in probabilities under different boundary conditions are exponentially small as in (1.1), one knows that ratios of such probabilities are exponentially close to 1. Problems of this type (with added complications) arise when one wishes to decompose a long open path in a percolation model, or a contour in the Ising or Potts model, into a number of segments, and express the probability of the path or contour as approximately the product of the probabilities of the segments, as in [2], [3], [5], [7] and [11]. We say that a measure P on $S^{\mathbb{Z}^d}$ has the *ratio weak mixing* property it from some $C, \lambda > 0$, for all sets $\Delta, \Gamma \subset \mathbb{Z}^d$,

$$\begin{aligned} &\sup\{|P(A\cap B)/P(A)P(B) - 1| : A \in \mathfrak{F}_\Delta, B \in \mathfrak{F}_\Gamma, P(A)P(B) > 0\} \\ &\quad \le C \sum_{x\in\Delta, y\in\Gamma} \exp(-\lambda|x-y|) \ , \end{aligned} \tag{1.2}$$

whenever the right side of (1.2) is less than 1. Ratio weak mixing appears considerably stronger than weak mixing when the event A has probability much smaller than the right side of (1.1). However, we will show that in fact, in all dimensions weak mixing implies ratio weak mixing for a large class of spin systems and percolation models. An analogous result for strong mixing (for general Λ, not restricted to squares) was established by Dobrushin and Shlosman [13] using completely different methods.

2. Background and definitions

Spin systems. A spin system with single-spin space $S = \{1, \ldots, q\}$ and range r is specified by an interaction $U = \{U_\Gamma : \Gamma \subset \mathbb{Z}^d, \text{diam}(\Gamma) \leq r\}$ where U_Γ is a real-valued function on $S^{\mathbb{Z}^d}$ determined on Γ. We assume U is translation invariant: if $\tau_k\sigma$ is given by $(\tau_k\sigma)_{x+k} = \sigma_x$ then $U_{\Gamma+k}(\tau_k\sigma) = U_\Gamma(\sigma)$. Let $(\sigma\eta)_\Lambda$ (or just $(\sigma\eta)$ if no confusion is likely) denote the configuration which coincides with σ on Λ and with η on Λ^C; we will call such a configuration a *blending* of σ and η. The Hamiltonian, or energy, of a configuration σ_Λ on a finite Λ subject to a boundary condition $\eta \in S^{\Lambda^C}$ is given by

$$H_{\Lambda,\eta}(\sigma_\Lambda) := \sum_{\Gamma:\Gamma\cap\Lambda\neq\phi} U_\Gamma((\sigma\eta)_\Lambda) \ .$$

The interaction U may depend on one or more parameters $p_1, \ldots, p_k$. The corresponding *Gibbs measure* on S^Λ at inverse temperature β is given by

$$P_{\Lambda,\eta,\beta}(\sigma_\Lambda) := \exp(-\beta H_{\Lambda,\eta}(\sigma_\Lambda))/Z_{\Lambda,\eta,\beta} \ ,$$

where

$$Z_{\Lambda,\eta,\beta} := \sum_{\sigma_\Lambda\in S^\Lambda} \exp(-\beta H_{\Lambda,\eta}(\sigma_\Lambda))$$

is the corresponding partition function. We say (*Gibbs*) *uniqueness holds* (at $(p_1, \ldots, p_k)$) if there is a unique limiting Gibbs measure as $\Lambda \nearrow \mathbb{Z}^d$.

The FKG property and FKG ordering. If S is $\{-1, 1\}$ or $\{0, 1\}$ then S^Λ has a natural partial ordering. A measure P on S^Λ has the *FKG property* if for every pair f, g of bounded nondecreasing functions on S^Λ, the covariance of f and g under P is nonnegative. P satisfies the *FKG lattice condition* if for every finite subset Γ of Λ, every $\eta \in S^{\Lambda\setminus\Gamma}$, and every $\sigma, \sigma' \in S^\Lambda$,

$$P_{\Gamma,\eta}(\sigma \vee \sigma')P_{\Gamma,\eta}(\sigma \wedge \sigma') \geq P_{\Gamma,\eta}(\sigma)P_{\Gamma,\eta}(\sigma') \ .$$

The FKG lattice condition implies the FKG property [17]. We say P *dominates* Q (in the FKG sense) if for every nondecreasing function f,

$$\int f\, dP \geq \int f\, dQ \ .$$

There then exists a coupling $\tilde{P}$ of P and Q such that $\tilde{P}(\{(\sigma, \sigma') : \sigma_x \geq \sigma'_x \text{ for all } x\}) = 1$.

Potts and Ising models. The q-state Potts model, with external field h applied to spin 1 is a spin system with $S = \{1, \ldots, q\}$ and Hamiltonian

$$H_{\Lambda,\eta}(\sigma_\Lambda) := \sum_{\langle xy \rangle : x \in \Lambda} \delta_{[(\sigma\eta)_\Lambda(x) = (\sigma\eta)_\Lambda(y)]} - h \sum_{x \in \Lambda} \delta_{[\sigma_x = 1]} \quad , \tag{2.1}$$

where the first sum is over all nearest-neighbor bonds $\langle xy \rangle$, i.e. unordered pairs with $|x - y| = 1$. We denote the corresponding Gibbs distribution on S^Λ by $P_{\Lambda,\eta,q,\beta,h}$.

Let η^s denote the configuration with every site having spin s, and define ρ^s similarly for bond configurations.

The model with free boundary condition is obtained by including only $x, y \in \Lambda$ in the first sum in (2.1). Alternatively, one can allow η to be an element of $\{0, 1, \ldots, q\}^{\Lambda^C}$; there is effectively no interaction of σ_Λ with those sites $x \in \Lambda^C$ where $\eta_x = 0$, which we call *empty sites*. (Note we allow empty sites only in the boundary condition; we are not considering site-diluted models here.)

For each q and d there is a critical point $0 < \beta_C(q, d) < \infty$ such that, for $h = 0$, uniqueness holds when $\beta < \beta_C(q, d)$ and not when $\beta > \beta_C(q, d)$.

The Ising model with external field h has $S = \{-1, 1\}$ and Hamiltonian.

$$H_{\Lambda,\eta,h}(\sigma_\Lambda) := - \sum_{\langle xy \rangle : x \in \Lambda} (\sigma\eta)_\Lambda(x)(\sigma\eta)_\Lambda(y) - h \sum_{x \in \Lambda} \sigma_x \quad ,$$

which is equivalent to the Potts model with $q = 2$ and external field $2h$.

The FK model. The facts given here, with additional details, may be found in [1], [16] or [19]. For $\Lambda \subset \mathbb{Z}^d$ let $\mathscr{B}(\Lambda)$ denote the set of all nearest-neighbor lattice bonds $\langle xy \rangle$ with $x, y \in \Lambda$, and let $\overline{\mathscr{B}}(\Lambda)$ denote the set of all nearest-neighbor lattice bonds $\langle xy \rangle$ with x or y in Λ. Define

$$\begin{aligned} \partial_{\text{in}}\Lambda &:= \{x \in \mathbb{Z}^d : x \in \Lambda, x \text{ adjacent to } \Lambda^C\}, \\ \partial_{\text{ex}}\Lambda &:= \{x \in \mathbb{Z}^d : x \notin \Lambda, x \text{ adjacent to } \Lambda\}, \\ \bar{\Lambda} &:= \Lambda \cup \partial_{\text{ex}}\Lambda \ ; \end{aligned}$$

here *adjacent* means separated by distance 1. We call $\bar{\Lambda}$ the *closure* of Λ. We let 1 and 0 stand for the open and closed states, respectively, of a bond, so the configuration space for the model with set $\mathscr{G}$ of bonds is $\{0, 1\}^{\mathscr{G}}$. The notation $(\omega\rho)_{\mathscr{C}}$ applies to blendings as for site models, for $\mathscr{C} \subset \mathscr{G}$. For Λ finite, $\rho \in \{0, 1\}^{\mathscr{B}(\Lambda^C)}$ and $\omega \in \{0, 1\}^{\overline{\mathscr{B}}(\Lambda)}$ let $N_{\Lambda,\rho}(\omega)$ denote the number of clusters (that is, connected components) in

$(\omega\rho)_{\overline{\mathscr{B}}(\Lambda)}$ which intersect Λ. Let $B(\omega)$ denote the number of open bonds in the configuration ω. We can construct an independent measure $P_{\text{ind},\Lambda,p}$ on $\{0,1\}^{\overline{\mathscr{B}}(\Lambda)}$ by taking each bond to be independently open with probability p, that is,

$$P_{\text{ind},\Lambda,p}(\omega) := p^{B(\omega)}(1-p)^{|\overline{\mathscr{B}}(\Lambda)|-B(\omega)} \ .$$

For $q \geq 1$ and $p \in [0,1]$ the *FK measure* $P_{\Lambda,\rho,q,p}$ on $\{0,1\}^{\overline{\mathscr{B}}(\Lambda)}$ is given by

$$P_{\Lambda,\rho,q,p}(\omega) := P_{\text{ind},\Lambda,p}(\omega)q^{N_{\Lambda,\rho}(\omega)}/Z_{\Lambda,\rho,q,p}, \quad \omega \in \{0,1\}^{\overline{\mathscr{B}}(\Lambda)} \ ,$$

where $Z_{\Lambda,\rho,q,p}$ is the partition function. When ρ is ρ^1 or ρ^0, we sometimes write w or f, for wired or free, in place of ρ^1 or ρ^0 respectively. The set $\{0,1\}^{\overline{\mathscr{B}}(\Lambda)}$ endowed with this measure is called the *FK model on* $\overline{\mathscr{B}}(\Lambda)$ *with parameters* (q,p) *and boundary condition* ρ. The measures

$$P_{*,q,p} := \lim_{\Lambda \nearrow \mathbb{Z}^d} P_{\Lambda,*,q,p}$$

on $\{0,1\}^{\mathscr{B}(\mathbb{Z}^d)}$ exist for $* = w$ or f, and are translation-invariant FKG measures. The set $\{0,1\}^{\mathscr{B}(\mathbb{Z}^d)}$ together with the measure $P_{w,q,p}$ (resp. $P_{f,q,p}$) forms the *FK model with parameters* (q,p) *and wired* (resp. *free*) *boundary condition*. Gibbs uniqueness holds for this model if and only if $P_{w,q,p} = P_{f,q,p}$. The measures $P_{*,q,p}$ satisfy the FKG lattice condition.

The FK model with external field. An external field can be introduced into the FK model as follows. Let $h \geq 0$. We append to the integer lattice a single *ghost size* z, connected by an *external bond* $\langle xz \rangle$ to each site x of $\mathbb{Z}^d$. The bonds of $\mathscr{B}(\mathbb{Z}^d)$ are called *internal bonds*. Under the independent measure, denoted $P_{\text{ind},\Lambda,p,h}$, each external bond is open with probability $1-(1-p)^h$, and each internal bond is open with probability p. Let $\mathscr{E}(\Lambda)$denote the set of all external bond with an endpoint in $\Lambda \subset \mathbb{Z}^d$. For Λ finite, in a slight abuse of notation we let Λ^c denote the complement of Λ in $\mathbb{Z}^d$, that is, excluding the ghost site. For $\rho \in \{0,1\}^{\mathscr{B}(\Lambda^c)\cup\mathscr{E}(\Lambda^c)}$ and $\omega \in \{0,1\}^{\overline{\mathscr{B}}(\Lambda)\cup\mathscr{E}(\Lambda)}$ let $N^+_{\Lambda,\rho}(\omega)$ denote the number of clusters in $(\rho\omega)_{\mathscr{B}(\Lambda^c)\cup\mathscr{E}(\Lambda^c)}$ which intersect Λ and do not contain z. We then have a corresponding FK measure

$$P_{\Lambda,\rho,q,p,h}(\omega) := P_{\text{ind},\Lambda,\text{p},h}(\omega)q^{N^+_{\Lambda,\rho}(\omega)}/Z_{\Lambda,\rho,q,p,h}, \quad \omega \in \{0,1\}^{\overline{\mathscr{B}}(\Lambda)\cup\mathscr{E}(\Lambda)} \ .$$

As for $h = 0$, the measures

$$P_{*,q,p,h} := \lim_{\Lambda \nearrow \mathbb{Z}^d} P_{\Lambda,*,q,p,h}$$

on $\{0,1\}^{\mathscr{B}(\mathbb{Z}^d)\cup\mathscr{E}(\mathbb{Z}^d)}$ exist for $* = w$ or f.

