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On local behaviour of the phase separation line in the 2D Ising model

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Summary. The aim of this note is to discuss some statistical properties of the phase separation line in the 2D low-temperature Ising model. We prove the functional central limit theorem for the probability distributions describing fluctuations of the phase boundary in the direction orthogonal to its orientation. The limiting Gaussian measure corresponds to a scaled Brownian bridge with direction dependent parameters. Up to the temperature factor, the variances of local increments of this limiting process are inversely proportional to the stiffness.

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1 Introduction

Fluctuations of the phase boundary in the two-dimensional (2D) Ising ferromagnet are known to be asymptotically Gaussian. Some interesting results in this area can be found in the literature (see, e.g., [8, 1, 9, 5, 6]), where, however, only vertical displacements of the phase separation line were investigated. Such a description is natural only if one considers horizontal or "almost horizontal" interfaces. For inclined interfaces with sufficiently large slope angles φ (say, $\varphi \approx \pi/4$) this is not more the case, and the approach becomes completely inadequate for "almost vertical" interfaces, since the latter tend to fluctuate mainly in "horizontal" direction. To study fluctuations of interfaces in the direction orthogonal to their orientation seems to be more appropriate.

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In fact, the formulas appearing here are of the simplest form and the corresponding parameters have a nice physical (and geometric) interpretation.

The main goal of the present paper is to discuss fluctuations of inclined Ising interfaces in the direction orthogonal to their orientation. More precisely, we consider the 2D Ising ferromagnet in a box with a (symmetric) two-component boundary conditions (see Sect. 2 for formal definitions) and study the limiting behaviour of the stochastic processes corresponding to orthogonal fluctuations of the phase boundaries. The probability distributions of these processes are shown to satisfy the functional central limit theorem. The limiting Gaussian measure presents the distribution of a (scaled) Brownian bridge with orientation dependent parameters. As it was predicted in [2], the variances of its local increments are inversely proportional to the stiffness[1], a well-known quantity in statistical mechanics.

Since the only condition imposed on interfaces is that of fixed endpoints, the situation under consideration is essentially local (i.e., we describe a microscopic piece of the phase boundary). Nevertheless, the estimates obtained below are uniform in inclination angle φ, $\varphi \in (0, \pi/2 - \Delta]$ with any fixed $\Delta > 0$ (provided only the inverse temperature β is large enough, $\beta \geq \beta_0(\Delta)$). Therefore, one can use the same approach to study such fluctuations of macroscopical pieces of the phase boundary.

The proof below is based on a similar result for the process of vertical fluctuations of the phase boundary in the 2D Ising model [5] and uses additionally some constructions and estimates from [7] and [6]. We show that asymptotically as $N \to \infty$ both processes are related by a simple change of variables.

The paper is organised as follows. Section 2 contains definitions and notations used later on. The main results are formulated in Sec. 3. In Sec. 4 some technical lemmas are collected, which form the basis of proofs of the main results in Sec. 5.

2 Definitions and notations

Lattices. Let $\mathbb{Z}^2$ be the two-dimensional integer lattice and $(\mathbb{Z}^2)^*$ be its dual, $(\mathbb{Z}^2)^* = (\mathbb{Z} + 1/2)^2$, both consisting of *sites*. These lattices are immersed into $\mathbb{R}^2$ equipped with the Euclidean distance $|\cdot|$, $|x - y| = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2}$, where $x = (x_1, x_2)$ and $y = (y_1, y_2)$. A *bond* is any segment of unit length connecting two neighbouring sites of the dual lattice. Let s, t be two neighbours in $\mathbb{Z}^2$; denote by f the unit segment connecting s and t. By definition, a bond e *separates* these sites if the segments f and e are orthogonal and meet at their midpoints.

Let s be any site. By definition, the *diagonal* at s is the straight line that passes through s and is orthogonal to the vector $(1, 1)$. A site $s \in \mathbb{Z}^2$ is

[1] i.e., the radius of curvature of the Wulff shape at the corresponding point

attached to $s^* \in (\mathbb{Z}^2)^*$ provided they share the diagonal and $|s - s^*| = \sqrt{2}/2$. A site $s \in \mathbb{Z}^2$ is attached to a bond e if s is attached to one end of e (see Fig. 1b below).

For a set $V \subset \mathbb{Z}^2$, $|V|$ denotes its cardinality and ∂V is its outer boundary,

$$\partial V = \left\{ s \in \mathbb{Z}^2 \setminus V : \exists t \in V \quad \text{with } |t - s| = 1 \right\} .$$

Configurations. For $V \subset \mathbb{Z}^2$ denote by $\Omega_V = \{-1, 1\}^V$ the set of all possible configurations $\sigma = \sigma_V$ in V. In the case $V = \{s\}$ the configuration σ_V is reduced to the *spin* at the site s and is denoted simply by σ_s.

Fix any $V \subset \mathbb{Z}^2$. A configuration $\overline{\sigma} = \overline{\sigma}_{\mathbb{Z}^2 \setminus V}$ in the complement $\mathbb{Z}^2 \setminus V$ is called a *boundary condition* (for V). Two kinds of boundary conditions will be used below: the constant *plus* boundary condition $\overline{\sigma}^+$,

$$\overline{\sigma}_t^+ = 1, \qquad \text{for all } t = (t_1, t_2) \in \mathbb{Z}^2 , \tag{2.1}$$

and the two-component boundary condition $\overline{\sigma}^\varphi$, $\varphi \in (-\pi/2, \pi/2)$,

$$\overline{\sigma}_t^\varphi = \begin{cases} 1, & \text{if } t_2 > t_1 \tan \varphi, \\ -1, & \text{otherwise} . \end{cases} \tag{2.2}$$

Contours. Let σ be a configuration in a set $V \subset \mathbb{Z}^2$ and $\overline{\sigma}$ be a boundary condition. The *boundary* $\Gamma(\sigma, \overline{\sigma})$ of the configuration σ under the boundary condition $\overline{\sigma}$ is the collection of all bonds separating the sites in $\mathbb{Z}^2$ with different values of spins. Then any site s^* of the dual lattice is the meeting point of an even number of such bonds. If four bonds meet at their common vertex we split them up into two pairs of linked bonds according to the rule of "rounding of corners" in Fig. 1a). Then the boundary $\Gamma(\sigma, \overline{\sigma})$ splits up into connected components to be called *contours*.

