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Averaged and quenched propagation of chaos for spin glass dynamics

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Summary. We study the asymptotic behaviour for both asymmetric and symmetric spin glass dynamics in a Sherrington-Kirkpatrick model as proposed by Sompolinsky-Zippelius. We prove, without any condition on time and temperature, averaged propagation of chaos results. Extending this result to replicated systems, we conclude that the law of a single spin converges to a non Markovian probability measure, in law with respect to the random interaction.

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1 Introduction

The goal of this paper is to study the asymptotic law of a particle for Langevin spin glass dynamics *with no restriction on time or temperature*. These dynamics, first introduced by Sompolinski and Zippelius [10], have been studied in [2] and [3] in a short time or high temperature regime. Since we consider particles in random interaction, this problem can be addressed in different terms. First, one could be willing to study the mean behaviour of the system, i.e the law of the system averaged on the interaction. As in [2], we will call this averaged law the annealed law of the system[1]. We will prove here

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[1]As stressed by a referee, we would like to point out that we may be using the term annealed (originally borrowed to the physics literature) in an improper manner. Nevertheless, we prefer to stick to this terminology to agree with [2].

that the annealed law of Langevin spin glass dynamics is chaotic in the sense that the averaged law of a finite number of spins converges to a product measure.

Furthermore, one could wonder if the same kind of results holds for almost all interaction (i.e if a quenched propagation of chaos occurs). We can only prove a weak version of this result by showing that the quenched law of the system is chaotic, in law with respect to the interaction. We have actually no idea weather this weak convergence can be improved in an almost sure convergence. Nevertheless, if an almost sure convergence holds, our result identifies the quenched limit behaviour. At this point, we would like to emphasize that, in the case where the system is not submitted to an external magnetic field, the quenched limit behaviour is identical to the annealed limit behaviour. In this setting, a true almost sure convergence might be easier to get. In general, it turns out that the quenched asymptotic law of a spin depends on a random magnetic field, and, as a consequence, that the random interaction is not completely averaged at the limit.

Let us now be more precise. We study the weak solution $P^N_\beta(J)$ on a finite time interval $[0, T]$ (which will be fixed throughout this paper) of the following system $\mathscr{S}^N_\beta(J)$ of randomly interacting diffusions:

$$\mathscr{S}^N_\beta(J) = \begin{cases} dx^j_t = -\nabla U(x^j_t)\,dt + dB^j_t + \frac{\beta}{\sqrt{N}}\sum_{i=1}^{N} J_{ji}\, x^i_t\,dt & \forall 1 \le i \le N \\ \text{Law of } x_0 = \mu_0^{\otimes N} \end{cases}$$

where $(B^j)_{1 \le j \le N}$ is a N dimensional Brownian motion.

As in [2] and [3], we will study only bounded spins, i.e we will assume that μ_0 is a probability measure on a bounded interval $[-A, A]$ which does not put mass on the boundary $\{-A, +A\}$ and that $U(x)$ is defined on $[-A, A]$ and tends to infinity when $|x| \to A$ sufficiently fast to insure that the spins x^j stay in the interval $[-A, A]$.

We will here be interested by the two models of couplings we have been considering so far in [2] and [3]. First, we will assume that the whole matrix $(J_{ij})_{i,j}$ is made of i.i.d $N(0, 1)$ random variables, i.e we will look at asymmetric dynamics. As already stressed in [2], these dynamics are not reversible for the S-K Gibbs measure but have the advantage to be easier to study. Furthermore, they seem to provide a satisfactory model for neural networks (see [5] and references therein). In a second time, we will impose the symmetry $J_{ij} = J_{ji}$ and therefore consider dynamics which are reversible for the Gibbs measure as studied by Sompolinski and Zippelius.

In both settings, we prove that the annealed law of the empirical measure $\hat{\mu}^N = \frac{1}{N}\sum_{i=1}^{N} \delta_{x^i}$ converges to δ_Q where Q is the non Markovian law described in [2] (resp. in [3]) for the asymmetric (resp. symmetric) model, with no restriction on time and temperature (see section 2.1 and section 3.1).

As a consequence, we deduce an annealed propagation of chaos, that is convergence of the annealed law of $(x^1, \ldots, x^m)$ to $Q^{\otimes m}$ (see Theorem 2.2 and Corollary 3.2).

For the asymmetric model, we have seen in [2], section 6, that Q can be seen as an average of non Markovian processes, namely that there exists a centered Gaussian process h and a non Markovian probability measure P_h such that $Q = \mathscr{E}^h[P_h]$, where $\mathscr{E}^h$ denotes the expectation over h. Thus, if we denote $\mathscr{E}$ the expectation over the random couplings J, we have proved that, for any bounded continuous function f, for any temperature $\frac{1}{\beta}$:

$$\lim_{N\to\infty} \mathscr{E}\left[\int f(x^1)\, dP_\beta^N(J)\right] = \mathscr{E}^h\left[\int f\, dP_h\right] .$$

Using replica, we improve this last result and deduce a quenched propagation of chaos, in the sense that, for any bounded continuous test functions $(f_1, .., f_m)$, $\int f_1(x^1)..f_m(x^m)dP_\beta^N(J)$ converges in law to $\prod_{i=1}^m \int f_i dP_{h_i}$ where h_i are i.i.d (see section 2.2). In the case where the system is not submitted to an external magnetic field (which correspond to the case where U is even), we also saw in [2] that h is almost surely null so that in fact the limit behaviour is not random.

This simplification as well occurs for the symmetric model (see [4]). In fact, if U is even and the initial law symmetric, we prove a quenched propagation of chaos result, i.e the convergence in probability of $\int f_1(x^1)..f_m(x^m)dP_\beta^N(J)$ towards $\prod_{i=1}^m \int f_i dQ$ in this setting too (see subsection 3.6).

We would like finally to stress that the two limiting processes for the averaged law of the asymmetric and symmetric dynamics are different (despite the fact that we use the same notation Q for both of them). Indeed, the symmetry of the couplings provide an additional drift to the diffusion, called the response function. This additional drift is a linear function of the whole past trajectory but depends as well on the whole past law of the process (see subsection 3.6).

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2 Asymmetric dynamics

In this first part, we will assume that the whole matrix $(J_{ij})_{i,j}$ is made of i.i.d $N(0,1)$ random variables. We will denote $\mathscr{E}$ the expectation over the $(J_{ij})_{i,j}$'s and, more generally, on every Gaussian process describing an interaction we will encounter.

2.1 Annealed propagation of chaos

The aim of this section is to prove:

Theorem 2.1 *The law of the empirical measure* $\hat{\mu}^N = \frac{1}{N}\sum_{i=1}^N \delta_{x^i}$ *under* $Q_\beta^N = \mathscr{E}\left[P_\beta^N(J)\right]$ *converges to* δ_Q.

The probability measure Q is described in Theorem 2.5.

According to lemma 3.1 of Sznitman [12], Theorem 2.1 gives the annealed propagation of chaos result:

Theorem 2.2 $Q_\beta^N = \mathscr{E}\left[P_\beta^N(J)\right]$ *is* Q*-chaotic, i.e for any bounded continuous functions* $(f_1, \ldots, f_m)$,

$$\lim_{N\to\infty} \int \prod_{j=1}^m f_j(x^j)\, dQ_\beta^N(x) = \prod_{j=1}^m \int f_j(x)\, dQ(x) \quad .$$

Theorem 2.1 was proved in [2] in the high temperature and small time regime $\beta^2 A^2 T < 1$ by use of large deviations. The advantage of this method is that it implies that the law of the empirical measure of the quenched system converges to a dirac measure at Q for almost all interaction (which of course does not imply a quenched propagation of chaos since the spins are not exchangeable for almost all interaction). Nevertheless, we needed in [2] a high temperature and small time restriction to get an exponential tightness property which was crucial in our strategy. Here, we deduce Theorem 2.1 from a weak large deviation principle and from a tightness (but not exponential) result. Namely, if W_T^A denotes the set of continuous functions from $[0, T]$ into $[-A, +A]$ and $\mathscr{M}_1^+(W_T^A)$ the set of probability measures on W_T^A, we prove that:

Theorem 2.3 *There exists a good rate function* H *such that, for any compact subset* K *of* $\mathscr{M}_1^+(W_T^A)$:

$$\overline{\lim_{N\to\infty}} \frac{1}{N} \log Q_\beta^N\left(\hat{\mu}^N \in K\right) \leq -\inf_K H \quad .$$

And:

Theorem 2.4 *For any real number* $\epsilon > 0$, *there exists a compact subset* K_ϵ *of* $\mathscr{M}_1^+(W_T^A)$ *such that, for any integer number* N,

$$Q_\beta^N\left(\hat{\mu}^N \in K_\epsilon^c\right) \leq \epsilon \quad .$$

Theorem 2.3 is proved in section 2.3 and Theorem 2.4 is proved in section 2.4.

To deduce Theorem 2.1 from Theorems 2.3 and 2.4, we need to recall that we characterized the minima of H in [2], section 5, and proved that:

Theorem 2.5 1) *H achieves its minimal value at the probability measures Q which satisfy*:

$$Q \ll P \qquad \frac{dQ}{dP} = \mathscr{E}\left[\exp\left\{\beta \int_0^T G_t^Q dB_t - \frac{\beta^2}{2}\int_0^T (G_t^Q)^2\, dt\right\}\right] \tag{1}$$

where $\mathscr{E}$ denotes the expectation over a centered Gaussian process G^Q with covariance

$$\mathscr{E}\left[G_t^Q G_s^Q\right] = \int x_s x_t \, dQ(x) \ .$$

2) *There exists a unique probability measure Q which satisfies (1).*

Let us mention that equation (1) is equivalent to the following non linear stochastic differential equation:

$$\begin{cases} dx_t = -\nabla U(x_t)\, dt + dB_t \\ dB_t = dW_t + \beta^2 \int_0^t dB_s \mathscr{E}\left[G_s^Q G_t^Q \frac{\exp\{-\frac{\beta^2}{2}\int_0^t (G_u^Q)^2 du\}}{\mathscr{E}\left[\exp\{-\frac{\beta^2}{2}\int_0^t (G_u^Q)^2\, du\}\right]}\right] dt \\ \text{Law of } (x) = Q, \text{Law of } (x_0) = \mu_0 \end{cases} \ . \tag{2}$$

where W is a Q Brownian motion.

Proof of Theorem 2.1 Let δ be a strictly positive real number and denote $B(Q,\delta)$ the open ball with respect to a metric compatible with the weak topology on $\mathscr{M}_1^+(W_T^A)$, for instance the Wasserstein metric (which definition is given in (7)). We prove that $Q_\beta^N(\hat{\mu}^N \in B(Q,\delta)^c)$ converges to zero for any positive real number δ. Indeed, if K_ϵ are the compact sets defined in Theorem 2.4, we have, for any $\epsilon > 0$:

$$\begin{aligned} Q_\beta^N\left(\hat{\mu}^N \in B(Q,\delta)^c\right) &\le Q_\beta^N\left(\hat{\mu}^N \in K_\epsilon^c\right) + Q_\beta^N\left(\hat{\mu}^N \in K_\epsilon \cap B(Q,\delta)^c\right) \\ &\le \epsilon + Q_\beta^N\left(\hat{\mu}^N \in K_\epsilon \cap B(Q,\delta)^c\right) \end{aligned} \tag{3}$$

But, since $B(Q,\delta)^c$ is closed, $K_\epsilon \cap B(Q,\delta)^c$ is compact so that we can use Theorem 2.3:

$$\overline{\lim} \frac{1}{N} \log Q_\beta^N\left(\hat{\mu}^N \in K_\epsilon \cap B(Q,\delta)^c\right) \le - \inf_{K_\epsilon \cap B(Q,\delta)^c} H \ .$$

But $\inf_{K_\epsilon \cap B(Q,\delta)^c} H$ is strictly positive according to Theorem 2.5. Hence, (3) implies that, for any $\epsilon > 0$,

$$\lim_{N\to\infty} Q_\beta^N\left(\hat{\mu}^N \in B(Q,\delta)^c\right) \le \epsilon \ ,$$

i.e

$$\lim_{N\to\infty} Q_\beta^N\left(\hat{\mu}^N \in B(Q,\delta)^c\right) = 0 \ .$$

□

2.2 Quenched propagation of chaos and replica

Theorem 2.2 can be extended to replicated systems as follows:

Let r be an integer number and denote $Q_\beta^{r,N}$ the annealed law of replicated spin glass dynamics:

$$Q_\beta^{r,N} = \mathscr{E}\left[P_\beta^N(J)^{\otimes r}\right]$$

Let Q_r be defined by:

$$Q_r \ll P^{\otimes r} \qquad \frac{dQ_r}{dP^{\otimes r}} = \mathscr{E}\left[\exp\left\{\beta\int_0^T \langle G_t^{Q_r}, dB_t\rangle - \frac{\beta^2}{2}\int_0^T ||G_t^{Q_r}||^2 dt\right\}\right] \tag{4}$$

where G^{Q_r} is a r-dimensional centered Gaussian process with covariance:

$$\mathscr{E}\left[G_t^{Q_r,i} G_s^{Q_r,j}\right] = \int x_s^j x_t^i dQ_r(x)$$

and where $\langle\ ,\ \rangle$ denotes the euclidean scalar product in $\mathbb{R}^r$, $\|\ \|^2 = \langle\ ,\ \rangle$.

