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Probability Theory and Related Fields

Superdiffusivity in first-passage percolation

C. Licea[1,⋆], **C.M. Newman**[2,⋆⋆], **M.S.T. Piza**[3,⋆⋆⋆]

[1] Courant Institute of Mathematical Sciences, New York University
[2] Courant Institute of Mathematical Sciences, New York University, 251 Mercer St., New York, NY 10012, USA (e-mail: newman@cims.nyu.edu)
[3] Department of Mathematics, University of California, Irvine, CA 92717, USA

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Summary. In standard first-passage percolation on $\mathbb{Z}^d$ (with $d \geq 2$), the time-minimizing paths from a point to a plane at distance L are expected to have transverse fluctuations of order L^ξ. It has been conjectured that $\xi(d) \geq 1/2$ with the inequality strict (superdiffusivity) at least for low d and with $\xi(2) = 2/3$. We prove (versions of) $\xi(d) \geq 1/2$ for all d and $\xi(2) \geq 3/5$.

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1. Introduction

Throughout this paper, we consider standard first-passage percolation on $\mathbb{Z}^d$ (with $d \geq 2$)[HW]. This is a model determined by i.i.d. non-negative random variables $\tau(e)$, indexed by $e \in \mathbb{E}^d$, the set of nearest neighbor edges $e = \{u, v\}$ with $u, v \in \mathbb{Z}^d$ separated by Euclidean distance $\|u - v\| = 1$. One defines the passage time for a finite nearest neighbor path r as

$$T(r) = \sum_{e \in r} \tau(e) \tag{1.1}$$

and the passage time between two sites $u, v \in \mathbb{Z}^d$ as

$$T(u, v) = \inf\{T(r) : r \text{ is a path from } u \text{ to } v\}. \tag{1.2}$$

The passage time between a site u and a subset Γ of $\mathbb{Z}^d$(e.g., between $u = 0$ and $\Gamma = \{(v_1, ..., v_d) : v_1 = n\}$) is

$$T(u, \Gamma) = \inf\{T(u, v) : v \in \Gamma\}. \tag{1.3}$$

⋆ *Present address:* Standard and Poor's, 25 Broadway, New York, NY 10014–1064, USA
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First-passage percolation is often regarded as a stochastic growth model by considering the growing random subset of $\mathbb{Z}^d$,

$$\widetilde{B}(t) = \{v : T(0, v) \leq t\}. \tag{1.4}$$

When $P(\tau(e) = 0) = 0$, $T(u, v)$ is a (random) metric on $\mathbb{Z}^d$ and $\widetilde{B}(t)$ is the ball of radius t centered at the origin. As long as $P(\tau(e) = 0) < p_c$, the critical value for standard independent undirected bond percolation on $\mathbb{Z}^d$, [K1] and $E(\tau(e)^2) < \infty$ (or under weaker moment conditions [CD]), the leading order large t-growth of $\widetilde{B}(t)$ is linear with a deterministic shape. (For a precise statement of the shape theorem, see [K1] or [K2].)

Fluctuations of $\widetilde{B}(t)$ are described in the physics literature (see, e.g., [KS]) by means of two exponents, χ and ξ, which describe respectively the longitudinal and transverse fluctuations of the growing surface of $\widetilde{B}(t)$. For example, it is expected that the time $T(0, \Gamma)$ when $\widetilde{B}(t)$ first reaches a plane Γ at distance L from the origin has a standard deviation of order L^χ while the place(s) on Γ first reached are contained (with high probability) within a deterministic subset of Γ whose diameter is of order L^ξ. The main results of this paper are lower bounds on the exponent ξ. We remark that heuristic arguments suggest a close relationship between lower bounds on ξ and nonexistence of (doubly infinite) geodesics (for the metric $T(u, v)$); some results on nonexistence (not based on bounds for ξ) appear in [LN].

There are, a priori, many possible mathematical definitions of the exponents χ and ξ, some based on point-to-plane and some based on point-to-point passage times. One of the open foundational problems of the subject, which is not attacked in this paper, is to prove that these various definitions all yield the same exponents. Rather, we will treat several different definitions of ξ (primarily of point-to-plane type) and obtain various lower bounds for these definitions.

The exponents χ and ξ are not expected to depend on the common distribution of the $\tau(e)$'s (nor on the direction of Γ from the origin), at least under a certain hypothesis on the common distribution. This hypothesis (which is discussed at greater length in [NP]) concerns the probability assigned to λ, the bottom of the support of the common distribution. The hypothesis, which will be assumed in two of our main results (see Theorems 1 and 3 below), is that one of the following two conditions be satisfied: either

$$\lambda = 0 \text{ and } P(\tau(e) = 0) < p_c \tag{1.5}$$

or else

$$\lambda > 0 \text{ and } P(\tau(e) = \lambda) < p_c^{dir}, \tag{1.6}$$

where p_c^{dir} is the critical value for independent *directed* bond percolation on $\mathbb{Z}^d$ (i.e., the model in which the only paths allowed are those along which every coordinate is nondecreasing). We remark that in [BK], distributions satisfying this hypothesis are said to be *useful*.

The exponents χ and ξ, however, *are* expected to depend on d, but nevertheless satisfy for all d the scaling identity $\chi = 2\xi - 1$ (see [KS]). The predicted values (for models whose exponents should have the same values as in first-passage percolation) for $d = 2$ are $\chi = 1/3$ and $\xi = 2/3$ [HH, K, HHF, KPZ]. There have been conflicting predictions about the qualitative nature of χ and ξ for higher dimensions ranging from lack of dependence on d [KZ] through their decreasing with d while $\chi > 0$ and $\xi > 1/2$ for all d [WK, KK] to the possibility that above some critical dimension, $\chi = 0$ and $\xi = 1/2$ [NR, H, CD]. Thus it is of interest to obtain rigorous bounds on (various definitions of) the exponents which go beyond the trivial bounds (assuming $E(\tau(e)^2) < \infty$ and $P(\tau(e) = 0) < p_c$), $0 \leq \chi \leq 1$ and $0 \leq \xi \leq 1$.

The first such bound, due to Kesten [K3], was $\chi \leq 1/2$, valid for all d. Although there is no proof of the scaling identity $\chi = 2\xi - 1$ (which would then yield $\xi \leq 3/4$), there is a rigorous inequality $\chi' \geq 2\xi - 1$ [NP], where χ' is an exponent closely related (and perhaps equal) to χ. Because Kesten's bound was extended to yield $\chi' \leq 1/2$ [K3, A], one does obtain the upper bound $\xi \leq 3/4$, for all d [NP]. We remark that an application of such upper bounds on ξ appears in [N].

The trivial bound $\chi \geq 0$ combined with the (nonrigorous) scaling identity would yield the nontrivial bound $\xi \geq 1/2$ (saying that the transverse spread of the growing $\widetilde{B}(t)$ is at least diffusive) for all d. One of the main results of this paper (see Theorem 2 of Sect. 2) is such a diffusive lower bound on ξ for all d. We remark that even a (subdiffusive) bound of the form $\xi > 0$ is nontrivial (from a mathematical, if not physical, point of view). In that spirit, we will also present (see Theorem 1 of Sect. 2) a bound of the form $\xi(d) \geq 1/(d+1)$, which uses a definition of ξ with certain advantages over the one used for the bound $\xi \geq 1/2$.

Of course, the physically most interesting bounds are superdiffusive ones, which via the scaling identity, correspond (nonrigorously) to bounds of the form $\chi > 0$. Such lower bounds on χ (and χ') have been obtained (but only for $d = 2$) in [NP]. By combining $\chi' \geq 2\xi - 1$ with an inequality $\chi \geq [1 - (d-1)\xi]/2$ first derived in [WA] (for a related model), it was shown in [NP] for $d = 2$, both that $\max(\chi, \chi') \geq 1/5$ and that $\chi \geq 1/8$. The Wehr-Aizenman type lower bound on χ combined with the (nonrigorous) scaling identity would yield $\xi \geq 3/5$ for $d = 2$. The second main result of this paper (see Theorem 3 of Sect. 2) is such a superdiffusive lower bound for $d = 2$. Indeed, part of our proofs of Theorems 2 and 3 may be regarded, in a sense we will not make more precise here, as yielding a rigorous version of $\chi \leq 2\xi - 1$, complementary to the inequality $\chi' \geq 2\xi - 1$ of [NP]. Theorem 2 then follows from the trivial bound $\chi \geq 0$ and Theorem 3 from $\chi \geq [1 - (d-1)\xi]/2$.

In the next section, we give precise statements of all our main results. We complete this section by giving a brief discussion of some physical background to the phenomenon of superdiffusivity.

Physically one may regard first-passage percolation as the zero temperature limit of a model of (undirected) polymers in a random environment. For example, let

$$\Gamma = \{(v_1, ... v_d) \in \mathbb{Z}^d : v_1 + ... + v_d = n\}$$

and let $\mathscr{R}$ be the set of nearest-neighbor paths (polymers) from the origin to (any point of) Γ. We consider the “energy” of a path $r \in \mathscr{R}$ to be the passage time $T(r)$ given in (1.1) and then define the Gibbs measure ν_β, at inverse temperature β, to be the probability measure on $\mathscr{R}$ which assigns to r a probability proportional to $\exp(-\beta T(r))$. The Gibbs measure ν_β is a random measure since it depends on the sample point ω in the underlying probability space $(\Omega, \mathscr{F}, P)$ for the $\tau(e)$'s. In the zero temperature limit ($\beta \to \infty$), the limiting measure ν_∞ assigns equal probability to all r's such that $T(r) = T(0, \Gamma)$. (We are assuming here that the inf in (1.2) and (1.3) is attained, which will be the case if $P(\tau(e) = 0) < p_c$.)

On the space $\Omega \times \mathscr{R}$ with probability measure $P(\omega)\nu_\infty(r|\omega)$, let V denote the endpoint (on Γ) of r. Consider two extreme cases. In the first case, the common distribution of the $\tau(e)$'s is continuous so that for (a.e.) ω, ν_∞ is supported on a single r; here, all the fluctuations in V are due to the random environment and their magnitude is described by the first-passage exponent ξ. In the second case the $\tau(e)$'s are all equal to a positive constant (i.e., the medium is nonrandom); here, all fluctuations are “thermal”– i.e., due to the ($\beta \to \infty$) Gibbs measure ν_∞. A moments thought shows that in this case, with our choice of Γ, ν_∞ corresponds to the distribution of an n-step simple random walk on $\mathbb{Z}^d$ in which only the d steps which increase a single coordinate are allowed and these are all equally likely. Clearly, in this case, the mean of V is $(n/d, ..., n/d)$ and its fluctuations are of order $n^{1/2}$. We see that $\xi > 1/2$ means that (at least at zero temperature) fluctuations due to the random environment dominate the thermal fluctuations in a nonrandom environment.

We finally note that for the case just discussed of a nonrandom environment, not only are the fluctuations of the endpoint V of order $n^{1/2}$, but also the entire path r is contained (with high probability) within a (deterministic) cylinder, centered on the straight line from 0 to $(n/d, ..., n/d)$, whose width is of order $n^{1/2}$. Thus our discussion of the meaning of $\xi > 1/2$ is also relevant for those definitions of ξ, soon to be introduced in Sect. 2, which are based on confinement of entire paths within cylinders rather than on confinement only of endpoints.

2. Main results

In this section, we state three theorems which give lower bounds on various definitions of the exponent ξ. Most of the section is taken up by the presentation and explanation of these definitions. The proofs of the three theorems are given in the following three sections.

Our first result has the form $\xi(d) \geq 1/(d+1)$ for two definitions of the exponent ξ, one of point-to-point type and one of point-to-plane type. Although this inequality appears weaker than the result presented later that $\xi(d) \geq 1/2$, we include it for two reasons. First, because the definitions of ξ used in it are, in certain senses, stronger than those used afterwards and second, because the arguments used for it will be used again later on to obtain the superdiffusive bound $\xi(2) \geq 3/5$. In fact, although it sounds strange, this superdiffusive bound is based on combining the arguments used to prove $\xi(2) \geq 1/(2+1)$ and those used to prove $\xi(2) \geq 1/2$.

For a nonzero vector x in $\mathbb{R}^d$ and for $w > 0$, we define L_x to be the line in $\mathbb{R}^d$, $\{\alpha x : \alpha \in \mathbb{R}\}$, and $\mathscr{C}(x, w)$ to be the cylinder in $\mathbb{R}^d$ of radius w and symmetry axis L_x, i.e.,

$$\mathscr{C}(x, w) = \{z \in \mathbb{R}^d : \operatorname{dist}(z, L_x) \leq w\}. \tag{2.1}$$

Here, $\operatorname{dist}(A, B)$ denotes $\inf\{\|x-y\| : x \in A, \ y \in B\}$. For $u, v \in \mathbb{Z}^d$, we denote the set of (time) minimizing paths between u and v by

$$\mathscr{M}(u, v) = \{r : r \text{ is a path from } u \text{ to } v \text{ with } T(r) = T(u, v)\}. \tag{2.2}$$

Similarly we define for a subset Γ of $\mathbb{Z}^d$

$$\mathscr{M}(u, \Gamma) = \{r : r \text{ is a path from } u \text{ to } \Gamma \text{ with } T(r) = T(u, \Gamma)\}. \tag{2.3}$$

$\mathscr{M}(u, v)$ and $\mathscr{M}(u, \Gamma)$ are (a.s.) nonempty if $P(\tau(e) = 0) < p_c$; they are (a.s.) singleton sets if the common distribution of the $\tau(e)$'s is continuous. We write (for a subset A of $\mathbb{R}^d$) that $\mathscr{M}(u, v)$ (resp. $\mathscr{M}(u, \Gamma)$) is in A to mean that every r in $\mathscr{M}(u, v)$ (resp. $\mathscr{M}(u, \Gamma)$) only touches sites in A.

Our point-to-point definition of ξ is

$$\xi^{(0)} = \sup\{\gamma \geq 0 : \limsup_{\|v\| \to \infty} P(\mathscr{M}(0, v) \text{ is in } \mathscr{C}(v, \|v\|^\gamma)) < 1\}. \tag{2.4}$$

According to this definition, in order that a number γ' exceeds $\xi^{(0)}$, there must exist some (deterministic) sequence v_n with $\|v_n\| \to \infty$, such that

$$P(\mathscr{M}(0, v_n) \text{ is in } \mathscr{C}(v_n, \|v_n\|^{\gamma'})) \to 1. \tag{2.5}$$

This particular definition will be quite useful in obtaining a lower bound for ξ because if the γ' in (2.5) is small, the time-minimizing paths will be contained (with high probability) in a narrow cylinder. This will imply a large variance for the passage time (as in [WA]) which will eventually lead to a contradiction by considering two adjacent cylinders and their respective passage times (see Fig. 1).