Phase boundary. Let $V_{NM} \subset \mathbb{Z}^2$ denote the box

$$V_{NM} = \left\{ t = (t_1, t_2) \in \mathbb{Z}^2 \colon |t_1| < N, |t_2| < M \right\}, \qquad M, N > 1 . \tag{2.3}$$

Fix any $\varphi \in (-\pi/2, \pi/2)$ and consider the boundary condition $\overline{\sigma}^\varphi$ from (2.2). Then the boundary $\Gamma_{V_{NM}}(\sigma, \overline{\sigma}) \equiv \Gamma_{NM}(\sigma, \overline{\sigma})$ of σ can be decomposed into certain amount of contours all of which but one being closed and a unique open contour being called the *phase boundary* (or the phase separation line). In the case $M > N(\tan \varphi + \varepsilon)$ with some fixed $\varepsilon > 0$ (the condition to be assumed everywhere below) the phase boundary forms a polygon connecting

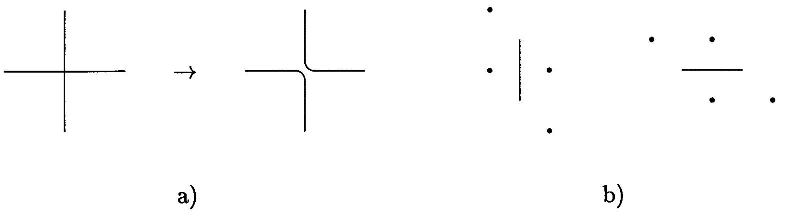

Fig. 1. a The rule of "rounding of corners"; **b** sites attached to vertical and horizontal bonds

the points $(-N, [-N\tan\varphi] + 1/2)$ and $(N, [N\tan\varphi] + 1/2)$. Let $\mathscr{T}^{\varphi}_{NM}$ denote the set of all polygons S that are phase boundaries for configurations $\sigma \in \Omega_{V_{NM}}$ under the boundary condition $\overline{\sigma}^{\varphi}$. If $S \in \mathscr{T}^{\varphi}_{NM}$, one says also that S is a phase boundary in V_{NM} consistent with the boundary condition $\overline{\sigma}^{\varphi}$.

Gibbs measures. Let V be a finite subset of $\mathbb{Z}^2$ and $\overline{\sigma}$ be a boundary condition. The *Gibbs distribution* $\mathbb{P}_{V,\beta}(\cdot|\overline{\sigma})$ in V with the boundary condition $\overline{\sigma}$ is the probability measure in Ω_V given by

$$\mathbb{P}_{V,\beta}(\sigma|\overline{\sigma}) = Z(V,\beta,\overline{\sigma})^{-1}\exp\{-\beta\mathscr{H}(\sigma|\overline{\sigma})\}, \qquad \sigma \in \Omega_V \ , \tag{2.4}$$

where the Hamiltonian $\mathscr{H}(\sigma|\overline{\sigma})$ is defined by

$$\mathscr{H}(\sigma|\overline{\sigma}) = -\sum_{\substack{s,t\in V,\\ |s-t|=1}} \sigma_s\sigma_t - \sum_{\substack{s\in V, t\in\partial V,\\ |s-t|=1}} \sigma_s\overline{\sigma}_t \ , \tag{2.5}$$

the partition function $Z(V,\beta,\overline{\sigma})$ is

$$Z(V,\beta,\overline{\sigma}) = \sum_{\sigma\in\Omega_V} \exp\{-\beta\mathscr{H}(\sigma|\overline{\sigma})\} \ , \tag{2.6}$$

and $\beta > 0$ denotes the inverse temperature. In what follows we will always assume that β is sufficiently large and (sometimes) drop the subscript β from notations.

Ensembles of phase boundaries. Let V_{NM} be the box defined in (2.3) and $\overline{\sigma}^{\varphi}$ be the boundary conditions fixed above. Denote by $\mathbb{P}_{N,M,\beta}(\cdot|\overline{\sigma}^{\varphi})$ the Gibbs distribution in $\Omega_{NM} = \{-1,1\}^{V_{NM}}$ defined as in (2.4)–(2.6). This Gibbs measure induces the probability distribution $\mathbf{P}^{\varphi}_{N,M}(\cdot)$ in the set $\mathscr{T}^{\varphi}_{NM}$ of all phase boundaries in V_{NM} consistent with the boundary condition $\overline{\sigma}^{\varphi}$,

$$\mathbf{P}^{\varphi}_{N,M}(S) = \mathbf{P}_{N,M,\beta}\Big(\Big\{\sigma \in \Omega_{NM} : \Gamma(\sigma,\overline{\sigma}^{\varphi}) \ni S\Big\} \ \Big|\ \overline{\sigma}^{\varphi}\Big), \qquad S \in \mathscr{T}^{\varphi}_{NM} \ .$$

Let $V_{N\infty}$ be the vertical strip (cf. (2.3))

$$V_{N\infty} = \Big\{t = (t_1,t_2) \in \mathbb{Z}^2 : |t_1| < N\Big\}, \qquad N > 1 \ . \tag{2.7}$$

Denote by $\mathscr{T}^{\varphi}_{N\infty}$ the set of all phase boundaries in $V_{N\infty}$ consistent with the boundary condition $\overline{\sigma}^{\varphi}$. Since $|V_{N\infty}| = \infty$, the corresponding Gibbs distribution in $\Omega_{N\infty} = \{-1,1\}^{V_{N\infty}}$ is not defined; nevertheless, for sufficiently large β the probability distribution $\mathbf{P}^{\varphi}_{N,\infty}(S)$, $S \in \mathscr{T}^{\varphi}_{N\infty}$, could be still defined (for details, see [6]). In what follows we will refer to this distribution as to the ensemble of phase boundaries in the vertical strip $V_{N\infty}$ (consistent with the boundary conditions $\overline{\sigma}^{\varphi}$).

Finally, let us introduce the ensemble of phase boundaries in $\mathbb{Z}^2$ consistent with the boundary condition $\overline{\sigma}^{\varphi}$ from (2.2). To do this, observe that the boundary of the configuration $\overline{\sigma}^{\varphi}$ itself consists of one infinite contour S^{φ}_{∞}. Let $\Delta(S^{\varphi}_{\infty}) \subset \mathbb{Z}^2$ be the set of all sites attached to bonds from S^{φ}_{∞}. By definition,

$$V_N = V_{N\infty} \cup \left(\mathbb{Z}^2 \setminus \Delta(S_\infty^\varphi)\right) ,$$

i.e., the set V_N consists of all sites in $\mathbb{Z}^2$ that are not attached to S_∞^φ outside $V_{N\infty}$. Let $\mathscr{T}_N^\varphi$ denote the set of all contours in V_N compatible with S_∞^φ outside the vertical strip $V_{N\infty}$ (in other words, every contour $S \in \mathscr{T}_N^\varphi$ is generated by some configuration $\sigma \in \{-1, 1\}^{V_N}$ and therefore passes through the points $(-N + 1/2, [-N \tan\varphi] + 1/2)$ and $(N - 1/2, [N \tan\varphi] + 1/2)$ to be called the beginning and the ending points of S respectively). The same arguments as above show that the probability distribution $\mathbf{P}_N^\varphi(S)$, $S \in \mathscr{T}_N^\varphi$, is well defined provided β is sufficiently large.

Surface tension. Let V_{NM} be the box from (2.3) and $Z(V_{NM}, \beta, \overline{\sigma})$ denote the partition function with the boundary condition $\overline{\sigma}$ (recall (2.6)). For any fixed $\varphi \in (-\pi/2, \pi/2)$ the unit vector $\mathbf{n} = \mathbf{n}_\varphi = (-\sin\varphi, \cos\varphi)$ is orthogonal to the graph of the straight line $t_2 = t_1 \tan\varphi$ in $\mathbb{R}^2$. By definition, the *surface tension* in the direction of $\mathbf{n}$ is given by

$$\tau_\beta(\varphi) = \tau_\beta(\mathbf{n}_\varphi) = \lim_{N\to\infty} \lim_{M\to\infty} \frac{\cos\varphi}{2\beta N} \log \frac{Z(V_{NM}, \beta, \overline{\sigma}^+)}{Z(V_{NM}, \beta, \overline{\sigma}^\varphi)} , \tag{2.8}$$

where the boundary conditions $\overline{\sigma}^\varphi$ and $\overline{\sigma}^+$ are defined by (2.2) and (2.1) respectively.

