Probab. Theory Relat. Fields 117, 361–386 (2000)
Digital Object Identifier (DOI) 10.1007/s004400000052

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Snakes and spiders: Brownian motion on $\mathbb{R}$-trees

Received: 27 April 1999 / Revised version: 14 June 1999 /
Published online: 31 May 2000 –

Abstract. We consider diffusion processes on a class of $\mathbb{R}$-trees. The processes are defined in a manner similar to that of Le Gall's Brownian snake. Each point in the tree has a real-valued "height" or "generation", and the height of the diffusion process evolves as a Brownian motion. When the height process decreases the diffusion retreats back along a lineage, whereas when the height process increases the diffusion chooses among branching lineages according to relative weights given by a possibly infinite measure on the family of lineages. The class of $\mathbb{R}$-trees we consider can have branch points with countably infinite branching and lineages along which the branch points have points of accumulation.

We give a rigorous construction of the diffusion process, identify its Dirichlet form, and obtain a necessary and sufficient condition for it to be transient. We show that the tail σ-field of the diffusion is always trivial and draw the usual conclusion that bounded space-time harmonic functions are constant. In the transient case, we identify the Martin compactification and obtain the corresponding integral representations of excessive and harmonic functions. Using Ray–Knight methods, we show that the only entrance laws for the diffusion are the trivial ones that arise from starting the process inside the state–space. Finally, we use the Dirichlet form stochastic calculus to obtain a semimartingale description of the diffusion that involves local time additive functionals associated with each branch point of the tree.

1. Introduction

Consider the collection $\mathscr{T}$ of bounded subsets of $\mathbb{R}$ that contain their supremum. We can think of the elements of $\mathscr{T}$ as being arrayed in a tree-like structure in the following way. Using genealogical terminology, $h(B) := \sup B$ is the real-valued generation to which $B \in \mathscr{T}$ belongs and $B|t := (B \cap]-\infty, t]) \cup \{t\} \in \mathscr{T}$ for $t \le h(B)$ is the ancestor of B in generation t. For $A, B \in \mathscr{T}$ the generation of the most recent common ancestor of A and B is $\tau(A, B) := \sup\{t \le h(A) \wedge h(B) : A|t = B|t\}$. That is, $\tau(A, B)$ is the generation at which the lineages of A and B diverge. There is a natural genealogical distance on $\mathscr{T}$ given by

$$D(A, B) := [h(A) - \tau(A, B)] + [h(B) - \tau(A, B)].$$

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Research supported in part by NSF grant DMS-9703845.

Mathematics Subject Classification (1991): 60J60, 60J65, 60J50, 60J35

Key words and phrases: Brownian motion – Diffusion – Tree – Dirichlet form – Martin boundary – Entrance boundary – Ray–Knight compactification – Local time – Semimartingale decomposition

Equivalently,

$$D(A, B) := \sup A + \sup B - 2\min\{\sup A, \sup B, \inf(A \triangle B)\},$$

where $\triangle$ denotes the *symmetric difference*.

The metric space $(\mathscr{T}, D)$ is essentially the *real tree* of [DT96, Ter97] (the latter space has as its points the bounded subsets of $\mathbb{R}$ that contain their infimum and the corresponding metric is such that the map from $(\mathscr{T}, D)$ into this latter space given by $A \mapsto -A$ is an isometry). With a slight abuse of nomenclature, we will refer here to $(\mathscr{T}, D)$ as the real tree.

The real tree is an example of an $\mathbb{R}$-*tree*: that is, a metric space $(\mathbf{T}, d)$ satisfying the following axioms (see [DMT96, DT96, Ter97] for an overview of the theory $\mathbb{R}$-trees and [Sha91] for a review of the extensive uses of $\mathbb{R}$-trees in group theory).

Axiom I: For all $x, y \in \mathbf{T}$ there exists a unique isometric embedding $\phi_{x,y}\colon [0, d(x, y)] \to \mathbf{T}$ such that $\phi_{x,y}(0) = x$ and $\phi_{x,y}(d(x, y)) = y$.

Axiom II: For every injective continuous map $\psi\colon [0, 1] \to \mathbf{T}$ one has $\psi([0, 1]) = \phi_{\psi(0),\psi(1)}([0, d(\psi(0), \psi(1))])$.