Our (first) point-to-plane definition of ξ is more complicated. For $\hat{x}$ a unit vector in $\mathbb{R}^d$ and $L > 0$, let $\Lambda(\hat{x}, L)$ be the half-space in $\mathbb{Z}^d$,

$$\Lambda(\hat{x}, L) = \{u \in \mathbb{Z}^d : u \cdot \hat{x} < L\}, \tag{2.6}$$

where $u \cdot \hat{x}$ denotes the standard inner product in $\mathbb{R}^d$. This is the intersection of $\mathbb{Z}^d$ with the half-space in $\mathbb{R}^d$ containing the origin whose boundary is the plane perpendicular to $\hat{x}$ at distance L from the origin. The $\mathbb{Z}^d$-boundary of $\Lambda(\hat{x}, L)$, denoted $\partial\Lambda(\hat{x}, L)$, is the set of v in $\mathbb{Z}^d \backslash \Lambda(\hat{x}, L)$ which are the nearest neighbors of some u in $\Lambda(\hat{x}, L)$. Our point-to-plane definition of ξ involves the containment for large L of $\mathscr{M}(0, \partial\Lambda(\hat{x}, L))$ in some cylinder $\mathscr{C}(X, L^\gamma)$. Since the endpoints of paths in $\mathscr{M}(0, \partial\Lambda(\hat{x}, L))$ need not be anywhere near $L\hat{x}$, there is no restriction on X other than it be a nonzero vector in $\mathbb{R}^d$ for each unit vector $\hat{x}$ in $\mathbb{R}^d$ and $L > 0$. The definition is

$$\xi^{(1)} = \sup\{\gamma \geq 0 : \begin{array}{l} \limsup_{L\to\infty} \sup_{\hat{x}} \sup_X P(\mathscr{M}(0, \partial\Lambda(\hat{x}, L)) \\ \text{is in } \mathscr{C}(X, L^\gamma)) < 1\}. \end{array} \tag{2.7}$$

According to this definition, in order that a number γ' exceeds $\xi^{(1)}$, there must exist some (deterministic) sequences $\hat{x}_n, X_n$ and $L_n \to \infty$ such that

$$P(\mathscr{M}(0, \partial\Lambda(\hat{x}_n, L_n)) \quad \text{is in} \quad \mathscr{C}(X_n, L_n^{\gamma'})) \to 1. \tag{2.8}$$

As in the point-to-point $\xi^{(0)}$, this definition is well suited to obtain a lower bound for ξ because a small γ' in (2.8) restricts the time-minimizing paths to a narrow cylinder. We can now state our first result.

Theorem 1. *For any $d \geq 2$, assume that either (1.5) or (1.6) is valid and that $E(\tau(e)^2) < \infty$. Then*

$$\xi^{(0)} \geq 1/(d+1) \quad \textit{and} \quad \xi^{(1)} \geq 1/(d+1). \tag{2.9}$$

Remark. We note that the hypothesis that either (1.5) or (1.6) is valid implies $\mathrm{var}(\tau(e)) > 0$. We also remark that the conclusions of Theorems 2 and 3 should remain valid without this hypothesis (although our proofs are not applicable).

Our second result has the form $\xi(d) \geq 1/2$ for a second point-to-plane definition $\xi^{(2)}$ of the exponent ξ. The methods we use for that bound are designed specifically for point-to-plane passage and we have no further results for point-to-point definitions of ξ. The definition of $\xi^{(2)}$ is weaker (i.e., a lower bound using this definition is weaker) in two aspects and stronger in one aspect than the definition of $\xi^{(1)}$.

The first weakening is that (except for distributions where $P(\tau(e) = 0) > 0$) we replace minimizing paths by “almost-minimizing” paths. That is, we replace $\mathscr{M}(u, \Gamma)$ by

$$\mathscr{M}(u, \Gamma; K) = \{r : r \text{ is a path from } u \text{ to } \Gamma \text{ with } T(r) \leq T(u, \Gamma) + K\}, \tag{2.10}$$

where $K \geq 0$ must have $P(\tau(e) \leq K) > 0$. Note that this will yield a $\xi^{(2)}$ which (in principle) depends on K. The strengthening is that (like in most of Sect. 1) $\xi^{(2)}$ is defined in terms of confinement of endpoints of (almost) minimizing paths rather than of the entire paths. Thus (restricting attention to $u = 0$) we define

$$R(\Gamma; K) = \{v \in \Gamma : T(0, v) \leq T(0, \Gamma) + K\}, \tag{2.11}$$

and consider whether $R(\partial\Lambda(\hat{x}, L); K)$ is probably confined in some deterministic $A(\hat{x}, L) \subseteq \partial\Lambda(\hat{x}, L)$ of diameter L^γ. For a subset A of $\mathbb{R}^d$, its diameter, diam(A), is the sup over $x, y \in A$ of $\|x - y\|$.

The second weakening can be explained as follows. For the definition of $\xi^{(1)}$, when $\gamma < \xi^{(1)}$, we are guaranteed that $\mathscr{M}(0, \partial\Lambda(\hat{x}, L))$ is not confined on the scale L^γ as $L \to \infty$ for *any* choice of $\hat{x}$. On the other hand for our upcoming definition of $\xi^{(2)}$, we will only be guaranteed that $R(\partial\Lambda(\hat{x}, L); K)$ is not confined on the scale L^γ for *some* sequences $\hat{x} = \hat{x}_n$ and $L = L_n \to \infty$. Although $\xi^{(2)}$ can be defined in a style more similar to the definition (2.7) of $\xi^{(1)}$, we find the following less confusing.

$$\begin{aligned} \xi_K^{(2)} = \sup \quad & \{\gamma \geq 0 : \exists \text{ sequences } \hat{x}_n \text{ and } L_n \to \infty \text{ such that} \\ & \text{there is no deterministic } A_n \text{ with } \operatorname{diam}(A_n) \leq L^\gamma \text{ such that} \\ & P(R(\partial\Lambda(\hat{x}_n, L_n); K) \subseteq A_n) \to 1\}. \end{aligned} \tag{2.12}$$

Before stating our second result, we comment on why this definition is suited to obtaining the lower bound 1/2 for ξ. We first comment on why the "strengthening" of the definition creates no problem. Unlike Theorem 1, the derivation of this lower bound will not involve estimates on the variance of passage times and so there is no need for narrow confining cylinders; confinement of the endpoints will suffice. Why are the two "weakenings" of the definition needed? The first plays a technical role that is hard to motivate in advance of the actual proof. The second weakening is important because for $\gamma' > \xi_K^{(2)}$, there will be confinement of endpoints on the scale $L^{\gamma'}$ simultaneously for *all* $\hat{x}$'s and large L; this is necessary due to our lack of control over $\hat{x}$-dependence (caused by the anisotropy of $\mathbb{Z}^d$). We can now state our second result.

Theorem 2. *For any $d \geq 2$, assume $P(\tau(e) = 0) < p_c$ and choose $K \geq 0$ such that $P(\tau(e) \leq K) > 0$. Then*

$$\xi_K^{(2)} \geq 1/2. \tag{2.13}$$

Remark. The proof of Theorem 2 actually yields a slightly stronger result than (2.13). Namely, that the set of γ's given in the definition (2.12) of $\xi_K^{(2)}$ includes $\gamma = 1/2$. In fact, slight changes of the arguments show that for any $C < \infty$, $R(\partial\Lambda(\hat{x}_n, L_n); K)$ cannot be confined (with probability $\to 1$) within any A_n of diameter $\leq CL^{1/2}$.

Our third and final result has the form $\xi(2) \geq 3/5$ for yet a third point-to-plane exponent $\xi^{(3)}$. Since the proof is based on a combination of the arguments used for our first two results, the definition of $\xi^{(3)}$ combines the weaknesses of both $\xi^{(1)}$ and $\xi^{(2)}$, as follows.

$$\begin{aligned} \xi_K^{(3)} = \sup \quad & \{\gamma \geq 0 : \exists \text{ sequences } \hat{x}_n \text{ and } L_n \to \infty \text{ such that} \\ & \text{there is no deterministic } X_n \text{ such that} \\ & P(\mathscr{M}(0, \partial\Lambda(\hat{x_n}, L_n)); K) \text{ is in } \mathscr{C}(X_n, L_n^\gamma)) \to 1\}. \end{aligned} \tag{2.14}$$

Note that $\xi_K^{(3)} \geq \max(\xi_K^{(2)}, \xi^{(1)})$.

Theorem 3. *Assume that either (1.5) or (1.6) is valid and that $E(\tau(e)^2) < \infty$. Choose K such that $P(\tau(e) \leq K) > 0$. Then*

$$\text{for } d = 2, \quad \xi_K^{(3)} \geq 3/5. \tag{2.15}$$

Remark. Our proofs of Theorems 2 and 3 can be adapted to obtain versions of these theorems with modified definitions of the exponents $\xi_K^{(2)}$ and $\xi_K^{(3)}$ (see (2.12) and (2.14)) where one chooses in advance a limit $\hat{x}$ of the sequences $\hat{x}_n$. The additional hypothesis needed in this case for the conclusions to hold is, roughly speaking, that the first-passage asymptotic shape (given by the shape theorem) should not have a "corner" at a point where its boundary has a tangent vector perpendicular to $\hat{x}$. If there is finite curvature at that point, then it should be possible to take $\hat{x}_n \equiv \hat{x}$ in (2.12) and (2.14) (and thus obtain lower bounds for ξ more like the fixed $\hat{x}$ upper bond of $3/4$ of [NP]), but we have not been able to do so. It is expected (and in fact it is a basic assumption behind the heuristic derivation of the relation $\chi = 2\xi - 1$ [KS]) that (assuming conditions (1.5) and (1.6)) the asymptotic shape boundary should actually have curvature bounded away from zero and infinity in all directions, but this has not yet been proven.

3. The bound $\xi(d) \geq 1/(d+1)$

In this section, we prove Theorem 1. The proof is based on an extension of the arguments used in [WA] and [NP] to obtain the inequality $\chi \geq [1-(d-1)\xi]/2$. We begin with a brief sketch of the proof, whose general structure is analogous to that used by Aizenman and Wehr [AW] in a different context. The strategy of the proof is to obtain a pair of bounds on the variance of the time difference δT between a passage time for length scale L and its spatial translate by length scale $L^{\gamma'}$ (see Figs. 1 and 2). One easily obtains an "a priori" upper bound, $\text{var}(\delta T) = O(L^{2\gamma'})$. The lower bound is based on a general result (see (3.4)) for functions of independent random variables, from which we obtain, under the assumption that the minimizing paths are confined within a cylinder of radius $O(L_n^{\gamma'})$ (see Figs. 1 and 2), a lower bound of the form $C'L_n^{1-(d-1)\gamma'}$. The assumption will be valid (with high probability, for large L) for any $\gamma' > \xi$. The lower bound is then consistent with the upper bound only if $\gamma' \geq 1/(d+1)$. Since this is true for every $\gamma' > \xi$, we obtain $\xi \geq 1/(d+1)$ as desired.

For ease of exposition, we begin the actual proof with the case of 0 or 1 valued $\tau(e)$'s, first handling the point-to-point exponent $\xi^{(0)}$ and then the point-to-plane exponent $\xi^{(1)}$. After that, we extend the argument to general distributions for the $\tau(e)$'s.

Proof of $\xi^{(0)} \geq 1/(d+1)$ *for* 0 *or* 1 *valued* $\tau(e)$*'s.* Let p denote $P(\tau(e) = 0)$ and $q = 1 - p$. We may assume that the $\tau(e)$'s are the coordinate variables on the canonical probability space $(\Omega, \mathscr{F}, P)$ with $\Omega = \{0,1\}^{\mathbb{E}^d}$, $\mathscr{F}$ the standard σ-field generated by cylinder sets and $P = P_p$, the Bernoulli product measure. By utilizing the symmetries of $\mathbb{Z}^d$, we may restrict attention in the definition (2.4) of $\xi^{(0)}$ to the type of v's in $\mathbb{Z}^d$ whose first coordinate, which we denote k, is non-negative and at least as large as the absolute value of any other coordinate. Note that then k is between $\|v\|/d^{1/2}$ and $\|v\|$. To prove $\xi^{(0)} \geq 1/(d+1)$, we will assume that (2.5) is valid for some γ' and some $v = v_n$ of the type just described, and then show that $\gamma' \geq 1/(d+1)$.

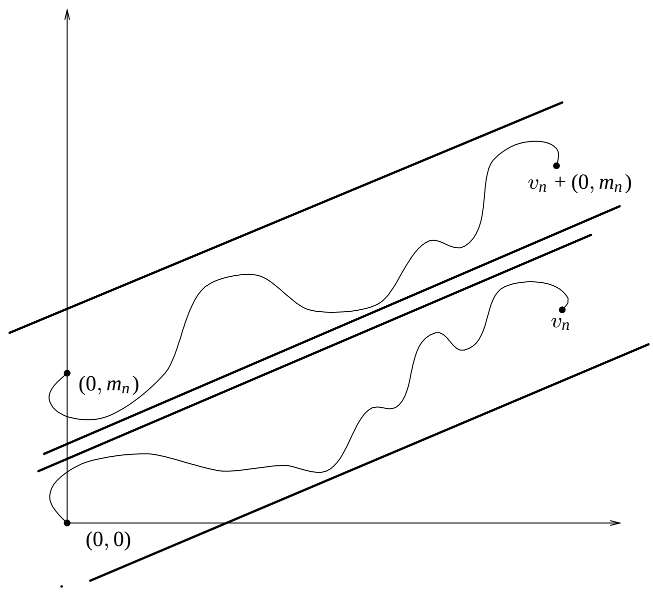

Fig. 1. Lower and upper point-to-point minimizing paths and the cylinders, $\mathscr{C}_n^l$ and $\mathscr{C}_n^u$, that contain each of them, for $d = 2$

For each $v_n \neq 0$, we consider two point-to-point passage times: a "lower" one, $T_n^l = T(0, v_n)$, and an "upper" one corresponding to a shift upward of the second coordinate by m_n units,

$$T_n^u = T((0, m_n, 0, ...0), v_n + (0, m_n, 0, ...0)). \tag{3.1}$$

The positive integer $m_n (\leq c_1 \|v_n\|^{\gamma'}$, where c_1 can be taken to be 3) will be chosen so that the two cylinders in $\mathbb{R}^d$,

$$\mathscr{C}_n^l = \mathscr{C}(v_n, \|v_n\|^{\gamma'}) \text{ and } \mathscr{C}_n^u = \mathscr{C}_n^l + (0, m_n, 0, ...0), \tag{3.2}$$

are (just barely) disjoint (see Fig. 1). We define δT_n to be the difference $T_n^l - T_n^u$ and note, by an obvious argument using the straight line paths between 0 and $(0, m_n, 0, ..., 0)$ and between v_n and $v_n + (0, m_n, 0, ..., 0)$, that $|\delta T_n| \leq 2m_n$ so that

$$\text{var}(\delta T_n) \leq (2m_n)^2 \leq c_2 \|v_n\|^{2\gamma'}. \tag{3.3}$$

(Here $c_2 = 4c_1 = 12$). The remainder of the proof consists in deriving a lower bound for var(δT_n), based on the assumed validity of (2.5), which would contradict (3.3) if γ' were below $1/(d+1)$.