Another quantity of interest, which will play an important role in the following, is the stiffness, $\tau_\beta(\varphi) + \frac{d^2}{d\varphi^2}\tau_\beta(\varphi)$. It is known ([2]), that the stiffness in the Ising model is positive for all subcritical temperatures.

Free energy. The surface tension $\tau_\beta(\varphi)$ is closely related to another important function, the so-called free energy $F(H) = F_\beta(H)$. This function is determined for all complex H satisfying the condition ([7, Sect. 4.8])

$$|\Re H| < 2 - \delta/\beta , \tag{2.9}$$

where $\delta > 0$ is any fixed constant, the inverse temperature β is sufficiently large, $\beta \geq \beta_0(\delta)$, and $\Re H$ stands for the real part of H. The free energy $F(H)$ is analytical in H satisfying (2.9); for real H it is a strictly convex function. Let $F^*(\cdot)$ be the Legendre transformation of $F(\cdot)$,

$$F^*(x) = \sup_H \big(Hx - F(H)\big) .$$

Then the following duality relation holds

$$\tau_\beta(\varphi) = \frac{1}{\beta} F^*(\beta \tan\varphi) \cos\varphi . \tag{2.10}$$

Additional notations. For a real number x denote by $[x]$ its integral part and by $\{x\} = x - [x]$ its fractional part. $\mathbf{C}[a, b]$ stands always for the space of continuous functions on the segment $[a, b]$.

3 Results

Fix some[2] $\varphi \in (0, \pi/2)$ and consider the set $\mathscr{T}^{\varphi}_{NM}$ of the phase boundaries described above. As it was shown in [5] the typical vertical deviation of $S \in \mathscr{T}^{\varphi}_{NM}$ from the segment connecting the initial and the ending points of S is of order $\sqrt{N}$. The aim of the present note is to study fluctuations of the phase boundary in the direction orthogonal to its orientation (for all three ensembles defined above). We prove that the corresponding distributions converge in $\mathbf{C}[0, l]$, $l = 1/\cos\varphi$, to a certain Gaussian measure. This limiting measure presents the distribution of some scaled Brownian bridge on $[0, l]$ with orientation dependent scaling factor determined in terms of the stiffness. The proof below is similar to that in the case of the one-dimensional Solid-On-Solid (SOS) model ([10]) and is based on the related result from [5] and some estimates from [7] and [6].

We start with the following definition. Fix any two numbers $\varphi \in (0, \pi/2)$ and $\alpha \in (0, 1)$, and consider arbitrary phase boundary $S \in \mathscr{T}^{\varphi}_{N\infty}$. As it was mentioned before, the typical polygon S is oriented along the line $y = x\tan\varphi$ in $\mathbb{R}^2$, namely S is "close" to the segment L_N of this line with the ending points $(-N, -N\tan\varphi)$ and $(N, N\tan\varphi)$. Put

$$J_\alpha = \left\{1, 2, \ldots, [N^\alpha] - 1\right\} . \tag{3.1}$$

Then the points $r_j = (x_j, x_j\tan\varphi)$, $x_j = (2j/[N^\alpha] - 1)N$, $j \in J_\alpha$, form the partition of the segment L_N into $[N^\alpha]$ congruent parts.

For any point $r_j \in L_N$, denote by $\mathfrak{n}_j(x)$ the normal line to L_N at r_j,

$$\mathfrak{n}_j(x) = \frac{x_j - x}{\tan\varphi} + x_j\tan\varphi = -x\cot\varphi + \frac{2x_j}{\sin 2\varphi} . \tag{3.2}$$

The line $\mathfrak{n}_j(x)$ intersects any phase boundary S at some number of points; choose two extremal of them, the most upper and the most lower one, and denote their abscissas by $\tilde{x}^+_j = \tilde{x}^+_{j,N}$ and $\tilde{x}^-_j = \tilde{x}^-_{j,N}$ correspondingly, $\tilde{x}^+_j \le \tilde{x}^-_j$. Let $\tilde{\zeta}^\pm_N(s)$, $s \in [0, l]$, be the continuous random processes such that

$$\tilde{\zeta}^\pm_N(s_j) = \frac{x_j - \tilde{x}^\pm_j}{\sqrt{N}\sin\varphi}, \qquad s_j = \frac{jl}{[N^\alpha]} = \frac{l}{2}\left(\frac{x_j}{N} + 1\right), \qquad j \in J_\alpha , \tag{3.3}$$

and which are linearly interpolated elsewhere (we put by definition $\tilde{\zeta}^\pm_N(0) = \tilde{\zeta}^\pm_N(l) = 0$). Observe that $\left|\tilde{\zeta}^+_N(jl/[N^\alpha])\right|\sqrt{N}$ presents the distance between r_j and the most upper common point of the graph of $\mathfrak{n}_j(x)$ and the phase boundary S. Clearly, the distributions $\tilde{\nu}^\pm_{N,\infty}$ of the processes $\tilde{\zeta}^\pm_N(s)$ in $\mathbf{C}[0, l]$ are uniquely determined by the probability measure $\mathbf{P}^{\varphi}_{N,\infty}(\cdot)$ in $\mathscr{T}^{\varphi}_{N\infty}$.

The main result of the present paper is given by the following

[2] Due to the symmetry the cases $\varphi > 0$ and $\varphi < 0$ are identical; the situation with $\varphi = 0$ corresponds to the vertical fluctuations of the horizontal phase boundary and was already considered in [9].

Theorem 1 *The distribution $\tilde{\nu}^{+}_{N,\infty}$ of the process $\tilde{\zeta}^{+}_{N}(s)$ converges weakly in* $\mathbf{C}[0,l]$ *to the distribution of the process*

$$\tilde{\zeta}(s) = \frac{w_{l,0}(s)}{\sqrt{\beta\big(\tau(\varphi)+\tau''(\varphi)\big)}}, \qquad s \in [0,l] \ , \tag{3.4}$$

where $w_{l,0}(s) = w(s) - \frac{s}{l}w(l)$ denotes the Brownian bridge on $[0,l]$ and $\tau(\varphi)$ is the surface tension defined in (2.8). *The same is true for the distribution $\tilde{\nu}^{-}_{N,\infty}$ of $\tilde{\zeta}^{-}_{N}(s)$. Finally, the process $\tilde{\zeta}^{+}_{N}(s) - \tilde{\zeta}^{-}_{N}(s)$ vanishes in probability as $N \to \infty$.*

The statistical properties of the phase boundaries from the sets $\mathscr{T}^{\varphi}_{NM}$ and $\mathscr{T}^{\varphi}_{N}$ have the same limiting behaviour. Namely, one proves

Theorem 2 *Let $\tilde{\nu}^{\pm}_{NM}$ and $\tilde{\nu}^{\pm}_{N}$ be the measures constructed as described above from the distributions $\mathbf{P}^{\varphi}_{N,M}(\cdot)$ in $\mathscr{T}^{\varphi}_{NM}$ and $\mathbf{P}^{\varphi}_{N}(\cdot)$ in $\mathscr{T}^{\varphi}_{N}$ correspondingly. Then all the statements of Theorem* 1 *hold true for $\tilde{\nu}^{\pm}_{NM}$ and $\tilde{\nu}^{\pm}_{N}$.*

4 Preliminaries

We collect here some technical results to be used in the proofs of Theorems 1 and 2. In what follows we will always assume that for some fixed $\Delta > 0$

$$0 \le \varphi \le \frac{\pi}{2} - \Delta \ . \tag{4.1}$$

4.1 SOS-approximation

Let $S \in \mathscr{T}^{\varphi}_{N}$ be any fixed phase boundary in $\mathbb{Z}^2$ consistent with the boundary conditions $\overline{\sigma}^{\varphi}$ from (2.2). For $m = -N+1, \ldots, N-1$, the contour S is said to be regular in mth column if the set

$$S \cap \left\{ y = (y_1, y_2) \in \mathbb{R}^2 : y_1 = m \right\}$$

consists of one point. In the opposite case this set contains at least three points (clearly, this number is always odd) and we say that the overhang takes place in this column.