Then Q_r exists and is unique (see [2], section 6), and we have:

Theorem 2.6 *For any integer number r, the law of the empirical measure $\hat\mu^{r,N} = \frac{1}{N}\sum_{i=1}^N \delta_{x_1^i,\ldots,x_r^i}$ under $Q_\beta^{r,N}$ converges to δ_{Q_r}.*

The proof of Theorem 2.6 is very similar to that of Theorem 2.2. We omit it. □

As a consequence:

Theorem 2.7 *$Q_\beta^{r,N}$ is Q_r-chaotic, i.e for any bounded continuous functions $(F_1,\ldots,F_m)$ on $(W_T^A)^r$,*

$$\lim_{N\to\infty} \mathscr{E}\left[\int F_1(x_1^1,\ldots,x_r^1)\ldots F_m(x_1^m,\ldots,x_r^m)\, dP_\beta^N(J)^{\otimes r}(x_1,\ldots,x_r)\right]$$
$$= \prod_{i=1}^m \int F_i(x_1,\ldots,x_r)\, dQ_r \ .$$

In particular, for any bounded continuous functions $(f_1,\ldots,f_m)$ on W_T^A,

$$\lim_{N\to\infty} \mathscr{E}\left[\left(\int f_1(x^1)\ldots f_m(x^m)\, dP_\beta^N(J)(x)\right)^r\right] = \prod_{i=1}^m \int f_i(x_1)\ldots f_i(x_r)\, dQ_r \ .$$

To deduce a quenched propagation of chaos from Theorem 2.7, we need to identify the probability measures Q_r themselves as replicated laws. This was done in [2], section 6, where we proved that there exists a couple (h, P_h) of a Gaussian process h and a probability measure P_h on W_T^A (which depends on h) such that:

Theorem 2.8

$$\textit{For any integer } r, \qquad Q_r = \mathscr{E}^h\left[P_h^{\otimes r}\right] \ .$$

Moreover, the couple (h, P_h) is defined by the following non linear procedure:

For f in $L^2([0,T])$, let $P(f)$ be the restriction on $[0,T]$ of the law of the diffusion

$$\begin{cases} dx_t = -\nabla U(x_t)\,dt + dB_t + \beta f(t)\,dt \\ \text{Law of } x_0 = \mu_0 \end{cases}$$

Let (h,g) be two independent centered Gaussian processes and denote $\mathscr{E}^h$ (resp. $\mathscr{E}^g$) the expectation over h (resp. g). We define non linearly the covariances of (h,g) by:

$$\mathscr{E}^g[g_t g_s] = \mathscr{E}^h\mathscr{E}^g\left[\int x_s x_t\, dP(g+h)\right] - \mathscr{E}^h\left[\mathscr{E}^g\left[\int x_s\, dP(g+h)\right]\mathscr{E}^g\left[\int x_t\, dP(g+h)\right]\right]$$

$$\mathscr{E}^h[h_t h_s] = \mathscr{E}^h\left[\mathscr{E}^g\left[\int x_s\, dP(g+h)\right]\mathscr{E}^g\left[\int x_t\, dP(g+h)\right]\right]$$

Finally P_h is given by:

$$P_h = \mathscr{E}^g[P(g+h)] \ . \tag{5}$$

Theorem 2.7 and Theorem 2.8 give:

Corollary 2.9 *For any integer r, for any bounded continuous functions $(f_1,\dots,f_m)$ on W_T^A,*

$$\lim_{N\to\infty} \mathscr{E}\left[\left(\int f_1(x^1)\dots f_m(x^m)\,dP_\beta^N(J)(x)\right)^r\right] = \prod_{i=1}^m \mathscr{E}^h\left[\left(\int f_i(x)\,dP_h(x)\right)^r\right]$$

Since the random variables $\int f_1(x^1)\dots f_m(x^m)\,dP_\beta^N(J)(x)$ are bounded, corollary 2.9 is equivalent to the convergence in law of such random variables, which gives the quenched propagation of chaos result:

Theorem 2.10 *For any bounded continuous functions $(f_1,\dots,f_m)$ on W_T^A, $\int f_1(x^1)\dots f_m(x^m)dP_\beta^N(J)(x)$ converges in law, when N tends to infinity, to $\prod_{j=1}^m \int f_j dP_{h_j}$, where h_i are independent copies of the centered Gaussian process h described above.*

Moreover, we described in [2], section 6, the case where the limiting law is deterministic, i.e the case where h is null almost surely. Then $P_h = \mathscr{E}^h[P_h] = Q$. Roughly speaking, it is the case where the potential U is even and the initial law is symmetric. Then, Theorem 2.10 becomes:

Corollary 2.11 *If U is even and μ_0 is symmetric, for any bounded continuous functions $(f_j, 1\le j\le m)$, $\int f_1(x^1)\dots f_m(x^m)dP_\beta^N(J)$ converges in probability to $\prod_{j=1}^m \int f_j\, dQ$.*

Remark 2.12 1) Theorem 2.10 (and corollary 2.11) can also be stated for finite vectors $\left(\int f_1^i(x^1)\dots f_m^i(x^m)P_\beta^N(J)\right)_{1\le i\le n}$, where $(f_j^i, 1\le i\le n, 1\le j\le m)$ are bounded continuous functions, and one finds that $\big(\int f_1^i(x^1)\ \dots f_m^i(x^m)$ $dP_\beta^N(J)\big)_{1\le i\le n}$ converges in law to $\left(\prod_{j=1}^m \int f_j^i dP_{h_j}\right)_{1\le i\le n}$. This is obvious since Theorem 2.7 gives the convergence of every moments of these random variables.

2) As stressed by a referee, Theorem 2.10 (and corollary 2.11) raises several questions. Indeed, one would like the convergence to hold for almost all interaction and simultaneously for all bounded continuous functions. Despite our dedication to these problems, we have no answers.

3) It is easy to understand why the hypotheses of Corollary 2.11 (i.e U even and μ_0 symmetric) simplify so much the results. In fact, it is known that the static Sherrington Kirkpatrick model with an external magnetic field is much more difficult to study than the model without external magnetic field. Our dynamic model contains a priori such a magnetic field. Indeed, adding a magnetic field h to the *SK* model consists in adding hx to the potential $U(x)$. The simple situation where U is even and μ_0 is symmetric corresponds to the model without magnetic field. Thus, Theorem 2.1 shows that adding an external magnetic field makes the limit law depend on an additional exterior field.

4) Theorem 2.7 gives a complete description of the microscopic behaviour of asymmetric spin glass dynamics: it shows that the law of a finite number of spins converges to independent laws submitted to independent identically distributed Gaussian external fields. Moreover, it is clear that the limiting law P_h is non Markovian since it depends on all the past through the law of the Gaussian process g (see (5)). Because of this non Markovian property, the study of the static properties of these systems through dynamics does not seem to be an easy problem.

2.3 Annealed weak large deviation upper bound

To state precisely the main result of this section, i.e the weak large deviation upper bound for the annealed law of the empirical measure stated in Theorem 2.3, we first define the rate function H which governs these deviations.

Definition 2.13 *For any $\mu \in \mathcal{M}_1^+(W_T^A)$, we define a centered Gaussian process G^μ by its covariance:*

$$\mathcal{E}\left[G_s^\mu G_t^\mu\right] = \int x_s x_t \, d\mu(x) \tag{6}$$

For any probability measure μ which is absolutely continuous with respect to P, we then define $\Gamma(\mu)$ by:

$$\Gamma(\mu) = \int \log \mathcal{E}\left[\exp\left\{\beta \int_0^T G_t^\mu \, dB_t(x) - \frac{\beta^2}{2} \int_0^T (G_t^\mu)^2 \, dt\right\}\right] d\mu(x)$$

where $B_t(x) = x_t - x_0 + \int_0^t \nabla U(x_s) \, ds$ for any $x \in W_T^A$.

Let then $I(\mu \mid P)$ denotes the entropy of μ with respect to P, namely:

$$I(\mu \mid P) = \begin{cases} \displaystyle\int \log \frac{d\mu}{dP} \, d\mu & \text{if } \mu \ll P \\ +\infty & \text{otherwise} \end{cases} .$$

Then, we define a function H on $\mathcal{M}_1^+(W_T^A)$ by:

$$H(\mu) = \begin{cases} I(\mu \mid P) - \Gamma(\mu) & \text{if } I(\mu \mid P) < \infty \\ +\infty & \text{otherwise} \end{cases} .$$

Remark 2.14 1) One can construct the Gaussian processes $\{G^\mu, \mu \in W_T^A\}$ on the same probability space. For instance, let (Ω, γ) be a probability space and $(J_i)_{i\in\mathbb{N}}$ be i.i.d $N(0,1)$ on Ω, then, if $(e_i^\mu)_{i\in\mathbb{N}}$ is an orthonormal basis in $L^2_\mu(W_T^A)$,

$$G_s^\mu = \sum_{i\in\mathbb{N}} J_i \int x_s e_i^\mu(x)\, d\mu(x)$$

is a centered Gaussian process with covariance (6). In the following pages, we will suppose that we have constructed all the Gaussian processes $\{G^\mu, \mu \in W_T^A\}$ on the same probability space (Ω, γ) and denote $\mathscr{E}$ the expectation under γ.

2) For any $\mu \ll P$, $\Gamma(\mu)$ is well defined since B is a semimartingale. As a consequence, Γ is well defined on $\{\mu \in \mathscr{M}_1^+(W_T^A) \ / \ I(\mu \mid P) < \infty\}$, so that H is well defined.

Then:

Theorem 2.15 1) *H is a good rate function, i.e H is positive and, for any real number M, $\{\mu \in \mathscr{M}_1^+(W_T^A) \ / H(\mu) \le M\}$ is compact.*

2) *For any compact subset K of $\mathscr{M}_1^+(W_T^A)$:*

$$\overline{\lim_{N\to\infty}} \frac{1}{N} \log Q_\beta^N\left(\hat{\mu}^N \in K\right) \le -\inf_K H \ .$$

Proof of Theorem 2.15. The first point is proved in [2], section 4.

To prove the second point, we first notice that, according to lemma 3.6 of [2], we have:

$$dQ_\beta^N = \exp\{N\Gamma(\hat{\mu}^N)\}\, dP^{\otimes N}$$

where $P^{\otimes N} = P_0^N$ is the law of the system without interaction. Thus, if Γ was bounded and continuous, Theorem 2.15.2) would be clear. To circumvent the fact that none of these properties is satisfied, we shall approximate Γ by linear functions. More precisely, for any $\nu \in \mathscr{M}_1^+(W_T^A)$, we define a map Γ_ν from $\mathscr{M}_1^+(W_T^A)$ into $\mathbb{R}$ by:

$$\Gamma_\nu(\mu) := \int \log \mathscr{E}\left[\exp\left\{\beta \int_0^T G_s^\nu \, dB_s(x) - \frac{\beta^2}{2}\int_0^T (G_s^\nu)^2\, ds\right\}\right] d\mu(x) \ .$$

We denote d_t the Wasserstein distance on W_t^A, i.e:

$$d_t(\mu, \nu) = \inf\left\{\int \sup_{s\le t} |x_s - y_s| \, d\xi(x,y)\right\} \tag{7}$$

where the infimum is taken on the probability measure ξ with marginals μ and ν. In short, we will denote d for d_T.