Proceeding (for a while) as in [NP],we choose an ordering, $e_1, e_2, ...,$ of the edges in $\mathbb{E}^d$. Let $\mathscr{G}_j$ be the σ-field generated by $\tau(e_j)$ and let $\mathscr{F}_j$ be the σ-field generated by $\tau(e_1), ..., \tau(e_j)$ (with $\mathscr{F}_0$ the trivial σ-field). Then a martingale

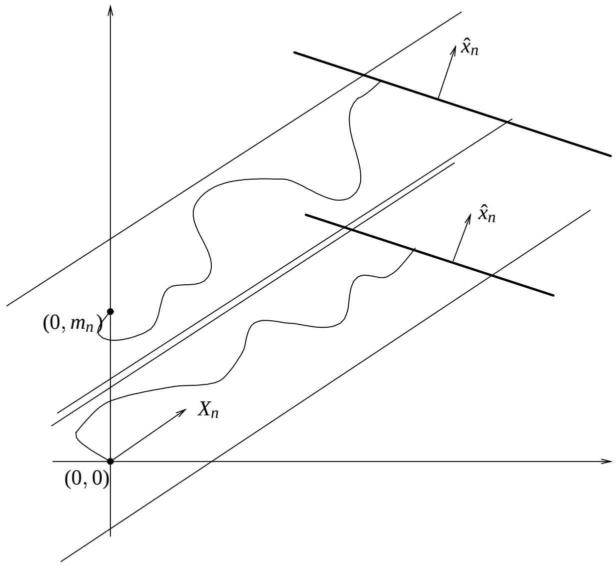

Fig. 2. Lower and upper point-to-plane minimizing paths and the cylinders, $\hat{\mathscr{C}}_n^l$ and $\hat{\mathscr{C}}_n^u$, that contain each of them, for $d = 2$. The lines in the figure perpendicular to $\hat{x}_n$ are given by the equations $u \cdot \hat{x}_n = L_n$ and $(u - (0, m_n)) \cdot \hat{x}_n = L_n$; these lines indicate the (approximate) location of the subsets of $\mathbb{Z}^2$, $\partial\Lambda(\hat{x}_n, L_n)$ and $\partial\Lambda(\hat{x}_n, L_n) + (0, m_n)$

identity and a standard inequality yield (as in Lemma 2 of [NP]) the following lower bound for $\delta T \equiv \delta T_n$:

$$\begin{aligned} \mathrm{var}(\delta T) &= \textstyle\sum_{j=1}^{\infty} \mathrm{var}[E(\delta T|\mathscr{F}_j) - E(\delta T|\mathscr{F}_{j-1})] \\ &\geq \textstyle\sum_{j=1}^{\infty} \mathrm{var}[E(E(\delta T|\mathscr{F}_j) - E(\delta T|\mathscr{F}_{j-1})|\mathscr{G}_j)] \\ &= \textstyle\sum_{j=1}^{\infty} \mathrm{var}[E(\delta T|\mathscr{G}_j)]. \end{aligned} \tag{3.4}$$

We express, for each j, $\omega \in \Omega = \{0, 1\}^{\mathbb{E}^d}$ as $(\omega^j, \hat{\omega}^j)$ where $\omega^j = \omega(e_j)$ and $\hat{\omega}^j$ is the restriction of ω to $\mathbb{E}^d \backslash \{e_j\}$. We define three random variables (depending only on $\hat{\omega}^j$): $\delta T_j^0 = \delta T((0, \hat{\omega}^j))$, $H_j^u = T_n^u((1, \hat{\omega}^j)) - T_n^u((0, \hat{\omega}^j))$ and $H_j^l = T_n^l((1, \hat{\omega}^j)) - T_n^l((0, \hat{\omega}^j))$. Then H_j^u (resp. H_j^l) is (in the language of [NP]) the indicator variable of the event that e_j "matters" for T_n^u (resp. for T_n^l) and we have

$$\delta T = \delta T_j^0 + [H_j^l - H_j^u]\omega^j \tag{3.5}$$

and

$$E(\delta T|\mathscr{G}_j) = E(\delta T_j^0) + [P(H_j^l = 1) - P(H_j^u = 1)]\omega^j. \tag{3.6}$$

Inserting this into (3.4) yields

$$\operatorname{var}(\delta T) \geq pq \sum_{j=1}^{\infty} [P(H_j^l = 1) - P(H_j^u = 1)]^2. \tag{3.7}$$

We next restrict the sum in the last inequality to j's with e_j in $\mathscr{C}_n'$, the intersection of the lower cylinder $\mathscr{C}_n^l$ given in (3.2) with the box $B_{2\|v_n\|} = \{z \in \mathbb{R}^d :| z_i | \leq 2\|v_n\| \text{ for } i = 1, \ldots, d\}$; we denote this restricted sum by $\sum_j{}'$. Applying the Cauchy-Schwarz inequality and denoting by $|\mathscr{C}_n'|$ the number of *edges* in $\mathscr{C}_n'$, we have

$$\begin{aligned} \operatorname{var}(\delta T) &\geq pq \textstyle\sum_j' [P(H_j^l = 1) - P(H_j^u = 1)]^2 \\ &\geq (pq/|\mathscr{C}_n'|)\{\textstyle\sum_j' P(H_j^l = 1) - \sum_j' P(H_j^u = 1)\}^2. \end{aligned} \tag{3.8}$$

The last two steps of the proof are to show that as $n \to \infty$,

$$\liminf_{n\to\infty} \|v_n\|^{-1} \sum_j{}' P(H_j^l = 1) \geq c_3 > 0, \tag{3.9}$$

while

$$\lim_{n\to\infty} \|v_n\|^{-1} \sum_j{}' P(H_j^u = 1) = 0. \tag{3.10}$$

Once this is done, it will follow from (3.8) that for some strictly positive c_4, and large n,

$$\operatorname{var}(\delta T) \geq 2pq{c_3}^2 \|v_n\|^2/|\mathscr{C}_n'| \geq c_4 \|v_n\|^2/\|v_n\|^{1+(d-1)\gamma'} \tag{3.11}$$

(where c_4 depends on p, d and the c_3 of (3.9)). This would contradict (3.3) if $2\gamma' < 1 - (d-1)\gamma'$, i.e., if $\gamma' < 1/(d+1)$, and the proof would be complete. We first show (3.9) and afterwards (3.10).

Let F_j^l denote the intersection of the two independent events $\{H_j^l = 1\}$ and $\{\tau(e_j) = 1\}$, or equivalently the event that $\tau(e_j) = 1$ and e_j belong to *some* minimizing path for T_n^l (such an e_j was called a "minimizing 1-edge" in [NP]). Then

$$\|v_n\|^{-1} \sum_j{}' P(H_j^l = 1) = \|v_n\|^{-1} q^{-1} \sum_j{}' P(F_j^l) \geq q^{-1} \|v_n\|^{-1} E(T_n^l 1_{\mathscr{M}_n^l \text{ is in } \mathscr{C}_n'}), \tag{3.12}$$

where $\mathscr{M}_n^l = \mathscr{M}(0, v_n)$ and the inequality follows because the total number of minimizing 1-edges is at least as big as their number in a particular minimizing path. We claim that by (2.5) and the shape theorem (see [K1]), the RHS of (3.12) is bounded away from zero as $n \to \infty$, which verifies (3.9). To see this, let μ denote the time constant, the strictly positive limit of $E(T(0, (n, 0, \ldots, 0))/n)$, and then define three events: $G_n^1 = \{T_n^l/\|v_n\| \geq \mu/(2\sqrt{d})\}$, $G_n^2 = \{\mathscr{M}_n^l \text{ is in } B_{2\|v_n\|}\}$ and $G_n^3 = \{\mathscr{M}_n^l \text{ is in } \mathscr{C}_n^l\}$. The RHS of (3.12) is bounded below by $[q^{-1}\mu/(2\sqrt{d})]P(G_n^1 \cap G_n^2 \cap G_n^3)$. The shape theorem implies that $P(G_n^1) \to 1$ and $P(G_n^2) \to 1$ while (2.5) states that $P(G_n^3) \to 1$; our claim follows.

To verify (3.10), we denote by Y_n^u the number of edges e_j such that $H_j^u = 1$ and $\tau(e_j) = 0$, and note that such an edge has the property that it is passed through by *every* minimizing path for T_n^u. Thus $Y_n^u \le Z_n^u$, the minimum length (i.e., no. of edges) among paths in $\mathscr{M}_n^u$, the set of minimizing paths for T_n^u, and so

$$\begin{aligned}\|v_n\|^{-1}{\sum_j}' P(H_j^u=1) &= \|v_n\|^{-1}p^{-1}{\sum_j}' P(H_j^u=1 \text{ and } \tau(e_j)=0)\\ &\le p^{-1}\|v_n\|^{-1}E(Y_n^u 1_{\{\mathscr{M}_n^u \text{ is in } \mathscr{C}_n{}^u\}^c})\\ &\le p^{-1}\|v_n\|^{-1}[E((Y_n^u)^2)P(\{\mathscr{M}_n^u \text{ is in } \mathscr{C}_n{}^u\}^c)]^{1/2}\\ &\le p^{-1}[E((Z_n^u/\|v_n\|)^2)(1-P(\mathscr{M}_n^u \text{ is in } \mathscr{C}_n{}^u))]^{1/2},\end{aligned} \tag{3.13}$$

where we have used the Cauchy-Schwarz inequality. As $n \to \infty$, $P(\mathscr{M}_n^u$ is in $\mathscr{C}_n{}^u)$ converges to 1 by (2.5) and so to complete the proof we need only that $E((Z_n^u/\|v_n\|)^2)$ is bounded away from ∞. This is a consequence of a result, presented as Proposition B1 in Appendix B, which follows from the arguments of [K3].

We have now completed the proof that the point-to-point exponent $\xi^{(0)}$ satisfies $\xi^{(0)} \ge 1/(d+1)$ for 0 or 1 valued $\tau(e)$'s. We next consider the point-to-plane exponent $\xi^{(1)}$.

Proof that $\xi^{(1)} \ge 1/(d+1)$ *for* 0 *or* 1 *valued* $\tau(e)$*'s.* The proof is very similar to the one just given for $\xi^{(0)}$ and uses the same notation. We will assume that (2.8) is valid for some γ' and then show $\gamma' \ge 1/(d+1)$. Without loss of generality, we assume that each X_n of (2.8) is of the type with non-negative first coordinate, at least as large as the absolute value of any other coordinate. Although the unit vector $\hat{x}_n$ of (2.8) need not be asymptotically parallel to X_n (see Fig. 2), nor even of this type, it is true (e.g., by the shape theorem) that the angle between $\hat{x}_n$ and X_n must be bounded away from $\pi/2$ as $n \to \infty$.

Again we consider lower and upper passage times, $\hat{T}_n^l = T(0, \partial\Lambda(\hat{x}_n, L_n))$ and

$$\hat{T}_n^u = T((0, m_n, 0, \ldots, 0), \partial\Lambda(\hat{x}_n, L_n) + (0, m_n, \ldots, 0)), \tag{3.14}$$

with m_n ($\le c_5 L_n^{\gamma'}$, where c_5 can be taken to be 3) chosen so that the two cylinders

$$\hat{\mathscr{C}}_n{}^l = \mathscr{C}(X_n, L_n^{\gamma'}), \quad \hat{\mathscr{C}}_n{}^u = \hat{\mathscr{C}}_n{}^l + (0, m_n, 0, \ldots, 0) \tag{3.15}$$

are (just barely) disjoint (see Fig. 2). Then (as in (3.3)), the variance of $\delta\hat{T}_n \equiv \hat{T}_n^l - \hat{T}_n^u$ is bounded by $c_6 L_n^{2\gamma'}$ (where $c_6 = 4c_5 = 12$). The remainder of the proof is to show that (as in (3.11)) $\mathrm{var}(\delta\hat{T}_n)$ is bounded below by a constant times $(L_n)^{1-(d-1)\gamma'}$, which contradicts the upper bound on the variance unless $\gamma' \ge 1/(d+1)$.

To obtain this lower bound, we use essentially the same arguments as before to obtain

$$\begin{aligned}\mathrm{var}(\delta\hat{T}_n) &\ge pq\sum_{j=1}^{\infty}[P(\hat{H}_j^l=1)-P(\hat{H}_j^u=1)]^2\\ &\ge (pq/|\mathscr{C}_n{}^*|)\{{\sum_j}^* P(\hat{H}_j^l=1)-{\sum_j}^* P(\hat{H}_j^u=1)\}^2\\ &\ge c_7(L_n)^{1-(d-1)\gamma'}\{L_n^{-1}{\sum_j}^* P(\hat{H}_j^l=1)-L_n^{-1}{\sum_j}^* P(\hat{H}_j^u=1)\}^2,\end{aligned} \tag{3.16}$$

where $\hat{H}_j^u = 1$ (resp. $\hat{H}_j^l = 1$) is the event that e_j matters for $\hat{T}_n^u$ (resp. $\hat{T}_n^l$), $\mathscr{C}_n^*$ denotes the intersection of $\hat{\mathscr{C}}_n^l$ with the box B_{2L_n}, and $\sum_j^*$ denotes the sum over j with e_j in $\mathscr{C}_n^*$ (and where c_7 depends on p, d, and c_6). Letting $\hat{\mathscr{M}}_n^l$ (resp. $\hat{\mathscr{M}}_n^u$) denote the set of minimizing paths for $\hat{T}_n^l$ (resp. $\hat{T}_n^u$), we obtain as analogues of (3.12) and (3.13),

$$L_n^{-1}\sum_j{}^* P(\hat{H}_j^l = 1) \geq q^{-1}L_n^{-1}E(\hat{T}_n^l 1_{\hat{\mathscr{M}}_n^l \text{ is in } \mathscr{C}_n^*}) \tag{3.17}$$

and

$$L_n^{-1}\sum_j{}^* P(\hat{H}_j^u = 1) \leq p^{-1}[E(\hat{Z}_n^u/L_n)^2(1 - P(\hat{\mathscr{M}}_n^u \text{ is in } \hat{\mathscr{C}}_n^u))]^{1/2}, \tag{3.18}$$

where $\hat{Z}_n^u$ is the minimum length of paths in $\hat{\mathscr{M}}_n^u$.

The RHS of (3.17) is bounded away from zero as $n \to \infty$ by (2.8) and the shape theorem like in the analogous bound for the RHS of (3.12). Because $\hat{Z}_n^u \leq Z^*(v_n)$ (see (B.1)) for v_n *any* point on $\partial\Lambda(\hat{x}_n, L_n) + (0, m_n, 0, \dots, 0)$, we see that the RHS of (3.18) tends to zero as $n \to \infty$ by (2.8) and Proposition 3.1. Thus by (3.16), $\mathrm{var}(\delta\hat{T}_n)$ is bounded below by a constant times $(L_n)^{1-(d-1)\gamma'}$ and the proof is complete.

Proof that $\xi^{(0)} \geq 1/(d+1)$ *for general* $\tau(e)$*'s.* The extension from 0 or 1 valued $\tau(e)$'s proceeds along the lines of an analogous extension in Sect. 4 of [NP]. The $\tau(e)$'s are now the coordinate variables on the space $(\Omega, \mathscr{F}, P)$ with $\Omega = \mathbb{R}^{\mathbb{E}^d}$ (or $[0,\infty)^{\mathbb{E}^d}$), $\mathscr{F}$ is the standard σ-field generated by cylinder sets and P is the product measure of the common distribution of the $\tau(e)$'s. We define, as in the 0 or 1 valued case: the random variables T_n^l, T_n^u, $\delta T_n = T_n^l - T_n^u$; the cylinders $\mathscr{C}_n^l$, $\mathscr{C}_n^u$ and $\mathscr{C}_n'$; the σ-fields $\mathscr{G}_j$ and $\mathscr{F}_j$; and we express, for each j, $\omega \in \Omega$ as $(\omega^j, \hat{\omega}^j)$.

Although $|\delta T_n|$ is no longer bounded (pointwise) by a multiple of m_n, its variance is still $O({m_n}^2)$ because $E(\tau(e)^2) < \infty$, and so $\mathrm{var}(\delta T_n) = O(\|v_n\|^{2\gamma'})$ as in (3.3). A lower bound for the variance of $\delta T = \delta T_n$ can be obtained by starting with the martingale bound (3.4).

