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Stochastic cascades and 3-dimensional Navier–Stokes equations

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Summary. In this article, we study the incompressible Navier–Stokes equations in $\mathbb{R}^3$. The non linear integral equation satisfied by the Fourier transform of the Laplacian of the velocity field can be interpreted in terms of a branching process and a composition rule along the associated tree. We derive from this representation new classes where global existence and uniqueness can be proven.

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0 Introduction

The motion of a viscous incompressible fluid in the whole space is described by the Navier–Stokes equations:

$$(0.1) \qquad \begin{cases} \partial_t u + u \cdot \nabla u = \nu \Delta u - \nabla p + f \ , \\ u_{t=0} = u_0, \ \text{div } u = 0 \ , \end{cases}$$

where $u(t,x) \in \mathbb{R}^3$, denotes the velocity field, $p(t,x)$ is the pressure, $f(t,x) \in \mathbb{R}^3$, the force field and $\nu > 0$, the kinematic viscosity. These equations were introduced in the works of Navier [9] and Stokes [11]. They modify Euler's equation, which is Newton's law for an infinitesimal volume element of the fluid, by the addition of a dissipative term $\nu\Delta u$, corresponding to friction forces. The mathematical study of these equations began in the thirties, in particular with the seminal paper [8] of Leray. Since then, there has been an enormous literature on the subject, see for instance [1], [3], [4], [5], [7], [13], [14] and references therein. Nevertheless some of the questions raised in [8] are to this day open. A notion of weak solution for (0.1) has been developed, see Temam [13]. Although one has a global existence result for weak solutions, one can in general only prove uniqueness for a more restrictive class of solutions, where a global existence result is not available. However, in the case of "suitably small data", one has global existence of a "good solution" and uniqueness of weak solutions, see Kiselev–Ladyzhenskaya [6], Serrin [10], p. 76, 83, 86.

The purpose of this article is to introduce a probabilistic interpretation of (0.1) and to use this to derive certain global existence results in classes where uniqueness holds. To this end we investigate a Fourier representation of (0.1), namely the integral equation:

$$\chi_t(\xi) = \exp\{-\nu|\xi|^2 t\}\chi_0(\xi) + \int_0^t \nu|\xi|^2 e^{-\nu|\xi|^2(t-s)}\left(\frac{1}{2}\chi_s \circ \chi_s(\xi) + \frac{1}{2}\varphi(s,\xi)\right)ds,$$
$$t \geq 0, \quad \xi \in \mathbb{R}^3\backslash\{0\}\ , \text{ with} \tag{0.2}$$

$$\chi = \frac{2}{\nu}\left(\frac{\pi}{2}\right)^{3/2}|\xi|^2\hat{u}\ , \tag{0.3}$$

$$\varphi = \frac{4}{\nu^2}\left(\frac{\pi}{2}\right)^{3/2}\hat{f}\ , \tag{0.4}$$

and the $\circ$-operation is defined for $\mathbb{C}^3$-valued functions f_1, f_2 on $\mathbb{R}^3\backslash\{0\}$, via:

$$f_1 \circ f_2(\xi) = -\mathrm{i}\int_{(\mathbb{R}^3\backslash\{0\})^2}\left(f_1(\xi_1)\cdot\frac{\xi}{|\xi|}\right)p(\xi)\big(f_2(\xi_2)\big)K_\xi(d\xi_1, d\xi_2)\ , \tag{0.5}$$

where $p(\xi)$ is the projection on the orthogonal of ξ and K_ξ the kernel:

$$K_\xi(h) = \frac{1}{\pi^3}\int_{\mathbb{R}^3\backslash\{0\}} h(\xi_1, \xi - \xi_1)\frac{|\xi|\,d\xi_1}{|\xi_1|^2|\xi - \xi_1|^2}\ . \tag{0.6}$$

Thanks to the three dimensional situation, it turns out that K has a number of remarkable properties (see Proposition 1.2). In particular, it is Markovian (i.e. a probability kernel). As a result of the special features of K, we are able to study existence and uniqueness questions for (0.2), with the help of a critical branching process on $\mathbb{R}^3\backslash\{0\}$, which we call the "stochastic cascade". For this process, a particle located at ξ, after an exponential holding time of parameter $\nu|\xi|^2$, with equal probability either dies out or gives birth to two descendants, distributed according to K_ξ.

We are able in section II to develop in a suitable setting, a representation formula for solutions of (0.2), as the expectation of the result of a certain operation performed "along the branching tree generated by the stochastic cascade". This is somewhat reminiscent of the Wild sums for Boltzmann equation (see Wild [15], [12] chapter IV, and also [16]).

We obtain a "domination principle" for existence and uniqueness questions in (0.2), see Theorem 2.2 and 2.4. Roughly speaking, if the scalar integral equation

$$X_t(\xi) = \exp\{-\nu|\xi|^2 t\}X_0(\xi) + \int_0^t \nu|\xi|^2 e^{-\nu|\xi|^2(t-s)}\left(\frac{1}{2}K_\xi(X_s \otimes X_s) + \frac{1}{2}\Phi(s,\xi)\right)ds \tag{0.7}$$

with non negative data X_0 and Φ, for which $|\chi_0| \le X_0$, $|\varphi| \le \Phi$, has a finite (minimal) solution $X_t(\xi)$, we have existence and uniqueness of solutions of (0.2) in the class $|\chi_t(\xi)| \le X_t(\xi)$.

From this domination principle, we deduce concrete classes for which we have existence and uniqueness in (0.2), see Theorem 3.2 and also Theorem 3.4. It should be pointed out that these results do not rely on the formal energy identity satisfied by (0.1), (in our context see (4.12), (4.16)). In fact we have some well defined evolutions with possibly infinite energy.

We are then able to derive certain existence and uniqueness results for weak solutions of (0.1), stated for simplicity in the case $f = 0$, see Theorem 4.1. These statements seem to be new, see for instance Remark 4.3. Although not directly comparable, some of them share a common flavor with Cannone's existence and uniqueness results in Besov space, see [1].

Let us finally describe how the article is organized. In section I, starting from a weak solution setting for (0.1), we derive (0.2), see Proposition 1.1, and investigate some of the special properties of the kernel K.

In section II, we develop the domination principle and the representation formula for solutions of (0.2), in terms of the "stochastic cascade". Our main results are the existence Theorem 2.2 and uniqueness Theorem 2.4.

In section III, we give concrete examples for the results of section II, see Theorem 3.2 and 3.4.

In section IV, we return to the Cauchy problem (0.1), for simplicity in the case $f = 0$. We derive a certain global existence and uniqueness result in Theorem 4.1.

1 Fourier representation of the Navier–Stokes equation

The object of this section is to derive an appropriate formulation of the Fourier transform of the incompressible 3-d Navier–Stokes equation. Our starting point is the notion of weak solutions of the Navier–Stokes equation in $\mathbb{R}^3$, for instance as in Temam [13], p. 282. That is we consider the spaces

$$\begin{aligned} \mathscr{V} &= \{u \in C_c^\infty(\mathbb{R}^3)^3, \operatorname{div} u = 0\} , \\ H &= \text{closure of } \mathscr{V} \text{ in } L^2(\mathbb{R}^3)^3 , \\ V &= \text{closure of } \mathscr{V} \text{ in } H^1(\mathbb{R}^3)^3 . \end{aligned} \tag{1.1}$$

Given an initial condition $u_0 \in H$ and a force $f(t,x) \in L^2(0,T;V')$, $T > 0$, a weak solution of Navier–Stokes equation on the interval $[0,T]$ is a $u \in L^2(0,T;V)$ with $u' \in L^1(0,T;V')$ such that for any $g \in \mathscr{V}$ and $t \in [0,T]$:

$$\text{(NS)} \qquad \langle u_t, g\rangle = \langle u_0, g\rangle + \int_0^t \nu\langle u_s, \Delta g\rangle + \langle u_s, u_s \nabla g\rangle + \langle f_s, g\rangle \, ds ,$$

where the notation $\langle h, g\rangle$ stands for $\sum_1^3 \int h^\ell \bar{g}^\ell dx$, (we shall later use this notation for complex valued functions). The existence of solutions to (NS) is known, see Temam [13] Theorem 3.1. Whenever it makes sense, we shall denote by $\hat{h}$ the Fourier transform of h:

$$\hat{h}(\xi) = \frac{1}{(2\pi)^{3/2}} \int_{\mathbb{R}^3} \exp\{-\mathrm{i}x \cdot \xi\} h(x)\, dx \,.$$

For $w_1(\xi)$, $w_2(\xi)$ two $\mathbb{C}^3$ valued measurable functions on $\mathbb{R}^3 \backslash \{0\}$, we shall write

$$(1.2) \qquad w_1 \circ w_2(\xi) = -\frac{\mathrm{i}}{\pi^3} \int \big(w_1(\xi_1) \cdot e_\xi\big) p(\xi) w_2(\xi - \xi_1) \frac{|\xi|\, d\xi_1}{|\xi_1|^2 |\xi - \xi_1|^2} \,,$$

for $\xi \in \mathbb{R}^3 \backslash \{0\}$, for which the above integral is absolutely convergent, where $w \cdot w' = \sum_1^3 w_i \bar{w}'_i$, for $w, w' \in \mathbb{C}^3$, $e_\xi = \xi/|\xi|$, and

$$(1.3) \qquad p(\xi)w = w - e_\xi(w \cdot e_\xi), \quad w \in \mathbb{C}^3 \,,$$

stands for the projection on the orthogonal of ξ (in $\mathbb{C}^3$). Our representation of the Fourier transform of Navier–Stokes equation comes in the following:

Proposition 1.1: *If u is a solution of (NS), one can choose $\chi_t(\xi)$, $t \in [0, T]$, $\xi \in \mathbb{R}^3$, continuous in t, measurable in ξ, with*

$$(1.4) \qquad \chi_t(\xi) = \frac{2}{\nu} \left(\frac{\pi}{2}\right)^{3/2} |\xi|^2 \hat{u}_t(\xi), \ \text{a.e.} \ \ \xi, \quad \text{for } t \in [0, T] \,,$$

$$(1.5) \qquad \chi_t(\xi) \cdot \xi = 0, \quad \chi_t(-\xi) = \overline{\chi_t(\xi)}, \quad 0 \le t \le T, \ \ \xi \in \mathbb{R}^3 \,,$$

such that for a.e. ξ,

$$\chi_t(\xi) = \exp\{-\nu|\xi|^2 t\} \chi_0(\xi) + \int_0^t \nu|\xi|^2 \mathrm{e}^{-\nu|\xi|^2 (t-s)} \left[\frac{1}{2} \chi_s \circ \chi_s(\xi) + \frac{1}{2} \varphi(s, \xi)\right] ds \,,$$