The key of our proof is the following technical lemma:

Lemma 2.16 *For any real number* α, $\alpha > 1$, *there exists a strictly positive real number* δ_α *such that, for any* $\delta < \delta_\alpha$, *there exists a function* $C_\alpha(.)$ *in* $\mathbb{R}$ *such that* $\lim_{\delta\to 0} C_\alpha(\delta) = 0$ *and*:

$$\int_{d(\hat{\mu}^N,\nu)<\delta} \exp\{\alpha N(\Gamma - \Gamma_\nu)(\hat{\mu}^N) + N\Gamma_\nu(\hat{\mu}^N)\}\, dP^{\otimes N} \leq \exp C_\alpha(\delta) N \ . \tag{8}$$

Proof of lemma 2.16. Let

$$B^N = \int_{d(\hat{\mu}^N,\nu)<\delta} \exp\{\alpha N(\Gamma - \Gamma_\nu)(\hat{\mu}^N)\} \exp\{N\Gamma_\nu(\hat{\mu}^N)\}\, dP^{\otimes N}$$

Let Q_ν be the probability measure on W_T^A defined by:

$$dQ_\nu(x) = \mathscr{E}\left[\exp\left\{\beta \int_0^T G_s^\nu\, dB_s(x) - \frac{\beta^2}{2}\int_0^T (G_s^\nu)^2\, ds\right\}\right] dP(x) \tag{9}$$

Then:

$$d(Q_\nu)^{\otimes N} = \exp\{N\Gamma_\nu(\hat{\mu}^N)\}\, dP^{\otimes N}$$

Writing down the definitions of Γ and Γ_ν, we find:

$$B^N = \int_{d(\hat{\mu}^N,\nu)<\delta} \prod_{i=1}^N \left(\frac{\mathscr{E}\left[\exp\left\{\beta \int_0^T G_t^{\hat{\mu}^N} dB_t^i - \frac{\beta^2}{2}\int_0^T \left(G_t^{\hat{\mu}^N}\right)^2 dt\right\}\right]}{\mathscr{E}\left[\exp\left\{\beta \int_0^T G_t^\nu\, dB_t^i - \frac{\beta^2}{2}\int_0^T (G_t^\nu)^2\, dt\right\}\right]} \right)^\alpha d(Q_\nu)^{\otimes N} \ .$$

Assume that we have constructed on the same probability space (Ω, γ) the Gaussian processes $G^{\hat{\mu}^N}$ and G^ν and denote:

$$X_N^j = \beta \int_0^T G_s^{\hat{\mu}^N} dB_s^j - \frac{\beta^2}{2}\int_0^T \left(G_s^{\hat{\mu}^N}\right)^2 ds$$
$$X_\nu^j = \beta \int_0^T G_s^\nu\, dB_s^j - \frac{\beta^2}{2}\int_0^T \left(G_s^\nu\right)^2 ds \ .$$

Then

$$B^N = \int_{d(\hat{\mu}^N,\nu)<\delta} \prod_{j=1}^N \left(\frac{\mathscr{E}[\exp\{X_N^j\}]}{\mathscr{E}[\exp\{X_\nu^j\}]} \right)^\alpha dQ_\nu^{\otimes N}$$
$$= \int_{d(\hat{\mu}^N,\nu)<\delta} \prod_{j=1}^N \left(\mathscr{E}\left[\frac{\exp X_\nu^j}{\mathscr{E}[\exp X_\nu^j]} \exp(X_N^j - X_\nu^j) \right] \right)^\alpha dQ_\nu^{\otimes N}$$

Since $\alpha > 1$, we can apply Jensen inequality in the last equality:

$$B^N \leq \int_{d(\hat{\mu}^N, v)<\delta} \prod_{j=1}^{N} \mathscr{E}\left[\frac{\exp X_v^j}{\mathscr{E}\left[\exp X_v^j\right]} \exp \alpha\left(X_N^j - X_v^j\right)\right] dQ_v^{\otimes N}$$

Therefore, if $p^{-1} + q^{-1} = 1$,

$$B^N \leq \left\{\int \prod_{j=1}^{N} \frac{\left(\mathscr{E}\left[\exp pX_v^j\right]\right)}{\left(\mathscr{E}\left[\exp X_v^j\right]\right)^p} dQ_v^{\otimes N}\right\}^{\frac{1}{p}}$$

$$\times \left\{\int_{d(\hat{\mu}^N, v)<\delta} \prod_{j=1}^{N} \mathscr{E}\left[\exp \alpha q\left(X_N^j - X_v^j\right)\right] dQ_v^{\otimes N}\right\}^{\frac{1}{q}} \quad (10)$$

We first bound the first term in the right hand side of (10). Gaussian calculus shows that

$$\exists C < \infty \qquad \frac{\mathscr{E}\left[\exp\{pX_v^j\}\right]}{\left(\mathscr{E}\left[\exp X_v^j\right]\right)^p} \leq \exp\left\{C(p-1)\left(1 + \mathscr{E}\left[\left(\int_0^T G_t^v \, dB_t^j\right)^2\right]\right)\right\}$$

So that, if p is close enough to 1, we can find a finite constant $C(p)$, $C(p) \searrow 0$ when $p \searrow 1$, such that:

$$B_1^N := \int \prod_{j=1}^{N} \frac{\left(\mathscr{E}\left[\exp pX_v^j\right]\right)}{\left(\mathscr{E}\left[\exp X_v^j\right]\right)^p} dQ_v^{\otimes N} = \left(\int \frac{\mathscr{E}\left[\exp pX_v^j\right]}{\left(\mathscr{E}\left[\exp X_v^j\right]\right)^p} dQ_v\right)^N \leq e^{C(p)N} \ . \quad (11)$$

We now bound the second term in the right hand side of (10) using Hölder inequality with conjugate exponents (η, σ) $(\eta^{-1} + \sigma^{-1} = 1)$:

$$B_2^N := \int_{d(\hat{\mu}^N, v)<\delta} \left(\prod_{j=1}^{N} \mathscr{E}\left[\exp \alpha q\left(X^j - X_v^j\right)\right]\right) \exp N\Gamma_v\left(\hat{\mu}^N\right) dP^{\otimes N}(x)$$

$$\leq \left\{\int \exp\left\{N\sigma\Gamma_v\left(\hat{\mu}^N\right)\right\} dP^{\otimes N}\right\}^{\frac{1}{\sigma}}$$

$$\times \left\{\int_{d(\hat{\mu}^N, v)<\delta} \prod_{j=1}^{N} \mathscr{E}\left[\exp\left\{\alpha\eta q\left(X^j - X_v^j\right)\right\}\right] dP^{\otimes N}\right\}^{\frac{1}{\eta}} \ . \quad (12)$$

The first term in the right hand side of (12) can be bounded if $\sigma - 1$ is small enough:

$$\int \exp\left\{N\sigma\Gamma_v\left(\hat{\mu}^N\right)\right\} dP^{\otimes N}(x)$$

$$= \left(\int \mathscr{E}\left[\exp\left\{\beta \int_0^T G_t^v dB_t(x) - \frac{\beta^2}{2}\int_0^T \left(G_t^v\right)^2 dt\right\}\right]^\sigma dP(x)\right)^N$$

$$\leq \left(\mathscr{E}\left[\int \exp\left\{ \beta\sigma \int_0^T G_t^\nu dB_t(x) - \frac{\beta^2}{2}\sigma \int_0^T \left(G_t^\nu\right)^2 dt \right\} dP(x) \right] \right)^N$$
$$= \left(\mathscr{E}\left[\int \exp\left\{ \frac{\beta^2}{2}(\sigma^2 - \sigma) \int_0^T \left(G_t^\nu\right)^2 dt \right\} \right] \right)^N .$$

But Gaussian integrability properties imply, as detailed in appendix A of [2], that there exists a finite constant $C(\sigma)$, $\lim_{\sigma\to 1} C(\sigma) = 0$, such that:

$$\mathscr{E}\left[\int \exp\left\{ \frac{\beta^2}{2}(\sigma^2 - \sigma) \int_0^T (G_t^\nu)^2 dt \right\} \right] \leq \exp\{C(\sigma)\} \tag{13}$$

So that we have proved that:

$$\int \exp\left\{N\sigma\Gamma_\nu\left(\hat{\mu}^N\right)\right\} dP^{\otimes N}(x) = \left(\int \exp\{\sigma\Gamma_\nu(\delta_x)\}\, dP(x) \right)^N < e^{C(\sigma)N} \tag{14}$$

We bound the second term in the right hand side of (12). By Cauchy Schwarz inequality, if $\kappa = \alpha\eta q\beta$:

$$\int_{d(\hat{\mu}^N,\nu)<\delta} \prod_{j=1}^N \mathscr{E}\left[\exp \alpha\eta q\left(X_N^j - X_\nu^j\right)\right] dP^{\otimes N} \leq$$
$$\left\{ \int \prod_{j=1}^N \mathscr{E}\left[\exp\left\{ 2\kappa \int_0^T \left(G_s^{\hat{\mu}^N} - G_s^\nu\right) dB_s^j - 2\kappa^2 \int_0^T \left(G_s^{\hat{\mu}^N} - G_s^\nu\right)^2 ds \right\} \right] dP^{\otimes N} \right\}^{\frac{1}{2}}$$
$$\times \left\{ \int_{d(\hat{\mu}^N,\nu)<\delta} \mathscr{E}\left[\exp\left\{ 2\kappa^2 \int_0^T \left(G_s^{\hat{\mu}^N} - G_s^\nu\right)^2 ds \right.\right.\right.$$
$$\left.\left.\left. + \beta\kappa \int_0^T \left(\left(G_s^{\hat{\mu}^N}\right)^2 - \left(G_s^\nu\right)^2 \right) ds \right\} \right]^N dP^{\otimes N} \right\}^{\frac{1}{2}} \tag{15}$$

The first term is bounded by one by supermartingale properties. For the second term, we remark that:

$$\int_0^T \left(\left(G_s^{\hat{\mu}^N}\right)^2 - \left(G_s^\nu\right)^2 \right) ds \leq \frac{1}{2}\delta^{\frac{1}{2}} \left(\frac{1}{\delta} \int_0^T \left(G_s^{\hat{\mu}^N} - G_s^\nu\right)^2 ds + \int_0^T \left(G_s^{\hat{\mu}^N} + G_s^\nu\right)^2 ds \right) \tag{16}$$

Hence, we can apply as in (13) lemma A.3.2) of [2] to the right hand side of (16) so that we find that, for any real number κ, there exists $C_\kappa(\delta)$, $\lim_{\delta\to 0} C_\kappa(\delta) = 0$, such that, for any $(x^i)_{1\leq i\leq N}$ such that $d(\hat{\mu}^N, \nu) < \delta$:

$$\mathscr{E}\left[\exp\left\{ 2\kappa^2 \int_0^T \left(G_s^{\hat{\mu}^N} - G_s^\nu\right)^2 ds + \beta\kappa \int_0^T \left(\left(G_s^{\hat{\mu}^N}\right)^2 - \left(G_s^\nu\right)^2 \right) ds \right\} \right] \leq e^{C_\kappa(\delta)} .$$

Thus

$$\int_{d(\hat{\mu}^N,\nu)<\delta} \mathscr{E}\left[\exp\left\{2\kappa^2 \int_0^T \left(G_s^{\hat{\mu}^N} - G_s^{\nu}\right)^2 ds \right.\right.$$
$$\left.\left. + \beta\kappa \int_0^T \left(\left(G_s^{\hat{\mu}^N}\right)^2 - \left(G_s^{\nu}\right)^2\right) ds\right\}\right]^N dP^{\otimes N} \leq e^{C_\kappa(\delta)N} \quad . \tag{17}$$