The marginals on internal bonds of FK measures with an external field are described as follows. For $\rho \in \{0,1\}^{\mathscr{B}(\Lambda^c)}$ and $\omega \in \{0,1\}^{\overline{\mathscr{B}}(\Lambda)}$

let $\mathscr{C}(\Lambda, \rho, \omega)$ denote the set of clusters of $(\rho\omega)_{\mathscr{B}(\Lambda^c)}$ which intersect Λ. For C a connected subgraph of the integer lattice, let

$$\psi_{q,p,h}(C) := 1 + (q-1)(1-p)^{h|C|} \ ,$$

where $|C|$ denotes the number of sites in C. The marginal on $\overline{\mathscr{B}}(\Lambda)$ under boundary condition ρ is then

$$P^{\mathrm{in}}_{\Lambda,\rho,q,p,h}(\omega) := P_{\mathrm{ind},\Lambda,p}(\omega)\Pi_{C\in\mathscr{C}(\Lambda,\rho,\omega)}\psi_{q,p,h}(C)/Z_{\Lambda,\rho,q,p,h}, \quad \omega \in \{0,1\}^{\overline{\mathscr{B}}(\Lambda)} \ .$$

The marginal on $\{0,1\}^{\mathscr{B}(\mathbb{Z}^d)}$ of $P_{*,q,p,h}$ is denoted $P^{\mathrm{in}}_{*,q,p,h}$, for $* = w$ or f The set $\{0,1\}^{\mathscr{B}(\mathbb{Z}^d)}$ together with the measure $P^{\mathrm{in}}_{w,q,p,h}$ (resp. $P^{\mathrm{in}}_{f,q,p,h}$) forms the *FK model with parameters* (q,p,h) *wired* (resp. *free*) *boundary condition.* Gibbs uniqueness holds for this model if and only if $P^{\mathrm{in}}_{w,q,p,h} = P^{\mathrm{in}}_{f,q,p,h}$, in which case we omit the w or f from the notation.

Lemma 2.1. *The FK model with parameters* $(q,p,h)(h \geq 0, q \geq 1)$ *satisfies the FKG lattice condition*

Proof. Fix Λ finite, $\omega, \omega' \in \{0,1\}^{\overline{\mathscr{B}}(\Lambda)}$, and a boundary condition ρ. We write P for $P^{\mathrm{in}}_{\Lambda,\rho,q,p,h}$. Let $\mathscr{F} := \{b \in \mathscr{B}(\bar{\Lambda}) : \omega_b = 1, \omega'_b = 0\}$, $\mathscr{G} := \{b \in \mathscr{B}(\bar{\Lambda}) : \omega_b = 0, \omega'_b \equiv 1\}$ and $\mathscr{H} := \mathscr{B}(\bar{\Lambda})\backslash(\mathscr{F} \cup \mathscr{G})$, so that for a configuration $\alpha \in \{0,1\}^{\overline{\mathscr{B}}(\Lambda)}$ we can write $\alpha = (\alpha_{\mathscr{F}}, \alpha_{\mathscr{G}}, \alpha_{\mathscr{H}})$. Note $\omega_{\mathscr{H}} = \omega'_{\mathscr{H}}$. We wish to show that

$$P(\omega \vee \omega')/P(\omega) \geq P(\omega')/P(\omega \wedge \omega') \tag{2.2}$$

so it is sufficient to show that

$f(\alpha_{\mathscr{F}}) := P((\alpha_{\mathscr{F}}, \rho^1_{\mathscr{G}}, \omega_{\mathscr{H}}))/P((\alpha_{\mathscr{F}}, \rho^0_{\mathscr{G}}, \omega_{\mathscr{H}}))$ is an increasing function of $\alpha_{\mathscr{F}}$, as (2.2) is equivalent to $f((\rho^1)_{\mathscr{F}}) \geq f((\rho^0)_{\mathscr{F}})$. When an open bond b is added to $\alpha_{\mathscr{F}}$, there are three possibilities:

(i) the end points of b are in a single cluster, in both $(\alpha_{\mathscr{F}}, \rho^0_{\mathscr{G}}, \omega_{\mathscr{H}})$ and $(\alpha_{\mathscr{F}}, \rho^1_{\mathscr{G}}, \omega_{\mathscr{H}})$,
(ii) the endpoints of b are in distinct clusters C and D in $(\alpha_{\mathscr{F}}, \rho^0_{\mathscr{G}}, \omega_{\mathscr{H}})$, and in distinct clusters E and F in $(\alpha_{\mathscr{F}}, \rho^1_{\mathscr{G}}, \omega_{\mathscr{H}})$,
(iii) the endpoints of b are in distinct clusters C and D in $(\alpha_{\mathscr{F}}, \rho^0_{\mathscr{G}}, \omega_{\mathscr{H}})$, and in a single cluster in $(\alpha_{\mathscr{F}}, \rho^1_{\mathscr{G}}, \omega_{\mathscr{H}})$.

Under (i), adding b does not change the value of f. Under (iii) f gets multiplied by the factor $g(C,D) := \psi_{q,p,h}(C)\psi_{q,p,h}(D)/\psi_{q,p,h} \ (C \cup D \cup \{b\})$. Since $\psi_{p,q,h}(C)$ is a decreasing function of $|C|$, this factor is at least 1. Under (ii) f gets multiplied by the factor $g(C,D)/g(E,F)$. It is easily checked that $g(C,D)$ is a decreasing function of $(|C|,|D|)$. Since $|E| \geq |C|$ and $|F| \geq |D|$, this shows the factor is again at least 1. □

The FK model with parameters (q, p, h) dominates Bernoulli bond percolation at density $p/(p+(1-p)q)$ on internal bonds, that is, for every Λ and ρ and every bond $b \in \mathscr{B}(\bar{\Lambda})$,

$$P_{\Lambda,\rho,q,p,h}(\omega_b = 1|\omega_e, e \neq b) \geq p/(p+(1-p)q) \text{ a.s.} \tag{2.3}$$

Let $[x \leftrightarrow y]$ denote the event that x is connected to y by a path of open bonds, and $[x \leftrightarrow \infty]$ the event that there is an infinite path of open bonds starting at x. (Throughout this paper, by a path we always implicitly mean a self- avoiding one.) For each $q \geq 1$ there is a critical point $0 < p_C(q,d) < 1$ such that $P_{*,q,p}(0 \leftrightarrow \infty)$ is 0 for $p < p_C(q,d)$ and positive for $p > p_C(q,d)$, for both $* = f$ and $* = w$. For $p < p_C(q,d)$ we have $P_{w,q,p} = P_{f,q,p}$.

For bond configurations, in place of (1.1) a suitable definition of weak mixing requires that

$$\sup\Big\{\mathrm{Var}(P_{\Lambda,\rho}(\omega_{\mathscr{B}(\Delta)} \in \cdot), P_{\Lambda,\rho'}(\omega_{\mathscr{B}(\Delta)} \in \cdot)) : \rho, \rho' \in \{0,1\}^{\mathscr{B}(\Lambda^C)}\Big\} \\ \leq C\Sigma_{x\in\Delta, y\in\Lambda^C} \exp(-\lambda|x-y|) \ . \tag{2.4}$$

Relations between FK and Potts/Ising models. Edwards and Sokal [15] observed that it is possible to construct both an FK model and the Potts model with boundary condition η, with the same value of q, on a single probability space, when

$$p = 1 - e^{-\beta} \ .$$

The construction, adapted here to general $h \geq 0$, can be done via the joint site-bond measure on $S^\Lambda \times \{0,1\}^{\overline{\mathscr{B}}(\Lambda)\cup\mathscr{E}(\Lambda)}$ given for a boundary condition η by

$$\tilde{P}_{\Lambda,\eta,q,p,h}(\sigma,\omega) := P_{\mathrm{ind},\Lambda,p,h}(\omega)\delta_{D(\Lambda,\eta)}(\sigma,\omega)/\tilde{Z}_{\Lambda,\eta,q,p,h} \ ,$$

where the event

$$\begin{aligned} D(\Lambda,\eta) := \{(\sigma,\omega) \in S^\Lambda \times \{0,1\}^{\overline{\mathscr{B}}(\Lambda)\cup\mathscr{E}(\Lambda)} : \\ &\omega_{\langle xy\rangle} = 0 \text{ for all } \langle xy\rangle \in \mathscr{B}(\bar{\Lambda}) \text{ with } (\sigma\eta)_\Lambda(x) \neq (\sigma\eta)_\Lambda(y), \\ &\text{and } \omega_{\langle xz\rangle} = 0 \text{ for all external bonds } \langle xz\rangle \in \mathscr{E}(\Lambda) \text{ with } \sigma_x \neq 1\} \end{aligned}$$

prohibits open bonds with differing states at the two endpoints (here we are implicitly assigning state $\eta_z = 1$ to the ghost site), and $\tilde{Z}_{\Lambda,\eta,q,p,h}$ is the partition function. The marginal of $\tilde{P}_{\Lambda,\eta,q,p,h}$ on S^Λ is then $P_{\Lambda,\eta,q,\beta,h}$, the q-state Potts model on Λ with boundary condition η at inverse temperature β and external field h. The marginal on $\{0,1\}^{\overline{\mathscr{B}}(\Lambda)\cup\mathscr{E}(\Lambda)}$ is $P_{\Lambda,\rho^1,q,p,h}(.\,|A(\Lambda,\eta))$, where $A(\Lambda,\eta)$ is the event that no two sites $x, y \in \partial_{\mathrm{ex}}\Lambda \cup \{z\}$ with $\eta_x \neq \eta_y$ are connected by a path, entirely within $\overline{\mathscr{B}}(\Lambda) \cup \mathscr{E}(\Lambda)$, of open bonds. Let

$$K(\Lambda,\eta,\omega) := |\{x \in \Lambda : x \leftrightarrow y \text{ for some } y \in \partial_{\mathrm{ex}}\Lambda \text{ with } \eta_y \neq 1\}|,$$

$$D_0(\Lambda,\eta) := \{\omega \in \{0,1\}^{\overline{\mathscr{B}}(\Lambda)} : \omega_{\langle xy\rangle} = 0 \text{ for all } \langle xy\rangle \in \overline{\mathscr{B}}(\Lambda) \text{ with } (\sigma\eta)_\Lambda(x) \neq (\sigma\eta)_\Lambda(y)\}\ .$$

The marginal of $\tilde{P}_{\Lambda,\eta,q,p,h}$ on internal bonds is the measure

$$P^{\mathrm{in}}_{\Lambda,\eta,q,p,h}(\omega) := P_{\mathrm{ind},\Lambda,p}(\omega)(1-p)^{hK(\Lambda,\eta,\omega)}\delta_{D_0(\Lambda,\eta)}(\omega) \times \Pi_{C\in\mathscr{C}(\Lambda,\rho^1,\omega)}\psi_{q,p,h}(C)/Z^{\mathrm{in}}_{\Lambda,\rho,q,p,h}, \quad \omega \in \{0,1\}^{\overline{\mathscr{B}}(\Lambda)}\ .$$

The internal bonds together with this measure from the *conditioned FK model on $\overline{\mathscr{B}}(\Lambda)$ with parameters (q,p,h) and boundary condition η.*

Remark 2.2. The measure $P^{\mathrm{in}}_{\Lambda,\eta,q,p,h}$ is the wired-boundary measure $P^{\mathrm{in}}_{\Lambda,\rho^1,q,p,h}$ conditioned on $D_0(\Lambda,\eta)$ (a decreasing event) and weighted by $(1-p)^{hK(\Lambda,\eta,\cdot)}$ (a decreasing function), so it is dominated by the wired boundary measure. When $\eta = \eta^1$, it is equal to the wired boundary measure. When $\eta = \eta^0$, $P^{\mathrm{in}}_{\Lambda,\eta,q,p,h}$ is the free-boundary measure $P^{\mathrm{in}}_{\Lambda,\rho^0,q,p,h}$.

One could similarly construct a joint measure corresponding to nonnegative external fields applied to several spins simultaneously in the Potts model, or apply a negative external field to some spin i by applying the opposite positive field to every $j \neq i$, but we will not consider these variations here.

All of this may be interpreted in terms of constructions of the FK model from the Potts model and vice versa, as follows.