(1.6)

for $0 \le t \le T$, *where*

$$(1.7) \qquad \begin{aligned} &\varphi(s, \xi) = \frac{4}{\nu^2} \left(\frac{\pi}{2}\right)^{3/2} \hat{f}_s(\xi), \quad \text{a.e.} \ \ s, \xi, \quad \textit{and} \\ &\varphi(s, \xi) \cdot \xi = 0 \,, \quad \varphi(s, -\xi) = \overline{\varphi(s, \xi)} \,, \quad 0 \le s \le T, \ \ \xi \in \mathbb{R}^3 \,. \end{aligned}$$

Proof: We can choose a measurable version $\hat{f}_s(\xi)$ of the Fourier transform of the real valued, weakly divergence free f_s, such that:

$$(1.8) \qquad \begin{aligned} &\int_0^T \!\! \int (1 + |\xi|^2)^{-1} \left|\hat{f}_s(\xi)\right|^2 d\xi\, ds < \infty, \quad \text{and} \\ &\hat{f}_s(\xi) \cdot \xi = 0, \quad \hat{f}_s(-\xi) = \overline{\hat{f}_s(\xi)}, \quad 0 \le s \le T, \quad \xi \in \mathbb{R}^3 \,. \end{aligned}$$

Applying similar considerations to $u' \in L^1(0, T; V')$, we can find a measurable version $\hat{u}_s(\xi)$ continuous in s such that

$$(1.9) \qquad \begin{aligned} &\int_0^T \!\! \int (1 + |\xi|^2) |\hat{u}_s(\xi)|^2\, d\xi\, ds < \infty, \quad \text{and} \\ &\hat{u}_s(\xi) \cdot \xi = 0, \quad \hat{u}_s(-\xi) = \overline{\hat{u}_s(\xi)}, \quad 0 \le s \le T, \quad \xi \in \mathbb{R}^3 \,. \end{aligned}$$

It follows from (NS) that for $t \in [0, T]$:

$$(1.10)\qquad \begin{aligned}\langle \hat{u}_t, \hat{g}\rangle = \langle \hat{u}_0, \hat{g}\rangle &+ \int_0^t -\nu\langle|\xi|^2\hat{u}_s, \hat{g}\rangle + \langle \hat{f}_s, \hat{g}\rangle\, ds \\ &+ \frac{1}{(2\pi)^{3/2}}\int_0^t\!\!\int \hat{u}_s^\ell(\xi)\overline{\hat{u}_s^k}(\xi-\xi')\big(-\mathrm{i}\xi_k'\overline{\hat{g}^\ell}(\xi')\big)\, d\xi\, d\xi'\, ds\,.\end{aligned}$$

Since $\hat{u}_s(-\xi) = \overline{\hat{u}}_s(\xi)$, the last term of (1.10) equals

$$(1.11)\qquad \frac{-\mathrm{i}}{(2\pi)^{3/2}}\int_0^t \langle \xi_k(\hat{u}_s^k * \hat{u}_s), \hat{g}\rangle = \left(\frac{2}{\pi}\right)^{3/2}\frac{\nu^2}{2}\int_0^t \frac{1}{2}\langle \chi_s \circ \chi_s, \hat{g}\rangle\, ds\,,$$

using $\hat{g}(\xi)\cdot\xi = 0$ in the last step, with χ_s as in (1.4), and as a result of (1.9),

$$(1.12)\qquad \sup_\xi \int_0^T ds \int |\chi_s(\xi_1)\cdot e_\xi|\; |\chi_s(\xi-\xi_1)|\frac{|\xi|\, d\xi_1}{|\xi_1|^2|\xi-\xi_1|^2} < \infty\,.$$

It now follows from (1.10) that for $t \in [0, T]$:

$$(1.13)\qquad \chi_t(\xi) \overset{\text{a.e.}}{=} \chi_0(\xi) + \int_0^t \nu|\xi|^2\left[-\chi_s(\xi) + \frac{1}{2}\chi_s\circ\chi_s(\xi) + \frac{1}{2}\varphi(s,\xi)\right] ds\,.$$

Using the continuity in t of $\chi_t(\xi)$, the above equality holds a.e. in ξ for $t \in [0, T]$. It then follows by considering the derivative of the absolutely continuous function $\exp\{-\nu|\xi|^2(t-s)\}\chi_s(\xi)$ that (1.6) holds. □

We now introduce the kernel K from $\mathbb{R}^3\backslash\{0\}$ to $\big(\mathbb{R}^3\backslash\{0\}\big)^2$ defined through

$$(1.14)\qquad \int h(\xi_1,\xi_2)K_\xi(d\xi_1, d\xi_2) = \frac{1}{\pi^3}\int h(\xi_1, \xi-\xi_1)\frac{|\xi|\, d\xi_1}{|\xi_1|^2|\xi-\xi_1|^2}\,,$$

for $h \geq 0$ measurable on $(\mathbb{R}^3\backslash\{0\})^2$.

So with the notations of (1.3), for $\xi \neq 0$,

$$(1.15)\qquad w_1\circ w_2(\xi) = -\mathrm{i}\int \big(w_1(\xi_1)\cdot e_\xi\big)p(\xi)w_2(\xi_2)K_\xi(d\xi_1, d\xi_2)\,,$$

whenever the integral is absolutely convergent.

As we shall now see, the kernel K has some remarkable properties which are crucial for the later developments in this article. We introduce some "angular variables" to describe K. We define

$$(1.16)\qquad D = \big\{(\theta_1,\theta_2,\alpha) \in (0,\pi)\times(0,\pi)\times[0,2\pi];\, \theta_1+\theta_2 < \pi\big\}\,.$$

For $(\theta_1,\theta_2,\alpha) \in D$, we denote by $\xi_1(\theta_1,\theta_2,\alpha)$ and $\xi_2(\theta_1,\theta_2,\alpha)$ the unique vectors of $(\mathbb{R}^3\backslash\{0\})^2$ such that:

$$(1.17)\qquad \xi_1+\xi_2 = e_3,\ \big((e_i)_{1\leq i\leq 3}\ \text{stands for the canonical basis of}\ \mathbb{R}^3\big)\,,$$

ξ_1 has colatitude θ_1 and longitude α; ξ_2 has colatitude θ_2 and longitude $\alpha+\pi(\mathrm{mod}\ 2\pi)$.

We denote by Ψ: $D \to \big(\mathbb{R}^3\backslash\{0\}\big)^2$, the map

(1.18) $$(\theta_1,\theta_2,\alpha)\to\big(\xi_1(\theta_1,\theta_2,\alpha),\xi_2(\theta_1,\theta_2,\alpha)\big)\ .$$

It is immediate to argue that Ψ is a bijection from D onto the set of ξ_1,ξ_2 in $\mathbb{R}^3\backslash\mathbb{R}e_3$, with $\xi_1+\xi_2=e_3$, and the restriction of Φ to $\overset{\circ}{D}$ induces a diffeomorphism.

For any $\xi\neq 0$, we denote by S_ξ the similitude of $\mathbb{R}^3$ sending e_3 on ξ, obtained as a composition of a rotation in the e_3, ξ plane and an homothety when e_3 and ξ are not colinear, and multiplication by a scalar when e_3 and ξ are colinear. Finally we denote by A the set

(1.19) $$A=\big\{(r,r_1,r_2)\in(0,\infty)^3,\ r_1+r_2>r,\ r_1+r>r_2,\ r_2+r>r_1\big\}\ .$$

A is symmetric under permutation of coordinates in $(0,\infty)^3$; it describes the set of ordered triplets arising as side lengths of non degenerate triangles of the Euclidean plane. We now collect some useful properties of K in the following

Proposition 1.2: *K is a Markovian kernel and for $\xi\in\mathbb{R}^3\backslash\{0\}$,*

(1.20) $$K_\xi(d\xi_1,d\xi_2)\ \text{ is the image of the probability }\ \frac{1}{\pi^3}1_D\,d\theta_1\,d\theta_2\,d\alpha\ ,\quad\text{under the map } (S_\xi\otimes S_\xi)\circ\Psi\ .$$

(1.21) $$\text{Under } K_\xi,\ (|\xi_1|,|\xi_2|)\ \text{has law}\ \ \frac{2}{\pi^2}1_A(r,r_1,r_2)\frac{dr_1}{r_1}\frac{dr_2}{r_2},\quad \text{where}\ \ r=|\xi|>0\ .$$

The law of $|\xi_1|$ or $|\xi_2|$ under K_ξ is

(1.22) $$\frac{2}{\pi^2}\ \log\left|\frac{r+r_1}{r-r_1}\right|\frac{dr_1}{r_1}\ .$$

For h,g non negative measurable on $(0,\infty)$, $i,j\in\{1,2\}$

(1.23) $$\int\frac{d\xi}{|\xi|^3}h(|\xi|)\int g(|\xi_i|)K_\xi(d\xi_1,d\xi_2)=\int\frac{d\xi}{|\xi|^3}g(|\xi|)\int h(|\xi_j|)K_\xi(d\xi_1,s\xi_2)\ .$$

Proof: We begin with the proof of (1.20). With no loss of generality, we assume that $\xi=e_3$. For $(\theta_1,\theta_2,\alpha)\in D$, we write

(1.24) $$r_1=|\xi_1(\theta_1,\theta_2,\alpha)|\in(0,\infty),\quad r_2=|\xi_2(\theta_1,\theta_2,\alpha)|\in(0,\infty)\ .$$