To then obtain a lower bound for $\mathrm{var}[E(\delta T|\mathscr{G}_j)]$, we introduce the following random variables, where $\sharp$ denotes either l or u,

$$\mathscr{D}_j^\sharp = \mathscr{D}_j^\sharp(\hat{\omega}^j) = \inf\{\ \omega^j \geq \lambda : \text{for } \omega = (\omega^j, \hat{\omega}^j),\ \text{no minimizing path for } T_n^\sharp \text{ passes through } e_j\}. \tag{3.19}$$

Let λ denote the essential infimum of the $\tau(e)$'s (i.e., the bottom of the support of their common distribution as in (1.5) and (1.6)). Then (for $\omega^j \geq \lambda$),

$$T_n^\sharp(\omega) = T_n^\sharp((\omega^j, \hat{\omega}^j)) = T_n^\sharp((\lambda, \hat{\omega}^j)) + \begin{cases} \omega^j - \lambda, & \text{if } \omega^j < \mathscr{D}_j^\sharp, \\ \mathscr{D}_j^\sharp - \lambda, & \text{if } \omega^j \geq \mathscr{D}_j^\sharp. \end{cases} \tag{3.20}$$

Note that if *some* minimizing path for $T_n^\sharp$ passes through e_j, then $\mathscr{D}_j^\sharp \geq \omega^j$, but if $\mathscr{D}_j^\sharp > \omega^j$, then *every* minimizing path for $T_n^\sharp$ passes through e_j.

We choose constants $b' \geq b > a \geq \lambda'$ such that

$$p \equiv P(\lambda' \leq \tau(e) \leq a) > 0, \quad q \equiv P(b \leq \tau(e) \leq b') > 0. \tag{3.21}$$

More specific choices will be made below. For a given j, let us denote the events $\{\lambda' \leq \tau(e_j) \leq a\}$ and $\{b \leq \tau(e_j) \leq b'\}$ by D^0 and D^1 and define, for $\delta = 0$ or 1,

$$x_\delta = E(E(\delta T|\mathscr{G}_j)1_{D^\delta})/P(D^\delta). \tag{3.22}$$

Then by an elementary argument (see, e.g., Lemma 3 of [NP])

$$\mathrm{var}[E(\delta T|\mathscr{G}_j)] \geq \frac{pq}{p+q}(x_1 - x_0)^2. \tag{3.23}$$

We will use (3.20) to express $E(\delta T_n|\mathscr{G}_j)$ as a specific function $\hat{g}(\omega^j)$. Then by the definition of x_δ we have

$$x_1 - x_0 \geq \inf\{\hat{g}(\theta) : b \leq \theta \leq b'\} - \sup\{\hat{g}(\theta) : \lambda' \leq \theta \leq a\}. \tag{3.24}$$

Define the function g_θ (on $\mathbb{R}$) by

$$g_\theta(\mathscr{D}) = \begin{cases} \theta \text{ if } \mathscr{D} > \theta, \\ \mathscr{D} \text{ if } \mathscr{D} \leq \theta. \end{cases} \tag{3.25}$$

Then from (3.20) we have $\hat{g}$ given by

$$\hat{g}(\theta) = K + E(g_\theta(\mathscr{D}_j^l) - g_\theta(\mathscr{D}_j^u)), \tag{3.26}$$

where K is a constant (depending on j). Since g_θ is monotonic in θ, we have from (3.24)

$$\begin{aligned} x_1 - x_0 &\geq E(g_b(\mathscr{D}_j^l) - g_{b'}(\mathscr{D}_j^u) - g_a(\mathscr{D}_j^l) + g_{\lambda'}(\mathscr{D}_j^u)) \\ &= E(g_b(\mathscr{D}_j^l) - g_a(\mathscr{D}_j^l) - [g_{b'}(\mathscr{D}_j^u) - g_{\lambda'}(\mathscr{D}_j^u)]) \\ &\geq (b-a)P(\mathscr{D}_j^l \geq b) - (b' - \lambda')P(\mathscr{D}_j^u > \lambda'). \end{aligned} \tag{3.27}$$

and so

$$|x_1 - x_0| \geq [(b-a)P(\mathscr{D}_j^l \geq b) - (b' - \lambda')P(\mathscr{D}_j^u > \lambda')]_+ \tag{3.28}$$

where $[\theta]_+ = \theta 1_{\theta \geq 0}$ denotes the positive part of θ. Combining the martingale bound (3.4) with (3.23) and (3.28), we have

$$\mathrm{var}(\delta T) \geq c_8 \sum_{j=1}^{\infty} [c_9 P(\mathscr{D}_j^l \geq b) - P(\mathscr{D}_j^u > \lambda')]_+{}^2, \tag{3.29}$$

where $c_8 = (p+q)^{-1}pq(b' - \lambda')^2$ and $c_9 = (b' - \lambda')^{-1}(b-a)$.

Now we restrict the sum in the last expression to the edges in $\mathscr{C}_n'$ and use the Cauchy- Schwarz inequality to obtain the analogue of (3.8):

$$\begin{aligned}\mathrm{var}(\delta T) &\ge (c_8/|\mathscr{C}_n'|)\{\textstyle\sum_j'[c_9 P(\mathscr{D}_j^l \ge b) - P(\mathscr{D}_j^u > \lambda')]_+\}^2 \\ &\ge (c_8/|\mathscr{C}_n'|)[c_9 \textstyle\sum_j' P(\mathscr{D}_j^l \ge b) - \sum_j' P(\mathscr{D}_j^u > \lambda')]_+{}^2.\end{aligned} \tag{3.30}$$

As in the proof for 0 or 1 valued $\tau(e)$'s (see (3.9) and (3.10) and the subsequent discussion there), it only remains to show that

$$\liminf_{n\to\infty} \|v_n\|^{-1} {\sum_j}' P(\mathscr{D}_j^l \ge b) > 0 \tag{3.31}$$

and

$$\lim_{n\to\infty} \|v_n\|^{-1} {\sum_j}' P(\mathscr{D}_j^u > \lambda') = 0. \tag{3.32}$$

To obtain (3.32), we recall that if $\mathscr{D}_j^u > \omega^j$, then *every* minimizing path for T_n^u passes through e_j. We now choose $\lambda' = a = \lambda$ if $P(\tau(e) = \lambda) > 0$ and otherwise choose $\lambda' > \lambda$; in either case, $p' \equiv P(\tau(e) \le \lambda') > 0$ and we have

$$\begin{aligned}P(\mathscr{D}_j^u > \lambda') &= (p')^{-1} P(\tau(e) \le \lambda', \mathscr{D}_j^u > \lambda') \\ &\le (p')^{-1} P(\text{every minimizing path for } T_n^u \text{ passes through } e_j).\end{aligned} \tag{3.33}$$

Thus we have, exactly as in (3.13), that $\|v_n\|^{-1}$ times ${\sum_j}' P(\mathscr{D}_j^u > \lambda')$ is bounded by the last expression in (3.13) (except) with p replaced by p'. As in the 0 or 1 valued case, (3.32) now follows by (2.5) and Proposition B1.

It remains to verify (3.31). If an edge e_j has $\mathscr{D}_j^l \ge b$ and $\omega^j < b$, then it must be passed through by *every* minimizing path for T_n^l. Letting $\hat{W}_n^l$ denote the total number of such edges and $\hat{p} = P(\tau(e) < b)$, we thus have (as an initial replacement for (3.12))

$$\begin{aligned}{\sum_j}' P(\mathscr{D}_j^l \ge b) &= \hat{p}^{-1} {\sum_j}' P(\mathscr{D}_j^l \ge b, \tau(e_j) < b) \\ &\ge \hat{p}^{-1} E(\hat{W}_n^l 1_{\mathscr{M}_n^l \text{ is in } \mathscr{C}_n'}) \\ &\ge \hat{p}^{-1} E(\hat{W}_n^l) - \hat{p}^{-1} [E((\hat{W}_n^l)^2)(1 - P(\mathscr{M}_n^l \text{ is in } \mathscr{C}_n'))]^{1/2}.\end{aligned} \tag{3.34}$$

But $\hat{W}_n^l \le Z_n^l$, the minimum length among paths in $\mathscr{M}_n^l$, and so, by Proposition B1, $\|v_n\|^{-2} E((\hat{W}_n^l)^2)$ is bounded away from ∞, while $P(\mathscr{M}_n^l \text{ is in } \mathscr{C}_n') \to 1$ by (2.5). Thus

$$\begin{aligned}\liminf_{n\to\infty} \|v_n\|^{-1} {\sum_j}' P(\mathscr{D}_j^l \ge b) &\ge \liminf_{n\to\infty} \|v_n\|^{-1} \hat{p}^{-1} E(\hat{W}_n^l) \\ &= \liminf_{n\to\infty} \|v_n\|^{-1} \textstyle\sum_{j=1}^{\infty} P(\mathscr{D}_j^l \ge b).\end{aligned} \tag{3.35}$$

We say e is a *minimizing b-edge* for T_n^l if $\tau(e) \ge b$ and *some* minimizing path for T_n^l passes through e and we denote by W_n^l the total number of such edges. If e_j is a minimizing b-edge for T_n^l, then $\mathscr{D}_j^l \ge b$ and so the RHS of (3.35) is bounded below by the lim inf of $\|v_n\|^{-1} E(W_n^l)$.

Thus, to complete the verification of (3.31), it now suffices to have the expectation of $W_n^l / \|v_n\|$ bounded away from zero as $n \to \infty$. If $P(\tau(e) = \lambda) < p_c$, then λ', a, b and b' could have been chosen so that $P(\tau(e) < b) < p_c$ and such a lower bound on $W_n^l / \|v_n\|$ would easily follow from the shape theorem (as in case (a') of Sect. 4 of [NP]). Under the more general hypotheses of Theorem 1 (i.e., to include cases where $\lambda > 0$ and $p_c \le P(\tau(e) = \lambda) < p_c^{dir}$), such a lower bound on $W_n^l/\|v_n\|$ follows from the results of [BK], as stated in Proposition B2 of Appendix B below. This completes the proof that $\xi(0) \ge 1/(d+1)$ for general $\tau(e)$'s.

Proof that $\xi^{(1)} \ge 1/(d+1)$ *for general* $\tau(e)$*'s.* This proof is a combination of the appropriate arguments used for $\xi^{(0)} \ge 1/(d+1)$ for general $\tau(e)$'s with those used for $\xi^{(1)} \ge 1/(d+1)$ for 0 or 1 valued $\tau(e)$'s. We leave the details to the reader, but remark that for unbounded $\tau(e)$'s, it can be shown that $\text{var}(\delta\hat{T}_n) = O({m_n}^2) = O({L_n}^{2\gamma'})$ by a straightforward conditioning argument.

4. The bound $\xi(d) \ge 1/2$

In this section, we prove Theorem 2. We begin with a heuristic sketch of the proof, restricted (for simplicity) to $d = 2$ and distributions where $P(\tau(e) = 0) > 0$. Choose any $\gamma' > \xi$. The key ingredient of the proof is the construction, for each large L, of a closed polygon with certain properties (see Fig. 4): Every point on the polygon is at distance about L (i.e., up to a factor bounded away from 0 and ∞ as $L \to \infty$) from the origin. Each segment $\mathscr{L}_i$ is of length about $L^{\gamma'}$ and thus there are about $L^{1-\gamma'}$ segments. Associated to each $\mathscr{L}_i$ is a deterministic subset A_i of $\mathbb{Z}^2$ with diameter substantially smaller than the length of $\mathscr{L}_i$ such that (with probability close to 1) all time-minimizing paths (from the origin to $\partial\Lambda(\hat{x}_i, L_i)$ where $\mathscr{L}_i$ is a segment of the line, $\{y \in \mathbb{R}^2 : y \cdot \hat{x}_i = L_i\}$) terminate on A_i. A crucial property of the polygon is that each A_i is near the midpoint of $\mathscr{L}_i$. The construction of the polygon is done inductively (see Proposition 4.1) by adding one segment at a time and determining $\hat{x}_{i+1}$ (in order that this crucial property be valid) by using "continuity" in the dependence of A_{i+1} on $\hat{x}_{i+1}$ (see Appendix A).

To complete our heuristic sketch, once we have constructed the polygons, we note that the average angle between successive segments is about $1/L^{1-\gamma'}$ and thus at least one such angle is $O(L^{\gamma'-1})$. Concentrating on that pair of segments, $\mathscr{L}_j$ and $\mathscr{L}_{j+1}$, we may assume (without loss of generality) that the passage time to $\partial\Lambda(\hat{x}_j, L_j)$ is $\le$ that to $\partial\Lambda(\hat{x}_{j+1}, L_{j+1})$(with probability at least 1/2). From Fig. 6, we see that the distance from A_j (near the center of $\mathscr{L}_j$) to $\partial\Lambda(\hat{x}_{j+1}, L_{j+1})$ is $O(L^{\gamma'} \cdot L^{\gamma'-1})$. If $\gamma' \le 1/2$, this is $O(1)$ which means it takes only $O(1)$ edges to extend a time-minimizing path to $\partial\Lambda(\hat{x}_j, L_j)$ into a path to $\partial\Lambda(\hat{x}_{j+1}, L_{j+1})$. If these $O(1)$ edges all have $\tau(e) = 0$ (which happens with probability bounded away from 0 as $L \to \infty$), then the extended path will be a time-minimizing path to $\partial\Lambda(\hat{x}_{j+1}, L_{j+1})$ but will *not* terminate on A_{j+1}. This contradicts the defining property of A_{j+1} and shows that $\gamma' \le 1/2$ is impossible. We now begin the detailed proof.

For the fixed $K \ge 0$ of the theorem, chosen so that $P(\tau(e) \le K) > 0$, we choose a fixed $\epsilon > 0$ such that $\epsilon < (1/2)P(\tau(e) \le K)$. The proof of Theorem 2 (and eventually of Theorem 3 as well) is based on the construction, for each large L, of certain (polygonally) convex sets in $\mathbb{R}^d$ containing the origin with certain properties, some of which we now state. For $d = 2$, this set, divided by a constant times L, may be regarded, for large L, as an approximation to the asymptotic shape (of the shape theorem).

The sets will be the intersection of a finite number of half planes in $\mathbb{R}^d$,

$$\tilde{\Lambda}(\hat{x}_i, L_i) = \{y \in \mathbb{R}^d : y \cdot \hat{x}_i < L_i\}, \tag{4.1}$$

with each unit vector $\hat{x}_i$ in the 1-2 plane and each L_i in $[L/\sqrt{2}, \sqrt{2}L]$. The sets will be invariant under reflections of either (or both) of the 1 and 2 coordinates and under their interchange. For $d = 2$: The boundary of the set is a polygon $\mathscr{P}_L$ (see Fig. 3) each of whose segments $\mathscr{L}_i$ is an interval of one of the ($d = 2$) lines,

$$\partial\tilde{\Lambda}(\hat{x}_i, L_i) = \{y \in \mathbb{R}^2 : y \cdot \hat{x}_i = L_i\}; \tag{4.2}$$

four of the $\hat{x}_i$'s are $(\pm 1, 0)$, $(0, \pm 1)$ with corresponding L_i's equal to L; all the segments, except the four with $\hat{x}_i = (\pm 1, \pm 1)/\sqrt{2}$, have the same length $4D$ and those special four segments have length between $4D$ and $12D$. The value of D will be required to satisfy $D \geq \bar{D}_L$ with $\bar{D}_L$ defined below. For $d > 2$: The convex set is a "polygonal barrel" whose boundary is of the form $\mathscr{P}_L \times \mathbb{R}^{d-2}$ with $\mathscr{P}_L$ as in the two-dimensional case.