C1. *Construction of the Potts model from the conditioned FK model.* Given $\Lambda \subset \mathbb{Z}^d$, an integer $q \geq 2$, a boundary condition η on Λ^C for the q-state Potts model, an inverse temperature $\beta > 0$ and an external field $h \geq 0$ applied to spin 1, first construct a realization of the conditioned FK model with parameters (q,p,h) and boundary condition η, where $p = 1 - \mathrm{e}^{-\beta}$. For each $C \in \mathscr{C}(\Lambda,\rho^1,\omega)$ which contains a site $y \in \partial_{\mathrm{ex}}\Lambda$ with $\eta_y = i$, we assign spin i to each site $x \in C \cap \Lambda$. For each $C \in \mathscr{C}(\Lambda,\rho^1,\omega)$ with $C \cap \partial_{\mathrm{ex}}\Lambda = \phi$, select independently a spin in $\{1,..,q\}$, with probability proportional to 1 for spin 1 and to $\mathrm{e}^{-\beta h|C|}$ for each spin $i \geq 2$; assign this spin to each site $x \in C$. The resulting site configuration is the q-state Potts model on Λ with boundary condition η, inverse temperature β and external field h applied to spin 1.

C2. *Construction of the conditioned FK model from the Potts model.* Suppose $0 \leq p \leq 1$, $h \geq 0, q \geq 1$ is an integer and η is a boundary condition on Λ^C. Define β by $p = 1 - \mathrm{e}^{-\beta}$. First construct a realization of the q-state Potts model on Λ at (β, h) with boundary condition η.

Then do independent bond percolation at density p on all bonds in $\overline{\mathscr{B}}(\Lambda)$ for which the states at the two endpoints are equal. The resulting bond configuration is the conditioned FK model on $\overline{\mathscr{B}}(\Lambda)$ with parameters (q, p, h) and boundary condition η.

Covariance versus connectivity. As is well known, it follows easily from C1 that when $h = 0$ and $p = 1 - \mathrm{e}^{-\beta}$,

$$q^2 \mathrm{cov}_{f,q,\beta,h}(\delta_{[\sigma_x=1]}, \delta_{[\sigma_y=1]}) = (q-1) P^{\mathrm{in}}_{f,h,p,q}(x \leftrightarrow y) \ , \qquad (2.5)$$

where $cov_{f,q,\beta,h}$ denotes covariance for the q-state Potts model on the full lattice, with parameters (β, h)

Exponential decay of connectivity. By a *path* from x to y in a site configuration we mean a sequence of distinct sites $x = x_0, x_1, \ldots, x_n = y$ such that x_i is adjacent to x_{i+1} for each i. Let $I \subset S$. An I-*site* is a site at which the spin is an element of I. An I-*path* is a path consisting of I-sites. We write $[x \leftrightarrow_I y]$ for the event that there is a I-path from x to y. The I-*cluster* $C_I(\Gamma, \sigma)$ of a set Γ of sites in a configuration σ consists of those sites which are connected to Γ by an I-path. If P is a measure on $S^{\mathbb{Z}^d}$ for which there exist positive constants C and λ such that

$$P[x \leftrightarrow_I y] \leq C \exp(-\lambda |x - y|) \text{ for all } x, y \ ,$$

we say P has *exponential decay of I-connectivity*

The boundary coupling property. *A coupling* of two measures P_1 and P_2 on S^Λ is a measure $\tilde{P}$ on $S^\Lambda \times S^\Lambda$ with marginals P_1 and P_2 (in order). The set of all such couplings is denoted $\kappa(P_1, P_2)$. If P_1 and P_2 are conditional distributions of some P given boundary conditions η_1 and η_2, we also say $\tilde{P}$ is a coupling of η_1 and η_2 under P; we write $\kappa_P(\eta_1, \eta_2)$ for the set of all such couplings.

Let $s' \in S$ and let I be either $\{s'\}$ or $S \backslash \{s'\}$. We say that a measure P on $S^{\mathbb{Z}^d}$ has the *boundary coupling property with respect to I* if for some $s \in I$, for every finite Λ and every boundary condition η on Λ^C, there exists a coupling $\tilde{P} \in \kappa_P(\eta, \eta^s)$ with the property that

$$\begin{aligned} &\tilde{P}[\{(\sigma, \sigma') \in S^\Lambda \times S^\Lambda : \sigma_x = \sigma'_x \text{ for every} \\ &\quad x \in \Lambda \cap C_I(\partial_{\mathrm{ex}}\Lambda, \sigma)^C \cap C_I(\partial_{\mathrm{ex}}\Lambda, \sigma')^C\}] = 1 \ . \end{aligned} \qquad (2.6)$$

For bond models, we say that a measure P on $\{0,1\}^{\mathscr{B}(\mathbb{Z}^d)}$ has the *boundary coupling property* if for every finite Λ and every boundary condition ρ on $\mathscr{B}(\Lambda^C)$, there exists a coupling $\tilde{P} \in \kappa_P(\rho, \rho^1)$ with the property that

$$\tilde{P}[\{(\omega,\omega') \in \{0,1\}^{\overline{\mathscr{B}}(\Lambda)} \times \{0,1\}^{\overline{\mathscr{B}}(\Lambda)} : \omega_e = \omega'_e \text{ for every } \\ \text{e} \in C(\partial_{\text{ex}}\Lambda,\omega)^C \cap C(\partial_{\text{ex}}\Lambda,\omega')^C\}] = 1 \ , \tag{2.7}$$

where $C(\partial_{\text{ex}}\Lambda, \omega)$ denotes the boundary cluster in ω, that is, the union of the connected components of the sites of $\partial_{\text{ex}}\Lambda$ in the configuration ω.

The boundary coupling property says the two configurations can be made to coincide outside the combined I-clusters of the boundary in both configurations. For the Ising model with $I = \{1\}$ or $\{-1\}$, this property is a well-known consequence of the FKG and Markov properties of the model; in fact the two configurations can be made to coincide outside the I-cluster of the boundary in σ' alone. Similar ideas carry over to other models, as we summarize in the next lemma, which includes the Ising model as a special case.

Lemma 2.3. *For a measure P on $\{0,1\}^{\mathbb{Z}^d}$, suppose that*

(i) *for every finite Λ, every pair of boundary conditions η, η' on Λ^c with $\eta \le \eta'$, and every $x \in \Lambda$,*
$$P_{\Lambda,\eta}(\sigma_x = 1) \le P_{\Lambda,\eta'}(\sigma'_x = 1) \ ;$$

(ii) *for every finite Λ, and every boundary condition η on Λ^C with $\eta_x = 0$ for all $x \in \partial_{\text{ex}}\Lambda$,*
$$P_{\Lambda,\eta} = P(\cdot \mid \eta_x = 0 \text{ for all } x \in \partial_{\text{ex}}\Lambda) \ .$$

Then P has the boundary coupling property with respect to $I = \{1\}$.

Proof. Order the sites of $\Lambda = \{x_1, .., x_m\}$ in such a way that x precedes y in the ordering if $d(x,\Lambda^C) < d(y,\Lambda^C)$ (for example, spiraling inwards if Λ is a cube.) We select the pairs $(\sigma_{x_i}, \sigma'_{x_i})$ one at a time, as follows. Let $R_0 = \phi$ and suppose some set R_n of sites has been selected, and the corresponding values (σ_x, σ'_x) chosen, by time n. Suppose also that $\sigma_x \le \sigma'_x$ for all $x \in R_n$. At time $n+1$, if $R_n = \Gamma$ we let i be the least index, if any, such that

$$\text{site } x_i \text{ has not been selected and some site adjacent to } x_i \text{ is connected} \\ \text{to } \partial_{\text{ex}}\Lambda \text{ in } \sigma' \text{ by a } \{1\}\text{-path of previously selected sites.} \tag{2.8}$$

We then have $(\eta_{\Lambda^C}, \sigma_\Gamma) \le (\eta'_{\Lambda^C}, \sigma'_\Gamma)$, so from (i),

$$\begin{aligned} P_{\Lambda,\eta}(\sigma_{x_i} = 1 | \sigma_x, x \in \Gamma) &= P_{\Lambda\setminus\Gamma,(\eta_{\Lambda^C},\sigma_\Gamma)}(\sigma_{x_i} = 1) \\ &\le P_{\Lambda\setminus\Gamma,(\eta'_{\Lambda^C},\sigma'_\Gamma)}(\sigma_{x_i} = 1) = P_{\Lambda,\eta'}(\sigma'_{x_i} = 1 | \sigma'_x, x \in \Gamma) \ . \end{aligned} \tag{2.9}$$

Let p and p' denote the probabilities on the left and right sides of (2.9), respectively. Then let $(\sigma_{x_i}, \sigma'_{x_i})$ be $(0,0)$ with probability $1-p'$, $(0,1)$ with probability $p'-p$ and (1,1) with probability p. Let τ be the first time at which there are no longer any sites satisfying (2.8). Then R_τ is necessarily the closure $\overline{C}_{\{1\}}(\partial_{\text{ex}}\Lambda, \sigma')$, so $\sigma_x = \sigma'_x = 0$ for all $x \in \partial_{\text{ex}} C_{\{1\}}(\partial_{\text{ex}}\Lambda, \sigma')$. But then by (ii), the inequality in (2.9) becomes an equality from time τ onward. But this means the coupling we have constructed satisfies

$$\sigma_x = \sigma'_x \text{ for every } x \in \Lambda \cap C_I(\partial_{\text{ex}}\Lambda, \sigma')^c \ ,$$

which establishes the boundary coupling property. □

The analog of Lemma 2.3 for bond models if as follows; the proof is the same.

Lemma 2.4. *For a measure P on $\{0,1\}^{\mathscr{B}(\mathbb{Z}^d)}$, suppose that*

(i) *for every finite set $\mathscr{G}$ of bonds, every pair of configurations ρ, ρ' on $\mathscr{G}^c$ with $\rho \le \rho'$, and every $e \in \mathscr{G}$,*

$$P(\omega_e = 1 | \omega_{\mathscr{G}^C} = \rho) \le P(\omega_e = 1 | \omega_{\mathscr{G}^C} = \rho') \ ;$$

(ii) *for every finite Γ, every configuration ρ on $\mathscr{B}(\Lambda)^C$ with $\rho_{\langle xy\rangle} = 0$* for every $x \in \partial_{\text{in}}\Lambda$ and $y \in \partial_{\text{ex}}\Lambda$, $P(\cdot \mid \omega_{\mathscr{B}(\Lambda)^c} = \rho_{\mathscr{B}(\Lambda)^c}) \ = P(\cdot \mid \omega_{\langle xy\rangle} = 0$ for every $x \in \partial_{\text{in}}\Lambda$ and $y \in \partial_{\text{ex}}\Lambda)$.

Then P has the boundary coupling property.

For a spin system, a natural way to create a coupling which establishes the boundary coupling property is to first create a similar coupling for a graphical representation of the model. This is exactly what Lemma 6.2 below does for the Potts model and its graphical representation, the FK model. In [4] the same method is applied to establish the boundary coupling property with respect the set I of all "nonempty" spins in the Potts lattice gas. For a wide class of spin systems, there is a graphical representation in which the probabilites of bond configurations are given by weights which are products of weights of individual clusters; see [6], [10]. The FK model is an example of this, with the weight of an individual cluster C being $q(p/(1-p))^{B(C)}$, where $B(C)$ is the number of bonds in C. It is easy to see that assumption (ii) of Lemma 2.4 holds for any such model. Assumption (i), however, is essentially a special case of the FKG property, so it will hold less generally. When Lemma 2.4 cannot be used due to the failure of assumption (i), an alternative is to create a 3-way coupling $\tilde{P}$ of $P_{\Lambda,\rho}, P_{\Lambda,\rho^1}$, and a third measure P' which has the FKG property and which dominates both $P_{\Lambda,\rho}, P_{\Lambda,\rho^1}$, For example, for graphical representations of spin systems with group symmetry, P' can be an FK model with parameters chosen as in ([16], Proposition 1.) The 3-way coupling should satisfy

$$\tilde{P}(\{(\omega, \omega', \omega''): \omega \le \omega'' \text{ and } \omega' \le \omega''\}) = 1$$

and should be such that the three configurations coincide outside the boundary cluster of the largest configuration, ω''. This coupling does not quite establish the boundary coupling property as such, but its existence can be substituted for the boundary coupling property e.g. in our Theorem 3.1, provided that P' has exponential decay of connectivity.