Projecting the equality

$$e_3=\xi_1+\xi_2\ ,$$

on $\mathbb{R}e_3$ and on the e_1,e_2 plane we find:

$$r_1\cos\theta_1+r_2\cos\theta_2=1,\quad r_1\sin\theta_1-r_2\sin\theta_2=0\ ,$$

so that for $(\theta_1,\theta_2,\alpha)\in D$:

$$r_1 = \frac{\sin\theta_2}{\sin(\theta_1+\theta_2)}, \quad r_2 = \frac{\sin\theta_1}{\sin(\theta_1+\theta_2)} . \tag{1.25}$$

Now for $g \geq 0$ measurable on $(\mathbb{R}^3\backslash\{0\})^2$, using polar coordinates we find:

$$\begin{aligned}\int g(\xi_1,\xi_2)K_{e_3}(d\xi_1,d\xi_2) &= \frac{1}{\pi^3}\int g(\xi_1, e_3-\xi_1)\frac{d\xi_1}{|\xi_1|^2|\xi-\xi_1|^2}\\ &= \frac{1}{\pi^3}\int_{(0,\infty)\times(0,\pi)\times(0,2\pi)} g\big(\xi_1(r_1,\theta_1,\alpha),\, e_3-\xi_1(r_1,\theta_1,\alpha)\big)\\ &\quad\times r_1^2\sin\theta_1\frac{dr_1 d\theta_1 d\alpha}{r_1^2|e_3-\xi_1(r_1,\theta_1,\alpha)|^2}\end{aligned} \tag{1.26}$$

with obvious notations,

$$\begin{aligned}&\overset{(1.25)}{=} \frac{1}{\pi^3}\int_D g\circ\Psi(\theta_1,\theta_2,\alpha)\frac{\sin^2(\theta_1+\theta_2)}{\sin^2\theta_1}\sin\theta_1\frac{\partial r_1}{\partial\theta_2}\,d\theta_1\,d\theta_2\,d\alpha\\ &= \frac{1}{\pi^3}\int_D g\circ\Psi(\theta_1,\theta_2,\alpha)\,d\theta_1\,d\theta_2\,d\alpha .\end{aligned}$$

This proves our claim (1.20). Let us now prove (1.21). Just as for the proof of (1.20), it suffices to consider the case $\xi = e_3$, so that $r = |\xi| = 1$. Observe that for $(\theta_1,\theta_2,\alpha)\in D$, $\big(1, r_1 = |\xi_1(\theta_1,\theta_2,\alpha)|, r_2 = |\xi_2(\theta_1,\theta_2,\alpha)|\big)\in A$, where A is defined in (1.19). In fact r_1, r_2 do not depend on α (see (1.25)), and

$$(\theta_1,\theta_2,\alpha)\in D\to(r_1,r_2)\quad\text{with}\quad(1,r_1,r_2)\in A \tag{1.27}$$

is a bijection when restricted to $\alpha=\alpha_0\in[0,2\pi)$. This is geometrically clear and in fact analytically one has

$$\cos\theta_2 = \frac{1+r_2^2-r_1^2}{2r_2}, \quad \cos\theta_1 = \frac{1+r_1^2-r_2^2}{2r_1} ,$$

which uniquely determines θ_1, θ_2. Now a direct calculation shows that the Jacobian of the above (restricted) map is:

$$J = \frac{\partial r_1}{\partial\theta_1}\frac{\partial r_2}{\partial\theta_2} - \frac{\partial r_1}{\partial\theta_2}\frac{\partial r_2}{\partial\theta_1} = -\frac{\sin\theta_1\sin\theta_2}{\sin(\theta_1+\theta_2)^2} = -r_1r_2 . \tag{1.28}$$

For $g\geq 0$ measurable on $(0,\infty)^3$, we thus find:

$$\begin{aligned}\int g(|\xi_1|,|\xi_2|)K_{e_3}(d\xi_1,d\xi_2) &\overset{(1.20)}{=} \frac{1}{\pi^3}\int_D g(r_1,r_2)\,d\theta_1\,d\theta_2\,d\alpha\\ &= \frac{2}{\pi^2}\int_{(0,\infty)^2} 1_A(1,r_1,r_2)g(r_1,r_2)\frac{dr_1}{r_1}\frac{dr_2}{r_2} .\end{aligned}$$

This proves (1.21). As for (1.22), (1.23), they are immediate consequences of (1.21) and the symmetry of A (for (1.23)). □

2 Existence and uniqueness principle

The object of this section is to derive an existence and uniqueness result for the Fourier representation of the 3-d Navier–Stokes equation (see (2.4) below). It will be the consequence of a "domination principle". This domination will involve the "stochastic cascade", that is the continuous time critical branching process on ξ space, where a particle located at $\xi \neq 0$, with rate $\nu|\xi|^2$ and with equal probability 1/2 either dies out or gives birth to two descendents ξ_1, ξ_2 distributed according to $K_\xi(d\xi_1, d\xi_2)$.

Let us first precisely state what we mean by a solution of the Fourier representation of Navier–Stokes equation (FNS). The data are measurable functions:

$$\text{(2.1)}\quad \chi_0 : \mathbb{R}^3\backslash\{0\} \to \mathbb{C}^3, \quad \text{with} \quad \chi_0(\xi)\cdot\xi = 0,\ \chi_0(-\xi) = \overline{\chi_0(\xi)},\ \xi \in \mathbb{R}^3\backslash\{0\}\ ,$$

$$\text{(2.2)}\quad \begin{aligned} &\varphi(s,\xi) : [0,T]\times\mathbb{R}^3\backslash\{0\} \to \mathbb{C}^3,\ \text{with} \int_0^T |\varphi(s,\xi)|\,ds < \infty\ ,\\ &\varphi(s,\xi)\cdot\xi = 0,\ \varphi(s,-\xi) = \overline{\varphi(s,\xi)}, \quad \text{for} \ \ s,\xi \in [0,T]\times\mathbb{R}^3\backslash\{0\}\ . \end{aligned}$$

A function $\chi : [0,T]\times\mathbb{R}^3\backslash\{0\} \to \mathbb{C}^3$, continuous in time, measurable in space such that

$$\text{(2.3)}\quad \begin{aligned} &\int_0^T ds \int |\chi_s(\xi_1)\cdot e_\xi||p(\xi)\chi_s(\xi_2)|K_\xi(d\xi_1,d\xi_2) < \infty, \quad \text{for a.e.} \ \ \xi,\\ &\chi_s(\xi)\cdot\xi = 0,\ \chi_s(-\xi) = \overline{\chi_s(\xi)}, (s,\xi) \in [0,T]\times\mathbb{R}^3\backslash\{0\}\ , \end{aligned}$$

will be called solution of (FNS) if for a.e. ξ:

$$\text{(2.4)}\quad \begin{aligned} \chi_t(\xi) = \exp\{-\nu|\xi|^2 t\}\chi_0(\xi) &+ \int_0^t \nu|\xi|^2 \exp\{-\nu|\xi|^2(t-s)\}\\ &\times \left[\frac{1}{2}\chi_s\circ\chi_s(\xi) + \varphi(s,\xi)\right] ds\ , \quad \text{for} \ \ t\in[0,T]\ . \end{aligned}$$

Of course one can define the notion of solution on $[0,\infty)\times\mathbb{R}^3\backslash\{0\}$ in an analogous fashion (the integrability condition (2.3) being then required for arbitrary $T\in(0,\infty)$).

We have seen in Proposition 1.1 (and (1.12)) that a weak solution u of (NS) with data u_0 and f gives rise to a solution of (FNS) with data χ_0, φ.

We shall now investigate some existence and uniqueness results for solutions of (FNS). To this end we now introduce the "stochastic cascade". Loosely speaking, for $\xi_0 \in \mathbb{R}^3\backslash\{0\}$, $t_0 \in \mathbb{R}$, P_{ξ_0,t_0} will be the law of a critical continuous time branching process, "going backward in time". The common ancestor "generated" at time t_0 in ξ_0 will branch at time $t_\emptyset < t_0$, with $t_0 - t_\emptyset$ exponentially distributed with parameter $\nu|\xi|^2$. With probability $\frac{1}{2}$ it will disappear and with probability $\frac{1}{2}$ it will branch into two particles ξ_1, ξ_2 distributed according to $K_\xi(d\xi_1, d\xi_2)$. Then the process goes on.

Formally P_{ξ_0,t_0} is a probability on the space Ω of marked trees ω of the form.

(2.5) $$\omega = \big(t, (t_m, \xi_m, v_m)_{m\in I}\big) ,$$

where t is the time at which the common ancestor "$\emptyset$" is generated, I denotes the set $\bigcup_{\ell\geq 0}\{1,2\}^\ell$ of " labels", that is of finite sequences of 1,2, with variable length ℓ, ($\emptyset$ denotes the unique label of length $\ell = 0$). The variable $v_m \in \{0,1\}$ indicates that the particle ξ_m dies out at time t_m, when $v_m = 0$, or gives birth to two descendants of label ξ_{m1}, ξ_{m2}, when $v_m = 1$. Moreover for $\omega \in \Omega$, the collection $\{m \in I, v_m(\omega) = 1\}$ is finite and such that (using the natural concatenation order on I):

(2.6) $$m' \prec m \ \text{ and } v_m(\omega) = 1 \Longrightarrow v_{m'} = 1 ,$$

the sequence of branching times is decreasing, i.e.:

(2.7) $$m' \prec m, \quad v_{m'}(\omega) = 1 \Longrightarrow t(\omega) > t_{m'}(\omega) > t_m(\omega)$$

and "everything stops at the death time":

(2.8) $$m' \prec m, \quad v_{m'}(\omega) = 0 \Longrightarrow t_m(\omega) = t_{m'}(\omega), \ \xi_m(\omega) = \xi_m(\omega') .$$

Of course P_{ξ_0,t_0}, a.s. $\xi_0 = \xi_\emptyset(\omega)$, $t_0 = t(\omega)$.