At zero temperature ($\beta = \infty$) every phase boundary S has the smallest possible length, and therefore it is regular in any column m, $|m| \le N-1$ (and is restricted to the vertical strip, i.e., $S \in \mathscr{T}^{\varphi}_{N\infty}$). For positive temperatures ($\beta < \infty$) this is not more the case due to appearance of overhangs, but the small temperature picture (β large) could be still considered as an excitation of the zero-temperature one. Thus, it is naturally to compare the original phase boundary S to some SOS-like (i.e., regular in any column m, $|m| \le N-1$) approximating polygon.

Assume first that some $S \in \mathscr{T}^{\varphi}_{N\infty}$ is fixed and for any integer k, $|k| \le N-1$, define

$$g^{+}_{N}(k) = \max\{t_2 : (k, t_2) \in S\} \ , \tag{4.2}$$

i.e., $(k, g_N^+(k))$ is the most upper point of the polygon S in kth column (by definition, we put also $g_N^+(x) = [x \tan \varphi] + 1/2$ if $x = \pm N$). Consider the collection of all unit horizontal segments centered at the points $(k, g_N^+(k))$ and connect their endpoints by vertical segments. As a result, one obtains the continuous (regular in any column m, $|m| \leq N - 1$) polygon S^+ connecting the points $(-N + 1/2, [-N \tan \varphi] + 1/2)$ and $(N - 1/2, [N \tan \varphi] + 1/2)$. (More formally S^+ could be defined as the union of the horizontal segments connecting the points $(k - 1/2, g_N^+(k))$ and $(k + 1/2, g_N^+(k))$, $k = -N + 1, \ldots, N - 1$, and the vertical segments with the endpoints $(k - 1/2, g_N^+(k - 1))$ and $(k - 1/2, g_N^+(k))$, $k = -N + 1, \ldots, N$; see Fig. 2.)

For any $j \in J_\alpha$ (recall (3.1)), consider the line $\mathfrak{n}_j(x)$ orthogonal to L_N at r_j (recall (3.2)) and denote by $(\bar{x}_j^+, \mathfrak{n}_j(\bar{x}_j^+))$, $\bar{x}_j^+ = \bar{x}_{j,N}^+$, the most upper common point of this straight line and the upper approximating polygon S^+.

In a similar way, starting from the quantities

$$g_N^-(k) = \min\{t_2 \colon (k, t_2) \in S\} \tag{4.3}$$

one defines the lower approximating polygon S^- and its most lower point $(\bar{x}_j^-, \mathfrak{n}_j(\bar{x}_j^-))$ on the normal line $\mathfrak{n}_j(x)$. Observe that the boundary S is a subset of the figure bounded by $S^+ \cup S^-$ (i.e., the dotted area in Fig. 2). As a result, one obtains the relation

$$\bar{x}_j^+ \leq \tilde{x}_j^+ \leq \tilde{x}_j^- \leq \bar{x}_j^- \ . \tag{4.4}$$

The use of introducing the approximating polygons S^+ and S^- is clarified by the following statement which is a variant of Proposition 4.15 from [7] and

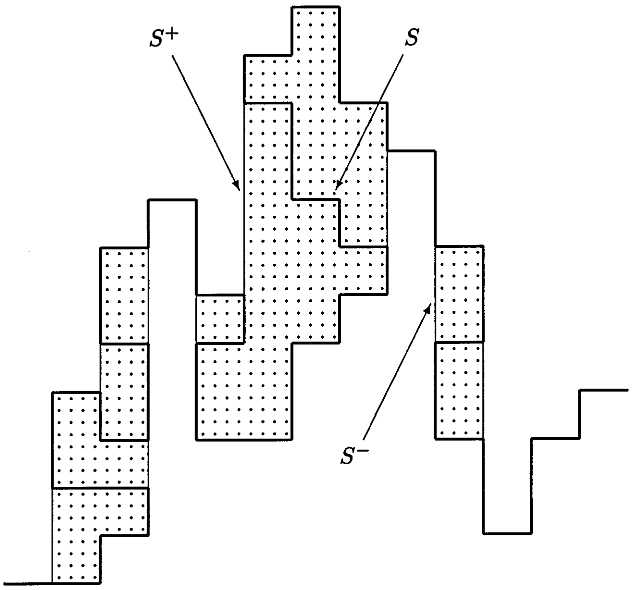

Fig. 2. Approximating polygons

can be proved by the same method. For any $\varphi \in [0, \pi/2)$ denote by $A^{\varphi}_{N,\infty}(k, \rho)$, $|k| \le N-1$, $\rho > 0$, the set

$$\Big\{S \in \mathscr{T}^{\varphi}_{N\infty} : g^{+}_{N}(k,S) \ge k\tan\varphi + (N-|k|)\rho \text{ or } g^{-}_{N}(k,S) \le k\tan\varphi - (N-|k|)\rho\Big\} .$$

Lemma 4.1 ([7]) *Assume that* φ *satisfies* (4.1) *and* $A^{\varphi}_{N,\infty}(k,\rho)$ *be as defined above. Then there exist positive constants* $\beta_0 = \beta_0(\Delta)$, $\rho_0 = \rho_0(\Delta)$, $N_0 = N_0(\Delta)$, $a = a(\Delta, \beta)$, $c = c(\Delta, \beta)$, *and* $C = C(\delta, \beta)$ *such that for all* $\beta \ge \beta_0$ *and* $N \ge N_0$ *one has*

$$\mathbf{P}^{\varphi}_{N,\infty}\Big(A^{\varphi}_{N,\infty}(k,\rho)\Big) \le C\sqrt{N}\exp\{-(N-|k|)G(\rho)\} \ , \tag{4.5}$$

where

$$G(\rho) = \begin{cases} a\rho^2, & \text{if} \quad |\rho| \le \rho_0, \\ a\rho_0^2 + c|\rho-\rho_0|, & \text{if} \quad |\rho| > \rho_0 \ . \end{cases} \tag{4.6}$$

Remark 4.1.1 All the constructions described above could be done also for $S \in \mathscr{T}^{\varphi}_{N}$ with the only difference that in this case such a polygon S is not necessarily a subset of the figure bounded by S^{+} and S^{-} (and thus the relation like (4.4) is not more valid). Nevertheless, the analogue of the formulated Lemma for the sets $A^{\varphi}_{N}(k,\rho)$ of phase boundaries $S \in \mathscr{T}^{\varphi}_{N}$ is also true. In fact, Proposition 4.15 in [7] was proved for sets like $A^{\varphi}_{N}(k,\rho)$.