For the real tree, $\phi_{A,B}(t)$ is given by $A|(h(A) - t)$ for $0 \le t \le h(A) - \tau(A, B)$ and $B|(t - h(A) + 2\tau(A, B))$ for $h(A) - \tau(A, B) \le t \le h(A) + h(B) - 2\tau(A, B) = D(A, B)$.

In this paper we study diffusion processes with state-spaces that are $\mathbb{R}$-trees. We can describe the sort of processes we have in mind very informally for the case of the real tree as follows.

Consider the collection $\mathscr{E}_+$ of subsets $B \subset \mathbb{R} \cup \{+\infty\}$ such that $-\infty < \inf B$ and $\sup B = +\infty \in B$. For $B \in \mathscr{E}_+$ and $t \in \mathbb{R}$, extend the notation introduced above by writing $B|t := (B \cap]-\infty, t]) \cup \{t\}$. We think of $\mathscr{E}_+$ as the collection of doubly infinite lineages in the real tree and of $B|t$ as the individual in the lineage $B \in \mathscr{E}_+$ in generation t.

We can equip $\mathscr{E}_+$ with a metric such that the balls in this metric are of the form $\{B \in \mathscr{E}_+ : B|t = A|t\}$ for some $A \in \mathscr{E}_+$ and $t \in \mathbb{R}$ (see §2). Let μ be a σ-finite Borel measure on $\mathscr{E}_+$ that assigns finite mass to all such balls. Write $\mathscr{E}_+^\mu$ for the (closed) support of μ and $\mathscr{T}^\mu$ for the subset of $\mathscr{T}$ consisting of points of the form $A|t$ for some $A \in \mathscr{E}_+^\mu$ and $t \in \mathbb{R}$. It is not hard to see that $\mathscr{T}^\mu$ is an $\mathbb{R}$-tree. As we observe in §4, $\mathscr{T}^\mu$ is necessarily separable whereas $\mathscr{T}$ is far from being separable - the removal of a single point disconnects $\mathscr{T}$ into a collection of components, the cardinality of which is that of the power set of the reals. Therefore $\mathscr{T}^\mu$ is a much "tamer" object than $\mathscr{T}$. However, $\mathscr{T}^\mu$ can still exhibit exotic phenomena such as points at which countably infinite branching occurs and lineages along which the branch points have points of accumulation.

We will be interested in the $\mathscr{T}^\mu$-valued process X that evolves in the following manner. The real-valued process H, where $H_t = h(X_t)$, evolves as a standard Brownian motion. For small $\epsilon > 0$ the conditional probability of the event $\{X_{t+\epsilon} \in C\}$ given X_t and H is approximately

$$\frac{\mu\{y : y|H_{t+\epsilon} \in C,\ y|H_t = X_t\}}{\mu\{y : y|H_t = X_t\}}.$$

In particular, if $H_{t+\epsilon} < H_t$, then $X_{t+\epsilon}$ is approximately $X_t|H_{t+\epsilon}$. This evolution is reminiscent of Le Gall's *Brownian snake* process (see, for example, [Le 93, Le 94a, Le 94b, Le 95]), with the difference that the "height" process H is a Brownian motion here rather than a reflected Brownian motion and the rôle of Wiener measure on $C(\mathbb{R}_+, \mathbb{R}^d)$ in the snake construction is played here by μ.

There is a large literature on random walks on trees and [Woe94, LP96] are excellent bibliographical references. In particular, there is a substantial amount of work on the Martin boundary of such walks beginning with [DM61, Car72, Saw78]. In the spirit of this paper, the Martin boundary of walks on non-locally finite graphs is dealt with in [CSW93].

The literature on diffusions on tree-like or graph-like structures is more modest. A general construction of diffusions on graphs using Dirichlet form methods is given in [Var85]. Diffusions on tree-like objects are studied in [DJ93, Kre95] using excursion theory ideas, local times of diffusions on graphs are investigated in [EK95, EK96], and an averaging principle for such processes is considered in [FW93]. One particular process that has achieved a substantial amount of attention is the so-called *Walsh's spider*, which is a diffusion on the tree consisting of a finite number of semi-infinte rays emanating from a single vertex (see [Wal78, BPY89, Tsi97, BEK$^+$98]).