We shall mainly be interested in the subset Ω_+ of marked trees ω with generation time $t(\omega) \geq 0$. A tree $\omega \in \Omega_+$, together with the data χ_0, φ will give rise to a $\mathbb{C}^3$ vector

(2.9) $$R(\chi_0, \varphi, \omega) ,$$

which is the result of an operation along the tree, which we now describe. For $\omega \in \Omega_+$, we have a finite collection (in fact binary tree) $N(\omega) \subseteq I$ of "operation nodes":

(2.10) $$N(\omega) = \big\{m \in I, \ t_m(\omega) > 0, \ \ v_m(\omega) = 1\big\} ,$$

and the finite collection $\partial N(\omega) \subseteq I$ of "input nodes":

$$\partial N(\omega) = \big\{m \in I\backslash N(\omega), \text{ the direct predecessor of } m \text{ belongs to } N(\omega)\big\} .$$
(2.11)

By convention, when $N(\omega) = \emptyset$, $\partial N(\omega) = \{\emptyset\}$. The set $\partial N(\omega)$ of input nodes is naturally partitioned into two disjoint subcollections:

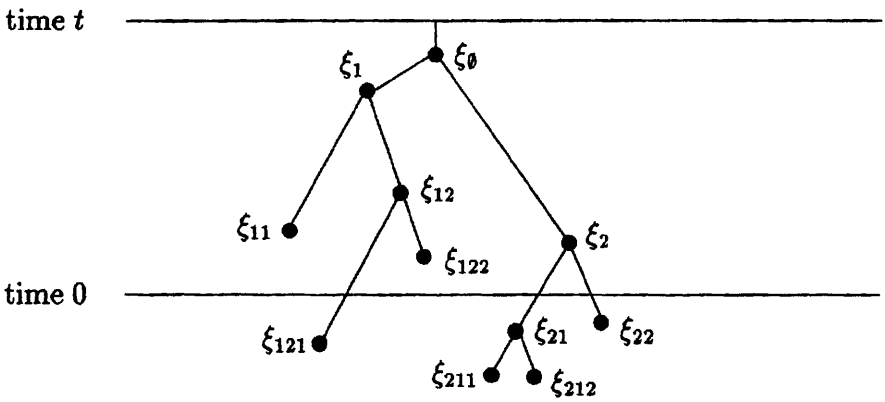
$$\begin{aligned} \partial_0 N(\omega) &= \{m \in \partial N(\omega), t_m(\omega) \leq 0\} \\ \partial_+ N(\omega) &= \{m \in \partial N(\omega), t_m(\omega) > 0\}, \text{ so that} \\ \partial N(\omega) &= \partial_+ N(\omega) \cup \partial_0 N(\omega), \quad \omega \in \Omega_+ . \end{aligned} \tag{2.12}$$

We have a (marked) node operation:

(2.13) $$w_1, w_2 \in \mathbb{C}^3, \ \ \xi \in \mathbb{R}^3\backslash\{0\} \longrightarrow w_1 \times^\xi w_2 = -\mathrm{i}(w_1 \cdot e_\xi)p(\xi)w_2 \in \mathbb{C}^3 ,$$

and to each input node $m \in \partial N(\omega)$ we attach an input which is defined as:

$$\begin{aligned} &\chi_0\big(\xi_m(\omega)\big), \text{ when } \in \partial_0 N(\omega), \text{ and} \\ &\varphi\big(t_m(\omega), \xi_m(\omega)\big), \text{ when } m \in \partial_+ N(\omega) . \end{aligned} \tag{2.14}$$



$R(\chi_0, \varphi, \omega)$ is then the result of the recursively performed operation (2.13) at each operation node in $N(\omega)$, with inputs given in (2.14). For instance

$$\begin{aligned} N(\omega) &= \{\emptyset, 1, 2, (1,2)\} \\ \partial_0 N(\omega) &= \{(1,2,1), (2,1), (2,2)\} \\ \partial_+ N(\omega) &= \{(1,1), (1,2,2)\} \\ R(\chi_0, \varphi, \omega) &= \big(\varphi(t_{11}, \xi_{11})\big) \times^{\xi_1} \big(\chi_0(\xi_{121}) \times^{\xi_{12}} \varphi(t_{122}, \xi_{122})\big) \\ &\quad \times^{\xi_\emptyset} \big(\chi_0(\xi_{21}) \times^{\xi_2} \chi_0(\xi_{22})\big) \ . \end{aligned}$$

Given non negative measurable functions X_0 on $\mathbb{R}^3\backslash\{0\}$ and Φ on $[0,T]\times \mathbb{R}^3\backslash\{0\}$, we also consider

$$M(X_0, \Phi, \omega) \ , \tag{2.15}$$

which is defined analogously to $R(\chi_0, \varphi, \omega)$, except that the operation performed at each operation node is the usual multiplication of numbers, instead of (2.13). We shall now begin with a lemma which collects some useful properties of the function $E_{\xi,t}[M]$, which will play an important role in this section.

Lemma 2.1: *Let X_0 and Φ be non negative measurable functions respectively defined on $\mathbb{R}^3\backslash\{0\}$ and $[0,T] \times \mathbb{R}^3\backslash\{0\}$. Then the function*

$$(t, \xi) \in [0, T] \times \mathbb{R}^3\backslash\{0\} \to X_t(\xi) = E_{\xi,t}\big[M(X_0, \Phi, \omega)\big] \in [0, \infty] \ , \tag{2.16}$$

is measurable and satisfies:

$$\begin{aligned} X_t(\xi) &= \exp\{-\nu|\xi|^2 t\} X_0(\xi) + \int_0^t \nu|\xi|^2 \mathrm{e}^{-\nu|\xi|^2(t-s)} \\ &\quad \times \left[\frac{1}{2}\int X_s(\xi_1) X_s(\xi_2) K_\xi(d\xi_1, d\xi_2) + \frac{1}{2}\Phi(s,\xi)\right] ds \ , \end{aligned}$$

$$\text{for } \ 0 \le t \le T, \ \ \xi \in \mathbb{R}^3\backslash\{0\}, \ \ (\text{with the convention } \ 0 \cdot \infty = 0) \ . \tag{2.17}$$

For each $\xi \in \mathbb{R}^3 \backslash \{0\}$

$$t \in [0,T] \to \exp\{\nu|\xi|^2 t\} X_t(\xi) \in [0,\infty] \text{ is non decreasing .} \tag{2.18}$$

Moreover, if $t \in (0,T]$ *and either* X_0 *or* $1_{[0,t]}\Phi$ *are symmetric (in* ξ*) and non negligible functions, then*:

$$X_t(\cdot) \text{ is bounded away from } 0 \text{ on compact of } \mathbb{R}^3 \backslash \{0\} . \tag{2.19}$$

Proof: The measurability is obvious, and for $t \in [0,T]$, $\xi \in \mathbb{R}^3 \backslash \{0\}$:

$$\begin{aligned} X_t(\xi) = E_{\xi,t}\big[M(X_0,\Phi,\omega)\big] &= E_{\xi,t}[M, t_\emptyset \le 0] \\ &+ E_{\xi,t}[M, t_\emptyset > 0, \nu_\emptyset = 1] + E_{\xi,t}[M, t_\emptyset > 0, \nu_\emptyset = 0] . \end{aligned}$$

Using the strong Markov property at time $t_\emptyset$, for the last two terms, we find

$$\begin{aligned} X_t(\xi) &= \exp\{-\nu|\xi|^2 t\} X_0(\xi) + \int_0^t \nu|\xi|^2 \mathrm{e}^{-\nu|\xi|^2(t-s)} \\ &\times \frac{1}{2}\int E_{\xi_1,s}[M] E_{\xi_2,s}[M] K_\xi(d\xi_1, d\xi_2)\, ds + \int_0^t \nu|\xi|^2 \mathrm{e}^{-\nu|\xi|^2(t-s)} \frac{1}{2}\Phi(s,\xi)\, ds . \end{aligned} \tag{2.20}$$

This proves (2.17). If we now multiply both members of (2.20) by $\exp\{\nu|\xi|^2 t\}$ we immediately obtain (2.18).

We shall not use (2.19) in the sequel. Nevertheless we present this non-degeneracy criterion for $X_t(\xi)$, having in mind the domination principle we develop in this section.

Let us now prove (2.19). It suffices to show that

$$X_t(\xi) > 0, \quad \text{for } |\xi| \neq 0 . \tag{2.21}$$

Indeed, applying this result to $s < t$ close to t, it will follow that

$$\int X_s(\xi_1) X_s(\xi_2) K_\xi(d\xi_1, d\xi_2) > 0 \text{ for } |\xi| \neq 0 . \tag{2.22}$$

Using the $L^2(d\xi_1)$ continuity of translations (see (1.14)), the above function remains bounded away from 0, when ξ varies in compact subsets of $\mathbb{R}^3 \backslash \{0\}$. Together with (2.18) and (2.20) we see that (2.19) follows from (2.21).

Let us now prove (2.21).

Without loss of generality we can assume that either

$$X_0 = c1_A, \quad \text{where } c > 0,\ A \subseteq \mathbb{R}^3 \backslash \{0\} \text{ is symmetric non negligible,} \tag{2.23}$$

or

$$\begin{aligned} &\Phi = c1_B, \quad \text{where } c > 0,\ B \subseteq [s_1, s_2] \times \mathbb{R}^3 \backslash \{0\} \\ &\text{is symmetric in the } \xi \text{ variable, non negligible and } [s_1, s_2] \subset (0,t) . \end{aligned} \tag{2.24}$$

Using a similar continuity argument as above, we see that under (2.23):

(2.25) $$K_\xi(A \times A) > 0 \text{ for small } |\xi| ,$$

and under (2.24)

(2.26) $$\int_{s_1}^{s_2} ds \int 1_B(s,\xi_1) 1_B(s,\xi_2) K_\xi(d\xi_1, d\xi_2) > 0, \quad \text{for small } |\xi| .$$

In the case of (2.23), it follows that for large n, small $\rho > 0$,

$$P_{\xi,t}\left[\bigcap_{|m|=n} \{v_m = 1, t_m = 1, |\xi_m| \le \rho, \xi_{m1} \in A, \xi_{m2} \in A, t_{m1} \le 0, t_{m2} \le 0\}\right] > 0 .$$

This now implies (2.21). An analogous argument can be used in the case of (2.24), (2.26). This finishes our proof. □

We are now ready to state an existence result for the solutions of (FNS).