Uniqueness of distributions with given specification. We say *uniqueness holds* for a measure P on $S^{\mathbb{Z}^d}$ if P is the only measure with conditional distributions $\{P_{\Lambda,\eta} : \Lambda \subset \mathbb{Z}^d \text{ finite}, \ \eta \in S^{\Lambda^C}\}$. If P is quasilocal and uniqueness holds, then for each finite Δ,

$$\sup\{\mathrm{Var}(P_{\Lambda,\eta}(\sigma_\Delta \in \cdot), P_{\Lambda,\eta'}(\sigma_\Delta \in \cdot)) : \ \eta, \eta' \in S^{\Lambda^C}\} \to 0 \quad \text{as } \Lambda \nearrow \mathbb{Z}^d \ ; \tag{2.10}$$

this follows from ([18], Theorem 4.17.) Conversely (2.10) implies that uniqueness holds. We will refer to (2.10) as *boundary negligibility*. For finite-range spin systems, then, boundary negligibility is equivalent to uniqueness of Gibbs distributions.

Bounded influence per site or bond. For $\Lambda \subset \mathbb{Z}^d$ and $r > 0$ let

$$\Lambda^r := \{x \in \mathbb{Z}^d : d(x, \Lambda) \le r\} \ .$$

We say that the measure P on $S^{\mathbb{Z}^d}$ has *bounded influence per site* if there exist $r \ge 1$ and $c > 0$ such that for every $n \ge 1$, every finite $\Lambda \subset \mathbb{Z}^d$, every $\eta, \eta' \in S^{\Lambda^C}$ which differ at atmost n sites in $\Lambda^r \backslash \Lambda$, and every $A \in \mathscr{F}_\Lambda$,

$$P_{\Lambda,\eta}(A) \le e^{cn} P_{\Lambda,\eta'}(A) \ . \tag{2.11}$$

The most obvious bond analog is not satisfied by the FK model due to lack of finite range, so for bond models we make a different definition as follows. Fix $r \ge 1$ and Λ finite. For $\rho, \rho' \in \{0,1\}^{\mathscr{B}(\Lambda^C)}$ let

$$R_{\Lambda,r}(\rho) := |\{x \in \partial_{\mathrm{ex}}\Lambda : x \leftrightarrow \partial_{\mathrm{ex}}(\Lambda^r) \text{ in } \mathscr{B}(\Lambda^C)\}|$$

and

$$D_{\Lambda,r}(\rho, \rho') := |\{e \in \overline{\mathscr{B}}(\Lambda^r) \backslash \overline{\mathscr{B}}(\Lambda) : \rho_e \ne \rho'_e\}|$$

We say that the measure P on $\{0,1\}^{\mathscr{B}(\mathbb{Z}^d)}$ has *bounded influence per bond* if there exist $r \ge 1$ and $c > 0$ such that for every finite $\Lambda \subset \mathbb{Z}^d$, every $\rho, \rho' \in \{0,1\}^{\mathscr{B}(\Lambda^C)}$, and every $A \in \mathscr{F}_\Lambda$,

$$P_{\Lambda,\rho}(A) \le \exp(c[D_{\Lambda,r}(\rho, \rho') + R_{\Lambda,r}(\rho) + R_{\Lambda,r}(\rho')]) P_{\Lambda,\rho'}(A) \ . \tag{2.12}$$

For the FK model with parameters $(q, p, \underline{h})$ this property is an easy consequence of the fact that for $\omega \in \{0,1\}^{\mathscr{B}(\Lambda)}$ and $r \ge 1$ we have

$$|N_{\Lambda,\rho'}(\omega) - N_{\Lambda,\rho}(\omega)| \le D_{\Lambda,r}(\rho,\rho') + R_{\Lambda,r}(\rho) + R_{\Lambda,r}(\rho') \ . \tag{2.13}$$

Controlling regions. For P a measure on $S^{\mathbb{Z}^d}$, $\Lambda \subset \mathbb{Z}^d$ finite, $\eta \in S^{\Lambda^C}$, we call $\Omega \subset A^C$ a *controlling region* for Λ and η if for every $\eta' \in S^{\Lambda^C}$ such that $\eta = \eta'$ on Ω, we have $P_{\Lambda,\eta} = P_{\Lambda,\eta'}$. We say P has *exponentially bounded controlling regions* if there exists constants $C, \lambda > 0$ such that for every choice of finite sets Λ and $\Omega \subset \Lambda^C$,

$$\begin{aligned} &P(\{\eta \in S^{\Lambda^C} : \Omega \text{ is not a controlling region for } \Lambda \text{ and } \eta\}) \\ &\quad \le C \sum_{x\in\Lambda, y\in\Omega^C\setminus\Lambda} \exp(-\lambda|x-y|) \ . \end{aligned}$$

(These definitions adapt straightforwardly to bond models.) In finite-range models $\Lambda^r\setminus\Lambda$ is always a controlling region, where r is the range. It follows easily from uniqueness of the infinite cluster (which is proved in [8]) that for the FK model on $\mathscr{B}(\mathbb{Z}^d)$ there is a.s. a finite controlling region for a given Λ, but this region is not uniformly bounded. However, it is easy to see that for

$$\Omega = \{y \in \Lambda^C : x \leftrightarrow y \text{ for some } x \in \partial_{\text{ex}}\Lambda\} \ ,$$

$\overline{\mathscr{B}}(\Omega)$ is a controlling region (see the proof of Theorem 4.1 below.) Therefore the FK model has exponentially bounded controlling regions whenever the connectivity decays exponentially. In two dimensions, it is also sufficient that the dual connectivity decay exponentially.

3. Statement of main results

Our first main result covers general two-dimensional models. The proof will be given in Section 4.

Theorem 3.1. (i) *Suppose S is finite or countably infinite, $s \in S, I = \{s\}$ or $S\setminus\{s\}$, and P is a measure on $S^{\mathbb{Z}^2}$ for which boundary negligibility, bounded influence per site, exponential decay of I-connectivity, and the boundary coupling property with respect to I all hold. Then P has the weak mixing property.*

(ii) *Suppose P is a measure on $\{0,1\}^{\mathscr{B}(\mathbb{Z}^2)}$ for which boundary negligibility, bounded influence per bond, exponential decay of connectivity, and the boundary coupling property all hold. Then P has the weak mixing property.*

For a finite-range spin system, boundary negligibility is equivalent to Gibbs uniqueness, so we have the following corollary.

Corollary 3.2. *Suppose S is finite or countably infinite, $s \in S, I = \{s\}$ or $S \backslash \{s\}$, and P is a Gibbs distribution on $S^{\mathbb{Z}^2}$ of a finite-range spin system. If Gibbs uniqueness, exponential decay of I-connectivity, and the boundary coupling property with respect to I all hold, then P has the weak mixing property.*

The next result strengthens the conclusion of Theorem 3.1 for many models of interest. Note that it is not restricted to two dimensions. The proof is in Section 5.

Theorem 3.3. *Suppose S is finite or countably infinite and P is a measure on $S^{\mathbb{Z}^d}$ or $\{0,1\}^{\mathscr{B}(\mathbb{Z}^d)}$ which has the weak mixing property and exponentially bounded controlling regions. Then P has the ratio weak mixing property.*

The remaining results in this section are for specific models. The proofs are in Section 6. We begin with the FK model. For the Potts model we assume the external field, if any, is nonnegative and is applied to spin to 1 only. Being finite-range spin systems, Potts and Ising models (i) have exponentially bounded controlling regions, (ii) are quasilocal so that uniqueness of Gibbs distributions implies boundary negligibility, and (iii) have bounded influence per site.

Theorem 3.4. *If the FK model on $\mathscr{B}(\mathbb{Z}^2)$ with parameters (q,p,h) and wired boundary condition has exponential decay of connectivity, then it has the ratio weak mixing property.*

Remark 3.5. It follows from the definitions that in two dimensions, weak mixing for the FK model with parameters (q,p) is equivalent to weak mixing for the corresponding measure on dual bonds, which is just the FK model with parameters (q,p^*), where p^* is dual to p. Therefore for $h = 0$, exponential decay of dual connectivity under free boundary conditions is also sufficient to yield ratio weak mixing.

Our next result, for the Potts model without external field, will be proved using Theorem 3.4 and the fact that exponential decay of correlations in the Potts model corresponds to exponential decay of connectivity in the FK model.

Theorem 3.6. *For the q-state Potts model on $\mathbb{Z}^2$ without external field at inverse temperature $\beta < \beta_C$, weak mixing holds if and only if correlations decay exponentially.*

When a positive external field is applied to one of the spins of the Potts model, exponential decay of correlations for the Potts model no longer corresponds to exponential decay of connectivity for the FK model, so Theorem 3.4 cannot be used. However, we can substitute the hypothesis of exponential decay of I-connectivity, as follows.

Theorem 3.7. *Suppose that for some $q \geq 1, \beta > 0$ and $h \geq 0$, the q-state Potts model on $\mathbb{Z}^2$ at inverse temperature β, with external field h applied to spin* 1, *has either exponential decay of* 1*-connectivity or exponential decay of $\{2, \ldots, q\}$-connectivity, and uniqueness holds at (β, h). Then ratio weak mixing holds at (β, h).*

For fixed β, as h varies one expects a sharp transition at some critical point $h_C(\beta, q) \geq 0$ from exponential decay of 1-connectivity to exponential decay of $\{2, \ldots, q\}$-connectivity, with $h_C(\beta, q) = 0$ for $\beta > \beta_C(q)$, where $\beta_C(q)$ denotes the critical point above which there is phase coexistence when $h = 0$. This has not yet been proven except for $q = 2$ ([9], [19]), but if it is true then Theorem 3.7 gives ratio weak mixing for all (β, h) with $h \geq 0$, except where $h = h_C(\beta, q)$.

For the Ising model on $\mathbb{Z}^2$, Chayes, Chayes and Schonmann [9] established exponential decay of (-1)-connectivity in the plus phase, and of 1-connectivity in the minus phase, for $h = 0$ and $\beta > \beta_C$. Since the model with $h = 0$ dominates the model for fixed $h < 0$ in the FKG sense, the 1-connectivity also decays exponentially when $h < 0$ and $\beta > \beta_C$. If follows from Lemma 1 of [26] that for $h < 0$ the model at (β_C, h) is FKG-dominated by the model at (β, h') for some $h' < 0$ and $\beta > \beta_C$, so exponential decay of the 1-connectivity also holds when $h < 0$ and $\beta = \beta_C$. The results for $h > 0$ are symmetric. For the Ising model on $\mathbb{Z}^d$, weak mixing when $\beta < \beta_C$, with h arbitrary, was proved by Higuchi [21]. Thus we obtain the following from Theorems 3.7 and 3.3.

Corollary 3.8. *For the Ising model at (β, h) on $\mathbb{Z}^d$, ratio weak mixing holds provided $\beta < \beta_C$. For $d = 2$, ratio weak mixing also holds whenever $h \neq 0$.*

For $d = 2, \beta \geq \beta_c$, and $h \neq 0$, this provides an alternative to the proof of weak mixing given by Schonmann and Shlosman in [26].

Finally, using Theorems 3.3 and 3.6 we will give an alternate proof, and an improvement, of the weak-mixing result of [28], as follows.

Corollary 3.9. *Suppose $q \geq 26$. For the q-state Potts model on $\mathbb{Z}^2$ without external field, ratio weak mixing holds at all inverse temperatures $\beta < \beta_C$.*

4. Proof of Theorem 3.1

For simplicity of exposition we will consider only translation-invariant P, but the proof works with no significant changes for general P.