The plan of the rest of the paper is as follows. We say some more about the structure of $\mathbb{R}$-trees in §2 and discuss a certain compactification for them in §3. We construct the process of interest to us in §4 and identify its Dirichlet form in §5. We give a necessary and sufficient condition for transience in §6 and observe that points are always regular for themselves. We present a class of examples in §7 that illustrate the transience/recurrence dichotomy. We use the Kolmogorov and Hewitt-Savage zero-one laws in §8 to show that the tail σ-field of the diffusion is always trivial and draw the usual conclusion that bounded space-time harmonic functions are constant. In §9 we construct a Martin compactification in the transient case and obtain corresponding integral representations for the excessive and harmonic functions. Using Ray-Knight methods, we establish in §10 that the only entrance laws are the "trivial" ones that arise from starting inside the state-space. Finally, we apply the Dirichlet form stochastic calculus in §11 to obtain a semimartingale decomposition of the diffusion that involves local time additive functionals associated with each branch point of the tree. This "infinitesimal" description of the diffusion's dynamics further confirms the informal one given above.

Notation 1.1. Given a metric space $\mathbf{U}$, we write $C(\mathbf{U})$, $\mathscr{B}(\mathbf{U})$, $bC(\mathbf{U})$, $b\mathscr{B}(\mathbf{U})$, $pC(\mathbf{U})$, and $p\mathscr{B}(\mathbf{U})$ for, respectively, the continuous, Borel, bounded continuous, bounded Borel, positive continuous, and positive Borel functions on $\mathbf{U}$.

2. More about $\mathbb{R}$-trees

We refer the reader to [DT96, DMT96, Ter97] for background on $\mathbb{R}$-trees and proofs of the facts that we outline below.

We will only consider $\mathbb{R}$-trees $(\mathbf{T}, d)$ that also satisfy the following additional axioms which are satisfied by the real tree.

Axiom III: The metric space $(\mathbf{T}, d)$ is *complete*.

Axiom IV: For each $x \in \mathbf{T}$ there is at least one isometric embedding $\theta : \mathbb{R} \to \mathbf{T}$ with $x \in \theta(\mathbb{R})$.

An *end* of $\mathbf{T}$ is an equivalence class of isometric embeddings from $\mathbb{R}_+$ into $\mathbf{T}$, where we regard two such embeddings ϕ and ψ as being equivalent if there exist $\alpha \in \mathbb{R}$ and $\beta \in \mathbb{R}_+$ such that $\alpha + \beta \geq 0$ and $\phi(t) = \psi(t + \alpha)$ for all $t \geq \beta$. Write $\mathbf{E}$ for the set of ends of $\mathbf{T}$.

By Axiom IV, $\mathbf{E}$ has at least 2 points. Fix a distinguished element $\dagger$ of $\mathbf{E}$. For each $x \in \mathbf{T}$ there is a unique isometric embedding $\kappa_x : \mathbb{R}_+ \to \mathbf{T}$ such that $\kappa_x(0) = x$ and κ_x is a representative of the equivalence class of $\dagger$. Similarly, for each $\xi \in \mathbf{E}_+ := \mathbf{E}\backslash\{\dagger\}$ there is at least one isometric embedding θ: $\mathbb{R} \to \mathbf{T}$ such that $t \mapsto \theta(t)$, $t \geq 0$, is a representative of the equivalence class of ξ and $t \mapsto \theta(-t)$, $t \geq 0$, is a representative of the equivalence class of $\dagger$. Denote the collection of all such embeddings by Θ_ξ. If $\theta, \theta' \in \Theta_\xi$, then there exists $\gamma \in \mathbb{R}$ such that $\theta(t) = \theta'(t + \gamma)$ for all $t \in \mathbb{R}$. It is thus possible to select an embedding $\theta_\xi \in \Theta_\xi$ for each $\xi \in \mathbf{E}_+$ in such a way that for any pair $\xi, \zeta \in \mathbf{E}_+$ there exists t_0 (depending on ξ, ζ) such that $\theta_\xi(t) = \theta_\zeta(t)$ for all $t \leq t_0$ (and $\theta_\xi(]t_0, \infty[) \cap \theta_\zeta(]t_0, \infty[) = \emptyset$). Extend θ_ξ to $\mathbb{R}^* := \mathbb{R} \cup \{\pm\infty\}$ by setting $\theta_\xi(-\infty) := \dagger$ and $\theta_\xi(+\infty) := \xi$.