Corollary 4.2 *Fix any* $\varepsilon > 0$. *Then there exist positive constants* $\beta_0 = \beta_0(\Delta)$, $C_i = C_i(\Delta, \beta, \varepsilon)$, $i = 1, 2$, *such that for all* $\beta \ge \beta_0$, $j \in J_\alpha$, *and* N *sufficiently large one has*

$$\mathbf{P}^{\varphi}_{N,\infty}\Big(|\bar{x}^{\pm}_{j} - x_j| > N^{(1+\varepsilon)/2}\Big) \le C_1\exp\{-C_2 N^{\varepsilon}\} \ .$$

In particular, the random variable $(x_j - \bar{x}^{\pm}_{j})/N$ *vanishes in probability as* $N \to \infty$ *(uniformly in* $j \in J_\alpha$*).*

Finally, let us estimate the difference $\bar{x}^{\pm}_{j} - \tilde{x}^{\pm}_{j}$. To this end, fix any $S \in \mathscr{T}^{\varphi}_{N\infty}$ and consider the set $\{m_1, \ldots, m_l\}$, $-N = m_0 < m_1 < \ldots < m_l < m_{l+1} = N$, of all m such that S is regular in column m. Cutting S by all vertical lines $y_1 = m_i$, $i = 1, \ldots, l$, one splits the polygon S into $l+1$ pieces $S_1, \ldots, S_{l+1}$ to be called *animals* (note that the definition of an animal in [7, Chap. 4] is more general than the definition here, nevertheless, all animals defined above are animals in the sense of [7]). For any $i = 1, \ldots, l+1$, the segment $[m_{i-1}, m_i]$ presents the horizontal projection of the animal S_i and is called the base of S_i. The key observation here is that for any j both numbers $\bar{x}^{+}_{j}$ and $\tilde{x}^{+}_{j}$ belong to the base $[m_{i-1}, m_i]$ of some animal S_i. As a result,

$$|\bar{x}^{+}_{j} - \tilde{x}^{+}_{j}| \le m_i - m_{i-1} \tag{4.7}$$

with some $i = i^{+}(j)$. A similar estimate holds for $\bar{x}^{-}_{j} - \tilde{x}^{-}_{j}$ as well.

Lemma 4.3 *There exist positive constants* $h_0 = h_0(\Delta)$, $\beta_0 = \beta_0(\Delta)$, *and* $C = C(\Delta, \beta)$ *such that for all* $\beta \geq \beta_0$ *and sufficiently large* N

$$\mathbf{E}^{\varphi}_{N,\infty} \exp\left\{ h \frac{|\bar{x}^{\pm}_j - \tilde{x}^{\pm}_j|}{\sin\varphi} \right\} \leq C \tag{4.8}$$

uniformly in $|h| \leq h_0$, $j \in J_\alpha$, *and* φ *satisfying* (4.1). *As a result, for any fixed* $\varepsilon > 0$ *one has*

$$\mathbf{P}^{\varphi}_{N,\infty}\left(|\bar{x}^{\pm}_j - \tilde{x}^{\pm}_j| \geq N^{\varepsilon} \sin\varphi \right) \leq C \exp\left\{ -h_0 N^{\varepsilon} \right\} \tag{4.9}$$

uniformly in $j \in J_\alpha$.

Proof. For any fixed $\varphi \in (0, \pi/2 - \Delta]$ estimate (4.8) follows from (4.7) and the analogue of Corollary 7.5 in [6]. To check the uniformity in φ, observe that $|\bar{x}^{\pm}_j - \tilde{x}^{\pm}_j| \cot\varphi$ gives the difference of ordinates of points $(\bar{x}^{\pm}_j, \mathfrak{n}_j(\bar{x}^{\pm}_j))$ and $(\tilde{x}^{\pm}_j, \mathfrak{n}_j(\tilde{x}^{\pm}_j))$. This quantity is bounded from above by the number of vertical bonds of the corresponding animal $S_{i(j)}$, $i = i^{\pm}(j)$. It remains to recall the remark after Lemma 7.3 in [6] and to observe the uniform positiveness of $1/\cos\varphi$ for $|\varphi| \leq \pi/2 - \Delta$. □

Fix now any $S \in \mathscr{T}^{\varphi}_N$ and construct the upper and the lower approximating polygons $S^{\pm}$ using the quantities $g^{\pm}_N(k)$, $|k| \leq N$, from (4.2) and (4.3). As before, determine the set $\{m_1, \ldots, m_l\}$, $-N < m_1 < \ldots < m_l < N$, of all m such that S is regular in column m and apply the animal decomposition $\{S_1, \ldots, S_{l+1}\}$ to S. Since the extremal animals S_1 and S_{l+1} are not necessarily located in the vertical strips $\{y \in \mathbb{R}^2 : y_1 \in [-N, m_1]\}$ and $\{y \in \mathbb{R}^2 : y_1 \in [m_l, N]\}$ respectively, the inequality (4.4) is no longer valid. Define

$$m_0 = \max\left\{ k \leq -N : S(k) = \emptyset \right\}, \qquad m_{l+1} = \min\left\{ k \geq N : S(k) = \emptyset \right\}$$

with $S(k)$ denoting the set of common points of S and the vertical line $y_1 = k$,

$$S(k) = S \cap \left\{ y = (y_1, y_2) \in \mathbb{R}^2 : y_1 = k \right\} . \tag{4.10}$$

Again we have the property $\tilde{x}^{\pm}_j$, $\bar{x}^{\pm}_j \in [m_{i-1}, m_i]$ for all $j \in J_\alpha$ and $i = i^{\pm}(j) \in \{1, \ldots, l+1\}$; therefore, (4.7) is valid for all such j. Taking into account Remark 4.1.1 one deduces

Lemma 4.4 *All the results listed in Corollary* 4.2 *and Lemma* 4.3 *are valid for the distributions* $\mathbf{P}^{\varphi}_N(\cdot)$ *in* $\mathscr{T}^{\varphi}_N$ *as well.*

4.2 Processes of vertical fluctuations

We discuss here the vertical fluctuations of the phase boundary S. To be specific we suppose that $S \in \mathscr{T}^{\varphi}_{N\infty}$, though all considerations will be also true for the ensembles $\mathscr{T}^{\varphi}_{NM}$ and $\mathscr{T}^{\varphi}_N$.

Fix arbitrary $S \in \mathscr{T}^{\varphi}_{N\infty}$ and for any integer k, $|k| \le N$, determine the quantities $g^+_N(k)$ as in (4.2). Let $g^+_N(x)$, $x \in [-N, N]$, be the piecewise linear interpolation of the values $g^+_N(k)$. Consider the random polygonal function

$$\theta^+_N(t) = \frac{1}{2}\Big(g^+_N\big((2t-1)N\big) - g^+_N(-N)\Big), \qquad t \in [0,1] \ ,$$

and denote by $\mu^+_N = \mu^+_{N,\infty}$ the corresponding measure in $\mathbf{C}[0,1]$ induced by the probability distribution $\mathbf{P}^{\varphi}_{N,\infty}(\cdot)$ in $\mathscr{T}^{\varphi}_{N\infty}$. Finally, let $\theta^{*,+}_N(t)$, $t \in [0,1]$, be the (upper) process of vertical fluctuations,

$$\theta^{*,+}_N(t) = \frac{1}{\sqrt{N}}\Big(\theta^+_N(t) - Nb_N t\Big), \qquad b_N = \frac{g^+_N(N) - g^+_N(-N)}{2N} \quad , \tag{4.11}$$

and let $\mu^{*,+}_N$ denote its distribution in $\mathbf{C}[0,1]$.