Further, we will only prove (ii), as the proof of (i) is similar but simpler. For bond models, let us write $P_{\Lambda,\rho}$ for $P(\cdot|\omega_{\overline{\mathscr{B}}(\Lambda)^C} = \rho_{\overline{\mathscr{B}}(\Lambda)^C})$. (This notation makes sense even if ρ is a configuration on a set larger than $\overline{\mathscr{B}}(\Lambda)^C$.) Given $A \subset S^{\mathbb{Z}^d}$, $\sigma \in S^{\mathbb{Z}^d}$ and $x \in \mathbb{Z}^d$ define $\tau_x\sigma$ by $(\tau_x\sigma)_y = \sigma_{y-x}$ and let $\tau_x A := \{\sigma : \tau_{-x}\sigma \in A\}$. Let

$$\Lambda(L,x) := \{y \in \mathbb{Z}^d : |y-x| \le L\}, \quad \Lambda(L) := \Lambda(L,0) \ .$$

We begin with a sketch of the proof. Consider a bond model P, and $\Lambda = \Lambda(L)$ for some $L > 0$. For $M < L$ the configuration in $\overline{\mathscr{B}}(\Lambda(L))\backslash\overline{\mathscr{B}}(\Lambda(L-M))$, together with the boundary condition on $\overline{\mathscr{B}}(\Lambda(L))^C$, acts as an "inner boundary condition" for events occurring in $\overline{\mathscr{B}}(\Lambda(L-M))$. We will show that, in the coupling of ρ^1 and arbitrary $\rho \in \{0,1\}^{\mathscr{B}(\Lambda(L)^C)}$ promised by the boundary coupling property, the inner boundary conditions seen by $\overline{\mathscr{B}}(\Lambda(L-M))$ in the two coupled configurations differ in a sense at only a small fraction of the sites in $\partial_{ex}\Lambda(L-M)$, except with exponentially small (in L, the size of the boundary) probability. This small probability is the first term on the right side of (4.10)–(4.12). For any event $A \in \mathscr{F}_{\overline{\mathscr{B}}(\Lambda(L-M))}$, conditionally on this similarity of inner boundary conditions, the probabilities of A under $P_{\Lambda(L),\rho^1}$ and under $P_{\Lambda(L),\rho}$ (or under P) differ by a factor of at most $\exp(108c\varepsilon L)$ for some constant c, by bounded influence per bond. This yields (4.10)–(4.12). In particular, since $P[0 \leftrightarrow \partial_{\text{in}}(\Lambda(L/2))]$ decays exponentially, by choosing ε small enough we can ensure that the factor of $\exp(108c\varepsilon L)$ does not destroy this exponential decay, so it holds uniformly over all ρ – see (4.13). Form this and the boundary coupling property one easily obtains weak mixing. Note it is essential for this argument that the size of the boundary be of order no more than L, which is what restricts us to two dimensions.

Thus suppose $\Lambda = \Lambda(L)$ for some $L > 0$. Fix $\varepsilon > 0$; from exponential decay of connectivity, there exists $m \ge 1$ such that $P[0 \leftrightarrow \partial_{\text{in}}(\Lambda(m))] < \varepsilon/m$. By boundary negligibility there exists $M > m$ such that

$$P_{\Lambda(M),\rho}(0 \leftrightarrow \partial_{\text{in}}(\Lambda(m))) < 2\varepsilon/m \quad \text{for all } \rho \in \{0,1\}^{\mathscr{B}(\Lambda(M)^C)} \ . \tag{4.1}$$

We need consider only $L > 4M$. Fix $\rho \in \{0,1\}^{\mathscr{B}(\Lambda(L)^C)}$. By the boundary coupling property there exists a coupling $\tilde{P} \in \kappa_P(\rho,\rho^1)$ such that (2.7) holds. Then for $A \in \mathscr{F}_{\overline{\mathscr{B}}(\Lambda(L-M))}$,

$$\begin{aligned} P_{\Lambda(L),\rho}(A) &= \int_{\{0,1\}^{\overline{\mathscr{B}}(\Lambda(L))}} P_{\Lambda(L-M),(\rho\alpha)}(A)\ P_{\Lambda(L),\rho}(d\alpha) \\ &= \int_{\{0,1\}^{\overline{\mathscr{B}}(\Lambda(L))}\times\{0,1\}^{\overline{\mathscr{B}}(\Lambda(L))}} P_{\Lambda(L-M),(\rho\alpha)}(A)\ \tilde{P}(d\alpha \times d\gamma) \end{aligned} \tag{4.2}$$

and similarly

$$P_{\Lambda(L),\rho^1}(A) = \int_{\{0,1\}^{\overline{\mathscr{B}}(\Lambda(L))} \times \{0,1\}^{\overline{\mathscr{B}}(\Lambda(L))}} P_{\Lambda(L-M),(\rho\gamma)}(A)\ \tilde{P}(d\alpha \times d\gamma)\ . \quad (4.3)$$

Essentially (4.2) says that one can compute the probability of A under $P_{\Lambda(L),\rho}$ by first choosing a realization (α, γ) on $\overline{\mathscr{B}}(\Lambda(L)) \times \overline{\mathscr{B}}(\Lambda(L))$ of the coupling of ρ and ρ^1, then throwing away both γ and $\alpha_{\mathscr{B}(\Lambda(L-M))}$, then using the remaining portion of α on $\overline{\mathscr{B}}(\Lambda(L)) \backslash \overline{\mathscr{B}}(\Lambda(L-M))$, together with ρ on $\mathscr{B}(\Lambda(L)^C)$, as a boundary condition to calculate the probability of A; this procedure is averaged over the choice of (α, γ). If we instead throw away both α and $\gamma_{\overline{\mathscr{B}}(\Lambda(L-M))}$, and use ρ^1 instead of ρ, the resulting probability is under $P_{\Lambda(L),\rho^1}$; this is (4.3). Let

$$T_{\Lambda(L-M),m}(\alpha) := |\{x \in \Lambda(L-M+m) \backslash \Lambda(L-M) : \\ x \leftrightarrow \partial_{\text{in}}(x + \Lambda(m)) \text{ in } \alpha\}|\ , \quad \alpha \in \{0,1\}^{\overline{\mathscr{B}}(\Lambda(L))}\ .$$

Then from (2.7), with probability 1,

$$\begin{aligned} &D_{\Lambda(L-M),m}(\alpha, \gamma) \\ &\quad \leq \left| [C(\partial_{ex}\Lambda(L), \alpha) \cup C(\partial_{ex}\Lambda(L), \gamma)] \cap \overline{\mathscr{B}}(\Lambda(L-M+m)) \backslash \overline{\mathscr{B}}(\Lambda(L-M)) \right| \\ &\quad \leq 2T_{\Lambda(L-M),m}(\alpha) + 2T_{\Lambda(L-M),m}(\gamma)\ , \end{aligned} \quad (4.4)$$

and

$$R_{\Lambda(L-M),m}(\alpha) \leq T_{\Lambda(L-M),m}(\alpha)\ , \quad (4.5)$$

and similarly for γ. From (4.4), (4.5) and bounded influence per site, for c as in (2.12),

$$\begin{aligned} &P_{\Lambda(L-M),(\rho\alpha)}(A) \\ &\quad \leq \exp\big(c\big[D_{\Lambda(L-M),m}(\alpha,\gamma) + R_{\Lambda(L-M),m}(\alpha) + R_{\Lambda(L-M),m}(\gamma)\big]\big) \\ &\qquad \times P_{\Lambda(L-M),(\rho\gamma)}(A) \\ &\quad \leq \exp\big(3c\big[T_{\Lambda(L-M),m}(\alpha) + T_{\Lambda(L-M),m}(\gamma)\big]\big)\ P_{\Lambda(L-M),(\rho\gamma)}(A)\ . \end{aligned} \quad (4.6)$$

With (4.2) and (4.3) this yields

$$\begin{aligned} P_{\Lambda(L),\rho}(A) &\leq \tilde{P}\big(T_{\Lambda(L-M),m}(\alpha) + T_{\Lambda(L-M),m}(\gamma) \geq 36\varepsilon L\big) \\ &\qquad + \exp(108c\varepsilon L) P_{\Lambda(L),\rho^1}(A) \\ &\leq P_{\Lambda(L),\rho}\big(T_{\Lambda(L-M),m} \geq 18\varepsilon L\big) + P_{\Lambda(L),\rho^1}\big(T_{\Lambda(L-M),m} \geq 18\varepsilon L\big) \\ &\qquad + \exp(108c\varepsilon L) P_{\Lambda(L),\rho^1}(A)\ . \end{aligned} \quad (4.7)$$

Note that (4.6), and thus also (4.7), remains true if ρ and ρ^1 are interchanged. Let us show that the first two terms on the right side of (4.7) each decay exponentially in L. The set

$\Omega := \Lambda\ (L - M + m)\backslash\Lambda(L - M)$ can be partitioned into 3Mm subsets $\Gamma_1, \ldots, \Gamma_{3Mm}$ in such a way that $\||\Gamma_i| - |\Gamma_j|\| \le 2$ for all $i, j, |\Gamma_i| \ge L/M$ (since $L > 4M$ gives $|\Omega| \ge 4m(L - M)$), and

$$x, y \in \Gamma_i, x \ne y \text{ implies } |y - x| \ge 2M \text{ and hence } d(\Lambda(m, y), x) > M \ . \tag{4.8}$$

Define the event

$$E := [0 \leftrightarrow \partial_{\text{in}}(\Lambda(m))]$$

and let

$$Y_i := |\Gamma_i|^{-1} \sum_{x \in \Gamma_i} \delta_{\tau_x E} \ .$$

Since $|\Omega| \le 6m(L - M)$ we have

$$\begin{aligned} &P_{\Lambda(L),\rho}\big(T_{\Lambda(L-M),m} \ge 18\varepsilon L\big) \\ &\quad \le P_{\Lambda(L),\rho}\left[|\Omega|^{-1} \sum_{x \in \Omega} \delta_{\tau_x E} \ge 3\varepsilon/m\right] \\ &\quad \le \sum_{i \le 3Mm} P_{\Lambda(L),\rho}[Y_i \ge 3\varepsilon/m] \ . \end{aligned} \tag{4.9}$$

By (4.8) and (4.1) we have

$$P_{\Lambda(L),\rho}(\tau_x E | \delta_{\tau_y E}, y \in \Gamma_i, y \ne x) \le 2\varepsilon/m \quad \text{a.s.}$$

Therefore $|\Gamma_i| Y_i$ under $P_{\Lambda(L),\rho}$ is stochastically smaller than a Binomial $(|\Gamma_i|, 2\varepsilon/m)$ random variable, which we denote X_i. Therefore by Bernstein's inequality (see [23]),

$$\begin{aligned} P_{\Lambda(L),\rho}[Y_i \ge 3\varepsilon/m] &\le P[|\Gamma_i|^{-1} X_i \ge 3\varepsilon/m] \le \exp(-\varepsilon|\Gamma_1|/3m) \\ &\le \exp(-\varepsilon L/3Mm) \ , \end{aligned}$$

which with (4.9) shows

$$P_{\Lambda(L),\rho}[T_{\Lambda(L-M),m} \ge 18\varepsilon L] \le 3Mm \exp(-\varepsilon L/3Mm) \ ,$$

and similarity for ρ^1. This and (4.7) show that for each $\varepsilon > 0$ there is an M such that

$$\begin{aligned} &P_{\Lambda(L),\rho}(A) \le 6Mm \exp(-\varepsilon L/3Mm) + \exp(108c\varepsilon L) P_{\Lambda(L),\rho^1}(A) \\ &\quad \text{for all } L > 4M, \rho \in \{0,1\}^{\mathscr{B}(\Lambda(L)^C)} \text{ and } A \in \mathscr{F}_{\overline{\mathscr{B}}(\Lambda(L-M))} \ . \end{aligned} \tag{4.10}$$

As noted above, we can interchange ρ and ρ^1 in this argument; we obtain that for each $\theta > 0$ there is an M such that for all $L > 4M$, and all ρ and A,

$$P_{\Lambda(L),\rho^1}(A) \leq 6Mr\exp(-\theta L/3Mm) + \exp(108c\theta L)P_{\Lambda(L),\rho}(A) \quad . \tag{4.11}$$

Averaging (4.11) over ρ under P we get

$$P_{\Lambda(L),\rho^1}(A) \leq 6Mr\exp(-\theta L/3Mm) + \exp(108c\theta L)P(A) \quad . \tag{4.12}$$

Now we focus on the particular event $A = A_L := [0 \leftrightarrow \partial_{\text{in}}(\Lambda(L/2))]$. From exponential decay of connectivity we have for some C, λ that $P(A_L) \leq CL\exp(-\lambda L)$ for all L. Choosing $\theta < \lambda/108c$ in (4.12) yields that for some C_1 and λ_1,

$$P_{\Lambda(L),\rho^1}(A_L) \leq C_1\exp(-\lambda_1 L) \quad \text{for all } L \ .$$

We can now apply (4.10) with $\varepsilon < \lambda_1/108c$ to obtain that for some C_2 and λ_2,

$$P_{\Lambda(L),\rho}(0 \leftrightarrow \partial_{\text{ex}}(\Lambda(L))) \leq P_{\Lambda(L),\rho}(A_L) \leq C_2\exp(-\lambda_2 L) \ \text{ for all } \eta \text{ and } L \ . \tag{4.13}$$