The ends of the real tree can be identified with the collection consisting of the empty set and the elements of $\mathscr{E}_+$. If we choose $\dagger$ to be the empty set so that $\mathscr{E}_+$ plays the role of $\mathbf{E}_+$, then we can define the isometric embedding θ_A for $A \in \mathscr{E}_+$ by $\theta_A(t) := (A \cap] - \infty, t]) \cup \{t\} = A|t$ in the notation of the Introduction.

The map $(t, \xi) \mapsto \theta_\xi(t)$ from $\mathbb{R} \times \mathbf{E}_+$ (resp. $\mathbb{R}^* \times \mathbf{E}_+$) into $\mathbf{T}$ (resp. $\mathbf{T} \cup \mathbf{E}$) is surjective. Moreover, if $\eta \in \mathbf{T} \cup \mathbf{E}$ is in $\theta_\xi(\mathbb{R}^*) \cap \theta_\zeta(\mathbb{R}^*)$ for $\xi, \zeta \in \mathbf{E}_+$, then $\theta_\xi^{-1}(\eta) = \theta_\zeta^{-1}(\eta)$. Denote this common value by $h(\eta)$, the *height* of η. In particular, $h(\dagger) := -\infty$ and $h(\xi) = +\infty$ for $\xi \in \mathbf{E}_+$. For the real tree with corresponding isometric embeddings defined as above, $h(B)$ is just $\sup B$, with the usual convention that $\sup \emptyset := -\infty$ (in accord with the notation of the Introduction).

Define a *partial order* $\leq$ on $\mathbf{T} \cup \mathbf{E}$ by declaring that $\eta \leq \rho$ if there exists $-\infty \leq s \leq t \leq +\infty$ and $\xi \in \mathbf{E}_+$ such that $\eta = \theta_\xi(s)$ and $\rho = \theta_\xi(t)$. For the real tree, this partial order is the usual inclusion partial order. Each pair $\eta, \rho \in \mathbf{T} \cup \mathbf{E}$ has a well-defined *greatest common lower bound* $\eta \wedge \rho$ in this partial order, with $\eta \wedge \rho \in \mathbf{T}$ unless $\eta = \rho \in \mathbf{E}_+$, $\eta = \dagger$ or $\rho = \dagger$. For $x, y \in \mathbf{T}$ we have

$$\begin{aligned} d(x, y) &= h(x) + h(y) - 2h(x \wedge y) \\ &= [h(x) - h(x \wedge y)] + [h(y) - h(x \wedge y)]. \end{aligned} \tag{2.1}$$

Therefore, $h(x) = d(x, y) - h(y) + 2h(x \wedge y) \leq d(x, y) + h(y)$ and, similarly, $h(y) \leq d(x, y) + h(y)$, so that

$$|h(x) - h(y)| \leq d(x, y), \tag{2.2}$$

with equality if $x, y \in \mathbf{T}$ are comparable in the partial order (that is, if $x \leq y$ or $y \leq x$).

If $x, x' \in \mathbf{T}$ are such that $h(x \wedge y) = h(x' \wedge y)$ for all $y \in \mathbf{T}$, then, by (2.1), $d(x, x') = [h(x) - h(x \wedge x')] + [h(x') - h(x \wedge x')] = [h(x) - h(x \wedge x)] + [h(x') - h(x' \wedge x')] = 0$, so that $x = x'$. Slight elaborations of this argument show that if $\eta, \eta' \in \mathbf{T} \cup \mathbf{E}$ are such that $h(\eta \wedge y) = h(\eta' \wedge y)$ for all y in some dense subset of $\mathbf{T}$, then $\eta = \eta'$.