Theorem 2.2: *Assume*

(2.27) $$E_{\xi,T}\left[M(X_0, \Phi, \omega)\right] < \infty, \quad \textit{for} \text{ a.e. } \xi ,$$

if χ_0, φ *satisfy* (2.1), (2.2) *and*

(2.28) $$|\chi_0| \le X_0, \ |\varphi| \le \Phi, \ \textit{then}$$

(2.29) $$\chi_t(\xi) = \begin{cases} E_{\xi,t}\left[R(\chi_0, \varphi, \omega)\right], & \textit{on the convergence set of } (2.27) \\ 0, & \textit{otherwise} \end{cases}$$

defines a solution of (*FNS*) *on* $[0, T] \times \mathbb{R}^3 \backslash \{0\}$, *such that*:

(2.30) $$|\chi_t(\xi)| \le X_t(\xi) \ \textit{ on } \ [0, T] \times \mathbb{R}^3 \backslash \{0\} .$$

Proof: Denote by E the (Borel) set of convergence in (2.27). From (2.18), we know that for $\xi \in E, t \in [0, T]$

(2.31) $$E_{\xi,t}[M] < \infty .$$

Observe that for $w_1, w_2 \in \mathbb{C}^3$, $\xi \in \mathbb{R}^3 \backslash \{0\}$,

(2.32) $$|w_1 \times^\xi w_2| \le |w_1||w_2|, \ \text{ and thus}$$

(2.33) $$|R(\chi_0, \varphi, \omega)| \le |M(X_0, \Phi, \omega)| .$$

It follows that $\chi_t(\xi)$ well defined in (2.29) and (2.30) holds. As a result of (2.17), (2.27), (2.33), when $\xi \in E$:

(2.34) $$\int_0^T \int |\chi_s(\xi_1)||\chi_s(\xi_2)| K_\xi(d\xi_1, d\xi_2) < \infty ,$$

and thus the integrability condition part of (2.3) holds. For any $\omega \in \Omega_+$, $R(\chi_0, \varphi, \omega) \cdot \xi_\emptyset = 0$, so that

(2.35) $$\chi_t(\xi) \cdot \xi = 0$$

on $[0,T] \times \mathbb{R}^3$ and in fact on $[0,T] \times \mathbb{R}^3\backslash\{0\}$ due to (2.29). Moreover, since $w_1 \times^{\xi} w_2 = \bar{w}_1 \times^{-\xi} \bar{w}_2$, and for $\xi \in \mathbb{R}^3\backslash\{0\}$; $K_{-\xi}(d\xi_1, d\xi_2) = -Id_{\mathbb{C}^3\times\mathbb{C}^3} \circ K_\xi(d\xi_1, d\xi_2)$, it follows that for $0 \le t \le T$, $|\xi| \neq 0$

(2.36) $\bar{R}(\chi_0, \varphi, \omega)$ under $P_{\xi,t}$ has same law as $R(\chi_0, \varphi, \omega)$ under $P_{-\xi,t}$.

It follows from this that E is symmetric and $\overline{\chi_t(\xi)} = \chi_t(-\xi)$ on $[0,T]\times \mathbb{R}^3\backslash\{0\}$, i.e. (2.3) holds.

An application of the strong Markov property at time $t_\emptyset$ shows that for $\xi \in E$, $t \in [0,T]$

$$\begin{aligned}\chi_t(\xi) &= E_{\xi,t}[R, t_\emptyset \le 0] + E_{\xi,t}[R, t_\emptyset > 0] \\ &= \exp\{-\nu|\xi|^2 t\}\chi_0(\xi) + \int_0^t ds\ \nu|\xi|^2 \exp\{-\nu|\xi|^2(t-s)\} \\ &\quad \times \left(\frac{1}{2}\varphi(s,\xi) + \frac{1}{2}\int -\mathrm{i}\big(E_{\xi_1,s}[R]\cdot e_\xi\big)p(\xi)E_{\xi_2,s}[R]K_\xi(d\xi_1,d\xi_2)\right) \\ &\qquad (E \times E \ \text{ has full } \ K_\xi(d\xi_1, d\xi_2) \text{ measure }) \\ &= \exp\{-\nu|\xi|^2 t\}\chi_0(\xi) + \int_0^t ds\ \nu|\xi|^2 \exp\{-\nu|\xi|^2(t-s)\} \\ &\quad \times \left(\frac{1}{2}\varphi(s,\xi) + \frac{1}{2}\chi_s \circ \chi_s(\xi)\right) ds \, . \end{aligned} \tag{2.37}$$

It now follows that $\chi_t(\xi)$ is continuous in the t variable and defines a solution of (FNS) with data χ_0, φ. □

Remark 2.3:

1) In the case when $E = \mathbb{R}^3\backslash\{0\}$, i.e. when $E_{\xi,T}[M] < \infty$ for all ξ, then $\chi_t(\xi)$ satisfies (2.4) for all $\xi \neq 0$.
2) It follows from a routine iteration argument that under (2.27), $E_{\xi,t}\big[M(X_0, \Phi, \omega)\big]$ defines the minimal non negative solution of (2.17).

   We shall not need this fact, but only state it for the sake of clarity. □

We now come to the uniqueness result.

Theorem 2.4: *If* $\chi_t(\xi)$ *is a solution of* (*FNS*) *on* $[0,T] \times \mathbb{R}^3\backslash\{0\}$ *with data* χ_0, φ, *and* (2.27), (2.28), (2.30) *hold*, *then for* a.e. ξ

$$\chi_t(\xi) = E_{\xi,t}\big[R(\chi_0, \varphi, \omega)\big], \quad t \in [0,T] \, . \tag{2.38}$$

Proof: We can choose a symmetric measurable set $F \subseteq \mathbb{R}^3\backslash\{0\}$, with $|F^c| = 0$, such that $E_{\xi,T}[M] < \infty$, on F, the integral in (2.3) is convergent and (2.4) holds for $\xi \in F$.

We shall represent $\chi_t(\xi)$, $\xi \in F$, as the expectation of what turns out to be a martingale. To this end, we define for $\omega \in \Omega_+$ and $n \ge 0$, in analogy with (2.10)–(2.11):

$$N_n(\omega) = \left\{ m \in \bigcup_{0 \le \ell \le n} \{1,2\}^{\ell},\ t_m(\omega) > 0,\ v_m(\omega) = 1 \right\} \tag{2.39}$$

and

$$\begin{aligned} \partial N_m(\omega) = \{ & m \in I \backslash N_n(\omega),\ \text{the direct predecessor of } m \text{ belongs to } N_n(\omega)\}, \\ & \text{with the convention } \partial N_n(\omega) = \{\emptyset\},\ \text{if } N_n(\omega) = \emptyset\ . \end{aligned} \tag{2.40}$$

It is also convenient to introduce the "truncated" and "untruncated" parts of ∂N_n:

$$\begin{aligned} \partial N_n^{(t)}(\omega) = \{ & m \in I,\ \text{the direct predecessor of } m \text{ belongs to } N_n(\omega) \\ & \text{and has length } n\} \end{aligned} \tag{2.41}$$

$$\partial N_n^{(u)}(\omega) = \partial N_n(\omega) \backslash \partial N_n^{(t)}(\omega)\ . \tag{2.42}$$

We now define for $n \ge 0$, $\omega \in \Omega_+$:

$$R_n(\chi_0, \varphi, \chi, \omega)\ . \tag{2.43}$$

The procedure is analogous to the definition of (2.9), except that we use different data at the input nodes. Namely, if $m \in \partial N_n^{(t)}(\omega)$, the input attached to m is

$$\chi_{t_{m'}}(\xi_m),\ \text{where } m' \text{ is the direct ancestor of } m \text{ (with length } n)\ , \tag{2.44}$$

and if $m \in \partial N_n^{(u)}(\omega)$, the input attached to m is the same as in (2.14).

$$\begin{cases} \chi_0(\xi_m), & \text{if } t_m \le 0, \\ \varphi(t_m, \xi_m), & \text{if } t_m > 0\ . \end{cases} \tag{2.45}$$

In a similar way we define for $n \ge 0$, $\omega \in \Omega_+$:

$$M_n(X_0, \Phi, X, \omega)\ . \tag{2.46}$$

with the usual multiplication replacing (2.13) and of course X determined by X_0, Φ via (2.16). As a result of the domination assumption (2.30)

$$|R_n| \le M_n\ . \tag{2.47}$$

We now introduce a filtration F_n, $n \ge 0$, on Ω_+:

$$F_n = \sigma\big(N_n(\omega), (t_m, \xi_m)_{m \in N_n(\omega) \cup \partial N_n^{(u)}(\omega)}, (\xi_m)_{m \in \partial N_n^{(t)}(\omega)}\big)\ , \tag{2.48}$$

where it should be observed that N_n determines $\partial N_n^{(t)}$ and $\partial N_n^{(u)}$.

Observe that R_n and M_n are F_n-adapted. Moreover, it is easy to deduce from the strong Markov property applied at times t_m, for m of length n, on the set $\{m \in N_n\} \in F_n$, that

$$M_n = E_{\xi,t}[M/F_n], \quad P_{\xi,t} \text{ a.s. for } t \in [0,T], \xi \in F\ . \tag{2.49}$$

This and (2.47) shows the $P_{\xi,t}$-integrability of R_n, for $t \in [0,T]$, $\xi \in F$. We shall now prove by induction the identity

$$\chi_t(\xi) = E_{\xi,t}\big[R_n(\chi_0, \varphi, \chi, \omega)\big], \quad n \geq 0, \ \ t \in [0,T], \ \ \xi \in F \ . \tag{2.50}$$

In fact we shall see a posteriori that R_n is an F_n-martingale.

For $n = 0$, (2.50) simply boils down to (2.4). Assume now (2.50) holds for n. Then from (2.4), when $\xi \in F$, $t \in [0,T]$:

$$\begin{aligned}\chi_t(\xi) &= \exp\{-\nu|\xi|^2 t\}\chi_0(\xi) + \int_0^t ds\ \nu|\xi|^2 \mathrm{e}^{-\nu|\xi|^2(\mathrm{t-s})} \\ &\quad \Big(\frac{1}{2}\varphi(s,\xi) + \frac{1}{2}\int -\mathrm{i}E_{\xi_1,s}[R_n]\cdot e_\xi p(\xi)\big(E_{\xi_2,s}[R_n]\big)K_\xi(d\xi_1, d\xi_2)\Big) \\ &\qquad \text{using the induction hypothesis} \ , \\ &= E_{\xi,t}\big[R_{n+1}(\chi_0,\varphi,\chi,\omega)\big] \ ,\end{aligned}$$

applying the strong Markov property at time $t_\emptyset$. This proves (2.50).