Next let us consider arbitrary finite sets $\Delta \subset \Lambda$, and $\rho \in \{0,1\}^{\mathscr{B}(\Lambda^C)}, A \in \mathscr{F}_{\overline{\mathscr{B}}(\Delta)}$. Let $\tilde{P} \in \kappa_P(\rho, \rho^1)$ be such that (2.6) holds. We have using (4.4), for some C_3

$$\begin{aligned}
&|P_{\Lambda(L),\rho}(A) - P_{\Lambda(L),\rho^1}(A)| \\
&\quad \leq \tilde{P}[\{(\omega,\omega')\in\{0,1\}^{\overline{\mathscr{B}}(\Lambda)}\times\{0,1\}^{\overline{\mathscr{B}}(\Lambda)} : \omega_e \neq \omega'_e \text{ for some e} \in \overline{\mathscr{B}}(\Delta)] \\
&\quad \leq \tilde{P}[\{(\omega,\omega')\in\{0,1\}^{\overline{\mathscr{B}}(\Lambda)}\times\{0,1\}^{\overline{\mathscr{B}}(\Lambda)} : x \leftrightarrow \partial_{\text{ex}}\Lambda \text{ in } \omega \\
&\qquad\quad \text{or in } \omega' \text{ for some } x \in \Delta] \\
&\quad \leq P_{\Lambda(L),\rho}[x \leftrightarrow \partial_{\text{ex}}(\Lambda(d(x,\Lambda^C) - 1, x)) \text{ for some } x \in \Delta] \\
&\qquad + P_{\Lambda(L),\rho^1}[x \leftrightarrow \partial_{\text{ex}}(\Lambda(d(x,\Lambda^C) - 1, x)) \text{ for some } x \in \Delta] \\
&\quad \leq 2\sum_{x\in\Delta} C_2\exp(-\lambda_2(d(x,\Lambda^C) - 1)) \\
&\quad \leq C_3 \sum_{x\in\Delta, y\in\Lambda^C} \exp(-\lambda_2|x-y|) \ ,
\end{aligned}$$

which proves weak mixing.

5. Proof of Theorem 3.3.

Define

$$t(\Delta,\Gamma,C,\lambda) := C\sum_{x\in\Delta,y\in\Gamma}\exp(-\lambda|x-y|) \ .$$

To prove Theorem 3.3, it is sufficient to establish (1.2) for Δ finite and A consisting of a single configuration α_Δ satisfying $P(\sigma_\Delta = \alpha_\Delta) > 0$. In that case, if Γ^C is infinite we can apply the result for $\Gamma \cup \Lambda(L)^C$ in place of Γ and let $L \to \infty$ to see that it is sufficient to prove (1.2) when Γ^C (which we call Λ) is also finite. The problem is reduced to showing that for some positive C and λ, for all finite sets $\Delta \subset \Lambda$, all $\alpha_\Delta \in S^\Delta$ satisfying $P(\sigma_\Delta = \alpha_\Delta) > 0$ and all $\eta, \eta' \in S^{\Lambda^C}$,

$$|P_{\Lambda,\eta}(\sigma_\Delta = \alpha_\Delta)/P_{\Lambda,\eta'}(\sigma_\Delta = \alpha_\Delta) - 1| \le t(\Delta, \Lambda^c, C, \lambda) \ , \tag{5.1}$$

whenever $t(\Delta, \Lambda^C, C, \lambda) < 1$ and at least one of the probabilities in (5.1) is positive.

Fix $\eta, \eta' \in S^{\Lambda^C}$ such that at least one of the probabilities in (5.1) is positive. Let $\tilde{P} \in \kappa_p(\eta, \eta')$ and let (σ, σ') denote a generic configuration in $S^\Lambda \times S^\Lambda$. Suppose $C, \lambda, \tilde{P}$ and an event $H \subset S^\Lambda \times S^\Lambda$ can be chosen so that C, λ do not depend on η, η' and so that, for $t = t(\Delta, \Lambda^C, C, \lambda)$, provided $t < 1$ we have

$$\tilde{P}(\sigma_\Delta = \alpha_\Delta) > 0 \text{ and } \tilde{P}(\sigma'_\Delta = \alpha_\Delta) > 0 \ , \tag{5.2}$$

$$\tilde{P}(H^C|\sigma_\Delta = \alpha_\Delta) \le t/2 \text{ and } \tilde{P}(H^C|\sigma'_\Delta = \alpha_\Delta) \le t/2 \ , \tag{5.3}$$

and

$$H \subset [\sigma_\Delta = \sigma'_\Delta] \ . \tag{5.4}$$

Then since $(1 - t/2)^{-1} \le 1 + t$ for $t < 1$, the quantity

$$\begin{aligned} &P_{\Lambda,\eta}(\sigma_\Delta = \alpha_\Delta)/P_{\Lambda,\eta'}(\sigma_\Delta = \alpha_\Delta) \\ &\quad = \tilde{P}(\sigma_\Delta = \alpha_\Delta)/\tilde{P}(\sigma'_\Delta = \alpha_\Delta) \\ &\quad = \tilde{P}(H \cap [\sigma_\Delta = \alpha_\Delta])\tilde{P}(H|\sigma'_\Delta = \alpha_\Delta)/\tilde{P}(H \cap [\sigma'_\Delta = \alpha_\Delta])\tilde{P}(H|\sigma_\Delta = \alpha_\Delta) \\ &\quad = \tilde{P}(H|\sigma'_\Delta = \alpha_\Delta)/\tilde{P}(H|\sigma_\Delta = \alpha_\Delta) \end{aligned}$$

is between $1 - t/2$ and $1 + t$, and (5.1) follows. Thus it remains to find $C, \lambda, \tilde{P}$ and H such that (5.2), (5.3) and (5.4) hold.

Since we do not a priori assume (5.2) we cannot assume that conditioning on the events in (5.2) is necessarily well-defined. Hence for this proof we make the following conditioning convention: if μ is a measure on configurations and A is a local event with $\mu(A) = 0$, then $\mu(.\,|A)$ means μ.

We divide the region between Δ and Λ^C into 3 "strips," as follows. Let

$$\begin{aligned} \Omega_1 &:= \{y \in \Lambda\backslash\Delta : d(y, \Delta)/d(y, \Lambda^C) < 1/2\}, \\ \Omega_2 &:= \{y \in \Lambda\backslash\Delta : 1/2 \le d(y, \Delta)/d(y, \Lambda^C) \le 2\}, \\ \Omega_3 &:= \{y \in \Lambda\backslash\Delta : d(y, \Delta)/d(y, \Lambda^C) > 2\} \ . \end{aligned} \tag{5.5}$$

Loosely, Ω_2 contains those points which are between 1/3 and 2/3 of the way from Δ to Λ^C. By weak mixing there exist constants $C_1 > 1$ and λ_1 as in (1.1); in particular,

$$\mathrm{Var}(P_{\Lambda,\eta}(\sigma_{\Omega_2} \in \cdot\,), P_{\Lambda,\eta'}(\sigma_{\Omega_2} \in \cdot)) \le t_1 := t(\Omega_2, \Lambda^C, C_1, \lambda_1) \ . \tag{5.6}$$

Therefore there exists a coupling $Q \in \kappa(P_{\Lambda,\eta}(\sigma_{\Omega_2} \in \cdot), P_{\Lambda,\eta'}(\sigma_{\Omega_2} \in \cdot))$ on $S^{\Omega_2} \times S^{\Omega_2}$ such that

$$Q[\sigma_{\Omega_2} \ne \sigma'_{\Omega_2}] \le t_1 \ .$$

Note that the event $R := [\Omega_2$ is a controlling region for $\Delta \cup \Omega_1] \subset S^{\Lambda^C}$ is in $\mathscr{F}_{\Omega_2}$. We now construct (σ, σ') as follows:

(i) choose $(\sigma_{\Omega_2}, \sigma'_{\Omega_2})$ under the distribution Q;
(ii) choose σ_{Ω_3} under $P_{\Lambda,\eta}(\cdot|\sigma_{\Omega_2})$ and σ'_{Ω_3} under $P_{\Lambda,\eta'}(\cdot|\sigma'_{\Omega_2})$ independently;
(iii) choose $\sigma_{\Delta\cup\Omega_1}$ under $P_{\Lambda,\eta}(\cdot|\sigma_{\Omega_2\cup\Omega_3})$;
(iv) if $\sigma_{\Omega_2} = \sigma'_{\Omega_2}$ and $\sigma_{\Omega_2} \in R$ then let $\sigma'_{\Delta\cup\Omega_1} = \sigma_{\Delta\cup\Omega_1}$; otherwise choose $\sigma'_{\Delta\cup\Omega_1}$ under $P_{\Lambda,\eta'}(.\,|\sigma'_{\Omega_2\cup\Omega_3})$, independent of $\sigma_{\Delta\cup\Omega_1}$.

Let $\tilde{P}$ denote the resulting distribution of (σ, σ') on $S^\Lambda \times S^\Lambda$. The fact that $\tilde{P} \in \kappa_P(\eta, \eta')$ follows from the fact that when $\sigma_{\Omega_2} = \sigma'_{\Omega_2}$ and $\sigma_{\Omega_2} \in R$, the conditional distributions $P_{\Lambda,\eta}(\cdot\,|\sigma_{\Omega_2\cup\Omega_3})$ and $P_{\Lambda,\eta'}(\cdot\,|\sigma'_{\Omega_2\cup\Omega_3})$ are the same. Note that despite the seeming asymmetry in (iii) and (iv), the construction is actually symmetric in the sense that the same $\tilde{P}$ would result if we interchanged the roles of η and η'. For every $A \in \mathscr{F}_{\Omega_2}$, by weak mixing,

$$\tilde{P}(\sigma_{\Omega_2} \in A|\sigma_\Delta = \alpha_\Delta) = P_{\Lambda,\eta}(\sigma_{\Omega_2} \in A|\sigma_\Delta = \alpha_\Delta) \le P_{\Lambda,\eta}(\sigma_{\Omega_2} \in A) + t_2 \ , \tag{5.7}$$

where $t_2 := t(\Lambda^C \cup \Delta, \Omega_2, C_1, \lambda_1)$, and similarly

$$\tilde{P}(\sigma'_{\Omega_2} \in A|\sigma'_\Delta = \alpha_\Delta) \le P_{\Lambda,\eta'}(\sigma'_{\Omega_2} \in A) + t_2 \ .$$

Next for $v_{\Omega_2} \in S^{\Omega_2}$ let

$$g(v_{\Omega_2}) := \tilde{P}(\sigma_{\Omega_2} \ne \sigma'_{\Omega_2}|\sigma_{\Omega_2} = v_{\Omega_2})$$

and

$$h(v_{\Omega_2}) := \tilde{P}(\sigma_{\Omega_2} \ne \sigma'_{\Omega_2}|\sigma'_{\Omega_2} = v_{\Omega_2}) \ .$$

Let m be the largest integer such that $3m < d(\Omega_2, \Lambda^C)$. Define $H \in \mathscr{F}_{\Omega_2\times\Omega_2}$ by

$$\begin{aligned} H :&= [\sigma_{\Omega_2} = \sigma'_{\Omega_2}] \cap [g(\sigma_{\Omega_2}) \le \exp(-\lambda_1 m)] \cap [h(\sigma'_{\Omega_2}) \\ &\le \exp(-\lambda_1 m)] \cap [\sigma_{\Omega_2} \in R] \cap [\sigma'_{\Omega_2} \in R] \ . \end{aligned}$$

Then (5.4) holds, so we must verify (5.2) and (5.3). Note that the last event in the definition of H is actually redundant; it is written only to emphasize the symmetry. We have

$$\begin{aligned} P_{\Lambda,\eta}[g(\sigma_{\Omega_2}) > \exp(-\lambda_1 m)] &= \tilde{P}[g(\sigma_{\Omega_2}) > \exp(-\lambda_1 m)] \\ &\le \exp(\lambda_1 m)\tilde{E}g(\sigma_{\Omega_2}) = \exp(\lambda_1 m)\tilde{P}[\sigma_{\Omega_2} \neq \sigma'_{\Omega_2}] \le \exp(\lambda_1 m)t_1 \le t_3 \ , \end{aligned} \tag{5.8}$$

where $t_3 := t(\Omega_2, \Lambda^C, C_1, 2\lambda_1/3)$ and $\tilde{E}$ denotes expectation with respect to $\tilde{P}$. Symmetrically,

$$P_{\Lambda,\eta'}[h(\sigma_{\Omega_2}) > \exp(-\lambda_1 m)] = \tilde{P}[h(\sigma'_{\Omega_2}) > \exp(-\lambda_1 m)] \le t_3 \ .$$