Because of the symmetries of our construction within the 1-2 plane, we will henceforth concentrate on the unit vectors $\hat{x}(\theta)$ of the form $(\cos\theta, \sin\theta, 0, ..., 0)$ with $0 \leq \theta \leq \pi/2$. For given θ and L (and our fixed K and ϵ), a deterministic subset $A = A(\theta, L)$ of the $\mathbb{Z}^d$-boundary $\partial\Lambda(\hat{x}(\theta), L)$ will be called *acceptable* (for θ, L) if

$$P(R(\partial\Lambda(\hat{x}, L); K) \subseteq A(\theta, L)) \geq 1 - \epsilon. \tag{4.3}$$

For given θ and L, we consider slabs in $\mathbb{Z}^d$ of thickness D' (see Fig. 4),

$$S(\theta, \gamma', D') = \{y \in \mathbb{Z}^d : -D'/2 \leq y_1(-\sin\theta) + y_2(\cos\theta) - \gamma' \leq D'/2\}, \tag{4.4}$$

and consider the smallest D' for which some such slab contains an acceptable A. This leads to our definition of $\hat{D}_L(\theta)$ and $\bar{D}_L$ as

$$\hat{D}_L(\theta) = c_{10} + \inf\{D : \quad \text{for some } \gamma', S(\theta, \gamma', D) \bigcap \partial\Lambda(\hat{x}(\theta), L) \text{ is acceptable for } \theta, L\}, \tag{4.5}$$

where $c_{10} = 2 + \sqrt{d+1}/2$, and

$$\bar{D}_L = \sup\{\hat{D}_{L'}(\theta) : 0 \leq \theta \leq \pi/2, \ \ L/\sqrt{2} \leq L' \leq \sqrt{2}L\}. \tag{4.6}$$

(The reason for this choice of c_{10} is explained later.) For each $0 \leq \theta \leq \pi/2$ and $L > 0$, we may then choose some $\bar{\gamma} = \bar{\gamma}(\theta, L)$ such that

$$\bar{A}(\theta, L) \equiv S(\theta, \bar{\gamma}(\theta, L), \hat{D}_L(\theta)) \bigcap \partial\Lambda(\hat{x}(\theta), L) \tag{4.7}$$

is acceptable for θ, L.

Because $\epsilon < 1/2$, it follows from the definition of acceptable and from the reflection and interchange symmetries of the 1-2 plane, that the sets $\overline{A}(\theta, L)$ for $\theta = 0$, $\pi/4$ and $\pi/2$ respectively intersect the sets $\{y \in \mathbb{Z}^d : y_2 = 0\}$, $\{y \in \mathbb{Z}^d : y_1 = y_2\}$ and $\{y \in \mathbb{Z}^d : y_1 = 0\}$. The crucial property of our constructed $\mathscr{R}_L$ (with $\hat{x}_i = \hat{x}(\theta_i)$) will be the similar requirement for *every* i that $\overline{A}(\theta_i, L_i)$ (almost) intersects the midpoint of the corresponding ith segment of the polygon ($\times\ \mathbb{R}^{d-2}$). The following proposition (see also Fig. 3) basically gives the intersection of $\mathscr{R}_L$ with the sector $\{\theta : 0 \leq \theta \leq \pi/4\}$ with the crucial property given as item (f); the remaining 7/8 of $\mathscr{R}_L$ is then determined by the symmetries in the 1-2 plane.

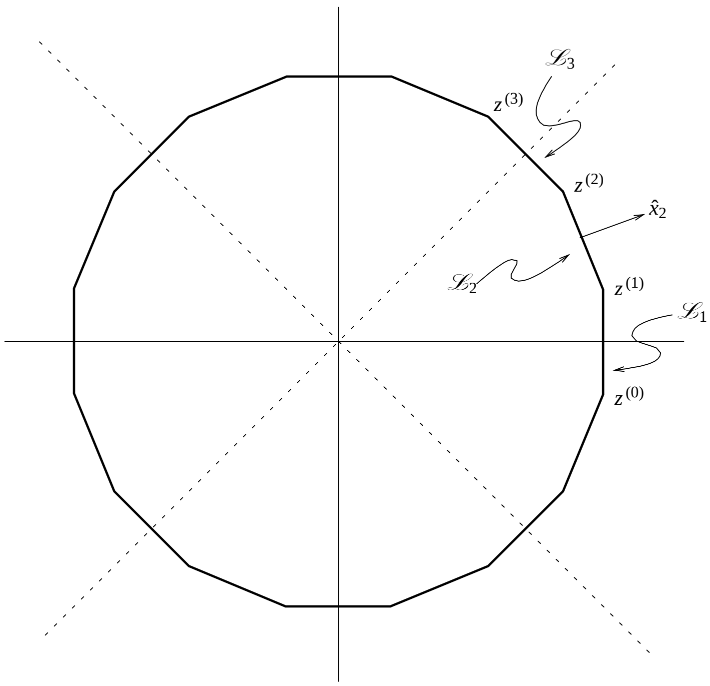

Fig. 3. A sketch of the polygon $\mathscr{R}_L$. In this example, $M_L = 3$ and the polygon is composed of $16 = (8M_L - 8)$ line segments, $\mathscr{L}_1, \mathscr{L}_2, \ldots, \mathscr{L}_{16}$. $\mathscr{L}_1$ is perpendicular to $\hat{x}_1 = (1, 0)$, has midpoint $(L, 0)$ and is of length $4D$. $\mathscr{L}_2$ is perpendicular to some unit vector $\hat{x}_2$, is of length $4D$ and is at some distance L_2 from the origin. $\mathscr{L}_3$ is perpendicular to $\hat{x}_3 = (1/\sqrt{2}, 1/\sqrt{2})$, has midpoint $(L_3, L_3)/\sqrt{2}$ for some L_3 and, in general, is of length between $4D$ and $12D$; in this sketch, it is of length $4D$

Proposition 4.1. *For any L such that $\overline{D}_L \leq (1/2)tan(\pi/8)L$ and any D in the interval $[\overline{D}_L, (1/2)tan(\pi/8)L]$, there exist $M_L + 1 \geq 3$ points in the 1-2 plane, $z^{(i)} = (z_1^{(i)}, z_2^{(i)}, 0, \ldots, 0)$, $0 \leq i \leq M_L$, with the following properties:*

(a) $(z_1^{(0)}, z_2^{(0)}) = (L, -2D)$, $\ (z_1^{(1)}, z_2^{(1)}) = (L, +2D)$,

(b) $z_1^{(i-1)} > z_1^{(i)} > 0$ *and* $0 < z_2^{(i-1)} < z_2^{(i)}$ *for* $2 \leq i \leq M_L$,

(c) $(z_1^{(M_L)}, z_2^{(M_L)}) = (z_2^{(M_L-1)}, z_1^{(M_L-1)})$,

(d) the line segments $\mathscr{L}_i$ *between* $z^{(i-1)}$ *and* $z^{(i)}$*, which are intervals of* $\{(y_1, y_2) : y_1 cos\theta_i + y_2 sin\theta_i = L_i\}$*, satisfy* $0 \le \theta_i < \theta_{i+1} \le \pi/4$ *for* $1 \le i \le M_L - 1$ *(with* $\theta_0 = 0$, $\theta_{M_L} = \pi/4$*),*

(e) the length of $\mathscr{L}_i$ *is* $4D$ *for* $1 \le i \le M_L - 1$ *and is in* $[4D, 12D]$ *for* $i = M_L$*,*

(f) for every $i = 1, \ldots, M_L$*, the set* $\bar{A}(\theta_i, L_i)$ *is within distance* $\bar{\alpha}/2$ *of the midpoint of* $\mathscr{L}_i$ *(here* $\bar{\alpha}$ *is the fixed geometrical constant* $\sqrt{d+1}$*).*

Proof. The $z^{(i)}$'s are constructed inductively with $z^{(0)}$ and $z^{(1)}$ given by property (a). We suppose we have so far constructed $z^{(0)}, \ldots, z^{(i)}$, satisfying all the properties up to that stage, with $\theta_i < \pi/4$ and with the distance Q_i from $z^{(i)}$ to the 45° line, $\{y \in \mathbb{Z}^d : y_1 = y_2\}$, satisfying $Q_i \ge 2D$. (We note that the condition that $D \le (1/2)\tan(\pi/8)L$ insures that $Q_i \ge 2D$ at least for $i = 1$). If $Q_i \le 6D$ then we set $M_L = i + 1$ and define $z^{(M_L)}$ by property (c); this will yield property (e) for $i = M_L$. If $Q_i > 6D$ then $z^{(i+1)}$ must be in the 1-2 plane exactly at distance $4D$ from $z^{(i)}$ with

$$z^{(i+1)} = z^{(i)} + 4D(-\sin\theta, \cos\theta, 0, \ldots, 0)$$

for some $\theta = \theta_{i+1}$ *to be chosen*. Let $L(\theta)(= L_{i+1})$ denote the distance between the origin and the line whose segment $\mathscr{L}_{i+1}$ connects $z^{(i)}$ and $z^{(i+1)}$. The induction will proceed, and this proof will be complete, if we can show that θ can be chosen in the open interval $(\theta_i, \pi/4)$ such that $\bar{A}(\theta, L(\theta))$ is within distance $\bar{\alpha}/2$ of the midpoint of $\mathscr{L}_{i+1}$.

To see that θ can be so chosen, consider the slab of width α, $S(\theta) \equiv S(\theta, \gamma'(\theta), \alpha)$ centered on this midpoint by choosing $\gamma'(\theta) = z_1^{(i)}(-\sin\theta) + z_2^{(i)}(\cos\theta) + 2D$ (see Fig. 4). Note that for $\theta = \theta_i$, $S(\theta)$ is at distance $2D - \frac{\alpha}{2} + 2D$ from the midpoint of $\mathscr{L}_i$, and so by property (f) for $\mathscr{L}_i$ (and the fact that $\bar{A}(\theta_i, L_i)$ is of width $\hat{D}_L(\theta_i) \le \bar{D}_L \le D$), we see that $A(\theta_i) \equiv \bar{A}(\theta_i, L(\theta_i))$ is *below* $S(\theta_i)$(see Fig. 5). For $\theta = \pi/4$, $S(\pi/4)$ is at distance at least $6D - 2D - \frac{\alpha}{2}$ below the 45° line; because the 45° line must intersect $A(\pi/4)$ (as noted in the paragraph preceding Proposition 4.1), we see that $A(\pi/4)$ is *above* $S(\pi/4)$. To find an angle θ in $(\theta_i, \pi/4)$ with the desired property, we need a kind of weak continuity result for $A(\theta)$. This is precisely given by Proposition A2 of Appendix A, which is applicable for α sufficiently large (so that each $\bar{A}(\theta, L)$ is α-connected); we can take $\alpha = \sqrt{d}$ and $\bar{\alpha} = \sqrt{\alpha^2 + 1}$. This completes the proof of Proposition 4.1.

To use Proposition 4.1 for the proof of Theorem 2, we first need a relation between the large L behavior of $\bar{D}_L$ (defined in (4.6)) and the value of the exponent $\xi_K^{(2)}$ (defined in (2.12)). Such a relation (see the discussion in Sect. 2 just preceding Theorem 2) is simply that for any $\gamma > \xi_K^{(2)}$, it must be (for any ϵ, as chosen at the beginning of this section) that $\bar{D}_L \le L^\gamma$ for all large L. To prove that $\xi_K^{(2)} \ge 1/2$, we will assume (for the chosen ϵ) that $\bar{D}_L = o(L^{1/2})$ and then apply Proposition 4.1 with $D = \bar{D}_L$ to derive a contradiction. Note that the hypothesis $\bar{D}_L \le \frac{1}{2}\tan(\pi/8)L$ of the proposition is valid for large L.

The application of Proposition 4.1 is based on an elementary argument that the changes in angle $\triangle\theta_i = \theta_{i+1} - \theta_i$ between succesive facets of the polygon cannot all be large. We first note that since each $\mathscr{L}_i$ has length $\le c_{11}\bar{D}_L$ (where

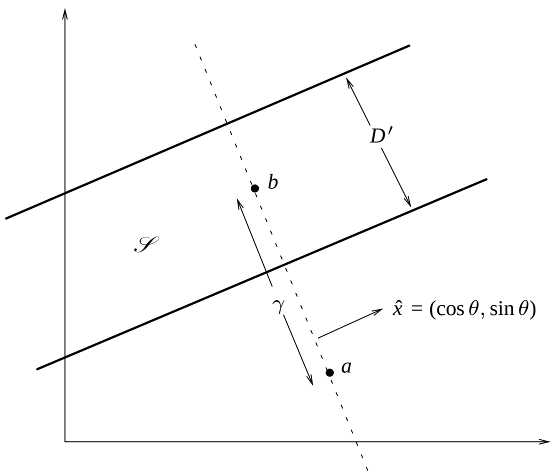

Fig. 4. The set $S(\theta, \gamma', D')$, in this $d = 2$ example, is the set of points in $\mathbb{Z}^2$ falling within the indicated slab $\mathscr{S}$ of width D'. The slab is parallel to $\hat{x} = (\cos\theta, \sin\theta)$ and perpendicular to the dashed line, which is at distance L from the origin. The point b on this line and in the center of the slab, is at distance γ' from the point $L(\cos\theta, \sin\theta)$; in the proof of Proposition 4.1 (and in Appendix A), as θ varies, b remains at a fixed distance $\gamma = 2D$ from a point a on this line which is also fixed

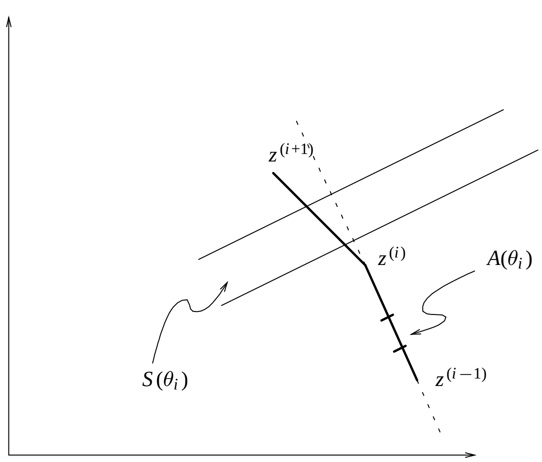

Fig. 5. The line segment $\mathscr{L}_i$, between $z^{(i-1)}$ and $z^{(i)}$ is perpendicular to $\hat{x}_i = (\cos\theta_i, \sin\theta_i)$. The set $A(\theta_i)$, whose approximate location is indicated, comes near the center of $\mathscr{L}_i$. The slab $S(\theta_i)$ is parallel to $\hat{x}_i$ but its center is about $2D$ above $z^{(i)}$ and hence $A(\theta_i)$ is below $S(\theta_i)$

$c_{11} = 12$), and the total length of the $\mathscr{L}_i$'s is at least $c_{12}L$ (where $c_{12} = 4\sqrt{2}$), it must be that $M_L \geq c_{13}L/\bar{D}_L$ (where $c_{13} = c_{12}/(8c_{11})$). We next note that the sum of the M_L angle changes $\triangle\theta_i$ is exactly $\pi/4$ and the smallest $\triangle\theta_i$ is bounded by the average $\triangle\theta_i$ which is $\pi/(4M_L)$. We conclude that there exists some $j \in \{1, ..., M_{L-1}\}$ such that

$$\triangle\theta_j = O(\bar{D}_L/L) = o(L^{-1/2}). \tag{4.8}$$

We now focus attention on the two half spaces (in $\mathbb{Z}^d$), $\Lambda = \Lambda(\hat{x}(\theta_j), L_j)$ and $\Lambda' = \Lambda(\hat{x}(\theta_{j+1}), L_{j+1})$ and the passage times $T = T(0, \partial\Lambda)$, $T' = T(0, \partial\Lambda')$. The (random) sets of endpoints of exactly minimizing paths are $R = R(\partial\Lambda; 0)$ and $R' = R(\partial\Lambda'; 0)$ and those of almost minimizing paths are $R(K) = R(\partial\Lambda; K)$ and $R'(K) = R(\partial\Lambda'; K)$. The deterministic replacements for $R(K)$ and $R'(K)$ are (see (4.7)) $\bar{A} = \bar{A}(\theta_j, L_j)$ and $\bar{A}' = \bar{A}(\theta_{j+1}, L_{j+1})$. $\bar{A}$ and $\bar{A}'$ are of diameter at most $\bar{D}_L$ and (by property (f) of Proposition 4.1) are within a bounded (as $L \to \infty$) distance of the midpoints of the respective segments $\mathscr{L} = \mathscr{L}_j$ and $\mathscr{L}' = \mathscr{L}_{j+1}$. Each of the two segments is of length at least $4\bar{D}_L$. Note that the distance between $\bar{A}$ and $\bar{A}'$ must be at least $\bar{D}_L - c_{14}$ with $c_{14} = \bar{\alpha}/2$. Since $\bar{D}_L \geq c_{10} = 2 + \bar{\alpha}/2$ (see (4.5)-(4.6)), this distance exceeds one.