We can now conclude the proof of Theorem 2.4. We denote by A_n, the F_n measurable event:

$$A_n = \{\omega \in \Omega_+, \ N_n(\omega) \text{ contains some } m \text{ of length } n\} \ . \tag{2.51}$$

Of course

$$R(\chi_0, \varphi, \omega) = R_n(\chi_0, \varphi, \chi, \omega) \ \ \text{on} \ \ A_n^c \ . \tag{2.52}$$

We thus see that for $\xi \in F$, $t \in [0,T]$:

$$\begin{aligned}|\chi_t(\xi) - E_{\xi,t}\big[R(\chi_0,\varphi,\omega)\big]| &\overset{(2.50)-(2.52)}{=} \big|E_{\xi,t}\big[(R_n - R)1_{A_n}\big]\big| \\ &\overset{(2.33)-(2.47)}{\leq} E_{\xi,t}\big[(M_n + M)1_{A_n}\big] \overset{(2.49)}{=} 2E_{\xi,t}[M 1_{A_n}] \to 0, \quad \text{as } n \to \infty \ ,\end{aligned}$$

since $\bigcap_n A_n = \emptyset$. This concludes our proof of (2.38). □

Remark 2.5:

1) If $E_{\xi,T}\big[M(X_0,\Phi,\omega)\big] < \infty$, for a.e. ξ for any $T > 0$, then Theorem 2.2 and 2.4 naturally provide global existence and uniqueness results.
2) Observe that it follows from the representation formula (2.38), and the same conditioning argument used for the proof of (2.49), that:

$$E_{\xi,t}\big[R(\chi_0,\varphi,\omega)|F_n\big] = R_n(\chi_0,\varphi,\chi,\omega) \ , \tag{2.53}$$

for $n \geq 0$, $\xi \in E$, $t \in [0,T]$ (in the notations of the beginning of the proof of Theorem 2.2).

We thus see a posteriori that $R_n(\chi_0,\varphi,\chi,\omega)$ defines an F_n-martingale, with expectation $\chi_t(\xi)$. □

We shall now close this section with a lemma which will be helpful in the discussion of examples in section III.

For $\omega \in \Omega$, we denote by $\bar{N}(\omega)$ the finite set of nodes $m \in I$ for which $v_m(\omega) = 1$, and define $\partial\bar{N}(\omega)$ as in (2.11), with the same convention that

$\partial\bar{N}(\omega) = \{\phi\}$, when $\bar{N}(\omega) = \phi$. Of course $\bar{N}, \partial\bar{N}$ have a distribution which is the same under each $P_{\xi,t}$.

For ψ a non negative measurable function on $\mathbb{R}^3\backslash\{0\}$, we introduce in analogy to (2.15)

$$\bar{M}(\psi,\omega) = \prod_{m\in\partial\bar{N}(\omega)} \psi(\xi_m)\ , \tag{2.54}$$

as well as

$$X^{\psi}(\xi) \overset{\text{def}}{=} E_{\xi,t}\big[\bar{M}(\psi,\omega)\big], \quad \xi\in\mathbb{R}^3\backslash\{0\},\ t\geq 0 \text{ arbitrary}\ . \tag{2.55}$$

We have

Lemma 2.6: *Assume that*

$$X^{\psi}(\xi) < \infty\ ,\ \textit{for all}\ \xi\in\mathbb{R}^3\backslash\{0\}\ , \tag{2.56}$$

then

$$X^{\psi}(\xi) = \tfrac{1}{2}\, K_{\xi}(X^{\psi}\otimes X^{\psi}) + \tfrac{1}{2}\,\psi(\xi), \quad \xi\in\mathbb{R}^3\backslash\{0\}\ , \tag{2.57}$$

and

$$X^{\psi}(\xi) = E_{\xi,t}\big[M(X^{\psi},\psi,\omega)\big], \quad \xi\in\mathbb{R}^3\backslash\{0\},\ t\geq 0\ . \tag{2.58}$$

Proof: Observe that (2.57) follows from the strong Markov property at time t_{ϕ}, whereas (2.58) is a consequence of the Markov property at time 0. □

Remark 2.7: Assume that ψ non negative measurable on $\mathbb{R}^3\backslash\{0\}$, is such that (2.56) holds, and φ measurable $\mathbb{R}^3\backslash\{0\}$ with values in $\mathbb{C}^3$ is such that:

$$\varphi(-\xi) = \overline{\varphi(\xi)}, \quad \varphi(\xi)\cdot\xi = 0, \text{ and } |\varphi| \leq \psi\ . \tag{2.59}$$

We can then define $\bar{R}(\varphi,\omega)$, for $\omega\in\Omega$, in analogy to (2.9), except that "operation nodes" are now $m\in\bar{N}(\omega)$, and "input data" are $\varphi(\xi_m)$, for $m\in\partial\bar{N}(\omega)$. If we now introduce

$$\chi^{\varphi}(\xi) = E_{\xi,t}\big[\bar{R}(\varphi,\omega)\big], \quad \xi\in\mathbb{R}^3\backslash\{0\},\ t\geq 0 \text{ arbitrary}\ , \tag{2.60}$$

we can argue as in (2.57), that:

$$\chi^{\varphi}(\xi) = \tfrac{1}{2}\,\chi^{\varphi}\circ\chi^{\varphi}(\xi) + \tfrac{1}{2}\,\varphi(\xi), \quad \xi\in\mathbb{R}^3\backslash\{0\}\ . \tag{2.61}$$

The uniqueness argument in Theorem 2.4, can in fact be easily modified to show that any measurable $\chi\colon \mathbb{R}^3\backslash\{0\}\to\mathbb{C}^3$, with $|\chi|\leq X^{\psi}$, which satisfies (2.61) in place of χ^{φ}, coincides with χ^{φ}. Note finally that the formula (2.60) defines a stationary solution of (FNS) with data χ^{φ}, φ, and that

$$\chi^{\varphi}(\xi) = E_{\xi,t}\big[R(\chi^{\varphi},\varphi,\omega)\big], \quad \text{for}\ \ \xi\in\mathbb{R}^3\backslash\{0\},\ t\geq 0\ . \tag{2.62}$$

□

3 Some examples

We shall now discuss some classes of examples where the results of the previous section apply.

For $\alpha \geq 0$, we define the functions ψ_α on $\mathbb{R}^3\backslash\{0\}$ as

$$\psi_0 = 1, \quad \text{if } \alpha = 0$$

(3.1) $$\psi_\alpha(\xi) = \frac{\pi^2}{2}\alpha|\xi|e^{-\alpha|\xi|}, \quad \text{if } \alpha > 0 .$$

The interest of these functions stems from the

Lemma 3.1: *For* $\alpha \geq 0$, $\xi \in \mathbb{R}^3\backslash\{0\}$, $t \geq 0$,

(3.2) $$\psi_\alpha(\xi) = K_\xi(\psi_\alpha \otimes \psi_\alpha) ,$$

(3.3) $$\psi_\alpha(\xi) = E_{\xi,t}\big[M(\psi_\alpha, \psi_\alpha, \omega)\big] .$$

Proof: In the case $\alpha = 0$, (3.2) and (3.3) are obvious. Let us prove (3.2) for $\alpha > 0$. If $|\xi| = r > 0$, it follows from (1.21) that

(3.4) $$K_\xi(\psi_\alpha \otimes \psi_\alpha) = \frac{\pi^2}{2}\int_{(0,\infty)^2} 1_A(r, r_1, r_2)\alpha^2 \mathrm{e}^{-\alpha(r_1+r_2)}\, dr_1\, dr_2$$

defining $u = r_1 + r_2, v = r_1 - r_2$

$$= \frac{\pi^2}{4}\int_{(r,\infty)\times(-r,r)} \alpha^2 \mathrm{e}^{-\alpha u}\, du\, dv = \frac{\pi^2}{2}\alpha r \mathrm{e}^{-\alpha r} = \psi_\alpha(\xi)$$

which proves (3.2).

Let us now prove (3.3). In view of Lemma 2.6, we only need to check that

(3.5) $$X^{\psi_\alpha} = \psi_\alpha .$$

Observe now that the conditional expectation

(3.6) $$E_{\xi,t}[\bar{M}|\bar{N}] = \psi_\alpha(\xi) \quad P_{\xi,t} \text{ a.s.}, \quad \xi \in \mathbb{R}^3\backslash\{0\}, \ t \geq 0 ,$$

as can be seen by recursively integrating out all variables ξ_m, $m \in N(\omega) \cup \partial N(\omega)$, in the order dictated by the length of the operation nodes in $N(\omega)$, and using (3.2). Our claim (3.3) then follows. □

We now consider χ_0 and φ, $\mathbb{C}^3$-valued respectively defined on $\mathbb{R}^3\backslash\{0\}$ and $\mathbb{R}_+ \times \mathbb{R}^3\backslash\{0\}$, for which (2.1) and the second line of (2.2) hold. As a result of Theorem 2.2 and 2.4, we have

Theorem 3.2: *Assume that for some* $\alpha \geq 0$:

(3.7) $$|\chi_0| \leq \psi_\alpha, \ |\varphi| \leq \psi_\alpha ,$$

(*with a slight abuse of notations*). *Then* (*FNS*) *has a unique*, *up to null set*, *globally defined solution in the class*

$$(3.8) \qquad |\chi| \le \psi_\alpha ,$$

which is given by

$$(3.9) \qquad \chi_t(\xi) = E_{\xi,t}\big[R(\chi_0, \varphi, \omega)\big], \; t \ge 0, \; \xi \in \mathbb{R}^3 \backslash \{0\} .$$

A second class of examples arises in the following situation. Consider a non negative measurable rotationally invariant function $\psi(\cdot)$ on $\mathbb{R}^3\backslash\{0\}$. We have