Therefore, inequalities (10), (11), (12), (14) and (17) show that, for any real number $\alpha > 1$, we can choose p and δ close enough to one so that, for δ small enough, we find a finite real number $C_\alpha(\delta)$, $\lim_{\delta\to 0} C_\alpha(\delta) = 0$, such that:

$$\frac{1}{N}\log \int_{d(\hat{\mu}^N,\nu)<\delta} \exp\left\{\alpha N(\Gamma - \Gamma_\nu)\left(\hat{\mu}^N\right)\right\} \exp N\Gamma_\nu\left(\hat{\mu}^N\right) dP^{\otimes N} < \exp C_\alpha(\delta)N \quad .$$

□

We finally prove the weak large deviation upper bound Theorem 2.15.2):

Let K be a compact set of $\mathscr{M}_1^+(W_T^A)$, K can be covered by a finite union of open balls for the Wasserstein's metric:

$$K \subset \bigcup_{i=1}^{M} B(\nu_i, \delta)$$

Where

$$B(\nu_i, \delta) = \left\{\mu \in \mathscr{M}_1^+(W_T^A) \ / \ d(\mu, \nu_i) < \delta\right\}$$

According to (24), we have:

$$Q_\beta^N\left(\hat{\mu}^N \in K\right) = \int_K \exp\left\{N\Gamma\left(\hat{\mu}^N\right)\right\} dP^{\otimes N}$$
$$= \sum_{i=1}^{M} \int_{K\cap B(\nu_i,\delta)} \exp N\left\{\Gamma\left(\hat{\mu}^N\right) - \Gamma_{\nu_i}\left(\hat{\mu}^N\right)\right\} \exp\left\{N\Gamma_{\nu_i}\left(\hat{\mu}^N\right)\right\} dP^{\otimes N}$$

With the definition of the probability measures Q_ν as in (9), we get:

$$Q_\beta^N\left(\hat{\mu}^N \in K\right) = \sum_{i=1}^{M} \int_{K\cap B(\nu_i,\delta)} \exp N\left\{\Gamma\left(\hat{\mu}^N\right) - \Gamma_{\nu_i}\left(\hat{\mu}^N\right)\right\} dQ_{\nu_i}^{\otimes N}$$

Hölder inequality shows that, for p, q conjugate exponents:

$$Q_\beta^N\left(\hat{\mu}^N \in K\right) \leq \sum_{i=1}^{M} \left(\int_{d(\hat{\mu}^N,\nu_i)<\delta} \exp qN\left\{\Gamma\left(\hat{\mu}^N\right) - \Gamma_{\nu_i}\left(\hat{\mu}^N\right)\right\} dQ_{\nu_i}^{\otimes N}\right)^{\frac{1}{q}}$$
$$\times\, Q_{\nu_i}^{\otimes N}(B(\nu_i, \delta) \cap K)^{\frac{1}{p}}$$

So that proposition 2.16 implies that:

$$Q_\beta^N\left(\hat{\mu}^N \in K\right) \leq \exp\frac{1}{q} NC_q(\delta) \times \left\{\sum_{i=1}^{M} Q_{\nu_i}^{\otimes N}(B(\nu_i, \delta) \cap K)^{\frac{1}{p}}\right\}$$

Thus Sanov Theorem implies that:

$$\overline{\lim_{N\to\infty}} \frac{1}{N} \log Q_\beta^N\left(\hat{\mu}^N \in K\right) \leq -\frac{1}{p} \inf_{1\leq i\leq M} \inf_{\overline{B(v_i,\delta)}\cap K} I\left(\,|\, Q_{v_i}\right) + \frac{1}{q} C_q(\delta) \tag{18}$$

But one can see (as proved in [2], appendix B) that

$$I\left(\,|\, Q_{v_i}\right) = \begin{cases} I\left(\,|\, P\right) - \Gamma_{v_i} & \text{if } I\left(\,|\, P\right) < \infty \\ +\infty & \text{otherwise.} \end{cases}$$

We could prove (see the proof of lemma 2.16) that there exists a finite constant C such that

$$|\Gamma_{v_i}(\mu) - \Gamma(\mu)| \leq C\left(1 + I\left(\mu\middle|P\right)\right) d(v_i, \mu) \tag{19}$$

So that (18) implies:

$$\overline{\lim_{N\to\infty}} \frac{1}{N} \log Q_\beta^N\left(\hat{\mu}^N \in K\right) \leq -\frac{1}{p} \inf_K \left\{(1 - C\delta) I\left(\,|\, P\right) - \Gamma\right\} + \frac{1}{q} C_q(\delta) + C\delta$$

Finally, we proved in [2] that there exists $\alpha < 1$ and a finite constant C such that $\Gamma \leq \alpha I\left(\,|\, P\right) + C$ so that:

$$\lim_{\delta\searrow 0} \inf_K \left\{(1 - C\delta) I\left(\,|\, P\right) - \Gamma\right\} = \inf_K \left\{I\left(\,|\, P\right) - \Gamma\right\} = \inf_K H$$

So that, letting $\delta \searrow 0$, and then $p \searrow 1$, we get:

$$\overline{\lim} \frac{1}{N} \log Q_\beta^N\left(\hat{\mu}^N \in K\right) \leq -\inf_K H \ .$$

2.4 Annealed tightness

In this section, we prove that the law of the empirical measure under Q_β^N is tight, i.e Theorem 2.4:

Theorem 2.17 *For any real number $\epsilon > 0$, there exists a compact subset K_ϵ of $\mathcal{M}_1^+(W_T^A)$ such that, for any integer number N,*

$$Q_\beta^N\left(\hat{\mu}^N \in K_\epsilon^c\right) \leq \epsilon \ .$$

Proof. To prove Theorem 2.15, we shall compare the annealed law Q_β^N and the law of the system without interaction $P^{\otimes N}$. To this end, first recall that, by definition of the relative entropy, for any integer N, for any bounded measurable function f on $(W_T^A)^N$,

$$\int f \, dQ_\beta^N \leq I\left(Q_\beta^N | P^{\otimes N}\right) + \log \int \exp\{f\} \, dP^{\otimes N}$$

Letting A be a measurable subset of $(W_T^A)^N$ and taking
$f = \log\left(1 + P^{\otimes N}(A)^{-1}\right) \mathbb{I}_A$, one finds that

$$Q_\beta^N(A) \le \frac{\log 2 + I\left(Q_\beta^N|P^{\otimes N}\right)}{\log\left(1 + P^{\otimes N}(A)^{-1}\right)} \tag{20}$$

But the law of the empirical measure under $P^{\otimes N}$ is exponentially tight (see lemma 3.2.7 of [7]) so that, for any real number $\epsilon > 0$, we can find a compact subset K_ϵ of $\mathscr{M}_1^+(W_T^A)$ such that

$$P^{\otimes N}\left(\hat{\mu}^N \in K_\epsilon^c\right) \le \exp\left\{-\frac{N}{\epsilon}\right\} \tag{21}$$

(20) and (21) imply that, for any real number $\epsilon > 0$,

$$Q_\beta^N\left(\hat{\mu}^N \in K_\epsilon^c\right) \le \frac{\log 2 + I\left(Q_\beta^N|P^{\otimes N}\right)}{\log\left(1 + \exp\{\frac{N}{\epsilon}\}\right)} \tag{22}$$

Thus, (22) implies Theorem 2.17 as soon as we have proved that there exists a finite constant C such that, for any integer number N,

$$I\left(Q_\beta^N|P^{\otimes N}\right) \le CN \tag{23}$$

To compute $I(Q_\beta^N|P^{\otimes N})$, we recall that we proved in lemma 3.6 of [2] that Girsanov Theorem implies that Q_β^N is absolutely continuous with respect to $P^{\otimes N}$ and that its Radon-Nykodim derivative is given by:

$$\frac{dQ_\beta^N}{dP^{\otimes N}} = \prod_{i=1}^N \mathscr{E}\left[\exp\left\{\beta\int_0^T G_t^{\hat{\mu}^N} dB_t^i - \frac{\beta^2}{2}\int_0^T \left(G_t^{\hat{\mu}^N}\right)^2 dt\right\}\right] \tag{24}$$

Where $G_t^{\hat{\mu}^N} = (1/\sqrt{N})\sum_{i=1}^N J_i x_t^i$ and $\mathscr{E}$ denotes the expectation on the i.i.d $N(0,1)$ random variables J_i (Remark here that $G^{\hat{\mu}^N}$ depends on the (x^i), even if we do not underline it in the notations).

Thus, by definition of the relative entropy and of Q_β^N, we have:

$$\begin{aligned} I\left(Q_\beta^N|P^{\otimes N}\right) &= \int \log \frac{dQ_\beta^N}{dP^{\otimes N}}\, dQ_\beta^N \\ &= \int \sum_{i=1}^N \log \mathscr{E}\left[\exp\left\{\beta\int_0^T G_t^{\hat{\mu}^N}\, dB_t^i - \frac{\beta^2}{2}\int_0^T \left(G_t^{\hat{\mu}^N}\right)^2 dt\right\}\right] dQ_\beta^N \end{aligned}$$

Since Q_β^N is exchangeable, we find:

$$I\left(Q_\beta^N|P^{\otimes N}\right) = N\int \log \mathscr{E}\left[\exp\left\{\beta\int_0^T G_t^{\hat{\mu}^N}\, dB_t^1 - \frac{\beta^2}{2}\int_0^T \left(G_t^{\hat{\mu}^N}\right)^2 dt\right\}\right] dQ_\beta^N \tag{25}$$

We now give another formula for the right hand side of (25). Namely:

$$\mathscr{E}\left[\exp\left\{\beta\int_0^T G_t^{\hat\mu^N}\,dB_t^1 - \frac{\beta^2}{2}\int_0^T \left(G_t^{\hat\mu^N}\right)^2 dt\right\}\right]$$
$$= \exp\left\{\beta^2\int_0^T \mathscr{E}\left[\Lambda_t^N G_t^{\hat\mu^N} \times \int_0^t G_s^{\hat\mu^N} dB_s^1\right] dB_t^1 - \frac{\beta^4}{2}\int_0^T \mathscr{E}\left[\Lambda_t^N G_t^{\hat\mu^N}\int_0^t G_s^{\hat\mu^N} dB_s^1\right]^2 dt\right\}$$

where

$$\Lambda_t^N = \frac{\exp\left\{-\frac{\beta^2}{2}\int_0^t\left(G_u^{\hat\mu^N}\right)^2 du\right\}}{\mathscr{E}\left[\exp\left\{-\frac{\beta^2}{2}\int_0^t\left(G_u^{\hat\mu^N}\right)^2 du\right\}\right]} \ .$$

The equality (26) is due to standard Gaussian computations and integration by parts formula (see lemma 5.14 in [2] and the related result (2)).