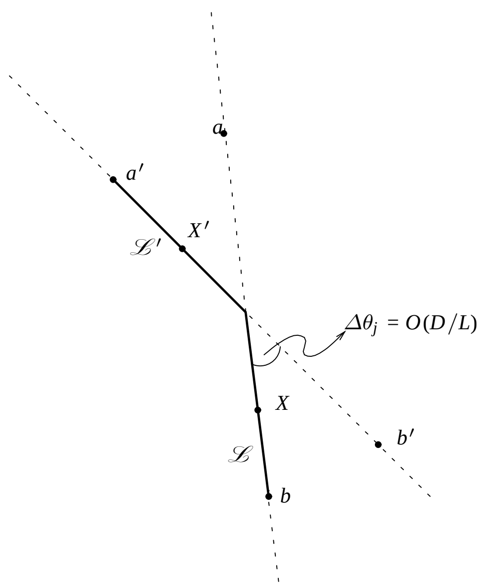

Fig. 6. The segments $\mathscr{L} = \mathscr{L}_j$ and $\mathscr{L}' = \mathscr{L}_{j+1}$ have lengths of order D. The sets $\bar{A}$ and $\bar{A}'$ are near the midpoints X and X'. The doubled segment $\hat{\mathscr{L}}$ (resp. $\hat{\mathscr{L}}'$) extends from a to b (resp. a' to b') . Every point in $\hat{\mathscr{L}}$ is within distance $O(D^2/L)$ of some point in $\hat{\mathscr{L}}'$ and vice-versa. In Sect. 4, $D = o(L^{1/2})$, and in Sect. 5, $D = O(L^{\gamma'})$

Because of (4.8), simple Euclidean geometry (see Fig. 6) shows that $\bar{A}$ must be close enough to $\partial\Lambda'$ so that every site in $\bar{A}$ is either in $\partial\Lambda'$ or has a neighbor in $\partial\Lambda'$ (with a similar closeness result for $\bar{A}'$ and $\partial\Lambda$). Let B denote the set of sites in $\partial\Lambda'$ which are in $\bar{A}\bigcup\partial\bar{A}$ and let $\mathscr{B}$ denote the set of edges from sites in $\bar{A}$ to sites in B. Similarly, let B' denote the set of sites in $\partial\Lambda$ which are in $\bar{A}'\bigcup\partial\bar{A}'$ and let $\mathscr{B}'$ denote the set of edges from sites in $\bar{A}'$ to sites in B'. Let $\mathscr{F}_0$ denote the σ-field generated by all $\tau(e)$'s *except* those in $\mathscr{B}\bigcup\mathscr{B}'$. Let T_0 (resp. $T_0{}'$) denote the inf of passage times $T(r)$ among all paths r from the origin to $\bar{A}$ (resp. to $\bar{A}'$) which only use edges *other* than those in $\mathscr{B}\bigcup\mathscr{B}'$. T_0 and $T_0{}'$ may be regarded as approximations to T and T' which (unlike T and T') are $\mathscr{F}_0$-measurable. By conditioning on $\mathscr{F}_0$, we have (compare (A.12) of Appendix A)

$$P(T(0,B) \le T_0 + K|\mathscr{F}_0) \ge P(\tau(e) \le K) \tag{4.9}$$

(with a similar inequality interchanging primed and unprimed quantities).

Now at least one of the two events, $T_0 \le T_0'$ or $T_0' \le T_0$ must occur with probability at least 1/2. Suppose (without loss of generality) that $P(T_0 \le T_0') \ge 1/2$. Then from (4.9),

$$P(T(0,B) \le T_0' + K) \ge (1/2)P(\tau(e) \le K). \tag{4.10}$$

Although T_0' may exceed $T(0,\bar{A}')$, it can only do so if a minimizing path for $T(0,\bar{A}')$ either passes through B, in which case $T(0,B) \le T(0,\bar{A}')$, or else passes through B', in which case $R'(K) \not\subseteq \bar{A}'$; we conclude that $T(0,B) \le T_0'+K$ implies that either $T(0,B) \le T(0,\bar{A}') + K$ or $R'(K) \not\subseteq \bar{A}'$ and so

$$P(T(0,B) \le T(0,\bar{A}') + K \text{ or } R'(K) \not\subseteq \bar{A}') \ge (1/2)P(\tau(e) \le K) > \epsilon. \tag{4.11}$$

We now show that this together with the fact (see (4.3)) that

$$P(R'(K) \subseteq \bar{A}') \ge 1 - \epsilon \tag{4.12}$$

leads to a contradiction.

The contradiction comes about because (4.11)-(4.12) imply

$$P(R'(K) \subseteq \bar{A}' \text{ and } T(0,B) \le T(0,\bar{A}') + K) > 0. \tag{4.13}$$

On the other hand, since $R' \equiv R(\partial\Lambda';0) \subseteq R(\partial\Lambda';K) \equiv R'(K)$, $R'(K) \subseteq \bar{A}'$ implies $R' \subseteq \bar{A}'$ so that then $T(0,\bar{A}') = T(0,\partial\Lambda')$ and so $T(0,B) \le T(0,\partial\Lambda')+K$ which implies (from the definition (2.11) of $R(\partial\Lambda';K)$) that some site in B must belong to $R'(K)$. But this contradicts $R'(K) \subseteq \bar{A}'$ since B and $\bar{A}'$ are disjoint. This is because all sites in B are within distance one of $\bar{A}$ and the distance from $\bar{A}$ to $\bar{A}'$ is (as mentioned above) strictly bigger than one. The proof of Theorem 2 is now complete.

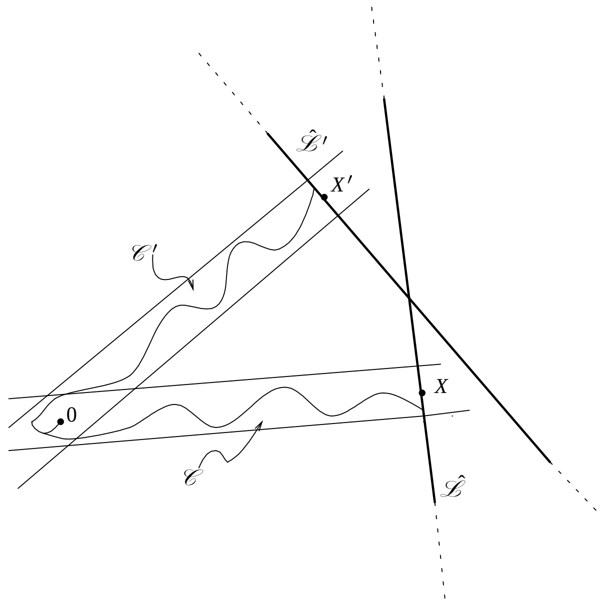

Fig. 7. The cylinders $\mathscr{C} \equiv \mathscr{C}(X, L^\gamma)$ and $\mathscr{C}' \equiv \mathscr{C}(X', L^\gamma)$ and minimizing paths confined to the cylinders $\mathscr{C}$ and $\mathscr{C}'$ respectively. The distance between X and X' is of order $L^{\gamma'}$ and thus the intersection of $\mathscr{C}$ and $\mathscr{C}'$ is within distance $O(L^{(1-(\gamma-\gamma')})$ of the origin

5. The bound $\xi(2) \geq 3/5$

In this section, we prove Theorem 3. Once more, we begin with a heuristic sketch of the proof, restricted to the case where $P(\tau(e) = 0) > 0$. This proof combines the geometric argument which showed $\xi(d) \geq 1/2$ (Theorem 2) with the martingale-based argument which showed $\xi(d) \geq 1/(d+1)$ (Theorem 1) to yield $\xi(2) \geq 3/5$, a seeming counterexample to the principle of "no free lunch". We choose any $\gamma' > \xi$ and use the same polygons and same specially chosen segments $\mathscr{L}_j$ and $\mathscr{L}_{j+1}$ as in our sketch of the proof of Theorem 2. But now we note that the time-minimizing paths are (with high probability) contained within narrow strips terminating near the centers of the two segments (see Fig. 7). As in Theorem 2, the distance of each termination region from the linear extension of the other segment is $O(L^{2\gamma'-1})$. Thus the difference δT between the two passage times has $\mathrm{var}(\delta T) = O([L^{2\gamma'-1}]^2)$. On the other hand, like in Theorem 1, there is a lower bound on this variance coming from a martingale-based inequality (see (3.4)) combined with the restriction (with high probability) of the respective minimizing paths to the mostly disjoint strips of Fig. 7 (as in [WA, AW]). This lower bound is of order $L^{1-(d-1)\gamma'} = L^{1-\gamma'}$, so that to avoid a contradiction, it must be that $2(2\gamma'-1) \geq 1-\gamma'$ or $\gamma' \geq 3/5$. Since this is valid for any $\gamma' > \xi^{(3)}$, we obtain $\xi^{(3)} \geq 3/5$. We now proceed with the detailed proof of Theorem 5.

As in our proof of $\xi(d) \geq 1/2$ in the last section, we choose a fixed positive $\epsilon < \frac{1}{2}P(\tau(e) \leq K)$. Because $\xi_K^{(3)} \geq \xi_K^{(2)}$, we see that for any $\gamma' > \xi_K^{(3)}$, the quantity $\bar{D}_L$, as defined in Sect. 4, is $o(L^{\gamma'})$. If $\gamma' < 1$, then for large L, the hypothesis of Proposition 4.1 that $\bar{D}_L \leq \frac{1}{2}\sin(\pi/8)L$ is valid and we will use the polygon construction of that proposition, but this time with $D = L^{\gamma'}$ (rather than $\bar{D}_L$).

As in the derivation of (4.8), there is some j such that

$$\triangle\theta_j = O(D/L) = O(L^{-(1-\gamma')}). \tag{5.1}$$

As in that part of Sect. 4, we focus on the half space $\Lambda = \Lambda(\hat{x}(\theta_j), L_j)$ and $\Lambda' = \Lambda(\hat{x}(\theta_{j+1}), L_{j+1})$ and the passage times T, T' along with the random sets R, R' and their deterministic replacements $\bar{A}$ and $\bar{A}'$.

Let X and X' respectively denote the midpoints of the line segments $\mathscr{L} = \mathscr{L}_j$ and $\mathscr{L}' = \mathscr{L}_{j+1}$ of the polygon. Note that because $\mathscr{L}$ and $\mathscr{L}'$ have lengths at least $4L^{\gamma'}$ and $\triangle\theta_j \to 0$ as $L \to \infty$ by (5.1), the distance between X and X' is at least $3L^{\gamma'}$ for large L. We choose some γ in $(\xi_K^{(3)}, \gamma')$. Because $\gamma > \xi_K^{(3)}$ and because $\bar{A}$ is at a bounded distance from X and has a diameter which is $o(L^{\gamma})$, it follows that the set of approximately minimizing paths, $\mathscr{M}((0, \partial\Lambda); K)$, and hence $\mathscr{M} \equiv \mathscr{M}(0, \partial\Lambda)$, the set of actually minimizing paths, is asymptotically contained in the cylinder $\mathscr{C} \equiv \mathscr{C}(X, L^{\gamma})$(see Fig. 7); i.e.,

$$P(\mathscr{M} \text{ is in } \mathscr{C}) \to 1 \text{ as } L \to \infty. \tag{5.2}$$

Similarly, $\mathscr{M}' \equiv \mathscr{M}(0, \partial\Lambda')$ is contained in $\mathscr{C}' \equiv \mathscr{C}(X', L^{\gamma})$:

$$P(\mathscr{M}' \text{ is in } \mathscr{C}') \to 1 \text{ as } L \to \infty. \tag{5.3}$$

To prove that $\xi_K^{(3)} \geq 3/5$, we will show that (5.1)-(5.3) imply that $\gamma' \geq 3/5$. The argument for doing that is a modification of the one used in Sect. 3 to prove $\xi^{(1)} \geq 1/(d+1)$, as we now explain.

Roughly speaking, we want to use T and T' as replacements for the $\hat{T}_n^l$ and $\hat{T}_n^u$ of Sect. 3 (see the discussion surrounding (3.14) there). But a basic ingredient in Sect. 3 was an upper bound on $\hat{T}_n^l - \hat{T}_n^u$ (or, in the case of unbounded $\tau(e)$'s, a bound on its variance) and we cannot easily get a useful upper bound directly for $T - T'$. So we will approximate T and T' by two other variables, as follows.

The endpoints of $\mathscr{L} = \mathscr{L}_j$ are $z^{(j-1)}$ and $z^{(j)}$ and those of $\mathscr{L}' = \mathscr{L}_{j+1}$ are $z^{(j)}$ and $z^{(j+1)}$. Let us extend $\mathscr{L}$ in the direction of $\mathscr{L}'$ by doubling its length; i.e., we define $\hat{\mathscr{L}}$ as the segment with endpoints $z^{(j-1)}$ and $z^{(j)} + (z^{(j)} - z^{(j-1)})$. Similarly we extend $\mathscr{L}'$ in the direction of $\mathscr{L}$ by defining $\hat{\mathscr{L}}'$ to have endpoints $z^{(j)} - (z^{(j+1)} - z^{(j)})$ and $z^{(j+1)}$. Let $\hat{\partial}\Lambda$ (resp., $\hat{\partial}\Lambda'$) be the set of sites in $\partial\Lambda$ (resp., $\partial\Lambda'$) within distance one of $\hat{\mathscr{L}}$ (resp. $\hat{\mathscr{L}}'$) and then define $\hat{T} = T(0, \hat{\partial}\Lambda)$ and $\hat{T}' = T(0, \hat{\partial}\Lambda')$ with corresponding sets of minimizing paths $\hat{\mathscr{M}}$ and $\hat{\mathscr{M}}'$. Because of (5.1) and the lengths of $\hat{\mathscr{L}}$ and $\hat{\mathscr{L}}'$, it follows that for large L, every site in $\hat{\partial}\Lambda$ is within distance $c_{15}L^{2\gamma'-1}$ of $\hat{\partial}\Lambda'$ (where c_{15} is a geometrical constant) and vice-versa (see Fig. 6). It is not hard to show that consequently (and analogously to the bound $\mathrm{var}(\delta\hat{T}_n) = O({m_n}^2)$ in the proof of Sect. 3 that $\xi^{(1)} \geq 1/(d+1)$ for general $\tau(e)$'s)

$$\mathrm{var}(\hat{T} - \hat{T}') = O([L^{2\gamma'-1}]^2). \tag{5.4}$$

The remainder of the proof is to obtain a lower bound on the variance, as a consequence of (5.2)-(5.3), of order $L^{1-\gamma}$ (i.e., $L^{1-(d-1)\gamma}$ with $d = 2$). This will imply a contradiction unless $2(2\gamma' - 1) \geq 1 - \gamma \geq 1 - \gamma'$, i.e., unless $\gamma' \geq 3/5$.