In a similar way, starting from the quantities $g^-_N(k)$ (recall (4.3)), one defines the (lower) process of vertical fluctuations $\theta^{*,-}_N(t)$, $t \in [0,1]$, with the distribution $\mu^{*,-}_N$ in $\mathbf{C}[0,1]$.

Proposition 4.5 ([5]) *Let $F(\cdot)$ denote the free energy and $\bar{H} = \bar{H}(\varphi)$ solve the equation $F'(\bar{H}) = \beta \tan\varphi$. Then the sequence of measures $\mu^{*,+}_N$ converges weakly in* $\mathbf{C}[0,1]$ *to the distribution $\bar{\mu}$ of the process*

$$\bar{\theta}(t) = \frac{1}{\beta}\sqrt{F''(\bar{H})}\, w_{1,0}(t), \qquad t \in [0,1] \ , \tag{4.12}$$

where $w_{1,0}(t)$ denotes the Brownian bridge on $[0,1]$. *The same is true for the measures $\mu^{*,-}_N$. Moreover, for any sequence α_N of real numbers such that* $\alpha_N \to 0$ as $N \to \infty$ *one has the convergence*

$$\alpha_N\big(\theta^+_N(t) - \theta^-_N(t)\big) \to 0 \tag{4.13}$$

in probability as $N \to \infty$.

Since $b_N \to b \equiv \tan\varphi$ as $N \to \infty$, the distributions of the random processes $N^{-1}\theta^{\pm}_N(t)$ converge weakly in $\mathbf{C}[0,1]$ (and in probability) to the distribution concentrated on the deterministic function $e(t) = bt$, $t \in [0,1]$. Its graph γ is a segment having the slope angle φ, $\tan\varphi = b$. For any $t \in [0,1]$ the quantity $s = s(t) = t/\cos\varphi$ presents the length of the segment on the graph γ of $e(t)$ with the endpoints $(0,0)$ and (t, bt); denote the inverse mapping $s \mapsto t_s = s\cos\varphi$ by $t(s)$. The quantity $l = s(1) = (\cos\varphi)^{-1}$ gives the total length of γ.

Simple geometrical considerations imply the estimate

$$\max\Big\{\big|y - g^+_N(x)\big| : y \in S^+(x)\Big\} \le \frac{1}{2}\Big|g^+_N([x+1]) - g^+_N([x])\Big| \ ,$$

where $S^+(x)$ is defined similarly to (4.10). Clearly, the same inequality holds for $g^-_N(\cdot)$ and S^-. The following statement presents the key estimate used in the proof of (4.13). Though it was not stated explicitly in [5], its proof could be established by a literal repetition of that of Lemma 7.3 in [6]. (A close result can be found in [4].)

Lemma 4.6 *There exist positive constants* $h_0 = h_0(\Delta)$, $\beta_0 = \beta_0(\Delta)$, *and* $C = C(\Delta, \beta)$ *such that for all* $\beta \geq \beta_0$ *and sufficiently large* N

$$\mathbf{E}^{\varphi}_{N,\infty} \exp\Big\{h|g^+_N(k+1) - g^+_N(k)|\Big\} < C$$

uniformly in $|h| \leq h_0$, $k = -N, \dots, N-1$, *and* φ *satisfying* (4.1).

As a direct implication, one deduces

Corollary 4.7 *For any* $\varepsilon > 0$ *one has*

$$\mathbf{P}^{\varphi}_{N,\infty}\Big(|\mathrm{n}_j(\bar{x}^{\pm}_j) - g^{\pm}_N(\bar{x}^{\pm}_j)| \geq N^{\varepsilon}\Big) \leq C \exp\{-2h_0 N^{\varepsilon}\}$$

uniformly in $j \in J_\alpha$.

5 Proof of Theorems 1 and 2

This section is devoted mainly to the proof of Theorem 1. Basically it consists of three steps. First, we compare the values $\tilde{\zeta}^{\pm}_N(s_j)$ of the process of interest (recall (3.3)) to the values $\theta^{*,\pm}_N(\bar{t}^{\pm}_j)$ of the process of vertical fluctuations (recall (4.9)) at the points

$$\bar{t}^{\pm}_j = \frac{1}{2}\Big(\frac{\bar{x}^{\pm}_j}{N} + 1\Big), \qquad j \in J_\alpha \ , \tag{5.1}$$

and show that as $N \to \infty$ the difference

$$\tilde{\zeta}^{\pm}_N(s_j) - \theta^{*,\pm}_N(\bar{t}^{\pm}_j)\cos\varphi$$

vanishes in probability (uniformly in $j \in J_\alpha$). Then, taking into account Corollary 4.2, we deduce the uniform in $j \in J_\alpha$ convergence

$$\theta^{*,\pm}_N(\bar{t}^{\pm}_j) - \theta^{*,\pm}_N(t_j) \to 0$$

in probability as $N \to \infty$ on every compact set in $\mathbf{C}[0,1]$, where (cf. (5.1))

$$t_j = \frac{1}{2}\Big(\frac{x_j}{N} + 1\Big) = \frac{j}{[N^\alpha]}, \qquad j \in J_\alpha \ , \tag{5.2}$$

and check that the family of random variables $\theta^{*,\pm}_N(t_j)$, $j \in J_\alpha$, has correct finite dimensional distributions, i.e., those prescribed by Theorem 1. Based on this, we prove the convergence of all finite-dimensional distributions of the processes $\tilde{\zeta}^{\pm}_N(s)$, $s \in [0, l]$, to that of $\tilde{\zeta}(s)$ from (3.4). Finally, we establish the weak compactness of the sequences of distributions $\tilde{v}^+_{N,\infty}$ and $\tilde{v}^-_{N,\infty}$. In what follows, we consider mainly the upper processes (and intersection points). Generalization to the lower case is straightforward.

We start with the following geometric observation (see Fig. 3 below). Let OC be the graph of the normal line $\mathrm{n}_j(x)$ to the "orientation line" OA of the phase boundary S at the point $O(x_j, bx_j)$. Consider the points B, C, and D having the following coordinates: $B(\bar{x}^+_j, g^+_N(\bar{x}^+_j))$, $C(\bar{x}^+_j, \mathrm{n}_j(\bar{x}^+_j))$, and $D(\tilde{x}^+_j, \mathrm{n}_j(\tilde{x}^+_j))$. Finally, define the point $A(\bar{x}^+_j, b\bar{x}^+_j)$ as the vertical projection of the points B and C on the "orientation line" OA of S. Recalling (3.3), we rewrite