For $x, x', z \in \mathbf{T}$ we have that if $h(x \wedge z) < h(x' \wedge z)$, then $x \wedge x' = x \wedge z$ and a similar conclusion holds with the rôles of x and x' reversed; whereas if $h(x \wedge z) = h(x' \wedge z)$, then $x \wedge z = x' \wedge z \le x \wedge x'$. Using (2.1) and (2.2) and checking the various cases one finds that

$$|h(x \wedge z) - h(x' \wedge z)| \le d(x \wedge z, x' \wedge z) \le d(x, x'). \tag{2.3}$$

For $\eta \in \mathbf{T} \cup \mathbf{E}$ and $t \in \mathbb{R}^*$ with $t \le h(\eta)$, let $\eta | t$ denote the unique $\rho \in \mathbf{T} \cup \mathbf{E}$ with $\rho \le \eta$ and $h(\rho) = t$. Equivalently, if $\eta = \theta_\xi(u)$ for some $u \in \mathbb{R}^*$ and $\xi \in \mathbf{E}_+$, then $\eta | t = \theta_\xi(t)$ for $t \le u$. For the real tree, this definition coincides with the one given in the Introduction.

The metric space $(\mathbf{E}_+, \delta)$, where

$$\delta(\xi, \zeta) := 2^{-h(\xi \wedge \zeta)},$$

is complete. Moreover, the metric δ is actually an *ultrametric*; that is, $\delta(\xi, \zeta) \le \delta(\xi, \eta) \vee \delta(\eta, \zeta)$ for all $\xi, \zeta, \eta \in \mathbf{E}_+$.

3. A compactification

Suppose in this section that $(\mathbf{E}_+, \delta)$ is separable. For $t \in \mathbb{R}$ consider the set

$$\mathbf{T}_t := \{x \in \mathbf{T} : h(x) = t\} = \{\xi | t : \xi \in \mathbf{E}_+\} \tag{3.1}$$

of points in $\mathbf{T}$ that have height t. For each $x \in \mathbf{T}_t$ the set $\{\zeta \in \mathbf{E}_+ : \zeta | t = x\}$ is a ball in $\mathbf{E}_+$ of diameter at most 2^{-t} and two such balls are disjoint. The separability of $\mathbf{E}_+$ is thus equivalent to each of the sets $\mathbf{T}_t$ being countable. In particular, separability of $\mathbf{E}_+$ implies that $\mathbf{T}$ is also separable, with countable dense set $\{\xi | t : \xi \in \mathbf{E}_+,\ t \in \mathbb{Q}\}$, say.

We can, via a standard Stone–Čech-like procedure, embed $\mathbf{T} \cup \mathbf{E}$ in a compact metric space in such a way that for each $y \in \mathbf{T} \cup \mathbf{E}$ the map $x \mapsto h(x \wedge y)$ has a continuous extension to the compactification (as an extended real-valued function).

More specifically, let T be a countable dense subset of $\mathbf{T}$. Let π be a strictly increasing, continuous function that maps $\mathbb{R}$ onto $]0, 1[$. Define an injective map Π from $\mathbf{T}$ into the compact, metrisable space $[0, 1]^T$ by $\Pi(x) := \pi(h(x \wedge y))_{y \in T}$. Identify $\mathbf{T}$ with $\Pi(\mathbf{T})$ and write $\overline{\mathbf{T}}$ for the closure of $\mathbf{T} (= \Pi(\mathbf{T}))$ in $[0, 1]^T$. In other words, a sequence $(x_n)_{n \in \mathbb{N}} \subset \mathbf{T}$ converges to a point in $\overline{\mathbf{T}}$ if $h(x_n \wedge y)$ converges (possibly to $-\infty$) for all $y \in T$, and two such sequences $(x_n)_{n \in \mathbb{N}}$ and $(x'_n)_{n \in \mathbb{N}}$ converge to the same point if and only if $\lim_n h(x_n \wedge y) = \lim_n h(x'_n \wedge y)$ for all $y \in T$.

We can identify distinct points in $\mathbf{T} \cup \mathbf{E}$ with distinct points in $\overline{\mathbf{T}}$. If $(x_n)_{n \in \mathbb{N}} \subset \mathbf{T}$ and $\xi \in \mathbf{E}_+$ are such that for all $t \in \mathbb{R}$ we have $\xi | t \le x_n$ for all sufficiently large

n, then $\lim_n h(x_n \wedge y) = h(\xi \wedge y)$ for all $y \in T$. We leave the identification of † to the reader.