From (5.7) and (5.8),

$$\tilde{P}[g(\sigma_{\Omega_2}) > \exp(-\lambda_1 m) | \sigma_\Delta = \alpha_\Delta] \le t_3 + t_2 \tag{5.9}$$

and symmetrically,

$$\tilde{P}[h(\sigma'_{\Omega_2}) > \exp(-\lambda_1 m) | \sigma'_\Delta = \alpha_\Delta] \le t_3 + t_2 \ . \tag{5.10}$$

Since the controlling region is exponentially bounded, for some C_2 and λ_2 we have

$$P_{\Lambda,\eta}[\sigma_{\Omega_2} \notin R] \le P[\sigma_{\Omega_2} \notin R] + t_1 \le t_4 + t_1 \ , \tag{5.11}$$

where $t_4 := t(\Delta \cup \Omega_1, \Omega_3 \cup \Lambda^C, C_2, \lambda_2)$. From (5.7) and (5.11),

$$\tilde{P}[\sigma_{\Omega_2} \notin R | \sigma_\Delta = \alpha_\Delta] \le t_4 + t_1 + t_2 \ . \tag{5.12}$$

By assumption one of the probabilities in (5.2) is positive; we may assume it is the first one. Note that by (iii) in the construction of $(\sigma, \sigma'), \sigma_\Delta$ and σ'_{Ω_2} are conditionally independent given σ_{Ω_2}. Therefore

$$\begin{aligned} &\tilde{P}[\sigma_{\Omega_2} \neq \sigma'_{\Omega_2}, g(\sigma_{\Omega_2}) \le \exp(-\lambda_1 m) | \sigma_\Delta = \alpha_\Delta] \\ &\quad \le \tilde{P}[\sigma_{\Omega_2} \neq \sigma'_{\Omega_2} | g(\sigma_{\Omega_2}) \le \exp(-\lambda_1 m), \sigma_\Delta = \alpha_\Delta] \\ &\quad \le \max\{\tilde{P}[\sigma_{\Omega_2} \neq \sigma'_{\Omega_2} | \sigma_{\Omega_2} = v_{\Omega_2}] : g(v_{\Omega_2}) \le \exp(-\lambda_1 m)\} \\ &\quad \le \exp(-\lambda_1 m) \\ &\quad \le t_5 \ , \end{aligned} \tag{5.13}$$

where $t_5 := t(\Omega_2, \Lambda^C, C_4, \lambda_1/3)$ with $C_4 := \exp(\lambda_1)$, provided the LHS of (5.13) is positive so that our conditioning convention need not be used. If the LHS of (5.13) is 0, then of course the bound of t_5 is valid anyway.

Suppose now that we can show that C and λ can be chosen so that for $t = t(\Delta, \Lambda^C, C, \lambda)$, we have

$$t_i \le t/20 \quad \text{for } i = 1, .., 5 \ . \tag{5.14}$$

Observe that

$$[\sigma_\Delta = \alpha_\Delta] \cap [\sigma_{\Omega_2} = \sigma'_{\Omega_2}] = [\sigma'_\Delta = \alpha_\Delta] \cap [\sigma_{\Omega_2} = \sigma'_{\Omega_2}] \ . \tag{5.15}$$

From (5.9), (5.13), (5.14) and (5.15), assuming $t < 1$,

$$\tilde{P}[\sigma'_\Delta = \alpha_\Delta | \sigma_\Delta = \alpha_\Delta] \ge \tilde{P}[\sigma_{\Omega_2} = \sigma'_{\Omega_2} | \sigma_\Delta = \alpha_\Delta] > 0 \ ,$$

so that the second probability in (5.2) in positive. Therefore symmetrically to (5.13) we obtain

$$\tilde{P}[\sigma_{\Omega_2} \ne \sigma'_{\Omega_2}, h(\sigma'_{\Omega_2}) \le \exp(-\lambda_1 m) | \sigma'_\Delta = \alpha_\Delta] \le t_5 \ . \tag{5.16}$$

We have from (5.10) and (5.16),

$$\begin{aligned}
&\tilde{P}[h(\sigma'_{\Omega_2}) > \exp(-\lambda_1 m), \sigma_{\Omega_2} = \sigma'_{\Omega_2} | \sigma_\Delta = \alpha_\Delta] \\
&\quad \le \tilde{P}[h(\sigma'_{\Omega_2}) > \exp(-\lambda_1 m) | \sigma_{\Omega_2} = \sigma'_{\Omega_2}, \sigma'_\Delta = \alpha_\Delta] \\
&\quad \le \tilde{P}[h(\sigma'_{\Omega_2}) > \exp(-\lambda_1 m) | \sigma'_\Delta = \alpha_\Delta] / P[\sigma_{\Omega_2} = \sigma'_{\Omega_2} | \sigma'_{\Delta_2} = \alpha_\Delta] \\
&\quad \le (t_3 + t_2)/(1 - t_3 - t_2 - t_5) \ ,
\end{aligned} \tag{5.17}$$

provided the LHS of (5.17) is positive so that our conditioning convention need not be used. But again if the LHS of (5.17) is 0, then the bound on the RHS is valid anyway. Combining (5.9), (5.12), (5.13) and (5.17) we obtain

$$\tilde{P}(H^C | \sigma_\Delta = \alpha_\Delta) \le 2t_1 + 2t_2 + t_3 + t_4 + (t_3 + t_2)/(1 - t_3 - t_2 - t_5) \ , \tag{5.18}$$

and the first half of (5.3) follows; the second half is symmetric.

It therefore remains to establish (5.14). Let $\lambda < \min(\lambda_1/9, \lambda_2/6)$. If $y \in \Omega_2$ and $x \in \Delta$ then $|y - x| \ge d(x, \Lambda^C)/3$. Similarly if $y \in \Omega_2$ and $x \in \Lambda^C$ then $|y - x| \ge d(x, \Delta)/3$. Therefore we have for some constants C_5 and C_6

$$\begin{aligned}
t_2 &= t(\Lambda^C \cup \Delta, \Omega_2, C_1, \lambda_1) \\
&\le C_1 \sum_{x \in \Delta} \sum_{y : |y-x| \ge d(x, \Lambda^C)/3} \exp(-\lambda_1 |x - y|) \\
&\quad + C_1 \sum_{x \in \Lambda^C} \sum_{y : |y-x| \ge d(x, \Delta)/3} \exp(-\lambda_1 |x - y|) \\
&\le C_5 \sum_{x \in \Delta} \sum_{k \ge d(x, \Lambda^C)/3} k^{d-1} \exp(-\lambda_1 k) \\
&\quad + C_5 \sum_{x \in \Lambda^C} \sum_{k \ge d(x, \Delta)/3} k^{d-1} \exp(-\lambda_1 k) \\
&\le C_6 \sum_{x \in \Delta} \exp(-3\lambda d(x, \Lambda^C)) + C_6 \sum_{x \in \Lambda^C} \exp(-3\lambda d(x, \Delta))
\end{aligned}$$

$$\le 2t(\Delta, \Lambda^C, C_6, 3\lambda)$$
$$\le t(\Delta, \Lambda^C, 40C_6, \lambda)/20 \ .$$

By essentially the same argument we get

$$t_3 \le t(\Delta, \Lambda^C, 20C_6, 2\lambda)/20 \text{ and } t_5 \le t(\Delta, \Lambda^C, 20C_6, \lambda)/20 \ .$$

Note also that $t_1 \le t_2$. Next suppose that $x \in \Delta \cup \Omega_1$ and $y \in \Omega_3 \cup \Lambda^C$. We claim that

$$|y - x| \ge d(x, \Lambda^C)/2 \ . \tag{5.19}$$

To see this, observe that $d(x, \Lambda^C) \le |y - x| + d(y, \Lambda^C)$, so either (5.19) holds or $d(y, \Lambda^C) \ge d(x, \Lambda^C)/2$. But in the latter case we have

$$\begin{aligned} d(x, \Lambda^C)/2 &\le d(y, \Lambda^C) < d(y, \Delta)/2 \le |y - x|/2 + d(x, \Delta)/2 \\ &\le |y - x|/2 + d(x, \Lambda^C)/4 \ , \end{aligned}$$

so (5.19) holds anyway. Similarly, we have also

$$|y - x| \ge d(x, \Delta)/2 \ . \tag{5.20}$$

Using (5.19) and (5.20) we obtain that for some constants $C_7, \ldots, C_{10}$,

$$\begin{aligned} t_4 &= t(\Delta \cup \Omega_1, \Omega_3 \cup \Lambda^C, C_2, \lambda_2) \\ &\le C_2 \sum_{x \in \Delta \cup \Omega_1} \sum_{y : |y - x| \ge d(x, \Lambda^C)/2} \exp(-\lambda_2 |y - x|) \\ &\le C_7 \sum_{x \in \Delta \cup \Omega_1} \sum_{k \ge d(x, \Lambda^C)/2} k^{d-1} \exp(-\lambda_2 k) \\ &\le C_8 \sum_{x \in \Delta \cup \Omega_1} \exp(-3\lambda d(x, \Lambda^C)) \\ &\le C_8 \sum_{x \in \Delta \cup \Omega_1} \sum_{y \in \Lambda^C} \exp(-3\lambda |y - x|) \\ &\le C_8 \sum_{y \in \Lambda^C} \sum_{x : |y - x| \ge d(y, \Delta)/2} \exp(-3\lambda |y - x|) \\ &\le C_9 \sum_{y \in \Lambda^C} \sum_{k \ge d(y, \Delta)/2} k^{d-1} \exp(-3\lambda k) \\ &\le C_{10} \sum_{y \in \Lambda^C} \exp(-\lambda d(y, \Delta)) \\ &\le t(\Delta, \Lambda^C, 20C_{10}, \lambda)/20 \ . \end{aligned}$$

The proof of (5.14), and thus of the theorem, is complete.

6. Proofs for specific models

Proof of Theorem 3.4. It follows easily from the fact that probabilities in the model are defined in terms of a product over clusters that (ii) of Lemma 2.4 is satisfied. Since by Lemma 2.1 the FK model satisfies the FKG lattice condition, (i) also holds. Therefore by that lemma, the model has the boundary coupling property with respect to $\{1\}$.

Since there is no percolation under the wired boundary condition, there is an asymptotically negligible probability that the boundary cluster meets a fixed box, that is,

$$\lim_{n\to\infty} P^{\text{in}}_{\Lambda(n),\rho^1,q,p,h}(C_{\{1\}}(\partial_{\text{ex}}\Lambda(n),\cdot\,)\cap\Lambda(m)\neq\phi)=0 \quad \text{for all } m\ . \tag{6.1}$$

The FKG property (Lemma 2.1) ensures that the probability in (6.1) is not increased if ρ^1 is replaced with another boundary condition. Together with the boundary coupling property, this establishes boundary negligibility and hence uniqueness.

From (2.13) the model has bounded influence per site. Therefore weak mixing follows from Theorem 3.1 (ii).

For $\rho\in\{0,1\}^{\mathscr{B}(\Lambda^C)}$, it is easy to see that the closure of the cluster of $\partial_{\text{ex}}\Lambda$ in ρ is a controlling region for $\overline{\mathscr{B}}(\Lambda)$ and ρ. Therefore for some C and λ, for $\Omega\subset\mathscr{B}(\Lambda^C)$, and Γ the set of endpoints of bonds in Ω,

$$\begin{aligned} P^{\text{in}}_{q,p,h}(\{\omega\in\{0,1\}^{\mathscr{B}(\Lambda^C)} &: \Omega \text{ is not a controlling region for}\\ &\overline{\mathscr{B}}(\Lambda) \text{ and } \rho\})\\ &\leq \sigma_{x\in\partial_{\text{ex}}\Lambda, y\in\partial_{\text{in}}(\Lambda\cup\Gamma)} P^{\text{in}}_{q,p,h}(x\leftrightarrow z \text{ for some } z \text{ adjacent to } y)\\ &\leq \sum_{x\in\Lambda, y\in\Gamma^C\setminus\Lambda} C\exp(-\lambda|y-x|)\ , \end{aligned}$$

from which it follows that $P^{\text{in}}_{q,p,h}$ has exponentially bounded controlling regions. Ratio weak mixing then follows from Theorem 3.3. □

Does weak mixing in the q-state Potts model at (β,h) ensure weak mixing in the corresponding FK model with parameters (q,p,h), with $p=1-\mathrm{e}^{-\beta}$, and vice-versa? The general answer is no, because by Remark 3.5, we can have weak mixing in the FK model when there is not even uniqueness in the Potts model. However, we do have the following.