Thus, (25) and (26) imply that:

$$I\left(Q_\beta^N | P^{\otimes N}\right) = N\int\left\{\beta^2\int_0^T \mathscr{E}\left[\Lambda_t^N G_t^{\hat\mu^N}\int_0^t G_s^{\hat\mu^N} dB_s^1\right] dB_t^1 - \frac{\beta^4}{2}\int_0^T \mathscr{E}\left[\Lambda_t^N G_t^{\hat\mu^N}\int_0^t G_s^{\hat\mu^N} dB_s^1\right]^2 dt\right\} dQ_\beta^N \tag{27}$$

Moreover, Girsanov Theorem implies that, under Q_β^N, B^1 is a semimartingale, more precisely that there exists a Q_β^N Brownian motion W^1 such that, for any time $t \le T$:

$$B_t^1 = W_t^1 + \beta^2\int_0^t \mathscr{E}\left[\Lambda_s^N G_s^{\hat\mu^N}\int_0^s G_u^{\hat\mu^N} dB_u^1\right] ds \ . \tag{28}$$

Thus, (27) becomes:

$$I\left(Q_\beta^N | P^{\otimes N}\right) = \frac{1}{2}\beta^4 N\int\int_0^T \mathscr{E}\left[\Lambda_t^N G_t^{\hat\mu^N}\int_0^t G_s^{\hat\mu^N} dB_s^1\right]^2 dt\, dQ_\beta^N \tag{29}$$

We now bound $f(t) = \int\mathscr{E}\left[\Lambda_t^N G_t^{\hat\mu^N}\int_0^t G_s^{\hat\mu^N} dB_s^1\right]^2 dQ_\beta^N$ through a Gronwall lemma argument. Using (28), one finds that:

$$f(t) = \int\mathscr{E}\left[\Lambda_t^N G_t^{\hat\mu^N}\int_0^t G_s^{\hat\mu^N} dB_s^1\right]^2 dQ_\beta^N$$
$$\le 2\int\mathscr{E}\left[\Lambda_t^N G_t^{\hat\mu^N}\int_0^t G_s^{\hat\mu^N} dW_s^1\right]^2 dQ_\beta^N + 2\beta^4\int\left(\int_0^t \mathscr{E}\left[\Lambda_t^N G_t^{\hat\mu^N} G_s^{\hat\mu^N}\right]\mathscr{E}\left[\Lambda_s^N G_s^{\hat\mu^N}\int_0^s G_u^{\hat\mu^N} dB_u^1\right] ds\right)^2 dQ_\beta^N \tag{30}$$

But Cauchy Schwartz inequality in the first term in the right hand side of (30) gives:

$$\int \mathscr{E}\left[\left(\Lambda_t^N G_t^{\hat{\mu}^N} \int_0^t G_s^{\hat{\mu}^N} dW_s^1\right)^2\right] dQ_\beta^N \leq \int \mathscr{E}\left[\left(\Lambda_t^N G_t^{\hat{\mu}^N}\right)^2\right] \mathscr{E}\left[\int_0^t G_s^{\hat{\mu}^N} dW_s^1\right]^2 dQ_\beta^N \tag{31}$$

Moreover, classical Gaussian properties imply (see appendix A of [2] for details), that, for any $x \in (W_T^A)^N$:

$$\mathscr{E}\left[\left(\Lambda_t^N G_t^{\hat{\mu}^N}\right)^2\right] \leq \mathscr{E}\left[\left(G_t^{\hat{\mu}^N}\right)^2\right] = \frac{1}{N}\sum_{i=1}^N (x_t^i)^2 \leq A^2 \tag{32}$$

So that (31) is bounded, for any $t \leq T$:

$$\begin{aligned}\int \mathscr{E}\left[\Lambda_t^N G_t^{\hat{\mu}^N} \int_0^t G_s^{\hat{\mu}^N} dW_s^1\right]^2 dQ_\beta^N &\leq A^2 \int \left[\int_0^t G_s^{\hat{\mu}^N} dW_s^1\right]^2 dQ_\beta^N \\ &= A^2 \int \mathscr{E}\left[\int_0^t \left(G_s^{\hat{\mu}^N}\right)^2 ds\right] dQ_\beta^N \leq A^4 T \end{aligned} \tag{33}$$

Similarly, we can bound the second term in the right hand side of (30) and finally get:

$$f(t) \leq 2A^4 T + 2\beta^4 A^4 T \int_0^t f(s)\, ds$$

Since this inequality holds for any $t \leq T$, Gronwall lemma gives:

$$\sup_{t\leq T} f(t) = \sup_{t\leq T} \int \mathscr{E}\left[\Lambda_t^N G_t^{\hat{\mu}^N} \int_0^t G_s^{\hat{\mu}^N} dB_s^1\right]^2 dQ_\beta^N \leq 2A^4 T \exp\{2\beta^4 A^4 T^2\}$$

Thus, (29) implies that:

$$I(Q_\beta^N | P^{\otimes N}) \leq \left(\beta^4 A^4 T^2 \exp\{2\beta^4 A^4 T^2\}\right) N \tag{34}$$

which is the bound (23) we needed to get Theorem 2.17. □

3 Symmetric dynamics

Here, we will not assume as in the previous section that the whole matrix $(J_{ij})_{i,j}$ is made of i.i.d $N(0,1)$ random variables but rather assume the symmetry of the couplings, i.e we will here suppose that the random matrix $(J_{ij})_{i,j}$ is symmetric. More precisely, we will suppose that under the diagonal, the $J_{i,j}$'s are i.i.d $N(0,1)$, that $J_{j,i} = J_{i,j}$ and that the $J_{i,i}$'s are $N(0,2)$. The fact that the value of the covariance of the couplings on the diagonal is so specific is not reflected by the asymptotic behaviour. Nevertheless, it is a nice choice on the technical point of view since it makes the law of the couplings invariant by rotations. We will denote $(\Omega, \mathscr{B}, \mathscr{E})$ the probability space on which these random variables live.

Under this symmetry hypotheses, our dynamics described by $\mathscr{S}_\beta^N(J)$ are reversible with respect to the Gibbs measure μ_J^N.

3.1 Annealed propagation of chaos

Our main result in this case is the convergence of the annealed law of the empirical measure towards solutions of a non linear martingale problem, or, equivalently, towards the solutions of a non linear stochastic differential equation. A large deviation upper bound for this averaged law in a large temperature (or short time) regime, which entails a propagation of chaos result, has been proved in [3].

The existence and uniqueness problems for this limit law are not obvious and are the analogue here of the existence and uniqueness problem for asymmetric spin glass dynamics as in the previous section. We already proved the uniqueness result for any temperature and time in [3], but on the set of probability measures with finite entropy with respect to the law without interaction. Here, we will give another proof which shows the uniqueness on a bigger subset of $\mathscr{M}_1^+(W_T^A)$ (see section 3.5).

As in the previous section, these results can be generalized to replicated systems. This generalization and its consequences on quenched properties will be mentioned in the last part of this paper.

Let us now describe more precisely the results and, in particular, the non linear stochastic differential equation which describes the limit behaviour of the empirical measure.

The symmetry of the couplings is reflected, in the limiting dynamics, through "covariance operators" $(\mathscr{C}_s)_{s\le T}$. For any probability measure $\mu \in \mathscr{M}_1^+(W_s^A)$, $\mathscr{C}_s$ is the operator in $L_\mu^2(W_s^A)\otimes L_\mu^2(W_s^A)$ such that

$$\mathscr{C}_s = \left(\mathbb{I} + \beta^2\mathscr{B}_s\otimes I + \beta^2 I\otimes\mathscr{B}_s\right)^{-1}$$

if I denotes the identity in $L_\mu^2(W_s^A)$, $\mathbb{I} = I\otimes I$ and $\mathscr{B}_s$ is the integral operator on $L_\mu^2(W_s^A)$ with kernel

$$b_s(x,y) = \int_0^s x_t y_t \, dt \quad .$$

Let us notice that $\mathscr{B}_s$ is a symmetric positive Hilbert-Schmidt in $L_\mu^2(W_s^A)$ (for any $\mu \in \mathscr{M}_1^+(W_s^A)$) operator so that $\mathscr{C}_s$ is always well defined and with spectral radius bounded by one.

Moreover, let us mention that the asymmetric analogue of $\mathscr{C}_s$ is

$$\mathscr{C}_s' = \left(I + \beta^2\mathscr{B}_s\right)^{-1}$$

which describes the annealed asymptotic law of the asymmetric dynamics according to (2). Indeed, the non linear drift depends mainly on:

$$\mathscr{E}\left[G_s^Q G_t^Q \frac{\exp\left\{-\frac{\beta^2}{2}\int_0^s \left(G_u^Q\right)^2 ds\right\}}{\mathscr{E}\left[\exp\left\{-\frac{\beta^2}{2}\int_0^s \left(G_u^Q\right)^2 ds\right\}\right]}\right] = \int x_s(\mathscr{C}_s'.X_t)(x)\, dQ(x) \ .$$

For the symmetric dynamics, the non linear drift is described by a more complicated function F: $\mathbb{R}^+ \times \mathscr{M}_1^+(W_T^A) \times W_T^A \to \mathbb{R}$ defined by:

$$F_t^\mu(x) = \int y_t(\mathscr{C}_t.a_t)(x,y)\, d\mu(y) \ ,$$

where:

$$a_t(x,y) = x_t y_t - x_0 y_0 + \int_0^t x_s \nabla U(y_s)\, ds + \int_0^t y_s \nabla U(x_s)\, ds$$

If $\int\left(\int_0^T |\nabla U(x_s)| ds\right)^2 d\mu(x)$ is finite, it is quite clear that $\{F_t^\mu(x), t \le T\}$ is well defined for any path x such that $\int_0^T |\nabla U(x_s)| ds$ is finite.

The limiting processes will then be characterized as weak solutions Q to the non linear stochastic differential equation on W_T^A:

$$\begin{cases} dx_t = -\nabla U(x_t)\, dt + dB_t + \beta^2 F_t^Q(x)\, dt \\ Law\ of\ \ x = Q \qquad Q\big|_{\mathscr{F}_0} = \mu_0 \end{cases}$$

such that $\int\left(\int_0^T |\nabla U(x_s)| ds\right)^2 dP(x)$ is finite. We recall that our dynamical system is starting from the product measure $\mu_0^{\otimes N}$ so that the initial law of the limiting process has to be μ_0.

We will denote $\mathscr{E}_{\mu_0}$ the set of solutions to this non linear stochastic differential equation.

Our main theorem states as follows:

Theorem 3.1 *Let* Π_β^N *be the annealed law of the empirical measure on* W_T^A, *i.e*:

$$\Pi_\beta^N(\mu \in .) = \mathscr{E}\left[P_\beta^N(J)\left[\frac{1}{N}\sum_{i=1}^N \delta_{x^i} \in .\right]\right] \ .$$

Then:

1) Π_β^N *is tight.*
2) *If* Π *is a limit of* Π_β^N, *then*:

$$\Pi\left(\mathscr{E}_{\mu_0}\right) = 1 \ .$$

3) $\mathscr{E}_{\mu_0}$ *contains a unique probability measure* Q.
4) $\Pi_{\beta,T}^N$ *converges to a dirac measure at* Q.

As a consequence, an annealed propagation of chaos phenomenon occurs:

Corollary 3.2 *For any continuous bounded functions* $(f_i, 1 \le i \le m)$,

$$\lim_{N\to\infty} \mathscr{E}\left[\int f_1(x^1)\dots f_m(x^m)\, dP_\beta^N(J)\right] = \prod_{i=1}^m \int f_i\, dQ$$

Let us now describe the main steps which lead to Theorem 3.1.

The proof of this theorem follows the usual scheme: we first identify conveniently the annealed law of the empirical measure in section 3.2. We then check that it is tight by an entropy argument in section 3.3. We finally turn to the identification of the limit; we first show that the limit probability measures are solutions of a non linear martingale problem, i.e by weak solutions of a non linear stochastic differential equation in section 3.4. We prove that there exists a unique solution to these non linear problems in section 3.5.

3.2 Study of the law of the empirical measure

We are going to see that the annealed law of the system is in fact the law of a system of diffusions in mean field interaction. The main peculiarity of this system, which is due to the randomness of the interaction, is that the drift coming from the interaction depends on the whole past trajectories of the diffusions. Indeed, we have:

Theorem 3.3 *The annealed law Q^N_β of the spin system is described by the following stochastic differential system*:

$$\begin{cases} dx^i_t = -\nabla U(x^i_t)\,dt + dB^i_t + \beta^2 F^{\hat\mu^N}_t(x^i)\,dt + \frac{1}{N}\beta^2 G^{\hat\mu^N}_t(x^i)\,dt \\ Law\ of\ x_0 = \mu_0^{\otimes N} \end{cases}$$

where $(B^i_t, t \geq 0, 1 \leq i \leq N)$ is a N dimensional Brownian motion and:

$$G^\mu_t(x) = -t\big(I + 2\beta^2\mathscr{B}_t\big)^{-1} X_t(x) \ .$$

G^μ depends on μ via the operator $\mathscr{B}_t$ in $L^2_\mu(W^A_T)$.