By essentially the same arguments used in Sect. 3 for the proof that $\xi^{(0)} \geq 1/(d+1)$ for general $\tau(e)$'s, we obtain the bound

$$\mathrm{var}(\hat{T} - \hat{T}') \geq c_8 \sum_{j=1}^{\infty} [c_9 P(\hat{\mathscr{D}}_j \geq b) - P(\hat{\mathscr{D}}_j' \geq \lambda)]_+^2, \tag{5.5}$$

where $\hat{\mathscr{D}}_j$ and $\hat{\mathscr{D}}_j'$ are respectively exactly as in (3.19) but with $T_n^\sharp$ replaced respectively by $\hat{T}$ and $\hat{T}'$.

We next restrict the summation in (5.5) to a subset of the j's. We define $\hat{\mathscr{C}}$ as the intersection of $\mathscr{C}$ and the box $B_{2\|X\|}$; its area is of order $L^{1+\gamma}$. Denoting by $\hat{\sum}_j$ the restricted sum over j's with e_j in $\hat{\mathscr{C}}$, we then have from (5.5) (compare (3.30)),

$$\mathrm{var}(\hat{T} - \hat{T}') \geq (c_8/|\hat{\mathscr{C}}|)[c_9 \hat{\sum}_j P(\hat{\mathscr{D}}_j \geq b) - \hat{\sum}_j P(\hat{\mathscr{D}}_j' \geq \lambda)]_+^2, \tag{5.6}$$

where $|\hat{\mathscr{C}}|$, the number of edges in $\hat{\mathscr{C}}$, is of order $L^{1+\gamma}$. As in Sect. 3, it only remains to show that

$$\liminf_{L\to\infty} L^{-1} \hat{\sum}_j P(\hat{\mathscr{D}}_j \geq b) > 0, \tag{5.7}$$

and

$$\limsup_{L\to\infty} L^{-1} \hat{\sum}_j P(\hat{\mathscr{D}}_j' > \lambda') = 0. \tag{5.8}$$

To obtain (5.7), we begin by noting that if $\mathscr{M}$ is in $\mathscr{C}$ then any minimizing path for $T = T(0, \partial\Lambda)$ automatically has its right endpoint in (a subset of) the extended interval $\hat{\mathscr{L}}$ and so is a minimizing path for $\hat{T}$. Thus $\mathscr{M}$ in $\mathscr{C}$ implies $\hat{\mathscr{M}}$ is in $\hat{\mathscr{C}}$ and so by (5.2) and the shape theorem, $P(\hat{\mathscr{M}}$ is in $\hat{\mathscr{C}}) \to 1$ as $L \to \infty$. Our analogue of (3.34) in this case is then

$$\begin{aligned} \hat{\sum}_j P(\hat{D}_j \geq b) &= \hat{p}^{-1} \hat{\sum}_j P(\hat{D}_j \geq b, \tau(e_j) < b) \\ &\geq \hat{p}^{-1} E[\hat{W}_L 1_{\hat{\mathscr{M}} \text{ is in } \hat{\mathscr{C}}}] \\ &\geq \hat{p}^{-1} E[\hat{W}_L] - \hat{p}^{-1} [E(\hat{W}_L^2)(1 - P(\hat{\mathscr{M}} \text{ is in } \hat{\mathscr{C}}))]^{1/2}, \end{aligned} \tag{5.9}$$

where $\hat{W}_L$ denotes the total number of edges having $\hat{D}_j \geq b$ and $\tau(e_j) < b$. Now $\hat{W}_L \leq Z_L$, the minimum length among paths in $\hat{\mathscr{M}}$, and it follows from Proposition A1 that $L^{-2} E(\hat{W}_L^2)$ is bounded away from infinity. Because $P(\hat{\mathscr{M}}$ is in $\hat{\mathscr{C}}) \to 1$, we thus have

$$\begin{aligned} \liminf_{L\to\infty} L^{-1} \hat{\sum}_j P(\hat{D}_j \geq b) &\geq \liminf_{L\to\infty} \hat{p}^{-1} L^{-1} E(\hat{W}_L) \\ &= \liminf_{L\to\infty} L^{-1} \sum_{j=1}^{\infty} P(\hat{D}_j \geq b). \end{aligned} \tag{5.10}$$

As in the discussion following (3.35), to obtain (5.7) it now suffices to show that $E(W_L)/L$ is bounded away from zero, where W_L is the number of minimizing b-edges for $\hat{T}$. This now follows from part c) of Proposition B2.

To obtain (5.8), we define $\tilde{C}$ as $\hat{C}\backslash B_{L^{1-\delta}}$, where $0 < \delta < \gamma' - \gamma$, and denote by $\tilde{\sum}_j$ the restricted sum over j's with e_j in $\tilde{C}$. The removal of the box $B_{L^{1-\delta}}$ makes $\tilde{C}$ disjoint from $\mathscr{C}'$ for large L (see Fig. 7). We note that if $\mathscr{M}'$ is in $\mathscr{C}'$, then any minimizing path for $\hat{T}'$ is also a minimizing path for T' and so is in $\mathscr{C}'$ and hence (for large L) does not touch $\tilde{C}$. Thus, writing

$$L^{-1}\hat{\sum_j} P(\hat{D}_j' > \lambda') = L^{-1}\tilde{\sum_j} P(\hat{D}_j' > \lambda') + L^{-1}\sum_j^R P(\hat{D}_j' > \lambda'), \tag{5.11}$$

where $\sum_j{}^R$ denotes the sum over the remaining edges in $\hat{\mathscr{C}}$ which are not in $\tilde{\mathscr{C}}$, we have for the first term on the RHS, as in the discussion following (3.33), the bound

$$\begin{aligned} L^{-1}\tilde{\sum}_j P(\hat{D}_j' > \lambda') &\le L^{-1}(p')^{-1}\tilde{\sum}_j P(\text{every path in } \hat{\mathscr{M}}' \text{ passes through } e_j) \\ &\le (p')^{-1}L^{-1}\sum_{j=1}^{\infty} P(\{\text{every path in } \hat{\mathscr{M}}' \text{ passes through } e_j\} \\ &\cap\{\mathscr{M}' \text{ is in } \mathscr{C}'\}^c) \\ &\le (p')^{-1}[E((Z_L/L)^2)(1 - P(\mathscr{M}' \text{ is in } \mathscr{C}'))]^{1/2}, \end{aligned} \tag{5.12}$$

where Z_L is the minimum length among minimizing paths for $\hat{T}'$. Because of (5.3), to show that the first term on the RHS of (5.11) tends to zero, it suffices if $E((Z_L/L)^2)$ is bounded as $L \to \infty$. That follows from Proposition A1.

For the second term on the RHS of (5.11), we have, where we denote by $\partial(L)$ the boundary of $B_L \cap \mathbb{Z}^2$,

$$\begin{aligned} L^{-1}\sum_j{}^R P(\hat{D}_j' > \lambda') &\le (p')^{-1}L^{-1}E\inf_{r\in\hat{\mathscr{M}}'}[\text{no. of edges in } r \text{ touching } B_{L^{1-\delta}}] \\ &\le (p')^{-1}L^{-1}E\sup_{v\in\partial(L^{1-\delta})}\sup_{r\in\mathscr{M}(0,v)}[\text{no.of edges in } r] \\ &\le (p')^{-1}L^{-1}E\sup_{v\in\partial(L^{1-\delta})}Z^*(v) \\ &= (p')^{-1}L^{-1}E(Z^*_{L^{1-\delta}}), \end{aligned} \tag{5.13}$$

where $Z^*(v)$ is defined in (B.1) of Appendix B and Z_L^* is defined as in (B.1) but with $T(0, v)$ replaced by

$$T_L = \sup_{v\in\partial(L)} T(0, v). \tag{5.14}$$

We claim that $E(T_L) = O(L)$ so that, by the Remark following Proposition B1, $E(Z^*_{L^{1-\delta}}) = O(L^{1-\delta})$, which shows that the last expression in (5.13) tends to zero. The claim is justified by using the fact that for each $v \in \partial(L)$, $E(T(0, v)) \le c_{16}L$ and (from [K3]), $\text{var}(T(0, v)) \le c_{17}L$, so

$$\begin{aligned} E(T_L/L) &= \int_0^{\infty} P(T_L/L \ge y)dy \\ &\le c_{18} + \int_{2c_{16}}^{\infty}\{\textstyle\sum_{v\in\partial(L)} P(T(v)/L \ge y)\}dy \\ &\le c_{18} + \int_{2c_{16}}^{\infty}\{\textstyle\sum_{v\in\partial(L)} P[T(v) - E(T(0, v)) \ge (y - c_{16})L]\}dy \\ &\le c_{18} + \int_{c_{16}}^{\infty}\{\textstyle\sum_{v\in\partial(L)}(xL)^{-2}\text{var}(T(0, v))\}dx \\ &\le c_{18} + \int_{c_{16}}^{\infty} c_{19}x^{-2}dx \equiv c_{20}. \end{aligned} \tag{5.15}$$

Here, c_{16} through c_{20} depend only on the common distribution of the $\tau(e)$'s. Although this last series of bounds is far from optimal, it does serve to verify our claim (for $d = 2$) and complete the proof of Theorem 3.

Appendix A

Continuity for point-to-plane passage

In this appendix, we consider standard first-passage percolation on $\mathbb{Z}^d$ (with $d \geq 2$) determined by i.i.d. variables $\{\tau(e) : e \in \mathbb{E}^d\}$ with $P(\tau(e) = 0) < p_c$, the critical value for standard independent bond percolation on $\mathbb{Z}^d$. As usual, $T(u, v)$ denotes the passage time between sites $u, v \in \mathbb{Z}^d$ and for $\Gamma \subseteq \mathbb{Z}^d$, $T(u, \Gamma)$ denotes the passage time between u and Γ (see (1.2)-(1.3)).

For any deterministic Γ, it is not hard to see that (a.s.) the inf in the definition (1.3) of $T(u, \Gamma)$ is assumed, that is $\{v \in \Gamma : T(u, v) = T(u, \Gamma)\}$ is nonempty. More generally, we define for any $K \geq 0$,

$$R_u(\Gamma; K) = \{v \in \Gamma : T(u, v) \leq T(u, \Gamma) + K\}; \tag{A.1}$$

when $u = 0$, the origin, we will delete the subscript. We will focus on $R(\Gamma; K)$ when $\Gamma = \Gamma(\theta)$ is (roughly speaking) a $(d-1)$-dimensional hyperplane rotated by angle θ. Our purpose is to derive some "continuity" properties of $R(\Gamma(\theta); K)$ as a function of θ. For the purposes of this paper, we take fixed real $z_1 > 0$, $z_2 > 0$ and consider the half-space

$$\Lambda(\theta) = \{y = (y_1, \ldots, y_d) \in \mathbb{Z}^d : y_1 \cos\theta + y_2 \sin\theta < z_1 \cos\theta + z_2 \sin\theta\}; \tag{A.2}$$

and its boundary

$$\partial\Lambda(\theta) = \{x \in \mathbb{Z}^d \setminus \Lambda(\theta) : \|x - y\| = 1 \text{ for some } y \in \Lambda(\theta)\}. \tag{A.3}$$

As θ varies, $\partial\Lambda(\theta)$ rotates around the subspace $\{(z_1, z_2, \hat{z}) : \hat{z} \in \mathbb{R}^{d-2}\}$. We will restrict attention to $0 \leq \theta \leq \pi/2$.

Rather than deal directly with the random sets $R(\partial\Lambda(\theta); K)$, we will replace them with deterministic sets $A(\theta) \subseteq \partial\Lambda(\theta)$ such that

$$P(R(\partial\Lambda(\theta); K) \subseteq A(\theta)) \geq 1 - \epsilon, \tag{A.4}$$

where ϵ must be sufficiently small so that

$$0 < \epsilon < \frac{1}{2} P(\tau(e) \leq K). \tag{A.5}$$

Evidently, for such an ϵ and $A(\theta)$ to exist, K must be such that

$$P(\tau(e) \leq K) > 0. \tag{A.6}$$

Note that $K = 0$ is allowed if the common distribution of the $\tau(e)$'s has an atom at 0. For the remainder of the appendix, we suppose that fixed $K \geq 0$ and ϵ

satisfying (A.5)-(A.6) have been chosen and that for the fixed $z_1 > 0$ and $z_2 > 0$, a deterministic $A(\theta)$ satisfying (A.4) has been chosen for each $\theta \in [0, \pi/2]$.

The first continuity proposition is as follows. We postpone presenting the proof until stating the second continuity proposition, which is a consequence of the first.

Proposition A1. *If $\theta_n \to \bar{\theta}$, then*

$$\mathrm{dist}(A(\theta_n), A(\bar{\theta})) \leq 1, \text{ for all large } n. \tag{A.7}$$

As θ varies, we want to consider certain slabs $S(\theta)$ which rotate along with $\partial\Lambda(\theta)$. For a fixed real γ and $\alpha \geq 1$, the slabs $S(\theta)$ are defined as

$$S(\theta) = \{y \in \mathbb{Z}^d : -\frac{\alpha}{2} \leq [y_1(-\sin\theta)+y_2(\cos\theta)]-[z_1(-\sin\theta)+z_2(\cos\theta)]-\gamma \leq \frac{\alpha}{2}\}. \tag{A.8}$$

This is the same as the $S(\theta, \gamma', D')$ defined in (4.4) with $D' = \alpha$ and $\gamma' = z_1(-\sin\theta)+z_2(\cos\theta)+\gamma$ (see Fig. 4). Roughly speaking, this is a slab of thickness α centered on a $(d-1)$-dimensional plane which intersects the plane $\partial\Lambda(\theta)$ in the $(d-2)$-dimensional subspace $(z_1, z_2) \times \mathbb{R}^{d-2} + \gamma(-\sin\theta, \cos\theta, \hat{0})$. The next proposition will be used to choose a θ so that $A(\theta)$ intersects $S(\theta)$. We say $A(\theta)$ is α'-connected, where $\alpha' > 0$, if every pair of sites u and v in $A(\theta)$ is connected by a sequence of sites in $A(\theta)$, $u_0 = u, u_1, \ldots, u_n = v$, with $\|u_i - u_{i-1}\| \leq \alpha'$ for every i. We say $A(\theta)$ is below (resp. above) $S(\theta)$ if every $y \in A(\theta)$ satisfies

$$[y_1(-\sin\theta) + y_2(\cos\theta)] - [z_1(-\sin\theta) + z_2(\cos\theta)] - \gamma < -\frac{\alpha}{2} \text{ (resp. } > \frac{\alpha}{2}). \tag{A.9}$$

Proposition A2. *Suppose $0 \leq \theta_i < \theta_f \leq \pi/2$ are such that $A(\theta_i)$ is below $S(\theta_i)$ and $A(\theta_f)$ is above $S(\theta_f)$. Suppose also that each $A(\theta)$ is α'-connected with $\alpha' \leq \alpha$. Then there exists some $\theta \in (\theta_i, \theta_f)$ such that $A(\theta)$ intersects $S(\theta)$.*

Proof of Proposition A1. Let us denote $\Lambda(\theta_n)$ by Λ_n and $\Lambda(\bar{\theta})$ by Λ. It is not necessarily the case that $\Lambda_n \to \Lambda$, as can be seen, for example, in the case where z_1 and z_2 are integers and $\bar{\theta}$=0. By considering separately the subsequence with $\theta_n \geq \bar{\theta}$ and the one with $\theta_n \leq \bar{\theta}$, we may without loss of generality assume, say, that $\theta_n \leq \bar{\theta}$ for all n. Then, since Λ is an *open* half-space (intersected with $\mathbb{Z}^d$), we have that $\Lambda_n \to \Lambda'$ (i.e., for every finite $B \subset \mathbb{Z}^d$, $\Lambda_n \bigcap B = \Lambda \bigcap B$ for all large n) and $\partial\Lambda_n \to \partial\Lambda'$, where Λ' is some subset of $\mathbb{Z}^d$ satisfying

$$\Lambda \subseteq \Lambda' \subseteq \Lambda^* \equiv \Lambda \cup \partial\Lambda. \tag{A.10}$$

Note that this implies that $T(0, \partial\Lambda') \geq T(0, \partial\Lambda)$. Using (A.10) and the fact that every $y \in \partial\Lambda$ has some (nearest) neighbor in the complement of Λ^*, it follows that every $y \in \partial\Lambda$ is either in $\partial\Lambda'$ or else has a nearest neighbor in $\partial\Lambda' \backslash \partial\Lambda$. We let $\mathscr{E}(y)$ denote the edge leading from y to one such neighbor.