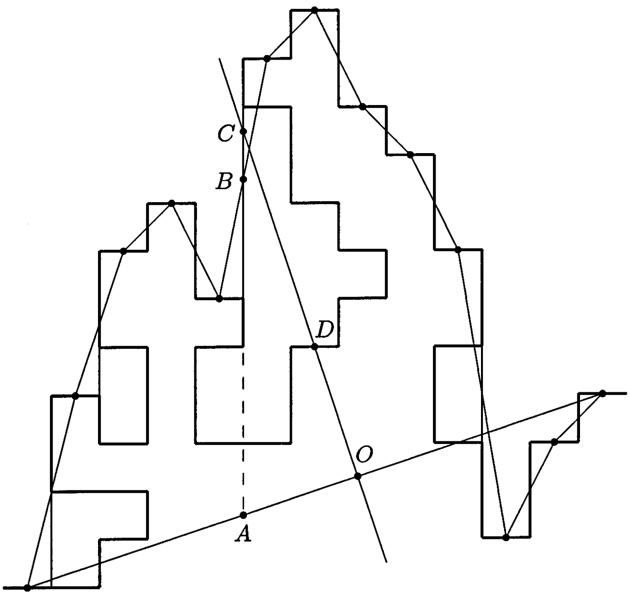

Fig. 3

$$\tilde{\zeta}^{+}_{N}(s_j) = \frac{x_j - \tilde{x}^{+}_{j}}{\sqrt{N}\sin\varphi} = \frac{\mathfrak{n}_j(\tilde{x}^{+}_{j}) - \mathfrak{n}_j(x_j)}{\sqrt{N}\cos\varphi}, \qquad j \in J_\alpha \ .$$

Now, elementary calculations lead to the relation

$$\tilde{\zeta}^{+}_{N}(s_j) = \theta^{*;+}_{N}(\bar{t}^{+}_{j})\cos\varphi + \frac{\bar{x}^{+}_{j} - \tilde{x}^{+}_{j}}{\sqrt{N}\sin\varphi} + \frac{\mathfrak{n}_j(\bar{x}^{+}_{j}) - g^{+}_{N}(\bar{x}^{+}_{j})}{\sqrt{N}}\cos\varphi \ , \tag{5.3}$$

where the following identity was used (cf. (4.11)):

$$\theta^{*;+}_{N}(\bar{t}^{+}_{j}) = \frac{g^{+}_{N}(\bar{x}^{+}_{j}) - \bar{x}^{+}_{j}\tan\varphi}{\sqrt{N}} \ .$$

Lemma 5.1 *There exist positive constants $\beta_0 = \beta_0(\Delta)$, $c = c(\Delta)$, and $C = C(\Delta, \beta)$ such that for all $\beta \geq \beta_0$ and any $\varepsilon > 0$ one has*

$$\mathbf{P}^{\varphi}_{N,\infty}\Big(\max_{j\in J_\alpha}\big|\tilde{\zeta}^{+}_{N}(s_j) - \theta^{*;+}_{N}(\bar{t}^{+}_{j})\cos\varphi\big| > N^{-(1/2-\varepsilon)}\Big) < C\exp\big\{-cN^{\varepsilon}\big\}$$

uniformly in natural N.

Proof. In view of relation (5.3) the statement of the lemma follows directly from Lemma 4.3 and Corollary 4.7. □

Consider the random processes (cf. (4.11))

$$\zeta^{\pm}_{N}(s) \equiv \theta^{*;\pm}_{N}(s\cos\varphi)\cos\varphi \tag{5.4}$$

and denote their distributions in $\mathbf{C}[0, l]$ by $\nu^{\pm}_{N,\infty}$.

Lemma 5.2 *The sequences of measures $\nu^{\pm}_{N,\infty}$ converge weakly in $\mathbf{C}[0,l]$ to the distribution of the random process (recall (3.4))*

$$\tilde{\zeta}(s) = \frac{w_{l,0}(s)}{\sqrt{\beta\big(\tau(\varphi)+\tau''(\varphi)\big)}}, \qquad s \in [0,l] \ ,$$

where $w_{l,0}(s)$ stands for the Brownian bridge in $[0,l]$ and $\tau(\varphi)$ is the surface tension from (2.8). Moreover, for any sequence of real numbers α_N such that $\alpha_N \to 0$ as $N \to \infty$ one has

$$\alpha_N \sqrt{N}\Big(\zeta^+_N(s) - \zeta^-_N(s)\Big) \to 0$$

in probability as $N \to \infty$.

Proof. Recall that $F'(\bar{H}) = \beta \tan\varphi$. Therefore,

$$\tau(\varphi) + \tau''(\varphi) = \beta F^{*\prime\prime}\big(\beta\tan\varphi\big)\cos^{-3}\varphi = \frac{\beta}{F''(\bar{H})\cos^3\varphi} \ , \tag{5.5}$$

where the first equality follows from relation (2.10) and the second one is implied by the duality relations for the Legendre transformation (see, e.g., Property A.1 in [6]). Changing the variables $t \mapsto s$ in (4.12) one immediately deduces the claim of the lemma from Proposition 4.5 and relation (5.5). □

Remark 5.2.1 Definition (5.4) induces the one-to-one correspondence ψ between $\mathbf{C}[0,1]$ and $\mathbf{C}[0,l]$, ψ: $f(t) \mapsto g(s) = f(s\cos\varphi)\cos\varphi$. Observe that ψ introduces a bijection between compact sets in these spaces.

Proof of Theorem 1. First, let us check that for every $\varepsilon > 0$

$$\mathbf{P}^{\varphi}_{N,\infty}\Big(\max_{j\in J_\alpha} |\theta^{*,+}_N(\bar{t}^+_j) - \theta^{*,+}_N(t_j)| \geq \varepsilon\Big) \to 0 \tag{5.6}$$

as $N \to \infty$, where $\bar{t}^+_j$ and t_j are defined in (5.1) and (5.2) respectively. To this end, choose arbitrary $\eta > 0$ and fix any compact set $\mathscr{K} \subset \mathbf{C}\ [0,1]$ such that

$$\mu^*_N\big(\mathbf{C}[0,1]\backslash\mathscr{K}\big) < \frac{\eta}{2} \tag{5.7}$$

for all $N \geq N_0$ with sufficiently large N_0 (such $\mathscr{K}$ always exists due to the weak compactness of the sequence $\mu^{*,+}_N$; recall Proposition 4.5). According to Arzelà's theorem [3, App. 1], all $f \in \mathscr{K}$ are equicontinuous,

$$\lim_{\delta\to 0} \sup_{f\in\mathscr{K}} \sup_{t',t''\in[0,1],|t'-t''|<\delta} \big|f(t') - f(t'')\big| = 0 \ . \tag{5.8}$$

Let $\delta > 0$ be such that $\sup_{|t'-t''|<\delta}\big|f(t') - f(t'')\big| < \varepsilon$ for all $f \in \mathscr{K}$. Then,

$$\Big\{\max_{j\in J_\alpha}\big|\theta^{*,+}_N(\bar{t}^+_j) - \theta^{*,+}_N(t_j)\big| \geq \varepsilon\Big\} \subset \Big\{\theta^{*,+}_N(\cdot) \in \mathbf{C}[0,1]\backslash\mathscr{K}\Big\} \cup \bigcup_{j\in J_\alpha}\Big\{\big|\bar{t}^+_j - t_j\big| \geq \delta\Big\} \tag{5.9}$$

and (5.6) follows directly from (5.9), (5.7), (5.1), (5.2), and Corollary 4.2. Consequently, in view of Lemma 5.1, definition (5.4), and the simple relation

$$\tilde{\zeta}_N^+(s_j) - \zeta_N^+(s_j) = \tilde{\zeta}_N^+(s_j) - \theta_N^{*,+}(\bar{t}_j^+)\cos\varphi + \left(\theta_N^{*,+}(\bar{t}_j^+) - \theta_N^{*,+}(t_j)\right)\cos\varphi \ ,$$

the inequality

$$\mathbf{P}_{N,\infty}^{\varphi}\left(\max_{j\in J_\alpha}\left|\tilde{\zeta}_N^+(s_j) - \zeta_N^+(s_j)\right| \geq \varepsilon/2\right) < \eta/4 \tag{5.10}$$

holds for any positive ε and η provided only $N \geq N_0 = N_0(\Delta, \varepsilon, \eta) > 0$ and $\beta \geq \beta_0(\Delta) > 0$. In the remaining part of the proof we will assume that $N \geq N_0$ with such N_0.