This theorem is proved by computing the Radon Nykodym derivative of Q^N_β with respect to $P^{\otimes N}_T$, i.e the analogue of lemma 3.6 in [2] for the symmetric case. A similar computation has been made in [3], Proposition 2.12. Nevertheless, since it is a key step in our approach, we wish to give some details. We will here state this result under the following form:

Lemma 3.4 *$P^{\otimes N}$ almost surely, we have*:

$$\frac{dQ^N_\beta}{dP^{\otimes N}_T} = \prod_{i=1}^N \exp\left(\beta^2\int_0^T H^{\hat\mu^N}_t(x^i,N)\,dB_t(x^i) - \frac{\beta^4}{2}\int_0^T H^{\hat\mu^N}_t(x^i,N))^2\,dt\right) \ ,$$

where $B_t(x^i) = x^i_t - x^i_0 + \int \nabla U(x_s)\ ds$ is a N dimensional Brownian motion under $P^{\otimes N}_T$ and:

$$H^{\hat\mu^N}_t(x^i,N) = F^{\hat\mu^N}_t(x^i) + \frac{1}{N}G^{\hat\mu^N}_t(x^i)$$

Theorem 3.3 can be deduced from Lemma 3.4 thanks to Girsanov Theorem. Indeed, $M^N_{\beta,t} = \frac{dQ^N_\beta}{dP_T^{\otimes N}}\big|_{\mathscr{F}_t}$ is a martingale since it is a supermartingale with mean 1 (since, for almost all $\mathbf{J}$, $\frac{dP^N_{\beta,T}(\mathbf{J})}{dP_T^{\otimes N}}\big|_{\mathscr{F}_t}$ is a martingale). Thus, Girsanov Theorem applies and, together with Lemma 3.4, shows Theorem 3.3.

Proof of Lemma 3.4. Since $(M^N_{\beta,t})_{t\geq 0}$ is a positive martingale (being the expectation of exponential martingales), it is enough (by uniqueness of the semimartingale decomposition) to show that the martingale part of $\log M^N_{\beta,t}$ is

$$\beta^2 \sum_{i=1}^N \int_0^T H_t^{\hat{\mu}^N}(x^i, N)\, dB_t(x^i) \ .$$

In the following, we will write that $A_t \underset{\text{mart}}{=} B_t$ when two semimartingales A and B have the same martingale parts.

Let us first recall that, if

$$\alpha_t\left(x^i, x^j\right) = \int_0^t x^i_s\, dB^j_s + \int_0^t x^j_s\, dB^i_s \ ,$$

we have shown in [3] that:

$$\log M^N_{\beta,t} \underset{\text{mart}}{=} \frac{\beta^2}{4} N\langle \alpha_t, (\mathscr{C}_t.\alpha_t)\rangle_{\hat{\mu}^N\otimes\hat{\mu}^N} \ . \tag{35}$$

thus, if we denote

$$Y_t(x,y) = (\mathscr{C}_t.\alpha_t)(x,y) \ ,$$

then we have:

$$\begin{aligned}
\log M^N_{\beta,t} &\underset{\text{mart}}{=} \frac{\beta^2}{4} N\langle \alpha_t, Y_t\rangle_{\hat{\mu}^N\otimes\hat{\mu}^N} \\
&= \frac{\beta^2}{4N}\sum_{i,j=1}^N \alpha_t\left(x^i, x^j\right) Y_t\left(x^i, x^j\right) \\
&\underset{\text{mart}}{=} \frac{\beta^2}{4N}\sum_{i,j=1}^N \int_0^t Y_s\left(x^i, x^j\right) d\alpha_s\left(x^i, x^j\right) \\
&\quad + \frac{\beta^2}{4N}\sum_{i,j=1}^N \int_0^t \alpha_s\left(x^i, x^j\right) dY_s\left(x^i, x^j\right)
\end{aligned} \tag{36}$$

Moreover, it is not hard to check that, for any functions f, g in $L^2(\hat{\mu}^N)\otimes L^2(\hat{\mu}^N)$,

$$\langle f, \mathscr{C}_t g\rangle_{\hat{\mu}^N\otimes\hat{\mu}^N} = \langle f, g\rangle_{\hat{\mu}^N\otimes\hat{\mu}^N} - \beta^2 \int_0^t \langle f, K_s g\rangle_{\hat{\mu}^N\otimes\hat{\mu}^N}\, ds \tag{37}$$

where

$$K_t = \mathscr{C}_t(I\otimes \mathscr{D}_t + \mathscr{D}_t\otimes I)\mathscr{C}_t$$

and D_t is the integral operator on $L^2(\hat{\mu}^N)$ with kernel $d_t(x,y) = x_t y_t$. Indeed, if one would consider a discretized version $\mathscr{C}_t^n$ of $\mathscr{C}_t$ given by:

$$\mathscr{C}_t^n := \left(\mathbb{I} + \frac{\beta^2}{n} \sum_{i=1}^{[nt]} I \otimes \mathscr{D}_{\frac{i}{n}} + \frac{\beta^2}{n} \sum_{i=1}^{[nt]} \mathscr{D}_{\frac{i}{n}} \otimes I \right)^{-1} ,$$

one would check that:

$$\mathscr{C}_t^n = I - \frac{\beta^2}{n} \sum_{j=1}^{[nt]} \mathscr{C}_{\frac{j}{n}}^n \left(I \otimes \mathscr{D}_{\frac{j}{n}} + \mathscr{D}_{\frac{j}{n}} \otimes I \right) \mathscr{C}_{\frac{j-1}{n}}^n$$

which is a discretized version of (37). By continuity of the trajectories, it is then not difficult to deduce (37).

Thus, $\mathscr{C}_t$ is a differentiable family of operators with respect to the time. Therefore:

$$dY_s\left(x^i, x^j\right) \underset{\text{mart}}{=} \left(\mathbb{I} + \beta^2 I \otimes \mathscr{B}_s + \beta^2 \mathscr{B}_s \otimes I\right)^{-1} d\alpha_s\left(x^i, x^j\right) .$$

As a consequence, we know that:

$$\begin{aligned} \log M_{\beta,t}^N &\underset{\text{mart}}{=} \frac{\beta^2}{2N} \sum_{i,j=1}^N \int_0^t Y_s\left(x^i, x^j\right) d\alpha_s\left(x^i, x^j\right) \\ &= \frac{\beta^2}{N} \sum_{i,j=1}^N \int_0^t Y_s\left(x^i, x^j\right) X_s(x^i)\, dB_s(x^j) \end{aligned} \tag{38}$$

Finally, it is not hard to check by Ito formula that:

$$\alpha_t(x^i, x^j) = a_t(x^i, x^j) - \delta_{i,j} t$$

where δ denotes the Kronecker symbol. As a consequence, we find that:

$$(38) = \beta^2 \sum_{i=1}^N \int_0^t H_s^{\hat{\mu}^N}(x^i, N) dB_s(x^i) \tag{39}$$

which ends the proof of the Lemma. □

3.3 Annealed tightness

In this section, we prove that the law of the empirical measure under Q_β^N is tight.

Theorem 3.5 *For any real number $\epsilon > 0$, there exists a compact subset K_ϵ of $\mathscr{M}_1^+(W_T^A)$ such that, for any integer number N,*

$$Q_\beta^N\left(\hat{\mu}^N \in K_\epsilon^c\right) \leq \epsilon$$

Proof. As for Theorem 2.17, we only need to bound:

$$\frac{1}{N} I(Q_\beta^N | P^{\otimes N}) = \frac{\beta^4}{2} \int_0^T \left(\iint H_t^{\hat{\mu}^N}(x,N)^2 \, d\hat{\mu}^N(x) \, dQ_\beta^N \right) dt \tag{40}$$

But, by definition of H^{μ}, for any time t

$$\begin{aligned}\iint H_t^{\hat{\mu}^N}(x,N)^2\,d\hat{\mu}^N(x)\,dQ_\beta^N &= \iint\left(\int y_t\mathscr{C}_t\alpha_t(x,y)\,d\hat{\mu}^N(y)\right)^2 d\hat{\mu}^N(x)\,dQ_\beta^N\\ &\le \iiint (y_t\mathscr{C}_t\alpha_t(x,y))^2\,d\hat{\mu}^N(y)\,d\hat{\mu}^N(x)\,dQ_\beta^N\\ &\le A^2\iiint (\mathscr{C}_t\alpha_t(x,y))^2 d\hat{\mu}^N(y)\,d\hat{\mu}^N(x)\,dQ_\beta^N\end{aligned}$$

where we have used Jensen inequality and bounded y_t by its supremum norm in the last line. Let us recall that $\mathscr{C}_t$ is a symmetric operator in $L^2(\hat{\mu}^N)\otimes L^2(\hat{\mu}^N)$ with eigenvalues smaller than one. Thus, we deduce that:

$$\iint H_t^{\hat{\mu}^N}(x,N)^2\,d\hat{\mu}^N(x)\,dQ_\beta^N \le A^2\iiint (\alpha_t(x,y))^2\,d\hat{\mu}^N(y)\,d\hat{\mu}^N(x)\,dQ_\beta^N \tag{41}$$

But α_t is defined by:

$$\alpha_t(x,y) = \int_0^t x_s\,dB_s(y) + \int_0^t y_s\,dB_s(x) \quad ,$$

if $B_t = x_t - x_0 + \int_0^t \nabla U(x_s)ds$. And, under Q_β^N, there exists a N Brownian motion W such that, for any $i\in[1,N]$:

$$B_t^i = W_t^i + \beta^2\int_0^t H_s^{\hat{\mu}^N}(x^i)\,ds$$

It is then a triviality to bound the left hand side of (41) and find, for any time $t\le T$:

$$\iint H_t^{\hat{\mu}^N}(x,N)^2\,d\hat{\mu}^N(x)\,dQ_\beta^N \le 2A^4T(1+\int_0^t\iint H_s^{\hat{\mu}^N}(x,N)^2\,d\hat{\mu}^N(x)\,dQ_\beta^N\,ds) \tag{42}$$

so that one can conclude that, for any time $t\le T$:

$$\iint H_t^{\hat{\mu}^N}(x,N)^2\,d\hat{\mu}^N(x)\,dQ_\beta^N \le 2A^4Te^{2A^4T} \quad ,$$

which finishes the proof of Theorem 3.5.

3.4 Identification of the limit probability measures

We are going to characterize any limit points of the law of the empirical measure as the solutions of a non linear martingale problem. To introduce this notion, we need to restrict ourselves to probability measures Q so that the function F_Q is well defined, namely to the set:

$$\mathscr{M} = \left\{P\in\mathscr{M}_1^+(W_T^A)\Big/\int\left(\int_0^T|\nabla U(x_s)|\,ds\right)^2 dP(x)<\infty\right\} .$$

For any probability measure Q in $\mathcal{M}$, let $\mathcal{L}_t(Q)$ be the generator defined by:

$$\mathcal{L}_t(f,Q)(x) = \frac{1}{2}f''(x) + \left(-\nabla U(x) + \beta^2 F_t^Q(x)\right)f'(x) \ .$$

We will then say that Q is solution of the non linear martingale problem $(\mathcal{L}_t, t \geq 0)$ with initial condition μ_0 if:

$$Q(X_0 \in B) = \mu_0(B) \qquad \forall\, B \in \mathcal{B}([-A,A]) \tag{43}$$

$$\text{and} \qquad \forall\, f \in \mathcal{C}^2([-A,A]) \quad f(X_t) - f(X_0) - \int_0^t \mathcal{L}_s(f,Q)(X)\,ds \tag{44}$$

is a Q-martingale.