Theorem 6.1. *Consider the q-state Potts model at* (β,h) on $\mathbb{Z}^d$ $(h\geq 0)$ *and the FK model with parameters* (q,p,h)*, with* $p=1-\mathrm{e}^{-\beta}$.

(i) If the Potts model is weak mixing, then the FK model is weak mixing.

(ii) If the FK model is weak mixing and has exponential decay of connectivity, then the Potts model is weak mixing.

Proof. Fix finite sets $\Delta \subset \Lambda \subset \bar{\Lambda} \subset \Gamma$. Suppose first that the Potts model is weak mixing. The main point is that under the joint site-bond measure, given the site configuration on Λ^C, the bonds there are independent of the site configuration on Δ, by the construction $C2$; similarly given the site configuration on Δ, the bond configuration there is independent of the site and bond configurations on Γ^C. For the details, fix $\rho, \rho' \in \{0,1\}^{\mathscr{B}(\Lambda^C)}, \eta \in S^{\Lambda^C}$, and $A \subset \{0,1\}^{\mathscr{B}(\Delta)}$. Then

$$\begin{aligned} &P^{\text{in}}_{\Lambda,\rho,q,p,h}(\omega_{\mathscr{B}(\Delta)} \in A) \\ &\quad = \lim\nolimits_{\Gamma \nearrow \mathbb{Z}^d} P_{\Gamma,\rho^0,q,p,h}\Big(\omega_{\mathscr{B}(\Delta)} \in A | \omega_{\overline{\mathscr{B}}(\Gamma)\setminus\overline{\mathscr{B}}(\Lambda)} = \rho_{\overline{\mathscr{B}}(\Gamma)\setminus\overline{\mathscr{B}}(\Lambda)}\Big) \\ &\quad = \lim\nolimits_{\Gamma \nearrow \mathbb{Z}^d} \tilde{P}_{\Gamma,\eta^0,q,p,h}\Big(\omega_{\mathscr{B}(\Delta)} \in A | \omega_{\overline{\mathscr{B}}(\Gamma)\setminus\overline{\mathscr{B}}(\Lambda)} = \rho_{\overline{\mathscr{B}}(\Gamma)\setminus\overline{\mathscr{B}}(\Lambda)}\Big) \ . \end{aligned} \tag{6.2}$$

Now by the above-mentioned conditional independence,

$$\begin{aligned} &\tilde{P}_{\Gamma,\eta^0,q,p,h}(\omega_{\mathscr{B}(\Delta)} \in A | \omega_{\overline{\mathscr{B}}(\Gamma)\setminus\overline{\mathscr{B}}(\Lambda)} = \rho_{\overline{\mathscr{B}}(\Gamma)\setminus\overline{\mathscr{B}}(\Lambda)}, \sigma_{\Gamma\setminus\Lambda} = \eta_{\Gamma\setminus\Lambda}) \\ &\quad = E_{\Lambda,\eta,q,\beta,h}(\tilde{P}_{\Delta,\eta^0,p,q,h}(\omega_{\mathscr{B}(\Delta)} \in A | \sigma_\Delta)) \ . \end{aligned}$$

We denote the last quantity by $\varphi(\eta)$. Then

$$\begin{aligned} f(\rho_{\overline{\mathscr{B}}(\Gamma)\setminus\overline{\mathscr{B}}(\Lambda)}) :&= \tilde{P}_{\Gamma,\eta^0,q,p,h}\Big(\omega_{\mathscr{B}(\Delta)} \in A | \omega_{\overline{\mathscr{B}}(\Gamma)\setminus\overline{\mathscr{B}}(\Lambda)} = \rho_{\overline{\mathscr{B}}(\Gamma)\setminus\overline{\mathscr{B}}(\Lambda)}\Big) \\ &= \tilde{E}_{\Gamma,\eta^0,q,p,h}\Big(\varphi(\sigma_{\Gamma\setminus\Lambda}) | \omega_{\overline{\mathscr{B}}(\Gamma)\setminus\overline{\mathscr{B}}(\Lambda)} = \rho_{\overline{\mathscr{B}}(\Gamma)\setminus\overline{\mathscr{B}}(\Lambda)}\Big) \ . \end{aligned}$$

Therefore since the Potts model is weak mixing, for some C and λ,

$$\Big| f\Big(\rho_{\overline{\mathscr{B}}(\Gamma)\setminus\overline{\mathscr{B}}(\Lambda)}\Big) - f\Big(\rho'_{\overline{\mathscr{B}}(\Gamma)\setminus\overline{\mathscr{B}}(\Lambda)}\Big)\Big| \le \sup \varphi - \inf \varphi \le t(\Delta, \Lambda^c, C, \lambda) \ ,$$

which with (6.2) establishes weak mixing for the FK model (cf. (2.4).)

Next suppose the FK model is weak mixing and has exponential decay of connectivity. Fix $U \subset S^\Delta$ and $\eta, \eta' \in S^{\Lambda^C}$. Then

$$P_{\Lambda,\eta,q,\beta,h}(\sigma_\Delta \in U) = E_{\Lambda,\eta,q,p,h}\Big(\tilde{P}_{\Lambda,\eta,q,p,h}\Big(\sigma_\Delta \in U | \omega_{\overline{\mathscr{B}}(\Lambda)}\Big)\Big) \ . \tag{6.3}$$

Let

$$F := \Big\{\omega_{\overline{\mathscr{B}}(\Lambda)} : C_I\Big(\partial_{ex}\Lambda, \omega_{\overline{\mathscr{B}}(\Lambda)}\Big) \cap \Delta = \phi\Big\} \ .$$

For $\omega_{\overline{\mathscr{B}}(\Lambda)} \in F$, the conditional probability on the right side of (6.3) is not affected by the boundary condition η, so that

$$\tilde{P}_{\Lambda,\eta,q,p,h}\Big(\sigma_\Delta \in U | \omega_{\overline{\mathscr{B}}(\Lambda)}\Big) = \tilde{P}_{\Lambda,\eta',q,p,h}\Big(\sigma_\Delta \in U | \omega_{\overline{\mathscr{B}}(\Lambda)}\Big) \text{ for all } \omega_{\overline{\mathscr{B}}(\Lambda)} \in F \ .$$

Therefore by (6.3) and Remark 2.2, since F^C is an increasing event,

$$\begin{aligned}&\left|P_{\Lambda,\eta,q,\beta,h}(\sigma_\Delta \in U) - P_{\Lambda,\eta',q,\beta,h}(\sigma_\Delta \in U)\right|\\&\quad\le P_{\Lambda,\eta,q,p,h}\left(F^C\right) + P_{\Lambda,\eta' q,p,h}\left(F^C\right)\\&\quad\le 2P^{\mathrm{in}}_{\Lambda,\rho^1,q,p,h}\left(F^C\right)\ .\end{aligned} \tag{6.4}$$

Let $\Omega_i,\ i = 1,2,3$, be as in (5.5). Then for some C_i and $\lambda_i, i = 1,2$,

$$\begin{aligned}P^{\mathrm{in}}_{\Lambda,\rho^1,q,p,h}\left(F^C\right) &\le P^{\mathrm{in}}_{\Lambda,\rho^1,q,p,h}(\Delta \leftrightarrow \partial_{\mathrm{ex}}(\Delta \cup \Omega_1))\\&\le P^{\mathrm{in}}_{w,q,p,h}(\Delta \leftrightarrow \partial_{\mathrm{ex}}(\Delta \cup \Omega_1)) + t\left(\Delta \cup \bar{\Omega}_1, \Lambda^C, C_1, \lambda_1\right)\\&\le t(\Delta, \Omega_2, C_2, \lambda_2) + t\left(\Delta \cup \bar{\Omega}_1, \Lambda^C, C_1, \lambda_1\right)\ .\end{aligned} \tag{6.5}$$

As in the proof of Theorem 3.3 (cf. the bounds on t_2 and t_4), there exist C and λ such that the right side of (6.5) is bounded by $t(\Delta, \Lambda^C, C, \lambda)$. With (6.4) this completes the proof. □

Lemma 6.2. *A Potts model on* $\mathbb{Z}^d$ *with external field* $h \ge 0$ *applied to spin* 1 *has the boundary coupling property, both with respect to* $\{1\}$ *and with respect to* $\{2,\dots,q\}$.

Proof. Fix Λ finite and $\eta \in S^{\Lambda^C}$, where $S = \{1,\dots,q\}$. Let $\beta > 0$ be the inverse temperature, and $p = 1 - e^{-\beta}$. By Remark 2.2, the conditioned FK measure $P^{\mathrm{in}}_{\Lambda,\eta,q,p,h}$ is FKG-dominated by the wired-boundary measure $P^{\mathrm{in}}_{\lambda,\eta^1,q,p,h}$. This remains true if we condition both measures on a fixed configurations on a subset of $\overline{\mathscr{B}}(\Lambda)$; more precisely if $\mathscr{D} \subset \overline{\mathscr{B}}(\Lambda)$ and $\rho_{\mathscr{D}} \in \{0,1\}^{\mathscr{D}}$, then $P^{\mathrm{in}}_{\Lambda,\eta^1,q,p,h}(\omega \in .|\omega_{\mathscr{D}} = \rho_{\mathscr{D}})$ dominates $P^{\mathrm{in}}_{\Lambda,\eta,q,p,h}(\omega \in .|\omega_{\mathscr{D}} = \rho_{\mathscr{D}})$. Further, since $P^{\mathrm{in}}_{\Lambda,\eta^1,q,p,h}$ satisfies the FKG lattice condition, if $\tau_{\mathscr{D}} \ge \rho_{\mathscr{D}}$ then $P^{\mathrm{in}}_{\Lambda,\eta^1,q,p,h}(\omega \in .|\omega_{\mathscr{D}} = \tau_{\mathscr{D}})$ dominates $P^{\mathrm{in}}_{\Lambda,\eta^1,q,p,h}(\omega \in .|\omega_{\mathscr{D}} = \rho_{\mathscr{D}})$ and hence also dominates $P^{\mathrm{in}}_{\Lambda,\eta,q,p,h}(\omega \in .|\omega_{\mathscr{D}} = \rho_{\mathscr{D}})$. Finally, conditionally on either A or B, condition (ii) of Lemma 2.4 is satisfied. With these observations it is easy to see that the sequential construction used in the proof of Lemma 2.3 yields a coupling in $\kappa(P^{\mathrm{in}}_{\Lambda,\eta,q,p,h}, P^{\mathrm{in}}_{\Lambda,\eta^1,q,p,h})$ such that the two configurations (ω,ω') agree outside the cluster C of $\partial_{\mathrm{ex}}\Lambda$ existing in the (larger) configuration ω'. In particular there are no bonds in either configuration connecting sites in C to sites outside C, and every cluster in ω is either contained in C or disjoint from C. When states are assigned to clusters as in construction $C1$, each cluster disjoint from C is assigned a state with probability proportional to 1 for state 1 and to $e^{-\beta hn}$ for each of states $2,\dots,q$, where n is the number of sites in the cluster; conditionally on the bond configuration, the assignments are independent for distinct clusters. Therefore the assignment can be done identically for those clusters of ω and ω' disjoint from C, yielding a coupling of the boundary conditions η and η^1 such that the two site configurations (σ, σ') agree outside C, hence also outside the 1-cluster of $\partial_{\mathrm{ex}}\Lambda$ in σ'. This establishes the boundary coupling property with respect to $\{1\}$.

The boundary coupling property with respect to $\{2, .., q\}$ is established in [4]. □

Theorem 3.7 follows directly from Lemma 6.2 and Theorems 3.1 and 3.3.

Proof of Theorem 3.6. That weak mixing implies exponential decay of correlations follows immediately from the definitions. So suppose correlations decay exponentially. From (2.5) and uniqueness, for $p = 1 - e^{-\beta}$, the FK model with parameters (q, p) has exponential decay of connectivity. From Theorem 3.4 this FK model is weak mixing, and it follows from Theorem 6.1 that the Potts model is weak mixing. □

Grimmett [20] proved that for $q > 25.72$ and $p \leq p_C(q, 2)$, the FK model on $\mathscr{B}(\mathbb{Z}^2)$ with parameters (q, p) and free boundary condition has exponential decay of connectivity. With (2.5) and Theorems 3.3 and 3.6, this proves Corollary 3.9.

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