Lemma 3.3: *Assume that* ψ *is a bounded non negative rotationally invariant measurable function and*

$$(3.10) \qquad \int \psi(\xi) \frac{d\xi}{|\xi|^3} \le 2\sqrt{2}\pi^2 ,$$

then X^ψ *is finite and for* $\xi \in \mathbb{R}^3\backslash\{0\}$ *and* $t \ge 0$,

$$(3.11) \qquad X^\psi(\xi) = E_{\xi,t}\big[M(X^\psi, \psi, \omega)\big] \le \max(1, \|\psi\|_\infty) .$$

Proof: Consider the norm $\| \; \|_1$ defined as

$$(3.12) \qquad \|f\|_1 = \frac{\sqrt{2}}{\pi} \int_0^\infty \operatorname*{ess\,sup}_{|e|=1} |f(re)| \frac{dr}{r}, \; f : \mathbb{R}^3\backslash\{0\} \to \mathbb{R} ,$$

as well as the supremum norm $\| \cdot \|_\infty$. Observe that for bounded rotationally invariant functions f_1, f_2 on $\mathbb{R}^3\backslash\{0\}$

$$(3.13) \qquad K(f_1 \otimes f_2)$$

is bounded rotationally invariant and using (1.21), (1.23)

$$\begin{aligned} &\|K(f_1 \otimes f_2)\|_\infty \le \|f_1\|_1 \|f_2\|_1 \\ &\|K(f_1 \otimes f_2)\|_\infty \le \|f_1\|_\infty \|f_2\|_\infty \\ &\|K(f_1 \otimes f_2)\|_1 \le \|f_1\|_\infty \|f_2\|_1 \\ (3.14) \qquad &\|K(f_1 \otimes f_2)\|_1 \le \|f_1\|_1 \|f_2\|_\infty \; . \end{aligned}$$

This can be encoded into

$$(3.15) \qquad \|K(f_1 \otimes f_2)\|_{a*b} \le \|f_1\|_a \|f_2\|_b \; ,$$

for $a, b \in \{1, \infty\}$, with a multiplication table

$$(3.16) \qquad \infty * \infty = \infty, \;\; \infty * 1 = 1, \;\; 1 * \infty = 1, \;\; 1 * 1 = \infty \; .$$

In other words $(\{\infty, 1\}, *)$ is the group on 2 elements with neutral element ∞.

Just as in (3.5), (3.6), for $\xi \in \mathbb{R}^3\backslash\{0\}, t \ge 0$, and a finite subset $\bar{N}_0$ of I such that $\{\bar{N}(\omega) = \bar{N}_0\}$ has positive probability under (any) $P_{\xi,t}$

$$E_{\xi,t}\big[\bar{M}(\psi, \omega) | \bar{N} = \bar{N}_0\big]$$

is obtained by recursively integrating out the variables ξ_m, $m \in \bar{N}_0 \cup \partial\bar{N}_0$. It now follows that if we control the $\| \; \|_1$ norm of all input data $\psi(\xi_m)$, $m \in \partial\bar{N}_0$, the rule (3.15) implies that

(3.17) $$\|E_{\cdot,t}[\bar{M}|\bar{N} = \bar{N}_0]\|_A \leq \prod_{m \in \partial\bar{N}_0} \|\psi\|_1 = 1$$

where A is the result of the operation corresponding to operation nodes $\bar{N}_0$, input nodes $\partial\bar{N}_0$ with value 1, and node operation $*$. In other words:

$$\begin{cases} A = \infty, & \text{if } \#\partial\bar{N}_0 \text{ is even} \\ A = 1, & \text{if } \#\partial\bar{N}_0 \text{ is odd} \end{cases} .$$

If in the case of an odd value for $\#\partial\bar{N}_0$, we control the $\| \ \|_\infty$ norm of $\psi(\xi_m)$, with $m \in \partial\bar{N}_0$ smallest in lexicographic order and the $\| \ \|_1$ norm of all other $\psi(\xi_{m'})$, $m' \in \partial\bar{N}_0$, we find

(3.19) $$\|E_{\cdot,t}[\bar{M}|\bar{N} = \bar{N}_0]\|_\infty \leq \|\psi\|_\infty, \text{ if } \#\partial\bar{N}_0 \text{ is odd} .$$

Our claim (3.11) now follows by integration, and Lemma 2.6. □

Theorem 3.4: *Consider* $\psi \geq 0$, *bounded measurable*, *rotationally invariant*, *for which* (3.10) *holds. If* χ_0, φ *are such that* (2.1) *and the second line of* (2.2) *hold*, *and*

(3.20) $$|\chi_0| \leq X^\psi, \quad |\varphi| \leq \psi ,$$

then (*FNS*) *has a unique*, *up to a null set*, *globally defined solution in the class*

(3.21) $$|\chi_t(\xi)| \leq X^\psi(\xi) \quad (\leq \max\{1, \|\psi\|_\infty\}), \text{ given by}$$

(3.22) $$\chi_t(\xi) = E_{\xi,t}\big[R(\chi_0, \varphi, \omega)\big], \ t \geq 0, \ \xi \in \mathbb{R}^3\backslash\{0\} .$$

Proof: This result follows directly from theorems 2.2, 2.4, and Lemma 3.3.

□

Remark 3.5: 1) In the case of Theorem 3.2 and 3.4, the solutions $\chi_t(\xi)$, $t \geq 0$, $\xi \in \mathbb{R}^3\backslash\{0\}$ are uniformly bounded, as well as $\varphi(s, \xi)$. If we now consider

(3.23) $$u_s(x) = \frac{\nu}{2\pi^3} \int e^{ix\cdot\xi} \chi_s(\xi) \frac{d\xi}{|\xi|^2}, \ s \geq 0, \ x \in \mathbb{R}^3 ,$$

(3.24) $$f_s(x) = \frac{\nu^2}{4\pi^3} \int e^{ix\cdot\xi} \varphi(s, \xi) d\xi, \quad s \geq 0, \ x \in \mathbb{R}^3 ,$$

we can write $\chi = \chi 1(|\cdot| \leq 1) + \chi 1(|\cdot| > 1)$. As a result of Young's inequality, (i.e. $\|\hat{w}\|_{p'} \leq \|w\|_p$, when $1 \leq p \leq 2$, $\frac{1}{p} + \frac{1}{p'} = 1$),

(3.25) $$u_s = v_s + \tilde{v}_s ,$$

where the $C_0(\mathbb{R}^3)$ and $L^{3+\varepsilon}$ norm of v_s are uniformly bounded ($\varepsilon > 0$), whereas the L^2 and $L^{3-\varepsilon}$ norm of $\tilde{v}_s$ are uniformly bounded, for $0 < \varepsilon < 1$. On the other hand,

(3.26) the H^{-2} norm of f_s is uniformly bounded .

Moreover, reversing the proof of Proposition 1.1, we see that for $g \in \mathscr{V}$ and $t \geq 0$:

$$\langle u_t, g\rangle = \langle u_0, g\rangle + \int_0^t \nu\langle u, \Delta g\rangle + \langle u, u\nabla g\rangle + \langle f, g\rangle\, ds, \tag{3.27}$$

where in view of (3.1) and the second line of (3.2),

$$u_s, f_s \text{ are weakly divergence free and real} \; . \tag{3.28}$$

So (3.27), (3.28) is a type of weak formulation of the 3-d incompressible Navier–Stokes equation.

2) It should be observed that the class of examples we have discussed in this section includes situations of infinite energy (which are beyond the formulation of (NS)). Indeed, if we choose in the setting of Theorem 3.2

$$\begin{aligned} &\chi_0(\xi) = p(\xi)\cdot e, \quad \text{with } |e| \le 1 \; , \\ &\varphi(s,\xi) = p(\xi)\cdot r(s), \text{ with } |r(s)| \le 1, r(s) \text{ measurable} \; , \end{aligned} \tag{3.29}$$

in this case (see Ladyzkenskaya [7] p. 51),

$$u_0(x) = \frac{\nu}{2\pi}\left[\frac{e}{|x|} - \frac{x\cdot(x\cdot e)}{|x|^3}\right] \; , \tag{3.30}$$

$$f_s(x) = 2\nu^2\delta r(s) - \frac{\nu^2}{2\pi}\,\mathrm{grad}\left(\frac{x\cdot r(s)}{|x|^3}\right), \text{ (distribution sense)} \; . \tag{3.31}$$

In a similar vein, assume χ_0, φ are measurable $\mathbb{C}^3$-valued functions on $\mathbb{R}^3\backslash\{0\}$, homogeneous of degree 0, (φ is time independent), for which (2.1) and the second line of (2.2) hold, as well as the inequalities

$$|\chi_0| \le 1, \; |\varphi| \le 1 \; .$$

The formula (3.9) then defines a solution $\chi_t(\xi)$ of (FNS) which enjoys the scaling relation:

$$\chi_{t\lambda^2}\left(\frac{\xi}{\lambda}\right) = \chi_t(\xi), \quad t \ge 0, \;\; \xi \in \mathbb{R}^3\backslash\{0\}, \;\; \lambda > 0 \; , \tag{3.32}$$

as can easily be deduced from the natural scaling property of the stochastic cascades. It then follows that $u_s(x)$ defined in (3.23), satisfies:

$$u_t(x) \overset{\text{a.e.}}{=} \frac{1}{\sqrt{t}} u_1\left(\frac{x}{\sqrt{t}}\right), \; t > 0 \; . \tag{3.33}$$

These are self similar solutions, which were considered in Cannone–Meyer–Planchon [2].

3) It follows from an easy calculation that no ψ_α is bigger than $\psi_{\alpha'}$, when $\alpha \neq \alpha'$. Moreover no ψ_α satisfies (3.10), even though $\int \psi_\alpha(\xi)\, \frac{d\xi}{|\xi|^3} < \infty$, when $\alpha > 0$. We thus see that Theorem 3.2 and 3.4 do offer distinct applications.