Next, let us prove the convergence of finite-dimensional distributions of the random process $\tilde{\zeta}_N^+(s)$ to that of $\tilde{\zeta}(s)$ from (3.4). Due to Lemma 5.2 it is enough to prove that

$$\sup_{s\in[0,l]}\left|\tilde{\zeta}_N^+(s) - \zeta_N^+(s)\right| \to 0 \tag{5.11}$$

in probability as $N \to \infty$. To do this, fix arbitrary $\varepsilon > 0$, $\eta > 0$ and consider any compact set $\mathscr{K} \subset \mathbf{C}[0,1]$ satisfying (5.7). For $s \in [0,l]$, denote by ρ_1, $\rho_2 \in \left\{jl/[N^\alpha], j \in J_\alpha\right\}$ the numbers such that $s \in [\rho_1, \rho_2]$ and $|\rho_1 - \rho_2| = l/[N^\alpha]$. Let $\lambda = \lambda(s) \in [0,1]$ be such that $s = \lambda\rho_1 + (1-\lambda)\rho_2$. Now, find $\bar{\delta} > 0$ with the property

$$\sup_{t',t''\in[0,1],|t'-t''|<\bar{\delta}}\left|f(t') - f(t'')\right| < \varepsilon/2 \tag{5.12}$$

uniformly in $f \in \mathscr{K}$ (recall (5.8)). Without loss of generality one may assume that $\bar{\delta}$ and N_0 fixed above are related via $\bar{\delta}[(N_0)^\alpha] > 1$. According to the definition of $\tilde{\zeta}_N^+(\cdot)$, one has

$$\tilde{\zeta}_N^+(s) = \lambda\tilde{\zeta}_N^+(\rho_1) + (1-\lambda)\tilde{\zeta}_N^+(\rho_2) \ . \tag{5.13}$$

Now, taking into account (5.4), rewrite

$$\begin{aligned}\left|\tilde{\zeta}_N^+(s) - \zeta_N^+(s)\right| &\leq \lambda\left|\tilde{\zeta}_N^+(\rho_1) - \zeta_N^+(\rho_1)\right| + (1-\lambda)\left|\tilde{\zeta}_N^+(\rho_2) - \zeta_N^+(\rho_2)\right| \\ &\quad + \lambda\left|\zeta_N^+(\rho_1) - \zeta_N^+(s)\right| + (1-\lambda)\left|\zeta_N^+(\rho_2) - \zeta_N^+(s)\right| \ .\end{aligned} \tag{5.14}$$

Then, the simple inclusion

$$\begin{aligned}&\left\{\left|\tilde{\zeta}_N^+(s) - \zeta_N^+(s)\right| \geq \varepsilon\right\} \\ &\subset \left\{\left|\tilde{\zeta}_N^+(\rho_1) - \theta_N^{*,+}(\rho_1\cos\varphi)\cos\varphi\right| \geq \varepsilon/2\right\} \cup \left\{\left|\rho_1 - s\right| \geq \bar{\delta}\right\} \\ &\cup \left\{\left|\tilde{\zeta}_N^+(\rho_2) - \theta_N^{*,+}(\rho_2\cos\varphi)\cos\varphi\right| \geq \varepsilon/2\right\} \cup \left\{\left|\rho_2 - s\right| \geq \bar{\delta}\right\} \\ &\cup \left\{\theta_N^{*,+}(\cdot) \in \mathbf{C}[0,1]\backslash\mathscr{K}\right\} \ ,\end{aligned}$$

and relations (5.7), (5.10), (5.12)–(5.14) imply that for N under consideration

$$\mathbf{P}_{N,\infty}^{\varphi}\left(\sup_{s\in[\rho_1,\rho_2]}\left|\tilde{\zeta}_N^+(s) - \zeta_N^+(s)\right| \geq \varepsilon\right) < \eta \ .$$

Observing that the last estimate is independent of $[\rho_1, \rho_2]$, one immediately deduces (5.11).

Finally, it remains to establish the weak compactness of the measures $\tilde{\nu}^+_{N,\infty}$. According to Theorem 8.2 from [3], one has to prove that for any $\varepsilon > 0$ and $\eta > 0$ there exist $\delta > 0$ and N_1 such that for all $N \geq N_1$

$$\mathbf{P}^{\varphi}_{N,\infty}\left(\sup_{s',s''\in[0,1],|s'-s''|<\delta} \left|\tilde{\zeta}^+_N(s') - \tilde{\zeta}^+_N(s'')\right| \geq \varepsilon \right) \leq \eta \ . \tag{5.15}$$

To do this, we observe that

$$\left|\tilde{\zeta}^+_N(s') - \tilde{\zeta}^+_N(s'')\right| \leq \left|\tilde{\zeta}^+_N(s') - \zeta^+_N(s')\right| + \left|\zeta^+_N(s') - \zeta^+_N(s'')\right| + \left|\zeta^+_N(s'') - \tilde{\zeta}^+_N(s'')\right| \ ,$$

and therefore

$$\mathbf{P}^{\varphi}_{N,\infty}\left(\sup_{|s'-s''|<\delta} \left|\tilde{\zeta}^+_N(s') - \tilde{\zeta}^+_N(s'')\right| \geq \varepsilon \right)$$
$$\leq \mathbf{P}^{\varphi}_{N,\infty}\left(\sup_{|s'-s''|<\delta} \left|\zeta^+_N(s') - \zeta^+_N(s'')\right| \geq \varepsilon/3 \right)$$
$$+ 2\mathbf{P}^{\varphi}_{N,\infty}\left(\sup_{s\in[0,l]} \left|\tilde{\zeta}^+_N(s) - \zeta^+_N(s)\right| \geq \varepsilon/3 \right) \ .$$

Now, (5.15) follows from the weak compactness of the sequence ν^+_N (recall Lemma 5.2), relation (5.11), and the last inequality.

The weak convergence of $\tilde{\nu}^+_{N,\infty}$ is proved. Clearly, the same arguments are applicable to $\tilde{\nu}^-_{N,\infty}$. Finally, the claim about the convergence

$$\tilde{\zeta}^+_N(s) - \tilde{\zeta}^-_N(s) \to 0$$

in probability as $N \to \infty$ follows from (5.11), its analogue for the lower process, and Lemma 5.2. □

Proof of Theorem 2. According to the assumption that $M > N(\tan\varphi + \varepsilon)$ with some $\varepsilon > 0$, the claim of the theorem for the measures $\tilde{\nu}^\pm_{NM}$ follows directly from Theorem 1 and Lemma 4.1.

Since all the statements in Sect. 4 are valid also for the distribution $\mathbf{P}^{\varphi}_N(\cdot)$ in $\mathscr{T}^{\varphi}_N$, the proof in the case $\tilde{\nu}^\pm_N$ distributions is a literal repetition of that of Theorem 1. □

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