Let us denote $\mathcal{E}_{\mu_0}$ the set of such solutions. Then, recalling that $\mu_0^{\otimes N}$ is the initial law of our dynamical system, we are going to prove that:

Theorem 3.6 *If* Π *is a limit point of* $(\Pi_{\beta,T}^N)_{N\geq 0}$,

$$\Pi(\mathcal{E}_{\mu_0}) = 1 \ .$$

Moreover, there exists a finite constant C such that, if Π *is a limit point of* $(\Pi_{\beta,T}^N)_{N\geq 0}$,

$$\int \left(\int \left(\int_0^T |\nabla U(x_s)|\,ds \right)^2 d\mu(x) \right) d\Pi(\mu) \leq C \ .$$

Since we have proved in Theorem 3.5 that the law of the empirical measure is tight, Theorem 3.6 shows that the law of the empirical measure concentrates on $\mathcal{E}_{\mu_0}$. To recover a result of the same flavor that those we got for the asymmetric dynamics, we translate this theorem in terms of non linear differential stochastic equations. Theorem 3.6 then states as follows:

Theorem 3.7 *The limit law of the empirical measure is supported by the weak solutions of the following non linear stochastic differential equation*:

$$\begin{cases} dx_t = -\nabla U(x_t)\,dt + dB_t + \beta^2 F_t^Q(x)\,dt \\ \text{Law of } x = Q \qquad Q\Big|_{\mathcal{F}_0} = \mu_0 \end{cases}$$

such that

$$\int \left(\int_0^T |\nabla U(x_s)|\,ds \right)^2 dQ(x) < \infty \ .$$

Proof of Theorem 3.6. Let us first notice that, since the spins are initially independent and with law μ, Cramer Theorem implies that any limit point Π of $\left(\Pi_{\beta,T}^N\right)_{N\geq 0}$ satisfies:

$$\Pi\left(\left\{\mu \in \mathcal{M}_1^+\left(W_T^A\right), \quad X_0 \circ \mu = \mu_0\right\}\right) = 1 \ ,$$

so that the problem boils down to show that the second point in (44) is satisfied for Π almost all measures on W_T^A. In other words, if we define, for any function f twice continuously differentiable on $[-A, +A]$ and any bounded continuous function ϕ $\sigma(X_u, u \le s)$ measurable, a function ψ: $\mathscr{M}_1^+\left(W_T^A\right) \to \mathbb{R}$ by:

$$\psi(\mu) = \int \left(f(X_t) - f(X_s) - \int_0^t \mathscr{L}_u(f,\mu)(X)\, du \right) \phi(X)\, d\mu(X) \ ,$$

we need to check that $\psi(\mu) = 0$ for Π-almost all probability μ. Since W_T^A is a metric space, it is not hard to see that this is enough to conclude that, for Π almost all μ, μ satisfies the non linear martingale problem $(\mathscr{L}_t, t \ge 0)$. Indeed, μ satisfies this martingale problem iff $\psi = \psi(t,s,f,\phi)$ is null for a countable number of times, a countable number of test functions ϕ and two functions f $\left(f(x) = x \text{ and } x^2\right)$.

This approach is very similar to that used by A.S. Sznitman in [11] and C. Leonard in [8] for diffusions in mean field interaction. We nevertheless have to be slightly more careful here since the non linear part of our generator depends on the potential U which is not bounded, and is therefore not a continuous function on $\mathscr{M}_1^+\left(W_T^A\right)$.

As in [8], we notice that, under Q_β^N, Ito's formula shows that, for given smooth functions f and ϕ:

$$\psi\left(\hat{\mu}^N\right) = \frac{1}{N}\sum_{i=1}^N \phi\left(x^i\right) \int_s^t f'\left(x_s^i\right)\left(dB_s^i + \frac{\beta^2}{N} G_s^{\hat{\mu}^N}\left(x_s^i\right) ds\right) \tag{45}$$

where B is a Q_β^N Brownian motion. But, let us first remark that a consequence of Lemma 3.22 in [3] is that there exists a finite constant such that:

$$\sup_{\mu \in \mathscr{M}_1^+\left(W_T^A\right)} \sup_{x \in W_T^A} \sup_{t \le T} |G_t^\mu(x)| \le CTA \ .$$

Thus, we find a finite constant $C = C(A, \beta, T)$ so that:

$$\int \left(\psi\left(\hat{\mu}^N\right)\right)^2 dQ_\beta^N(x) \le \frac{1}{N} C \|\phi\|_\infty^2 \|f'\|_\infty^2 \tag{46}$$

where we have assumed in the last line that t and s are smaller than T. If ψ were a bounded continuous function, (46) would imply that any limit point Π of $\Pi_{\beta,T}^N$ would satisfy $\int \psi(\mu)^2 d\Pi(\mu) = 0$. To circumvent the fact that ψ is nor bounded nor continuous, we follow the usual approximation scheme.

To this end, let us consider, for any integer M, a smooth function ρ_M on $\mathbb{R}^+$ such that:

$$\|\rho_M\|_\infty \le 1, \quad \rho_M(x) = \ 1 \ \text{if } x \le M, \quad \rho_M(x) = 0 \ \text{if } \ x \ge 2M \ .$$

We then let, for $t \le T$, $V_t^M\left(X_{[0,T]}\right) = \rho_M\left(\int_0^T |\nabla U(X_t)|\, dt\right) \nabla U(X_t)$. V_M is bounded and continuous for any finite integer number M. Let us then define:

$$a_T^M(x,y) = \left(x_T y_T - x_0 y_0 + \int_0^T x_s V_s^M(y)\, ds + \int_0^T y_s V_s^M(x)\, ds \right) ,$$

and:

$$F_t^M(Q,x) = \int y_s \mathscr{C}_s a_s^M(x,y)\, dQ(y) \ .$$

It is not difficult to check that, since the canonical process is uniformly bounded, $F_t^M : \mathscr{M}_1^+(W_T^A) \times W_T^A \to \mathbb{R}$ is a bounded continuous function.

We finally let $\mathscr{L}_t^M(Q)$ be the operator on smooth bounded functions such that

$$\mathscr{L}_t^M(Q,f)(X_{[0,T]}) = \tfrac{1}{2} f''(X_t) + \left(-V_t^M(X_{[0,T]}) + \beta^2 F_t^M(Q, X_{[0,T]})\right) f'(X_t)$$

and define accordingly:

$$\psi^M(\mu) = \int \left(f(X_t) - f(X_s) - \int_s^t \mathscr{L}_u^M(\mu,f)(X_{[0,T]})\, du \right) \phi(X)\, d\mu(X) \ .$$

It is clear that ψ^M is a bounded continuous function on $\mathscr{M}_1^+(W_T^A)$. Using (46), we are going to see that there exists a sequence ϵ_M going to zero when M goes to infinity and such that, for any limit point Π,

$$\int |\psi^M(\mu)|\, d\Pi(\mu) \le \epsilon_M \ . \tag{47}$$

We will then show that, for any such probability measure Π, we as well have that

$$\int |\psi(\mu)|\, d\Pi(\mu) \le \underline{\lim}_{M\to\infty} \int |\psi^M(\mu)|\, d\Pi(\mu) \ , \tag{48}$$

which implies that

$$\int |\psi(\mu)|\, d\Pi(\mu) = 0 \ , \tag{49}$$

and thus achieves the proof.

To prove (47), let us first remark that (46) implies that:

$$\begin{aligned}
\int |\psi^M(\mu)|\, d\Pi(\mu) &\le \overline{\lim}_{N\to\infty} \int |\psi^M(\mu)|\, d\Pi_{\beta,T}^N(\mu) \\
&\le \overline{\lim}_{N\to\infty} \sqrt{\int |\psi(\mu)|^2\, d\Pi_{\beta,T}^N(\mu)} \\
&\quad + \overline{\lim}_{N\to\infty} \int |\psi^M(\mu) - \psi(\mu)|\, d\Pi_{\beta,T}^N(\mu) \\
&= \overline{\lim}_{N\to\infty} \int |\psi^M(\mu) - \psi(\mu)|\, d\Pi_{\beta,T}^N(\mu) \ ,
\end{aligned}$$

so that the problem boils down to prove that there exists a sequence of real numbers ϵ_M going to zero when M goes to infinity such that:

$$\overline{\lim}_{N\to\infty} \int |\psi^M(\mu) - \psi(\mu)|\, d\Pi_{\beta,T}^N(\mu) \le \epsilon_M \ . \tag{50}$$

To prove this last inequality, let us first notice that there exists a finite constant C_T such that

$$\left|\psi^M(\mu) - \psi(\mu)\right| \leq C_T\|\phi\|_\infty\|f'\|_\infty\sqrt{\int\left(\int_0^T \left|\nabla U(x_s) - V_s^M\left(X_{[0,T]}\right)\right| ds\right)^2 d\mu(x)} \ . \quad (51)$$

But $\left|\nabla U(x_s) - V_s^M\left(X_{[0,T]}\right)\right| = |\nabla U(x_s)|\left(1 - \rho_M\left(\int_0^T |\nabla U(x_u)|\, du\right)\right)$, so that Cauchy Schwartz inequality implies that, for any positive integer number δ, we have:

$$\int\left|\psi^M(\mu) - \psi(\mu)\right| d\Pi_{\beta,T}^N(\mu)$$

$$\leq C_T\|\phi\|_\infty\|f'\|_\infty\delta\iint\left(\int_0^T |\nabla U(x_s)|\, ds\right)^2 d\mu(x)\, d\Pi_{\beta,T}^N(\mu)$$

$$+ C_T\|\phi\|_\infty\|f'\|_\infty\frac{1}{\delta}\iint\left(1 - \rho_M\left(\int_0^T |\nabla U(x_s)|\, ds\right)\right)^2 d\mu(x)\, d\Pi_{\beta,T}^N(\mu) \quad (52)$$

To bound the right hand side of (52) we use the entropy inequality (and monotone convergence Theorem). For the first term in the right hand side of (52), we have, for any $\alpha > 0$

$$\alpha N\int\int\left(\int_0^T |\nabla U(x_s)|\, ds\right)^2 d\mu(x)\, d\Pi_{\beta,T}^N(\mu)$$

$$\leq I\left(Q_\beta^N|P^{\otimes N}\right) + N\log\int\exp\left\{\alpha\left(\int_0^T |\nabla U(x_s)|\, ds\right)^2\right\} dP(x) \ . \quad (53)$$

But, one can check that, for α small enough, $\int\exp\left\{\alpha\left(\int_0^T |\nabla U(x_s)|\, ds\right)^2\right\} dP(x)$ is finite. Thus, since we have seen in the proof of Theorem 3.5 that $I\left(Q_\beta^N|P^{\otimes N}\right)$ grows at most linearly in N, (53) shows that

$$\int\left(\int_0^T |\nabla U(x_s)|\, ds\right)^2 d\mu(x)\, d\Pi_{\beta,T}^N(\mu) \quad (54)$$

is bounded uniformly on N. Similarly, we have that, for any positive real number α:

$$\alpha N\iint\left(1 - \rho_M\left(\int_0^T |\nabla U(X_s)|\, ds\right)\right)^2 d\mu(x)\, d\Pi_{\beta,T}^N(\mu)$$

$$\leq I\left(Q_\beta^N|P^{\otimes N}\right) + N\int\exp\left\{\alpha\left(1 - \rho_M\left(\int_0^T |\nabla U(X_s)|\, ds\right)\right)^2\right\} dP(x)$$

$$\leq I\left(Q_\beta^N|P^{\otimes N}\right) + N\left(1 + e^\alpha P\left(\int_0^T |\nabla U(X_s)|\, ds > M\right)\right) \quad (55)$$

As a consequence, since $P(\int_0^T |\nabla U(X_s)|\, ds > M)$ goes to zero when M goes to infinity, there exists a finite constant c_T and a sequence of real numbers $(\alpha_M)_{M\geq 0}$ going to infinity when M does and so that, for any integer M:

$$\alpha_M \iint \left(1 - \rho_M\left(\int_0^T |\nabla U(X_s)|\, ds\right)\right)^2 d\mu(x)\, d\Pi^N_{\beta,T}(\mu) \leq c_T \qquad (56)$$

Hence, (52), (54) and (56) imply (taking $\delta = (1/\sqrt{\alpha_M})$ in (52)), that there exists a sequence of real numbers $(\epsilon_M)_{M\geq 0}$ going to zero when M goes infinity and so that, for any integer number N:

$$\int |\psi^M(\mu) - \psi(\mu)|\, d\Pi^N_{\beta,T}(\mu) \ \leq \ \|\phi\|_\infty \|f'\|_\infty \epsilon_M$$

which finishes the proof of (50).