Let $\mathscr{F}$ denote the σ-field generated by the $\tau(e)$'s with at least one site of e in Λ. Conditional on $\mathscr{F}$, one can determine $T(0, \partial\Lambda)$ and find a path within

Λ (except for its last edge) from 0 to some $y \in \partial\Lambda$ whose passage time equals $T(0, \partial\Lambda)$. If $y \in \partial\Lambda'$ then $T(0, \partial\Lambda') = T(0, \partial\Lambda)$; if $y \notin \partial\Lambda'$, then still

$$T(0, \partial\Lambda') \leq T(0, \partial\Lambda) + \tau(\mathscr{E}(y)). \tag{A.11}$$

This immediately implies that (a.s.)

$$P(T(0, \partial\Lambda') \leq T(0, \partial\Lambda) + K | \mathscr{F}) \geq P(\tau(e) \leq K). \tag{A.12}$$

We further claim that if $T(0, \partial\Lambda') \leq T(0, \partial\Lambda) + K$, then

$$\text{dist}(R(\partial\Lambda'; 0), R(\partial\Lambda; K)) \leq 1. \tag{A.13}$$

To see this, find a path within Λ' (except for its last edge) from 0 to some $z \in \partial\Lambda'$ whose passage time equals $T(0, \partial\Lambda')$. If $z \notin \partial\Lambda$, then by (A.10), its predecessor on the path, which we denote y, is in $\partial\Lambda$; if $z \in \partial\Lambda$, then let $y = z$. In either case, we have

$$T(0, y) \leq T(0, z) = T(0, \partial\Lambda') \leq T(0, \partial\Lambda) + K, \tag{A.14}$$

and thus $y \in R(\partial\Lambda; K)$, which gives (A.13). Combining (A.12) and (A.13) yields that

$$P(\text{dist}(R(\partial\Lambda'; 0), R(\partial\Lambda; K)) \leq 1) \geq P(\tau(e) \leq K). \tag{A.15}$$

We next need to relate $R(\partial\Lambda_n; 0)$ to $R(\partial\Lambda'; 0)$. Since $P(\tau(e) = 0) < p_c$, it follows that for any $\delta > 0$, there is some finite $B_\delta \subset \mathbb{Z}^d$ so that with probability at least $1 - \delta$, $R(\partial\Lambda'; 0)$ and $R(\partial\Lambda(\theta); 0)$ for every $\theta \in [0, 2\pi]$ are contained within B_δ. Since $\partial\Lambda_n \to \partial\Lambda'$, it follows that for $n \geq n_0(\delta)$,

$$P(R(\partial\Lambda_n; 0) = R(\partial\Lambda'; 0)) \geq 1 - \delta. \tag{A.16}$$

Thus (A.15) shows that for $n \geq n_0(\delta)$

$$P(\text{dist}(R(\partial\Lambda_n; 0), R(\partial\Lambda; K)) \leq 1) \geq P(\tau(e) \leq K) - \delta. \tag{A.17}$$

Consider the event appearing on the LHS of (A.17) along with the event in (A.4) with $\theta = \underline{\theta}_n$ (which implies that $R(\partial\Lambda_n; 0) \subseteq A(\theta_n)$) and finally the event in (A.4) with $\theta = \overline{\theta}$. The intersection of these three events has (for large n) probability at least $P(\tau(e) \leq K) - 2\epsilon - \delta$. But intersection is impossible unless $\text{dist}(A(\theta_n), A(\overline{\theta})) \leq 1$. By (A.5), we may choose

$$0 < \delta < P(\tau(e) \leq K) - 2\epsilon, \tag{A.18}$$

so that the intersection has strictly positive probability for all large n, which implies (A.7) and completes the proof of Proposition A1.

Proof of Proposition A2. Let

$$\overline{\theta} = \sup\{\theta \in [\theta_i, \theta_f] : A(\theta) \text{ is below } S(\theta)\}. \tag{A.19}$$

Suppose we can prove that it is neither the case that $A(\bar{\theta})$ is above $S(\bar{\theta})$ nor that $A(\bar{\theta})$ is below $S(\bar{\theta})$. If so, then $A(\bar{\theta})$ must intersect $S(\bar{\theta})$ (proving the proposition), since otherwise $A(\bar{\theta})$ could not be α'-connected with $\alpha' \le \alpha$ (which was assumed in the proposition). If $A(\bar{\theta})$ were above $S(\bar{\theta})$, then $\bar{\theta} > \theta_i$ and there would exist some sequence θ_n (with $\theta_i \le \theta_n < \bar{\theta}$) such that $\theta_n \to \bar{\theta}$ with $A(\theta_n)$ below $S(\theta_n)$ for each n. But then since the width of $S(\bar{\theta})$ is $\alpha \ge 1$, it would follow that $\mathrm{dist}(A(\theta_n), A(\bar{\theta})) > 1$ for all n, which would contradict Proposition A1. If $A(\bar{\theta})$ were below $S(\bar{\theta})$, then $\bar{\theta} < \theta_f$ and there would exist some sequence θ_n (with $\bar{\theta} < \theta_n \le \theta_f$) such that $\theta_n \to \bar{\theta}$ with $A(\theta_n)$ above $S(\theta_n)$ for each n. As in the previous case, this would contradict Proposition A1. Thus $A(\bar{\theta})$ is neither above $S(\bar{\theta})$ nor is it below $S(\bar{\theta})$ and the proof is complete.

Appendix B

Some technical estimates

In this appendix, we present two propositions, concerning standard first-passage percolation on $\mathbb{Z}^d$. The first of these concerns the relation between the lengths and passage times of long paths. It follows from the arguments of [K3]. The second proposition concerns the number of edges in minimizing paths taking values in a specified subset A of the real line; it follows from the arguments of [BK].

Proposition B1 [K3]. *For any $d \ge 2$, assume $P(\tau(e) = 0) < p_c$ and $E(\tau(e)^2) < \infty$. Define for $v \in \mathbb{Z}^d$,*

$$Z^*(v) = \sup\{m : \ \exists \text{ a (site self-avoiding) path } r \text{ starting at 0 with } m \text{ edges such that } T(r) \le T(0, v)\}; \tag{B.1}$$

then there is a $K < \infty$ such that

$$E(Z^*(v)^2) \le K\|v\|^2 \text{ for all } v \in \mathbb{Z}^d. \tag{B.2}$$

Proof. We follow the proof of Eq. (2.25) of [K3]. For each nonzero $v = (v_1, \dots, v_d)$ choose a path $r(v)$ with $\|v\|_1 = |v_1| + \dots + |v_d|$ edges from the origin to v. Then $T(r(v))$, which is the sum of $\|v\|_1$ i.i.d. $\tau(e)$'s, is an upper bound for $T(0, v)$ and

$$\begin{aligned} E((Z^*(v)/\|v\|_1)^2) &= \int_0^\infty P(Z^*(v)^2 \ge y\|v\|_1{}^2)dy = \int_0^\infty 2zP(Z^*(v) \ge z\|v\|_1)dz \\ &\le \int_0^\infty 2zP(T(r(v)) \ge az\|v\|_1)dz + \\ &\qquad \int_0^\infty 2zP(Z^*(v) \ge z\|v\|_1 \text{ and } T(r(v)) < az\|v\|_1)dz \\ &\le (a\|v\|_1)^{-2}E(T(r(v))^2) + \int_0^\infty 2z\,g(z\|v\|_1, az\|v\|_1)dz \\ &= \{\|v\|_1^2[E(\tau(e))]^2 + \|v\|_1 \mathrm{var}(\tau(e))\}[a^2\|v\|_1{}^2]^{-1} \\ &\qquad +2[\|v\|_1{}^2]^{-1}\int_0^\infty z\,g(z, az)dz \end{aligned} \tag{B.3}$$

where

$$g(z,az) = P\{\ \exists \text{ a (site self-avoiding) path } r \text{ from the origin with } \geq z \text{ edges and } T(r) \leq az\}. \tag{B.4}$$

To complete the proof of the proposition, it suffices to show that the final integral in (B.3) is finite for small positive a. To see this, pick some $\delta > 0$ such that $P(\tau(e) \leq \delta) < p_c$. If we call an edge e open if $\tau(e) \leq \delta$ and closed if $\tau(e) > \delta$, we may regard this as a subcritical standard bond percolation model. In order that $T(r) \leq az$, the path r must clearly have $\leq az/\delta$ closed edges. Thus

$$g(z,az) \leq P(\ \exists \text{ a (site self-avoiding) path } r \text{ from the origin with } \geq z \text{ edges, at most } za/\delta \text{ of which are closed}). \tag{B.5}$$

It is known [K1, Proposition 5.8] that in subcritical percolation, for a/δ sufficiently small, the RHS of (B.5) is bounded by $\exp(-c_5 z)$ for some $c_5 > 0$.

Remark. A minor variation of the above proof shows that if T_n is a sequence of random variables with $E(T_n) = O(n)$, then Z_n^*, defined as in (B.1) but with $T(0,v)$ replaced by T_n, has $E(Z^*) = O(n)$.

Our next proposition states a result concerning the number of edges in a minimizing path that have passage times assuming values in a specified range. This result essentially follows from the arguments of van den Berg and Kesten [BK]. We start by ordering all paths on $\mathbb{Z}^d$ in some arbitrary (deterministic) way and for a subset $\Gamma \subset \mathbb{Z}^d$ we define $\pi(\Gamma)$ as the first path in that ordering such that $\pi(\Gamma) \in \mathscr{M}(0,\Gamma)$.

Proposition B2 [BK]. *For any $d \geq 2$, assume that either (1.5) or (1.6) is valid and that $E(\tau(e)^2) < \infty$. Let A be a Borel set on the real line with $P(\tau(e) \in A) > 0$. Then:*

a) $\liminf_{\|v\|\to\infty} \|v\|^{-1} E\{\text{no. of edges } e \in \pi(v) \text{ with } \tau(e) \in A\} > 0.$

b) $\liminf_{L\to\infty} L^{-1} \inf_{\hat{x}} E\{\text{ no. of edges } e \in \pi(\partial\Lambda(\hat{x},L)) \text{ with } \tau(e) \in A\} > 0.$

c) Let $\Gamma_L \subset \mathbb{Z}^d$ with $\operatorname{diam}\Gamma_L = o(L)$ and $\liminf_{L\to\infty} L^{-1} \operatorname{dist}(0,\Gamma_L) > 0$, then

$$\liminf_{L\to\infty} L^{-1} E\{\text{no. of edges } e \in \pi(\Gamma_L) \text{ with } \tau(e) \in A\} > 0.$$

Remark. In [BK] it is pointed out that part a) in the case where $v = n\hat{e}_1$, with $\hat{e}_1 = (1,0,\ldots,0)$, is a consequence of their Theorem 2.13. The general case stated above can be obtained from a simple extension of their arguments and we sketch the proof for completeness.

Proof. Let F denote the common distribution function of the passage times $\tau(e)$. For a unit vector $\hat{x}$ in $\mathbb{R}^d$, denote by $v(n,\hat{x})$, the closest site in $\mathbb{Z}^d$ to $n\hat{x}$ (or anyone of them in the event of a tie). Then the time constant along the direction of $\hat{x}$ is defined by

$$\mu_{\hat{x}}(F) = \lim_{n\to\infty} \frac{T(0,v(n,\hat{x}))}{n}, \tag{B.6}$$

where the existence of the limit a.s. and in L^1 is a consequence of the subadditive ergodic theorem. A crucial result is that if F and G are distribution functions with finite means satisfying (1.5) and (1.6) and if $F(x) \geq G(x)$ for all $x \in \mathbb{R}$, with $F \not\equiv G$, then for all unit vectors $\hat{x}$, $\mu_{\hat{x}}(G) > \mu_{\hat{x}}(F)$. This result is stated in [BK] (see their Theorem 2.13) for $\hat{x} = \hat{e}_1$ but the result for general $\hat{x}$ follows from the same arguments. In particular, the key step in the arguments of van den Berg and Kesten is a *local* modification argument (see p.73 and p.78 of [BK]) for certain boxes crossed by a minimizing path; the value of $\hat{x}$ is irrelevant to that argument. We also remark that similar modification arguments were used in the general $\hat{x}$ setting in [NP].

Proposition B2 is a now straightforward consequence of this general result of van den Berg and Kesten. To see this, assume that the conclusion of part a) does not hold. Then we can find a bounded A with $P(\tau(e) \in A) > 0$ and (by compactness) a sequence of points $v_n \in \mathbb{Z}^d$ with $\|v_n\| \to \infty$ and $v_n/\|v_n\| \to$ some $\hat{x}$ (so that by the continuity of $\mu_{\hat{x}}(F)$, $\lim_{n\to\infty}\|v_n\|^{-1}E[T(0,v_n)] = \mu_{\hat{x}}(F)$) such that

$$\liminf_{n\to\infty} \frac{E[\text{no. of } e \in \pi(v) \text{ with } \tau(e) \in A]}{\|v_n\|} = 0. \tag{B.7}$$

Consider now passage times having distribution G obtained from F by pushing the mass from the set A to the set $A+\delta$ with $\delta > 0$, as follows: G is the distribution of $h(\tau(e))$ where h is defined by $h(t) = t$ for $t \notin A$ and $h(t) = t+\delta$ for $t \in A$. Clearly, $G(x) \leq F(x)$ and $F \not\equiv G$ so we should have $\mu_{\hat{x}}(G) > \mu_{\hat{x}}(F)$. On the other hand a simple argument now shows that if (B.7) holds, then $\mu_{\hat{x}}(G) \leq \mu_{\hat{x}}(F)$, a contradiction. This proves part a).

To obtain part b), we notice that if $\tau(e)$ has distribution F, it follows from the shape theorem that for every unit vector $\hat{x}$, there exists a unit vector $\hat{y}(\hat{x})$, with $\hat{y}(\hat{x}) \cdot \hat{x} \geq 1/\sqrt{2}$, such that

$$\lim_{L\to\infty} \frac{T(0, \partial\Lambda(\hat{x}, L))}{L} = \frac{\mu_{\hat{y}}(F)}{\hat{y} \cdot \hat{x}}; \tag{B.8}$$

$\hat{y}(\hat{x})$ is a direction such that some tangent plane to the boundary of the asymptotic shape, where it intersects the $\hat{y}$-ray, is perpendicular to $\hat{x}$. The proof can now be completed by using an argument along the same lines as the one sketched for part a). We leave the details of the proof of part c) to the reader.

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