□

4 Back to Navier–Stokes equation

We shall now explain how the results we obtained for (FNS) apply to the study of (NS). We shall not seek the greatest generality, for the sake of simplicity, and will restrict to the case where $f = 0$. Our main object is

Theorem 4.1: *Assume that* $u_0 \in H$, $\chi_0 = \frac{2}{\nu}\left(\frac{\pi}{2}\right)^{3/2}|\xi|^2 \hat{u}_0$ *satisfies*

$$|\chi_0| \overset{\text{a.e.}}{\leq} \Psi \ , \tag{4.1}$$

where Ψ *is either one of the* ψ_α $(= X^{\psi_\alpha})$ *in* (3.1) *or* X^ψ *for* ψ *a bounded non negative rotationally invariant measurable function satisfying* (3.10)). *Then there exists a unique solution* u_t, $t \geq 0$, *of* (NS), *with initial data* u_0 *and force* $f = 0$, *for which*:

$$\frac{2}{\nu}\left(\frac{\pi}{2}\right)^{3/2}|\xi|^2|\hat{u}_t| \overset{\text{a.e.}}{\leq} \Psi(\xi), \ \textit{for } t \geq 0 \ . \tag{4.2}$$

Moreover for any $t \geq 0$:

$$\int u_0^2\, dx \geq \int u_t^2\, dx + 2\nu \int_0^t\!\!\int |\nabla u_s|^2\, dx \tag{4.3}$$

(*where* $|\nabla u|^2 = \sum_1^3 |\nabla u^i|^2$).

Proof: The uniqueness part of the statement follows immediately from Proposition 1.1, Theorem 3.2, 3.4.

We shall therefore mainly be concerned with the existence part. Consider χ_t, $t \geq 0$, the solution with initial condition χ_0, (which we choose so that (2.1) holds and $|\chi_0| \leq \Psi$) given by Theorem 3.2 or 3.4. If we define for $t \geq 0$

$$u_t = \frac{\nu}{2}\left(\frac{2}{\pi}\right)^{3/2}\left(\frac{\chi_t}{|\xi|^2}\right)^{\vee}, \tag{4.4}$$

(where $\check{h}$ denotes the inverse Fourier transform of h), we already know that (3.27) holds, and div $u_t = 0$, for $t \geq 0$. Our claim will then follow once we show that (4.3) holds. Indeed, it will then follow that $u \in L^2(0,T;V)$ for all $T > 0$, and the condition $u' \in L^1(0,T;V')$ is then known to be a consequence of (3.27), see Temam [13], p. 281.

We define for $n \geq 1, t \geq 0, \xi \in \mathbb{R}^3\backslash\{0\}$.

$$\chi_t^n(\xi) = E_{\xi,t}\big[R(\chi_0, 0, \omega), C_n\big] \ , \tag{4.5}$$

where

$$C_n = \big\{\omega \in \Omega_+, |\xi_m| \leq n, \ \text{for } m \in N(\omega) \cup \partial N(\omega)\big\} \ . \tag{4.6}$$

It follows from dominated convergence that for $\xi \in \mathbb{R}^3\backslash\{0\}, t \geq 0$,

$$\chi_t^n(\xi) \underset{n\to\infty}{\longrightarrow} \chi_t(\xi), \quad \text{and} \quad |\chi_t^n(\xi)| \leq X_t(\xi) \ . \tag{4.7}$$

Applying the strong Markov property at time $t_\emptyset$, we also see that for $t \geq 0$, $\xi \in \mathbb{R}^3\backslash\{0\}$:

$$\chi_t^n(\xi) = \exp\{-\nu|\xi|^2 t\} 1_{B_n}\chi_0(\xi) + \int_0^t \nu|\xi|^2 e^{-\nu|\xi|^2(t-s)} \frac{1}{2} \chi_s^n \circ_n \chi_s^n(\xi)\, ds \quad , \tag{4.8}$$

where

$$B_n = \{\xi \in \mathbb{R}^3\backslash\{0\}, |\xi| \leq n\} \quad , \tag{4.9}$$

and for bounded measurable $\mathbb{C}^3$-valued functions on $\mathbb{R}^3\backslash\{0\}$:

$$f_1 \circ_n f_2 = 1_{B_n}\left[(1_{B_n} f_1) \circ (1_{B_n} f_2)\right] \quad . \tag{4.10}$$

Lemma 4.2. *If* f_1, f_2, f_3 *are* $\mathbb{C}^3$ *valued bounded measurable functions on* $\mathbb{R}^3\backslash\{0\}$, *for which*

$$f_i(\xi) \cdot \xi = 0, \;\; f_i(-\xi) = \overline{f_i(\xi)}, \;\; i = 1, 2, 3, \;\; \xi \in \mathbb{R}^3\backslash\{0\}, \tag{4.11}$$

then

$$\int f_1 \circ_n f_2 \cdot f_3 \frac{d\xi}{|\xi|^2} = -\int f_1 \circ_n f_3 \cdot f_2 \frac{d\xi}{|\xi|^2}, \tag{4.12}$$

(*where we recall that* $a \cdot b = \sum_1^3 a_i \bar{b}_i$, *for* $a, b \in \mathbb{C}^3$).

Proof: In view of (4.10), both integrals in (4.12) are absolutely convergent. Moreover, for $w \in \mathbb{C}^3, \xi \in \mathbb{R}^3\backslash\{0\}$,

$$w \cdot f_i(\xi) = \big(p(\xi)w\big) \cdot f_i(\xi), \quad i = 2, 3.$$

Denoting by f_i^n the function $1_{B_n} f_i$, we thus find:

$$\int f_1 \circ_n f_2 \cdot f_3 \frac{d\xi}{|\xi|^2} = -\frac{i}{\pi^3} \iint \frac{f_1^n(\xi - \xi_2) \cdot \xi}{|\xi - \xi_2|^2} \frac{f_2^n(\xi_2) \cdot f_3^n(\xi)}{|\xi_2|^2 |\xi|^2}\, d\xi_2\, d\xi, \tag{4.13}$$

observe that in view of (4.11)

$$f_1^n(\xi - \xi_2) \cdot \xi = f_1^n(\xi - \xi_2) \cdot \big(\xi - (\xi - \xi_2)\big) = f_1^n(\xi - \xi_2) \cdot \xi_2.$$

The left hand side of (4.12) thus equals

$$\frac{i}{\pi^3} \iint \frac{f_1^n(\xi - \xi_2) \cdot (-\xi_2)}{|\xi - \xi_2|^2} \frac{f_2^n(\xi_2)}{|\xi_2|^2} \cdot \frac{f_3^n(\xi)}{|\xi|^2}\, d\xi_2\, d\xi \quad .$$

As a result of (4.11), we also have

$$f_2^n(-\xi_2) \cdot f_3^n(-\xi) = f_3^n(\xi) \cdot f_2^n(\xi_2) \quad ,$$

and the above integral therefore equals

$$\frac{i}{\pi^3} \iint \frac{f_1^n(\xi_2 - \xi) \cdot \xi_2}{|\xi_2 - \xi|^2} \frac{f_3^n(\xi)}{|\xi|^2} \cdot \frac{f_2^n(\xi_2)}{|\xi_2|^2}\, d\xi d\xi_2 = -\int f_1^n \circ_n f_3^n \cdot f_2^n \frac{d\xi}{|\xi|^2}.$$

This proves (4.12). □

Multiplying (4.8) by $\exp\{\nu|\xi|^2 t\}$ and differentiating, we find:

$$\frac{d}{dt}\chi_t^n(\xi) = \nu|\xi|^2\left(-\chi_t^n(\xi) + \tfrac{1}{2}\chi_t^n \circ_n \chi_t^n(\xi)\right), \tag{4.14}$$

$\xi \in \mathbb{R}^3\backslash\{0\}, t > 0$, so that:

$$\begin{aligned}\frac{d}{dt}\chi_t^n(\xi)\cdot\chi_t^n(\xi) = &-2\nu|\xi|^2\chi_t^n\cdot\chi_t^n + \frac{\nu}{2}|\xi|^2\left(\chi_t^n \circ_n \chi_t^n(\xi)\cdot\chi_t^n(\xi)\right.\\ &\left.+\chi_t^n(\xi)\cdot\chi_t^n \circ_n \chi_t^n(\xi)\right).\end{aligned} \tag{4.15}$$

Multiplying both members by $|\xi|^{-4}$, integrating in ξ and t, the last two terms of the right member vanish thanks to (4.12), so that for $t \geq 0$:

$$\int \frac{|\chi_0^n(\xi)|^2}{|\xi|^4}d\xi = \int \frac{|\chi_t^n(\xi)|^2}{|\xi|^4}d\xi + 2\nu\int_0^t\!\!\int \frac{|\chi_s^n(\xi)|^2}{|\xi|^2}\,d\xi\,ds\ , \tag{4.16}$$

where the left hand side of (4.16) is finite since $u_0 \in H$. Using (4.7) and Fatou's lemma in (4.16), we obtain

$$\int \frac{|\chi_0(\xi)|^2}{|\xi|^4}\,d\xi \geq \int \frac{|\chi_t(\xi)|^2}{|\xi|^4}\,d\xi + 2\nu\int_0^t\!\!\int \frac{|\chi_s(\xi)|^2}{|\xi|^2}\,d\xi\ . \tag{4.17}$$

This proves our claim (4.3). □

Remark 4.3:

1) The argument we just used shows that in fact

$$\int u_t^2\,dx \geq \int u_{t'}^2\,dx + 2\nu\int_t^{t'}\!\!\int |\nabla u_s|^2 dx\,ds, \text{ for any } 0 \leq t < t'\ . \tag{4.18}$$

2) The condition stated in Serrin [10] p. 76, 83, 86, for global existence and uniqueness in the class of weak solutions satisfying (4.3) is that $u_0 \in H^2$ and

$$|u_0|_2^3\,|u_{t0}|_2 < 768\nu^5\ ,$$

where $|w|_2^2 = \int |w(x)|^2 dx$.

Theorem 4.1 covers cases where this condition fails. For instance choosing $\psi = 1$ in (4.1) and

$$\chi_0(\xi) = 1\left(\frac{1}{n} \leq |\xi| \leq 1\right)p(\xi)e\ ,$$

with $n \geq 2$, $0 < |e| \leq 1$, we can make the corresponding

$$|u_0|_2 = \frac{\nu}{2}\left(\frac{2}{\pi}\right)^{3/2}\left|\frac{\chi_0}{|\xi|^2}\right|_2$$

arbitrarily large (by choosing n large), while keeping

$$|u_{t0}|_2 = \frac{\nu^2}{2}\left(\frac{2}{\pi}\right)^{3/2}\left|-\chi_0 + \frac{1}{2}\chi_0 \circ \chi_0\right|_2$$

away from 0. □

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