To achieve the proof of (49), we notice that, since $\psi^M(\mu)$ converges pointwise to $\psi(\mu)$ on $\mathscr{M}$, which can be seen to have probability one under Π according to (54), Fatou's Lemma implies that:

$$\int |\psi(\mu)|\, d\Pi(\mu) \leq \varliminf_{M\to\infty} \int |\psi^M(\mu)|\, d\Pi(\mu) \ .$$

The right hand side is null according to (47), so that we have proved Theorem 3.6. □

3.5 Uniqueness of the limit point and characterization

Our final goal is to prove the uniqueness of the solutions Q to the non linear stochastic differential system defined in Theorem 3.7 on $\mathscr{M}$.

Thus, since Theorem 3.7 and Theorem 3.5 implies that there exists at least such a solution Q, we can deduce that there is a unique solution to this problem. Then, Theorem 3.7 implies Theorem 3.1.

Let us recall that we already proved in [3] that there is at most one solution to the non linear stochastic differential system defined in Theorem 3.7 with finite entropy with respect to P. In our setting, it is not clear that, for any limit point Π, for Π almost all μ, $I(\mu|P)$ is finite (even if we know that $I(\int \mu\, d\Pi(\mu)|P)$ is finite). We could adapt the argument given in [2] to the probability measures such that (57) is satisfied. Nevertheless, we prefer to give a new argument which is simpler but only apply to the case where U can be written as the sum of a convex and a Lipschitz function. Since we already assumed that U is smooth, this assumption reduces to the fact that U blows up near the boundary points $\{-A\}$ and $\{A\}$ like a convex function. Let us first state the result:

Theorem 3.8 *If U is the sum of a convex and a Lipschitz function, for any finite time T there exists a unique solution Q to the non linear equation*:

$$\begin{cases} dX_t = -\nabla U(X_t)dt + dB_t + \beta^2 F_t^Q(X)\,dt & t \le T \\ Law\ of\ X = Q & Q\Big|_{\mathscr{F}_0} = \mu_0 \end{cases}$$

Proof. We will assume that U is convex to simplify the notations. As usual, we assume that we have two weak solutions Q_1 and Q_2 to this non linear stochastic differential equation and want to show that they have to be equal. We can construct these two solutions on the same probability space. Namely, if we consider the law of the coupled processes (X^1, X^2) defined by:

$$\begin{cases} dX_t^1 = -\nabla U(X_t^1)\,dt + dB_t + \beta^2 F_t^{Q_1}(X^1)\,dt \\ dX_t^2 = -\nabla U(X_t^2)\,dt + dB_t + \beta^2 F_t^{Q_2}(X^2)\,dt \\ Law\ of\ X^1 = Q_1 \quad Law\ of\ X^2 = Q_2 \quad X_0^1 = X_0^2 \quad Q_1\Big|_{\mathscr{F}_0} = \mu_0 \end{cases}$$

it is clear that the law of X^1 (resp. of X^2) is Q_1(resp. Q_2).

We then want to prove that

$$E\left[\sup_{t\le T}|X_t^1 - X_t^2|\right]$$

is null, which will imply that the Wasserstein distance between Q_1 and Q_2 is null and thus that $Q_1 = Q_2$.

Since ∇U is not Lipschitz, the strategy will be based on a contraction argument for the function:

$$H_s = E\left[\sup_{t\le s}|X_t^1 - X_t^2|\right] + E\left[\int_0^s |\nabla U(X_t^1) - \nabla U(X_t^2)|\,dt\right]$$

rather than on

$$E\left[\sup_{t\le s}|X_t^1 - X_t^2|\right]$$

only. We will show indeed that there exists a finite constant C such that:

$$H_s \le C\int_0^s H_u\,du\ , \tag{57}$$

which, according to Gronwall Lemma, guaranties that $H \equiv 0$, and thus gives the result.

Let us now go into the details of the proof. Its first step is to express H_s. To this end, let us first notice that the process $Y_t = X_t^1 - X_t^2$ satisfies the differential equation:

$$\begin{cases} dY_t = -(\nabla U(X_t^1) - \nabla U(X_t^2))\,dt + \beta^2\left(F_t^{Q_1}(X^1) - F_t^{Q_2}(X^2)\right)dt \\ Y_0 = 0 \end{cases}$$

Thus, $(Y_t, t \ge 0)$ is a process with finite variations and Ito formula gives:

$$\sup_{u\le s}|Y_u| + \int_0^s \mathrm{sgn}(Y_u)\big(\nabla U\big(X_t^1\big) - \nabla U\big(X_t^2\big)\big)\,dt$$
$$\le \beta^2 \int_0^s \left|F_t^{Q_1}(X^1) - F_t^{Q_2}(X^2)\right| dt \ ,$$

where $\mathrm{sgn}(0) = 0$. But, since we have assumed that U is convex, $\nabla U\big(X_t^1\big) - \nabla U\big(X_t^2\big)$ has the same sign as Y_t so that we have indeed:

$$\sup_{u\le s}|Y_u| + \int_0^s \left|\nabla U\big(X_t^1\big) - \nabla U\big(X_t^2\big)\right| dt \le \beta^2 \int_0^s \left|F_t^{Q_1}(X^1) - F_t^{Q_2}(X^2)\right| dt \tag{58}$$

Let us now focus on the right hand side of (58). We proved in [3] (see Lemma 3.21) that:

Lemma 3.9 *For any time T, there exists a finite constant A_T such that, for any paths x and y, for any probability measures μ and ν, for any time $t \le T$*

$$\begin{aligned}\left|F_t^{\mu}(x) - F_t^{\nu}(y)\right| \ &\le A_T \int_0^t \left(\langle|\nabla U(X_u)|\rangle_\mu + \langle|\nabla U(X_u)|\rangle_\nu\right) du \\ &\quad \times \left(\sup_{u\le t}|x_u - y_u| + d_t(\mu,\nu)\right) \\ &\quad + A_T\left(K_t(\mu,\nu) + \int_0^t |\nabla U(x_s) - \nabla U(y_s)|\,ds\right)\end{aligned}$$

where d_t is the Wasserstein distance between μ and ν and

$$K_T(\mu,\nu) = \inf\left\{\iint_0^T |\nabla U(X_u) - \nabla U(Y_u)|\,du\,d\xi(X,Y)\right\}$$

where the infimum is taken over the probability measures ξ with marginales ν and μ.

This Lemma is enough to get the contraction argument. Indeed, we obviously have the bounds:

$$d_T(Q_1,Q_2) \le E\left[\sup_{u\le T}\left|X_u^1 - X_u^2\right|\right]$$

and:

$$K_T(\mu,\nu) \le E\left[\int_0^T \left|\nabla U\big(X_u^1\big) - \nabla U\big(X_u^2\big)\right| du\right] \ .$$

Thus, taking the expectation on both sides of (58) and using Lemma 3.9 we get:

$$
\begin{aligned}
H_s \leq\ & \beta^2 \int_0^s E\left[\left|F_t^{Q_1}(X^1) - F_t^{Q_2}(X^2)\right|\right] dt \\
&\leq 2\beta^2 A_T \left(\int_0^T \left(\langle |\nabla U(X_u)| \rangle_{Q_1} + \langle |\nabla U(X_u)| \rangle_{Q_2} \right) du \right) \\
&\quad \times \int_0^s \quad E\left[\sup_{u \leq t} |X_u^1 - X_u^2| \right] dt \ + 2\beta^2 A_T \\
&\quad + \int_0^s E\left[\int_0^t |\nabla U(X_u^1) - \nabla U(X_u^2)| \, du \right] dt
\end{aligned}
$$

Thus, since we assumed that $\int_0^T < |\nabla U(X_u)| >_{Q_1} du$ and $\int_0^T \langle |\nabla U(X_u)| \rangle_{Q_2} du$ are finite, we find a finite constant B_T such that, for any $s \leq T$:

$$
\begin{aligned}
H_s &\leq B_T \int_0^s \left(E\left[\sup_{u \leq t} |X_u^1 - X_u^2| \right] + E\left[\int_0^t |\nabla U(X_u^1) - \nabla U(X_u^2)| \, du \right] \right) dt \\
&= B_T \int_0^s H_t \, dt \ ,
\end{aligned}
$$

that is (57). This finishes the proof of Theorem 3.8.

3.6 Quenched propagation of chaos

It is not hard to generalize our strategy to replicated systems as we did in section 2.2 for the asymmetric model. Let us summarize the analogue of Theorem 2.7 in the symmetric case:

Theorem 3.10 *Let Q_r be the unique solution of the non linear stochastic differential equation given by:*

$$
\begin{cases}
dx_t^i = -\nabla U(x_t^i) \, dt + dB_t^i + \beta^2 F_t^i(Q_r, x) \, dt \qquad 1 \leq i \leq r \\
Law \ of \ x = P \qquad P\big|_{\mathscr{F}_0} = \mu_0
\end{cases}
$$

such that $\int \left(\int_0^T \sum_{i=1}^r |\nabla U(x_s^i)| \, ds \right)^2 dQ_r(x)$ *is finite.* $F_t^i(Q_r, x)$ *is defined by:*

$$
F_t^i(Q_r, x) = \int y_t^i \left(\mathbb{I} + \beta^2 \mathscr{B}_t^r \otimes I + \beta^2 I \otimes \mathscr{B}_t^r \right)^{-1} a_t^r(x, y) \, dQ_r(y) \ ,
$$

where

$$
a_t^r(x, y) = \sum_{i=1}^r a_t(x^i, y^i) \ ,
$$

and $\mathscr{B}_t^r$ *is an integral operator in* $L^2(Q_r)$ *with kernel* b_t^r *given by:*

$$
b_t^r(x, y) = \sum_{i=1}^r b_t(x^i, y^i) \ .
$$

Then, for any bounded continuous functions $(f_1, \ldots, f_m)$ on W_T^A, *for any integer number* r,

$$\lim_{N\to\infty} \mathscr{E}\left[\left(\int f_1(x^1)\dots f_m(x^m)\, dP^N_\beta(J)(x)\right)^r\right] = \prod_{i=1}^m \int f_i(x_1)\dots f_i(x_r)\, dQ_r$$

We were not able to decouple the probability measures Q_r as in Theorem 2.8 in the general case. Nevertheless, we proved in [4] that:

Theorem 3.11 *If U is even and μ_0 symmetric,*

$$\int x_t \bar{x}_s\; dQ_2(x,\bar{x}) = 0 \qquad \forall\ (s,t)\ .$$

As a consequence,

$$Q_r = (Q_1)^{\otimes r}\ .$$

In particular, for any bounded continuous functions $(f_1,\dots,f_m)$ on W^A_T, for almost all J, $\int f_1(x^1)\dots f_m(x^m)\, dP^N_\beta(J)(x)$ converges in law to $\prod_{i=1}^m \int f_i\, dQ$.

Thus, the overlap goes to zero when the number of particles goes to infinity even at very low temperature. This is surprising at the first glance since the overlap is supposed to exhibit a very interesting behaviour at low temperature. Nevertheless, we understood that it was expected by the Physicists community. Let us finally compare in this case the limit process Q to the limit process obtained for asymmetric dynamics.

Indeed, we can see that Q can be written as:

$$Q = \mathscr{E}\left[P^Q_g\right]$$

where P_g is described as the weak solution of:

$$\begin{cases} dx_t = dB_t - \nabla U(x_t)\, dt + \beta g_t\, dt + \beta^2 \int_0^t x_s R_Q(t,s)\, ds\, dt \\ Law\ of\ x_0 = \mu_0\ . \end{cases}$$

and g is a centered Gaussian process with covariance:

$$\mathscr{E}[g_s g_t] = \int x_s x_t\, dQ(x)\ .$$

R_Q, which was null for asymmetric dynamics, is described here by the following formula:

$$\begin{aligned} R_Q(t,s) = {} & \frac{\partial}{\partial s}\langle (I+\beta^2\mathscr{B}_t)^{-1} X_t, B_s\rangle_Q \\ & - \beta^2 \int\!\!\int y_t z_s \left(\mathscr{C}_t (I+\beta^2\mathscr{B}_t)^{-1}.a_t\right)(y,z)\, dQ(y)\, dQ(z)\ . \end{aligned} \tag{59}$$

The proof of this result is given in [4].

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