In fact, we have $\overline{\mathbf{T}} = \mathbf{T} \cup \mathbf{E}$. To see this, suppose that $(x_n)_{n\in\mathbb{N}} \subset \mathbf{T}$ converges to $x_\infty \in \overline{\mathbf{T}}$. Put $h_\infty := \sup_{y\in T} \lim_n h(x_n \wedge y)$. Assume for the moment that $h_\infty \in \mathbb{R}$. We will show that $x_\infty \in \mathbf{T}$ with $h(x_\infty) = h_\infty$. For all $k \in \mathbb{N}$ we can find $y_k \in T$ such that

$$h_\infty - \frac{1}{k} \le \lim_n h(x_n \wedge y_k) \le h(y_k) \le h_\infty + \frac{1}{k}.$$

Observe that

$$\begin{aligned} d(y_k, y_\ell) &\le \limsup_n \Big(d(y_k, x_n \wedge y_k) + d(x_n \wedge y_k, x_n \wedge y_\ell) \\ &\qquad + d(x_n \wedge y_\ell, y_\ell) \Big) \\ &= \limsup_n \Big([h(y_k) - h(x_n \wedge y_k)] + |h(x_n \wedge y_k) - h(x_n \wedge y_\ell)| \\ &\qquad + [h(y_\ell) - h(x_n \wedge y_\ell)] \Big) \\ &\le \frac{2}{k} + \left(\frac{1}{k} + \frac{1}{\ell}\right) + \frac{2}{\ell}. \end{aligned}$$

Therefore, $(y_k)_{k\in\mathbb{N}}$ is a d-Cauchy sequence and, by Axiom III, this sequence converges to $y_\infty \in \mathbf{T}$. Moreover, by (2.2) and (2.3), $\lim_n h(x_n \wedge y_\infty) = h(y_\infty) = h_\infty$.

We claim that $y_\infty = x_\infty$; that is, $\lim_n h(x_n \wedge z) = h(y_\infty \wedge z)$ for all $z \in T$. To see this, fix $z \in T$ and $\epsilon > 0$. If n is sufficiently large, then

$$h(x_n \wedge z) \le h(y_\infty) + \epsilon \tag{3.2}$$

and

$$h(y_\infty) - \epsilon \le h(x_n \wedge y_\infty) \le h(y_\infty). \tag{3.3}$$

If $h(y_\infty \wedge z) \le h(y_\infty) - \epsilon$, then (3.3) implies that $y_\infty \wedge z = x_n \wedge z$. On the other hand, if $h(y_\infty \wedge z) \ge h(y_\infty) - \epsilon$, then (3.3) implies that

$$h(x_n \wedge z) \ge h(y_\infty) - \epsilon, \tag{3.4}$$

and so, by (3.2) and (3.3),

$$\begin{aligned} |h(y_\infty \wedge z) - h(x_n, z)| &\le [h(y_\infty) - (h(y_\infty) - \epsilon)] \\ &\quad \vee [(h(y_\infty) + \epsilon) - (h(y_\infty) - \epsilon)] \\ &= 2\epsilon. \end{aligned} \tag{3.5}$$

We leave the analogous arguments for $h_\infty = +\infty$ (in which case $x_\infty \in \mathbf{E}_+$) and $h_\infty = -\infty$ (in which case $x_\infty = \dagger$) to the reader.

We just seen that the construction of $\overline{\mathbf{T}}$ does not depend on T (more precisely, any two such compactifications are homeomorphic). Moreover, a sequence $(x_n)_{n\in\mathbb{N}} \subset \mathbf{T} \cup \mathbf{E}$ converges to a limit in $\mathbf{T} \cup \mathbf{E}$ if and only if $\lim_n h(x_n \wedge y)$ exists for all $y \in \mathbf{T}$, and two convergent sequences $(x_n)_{n\in\mathbb{N}}$ and $(x'_n)_{n\in\mathbb{N}}$ converge to the same limit if and only if $\lim_n h(x_n \wedge y) = \lim_n h(x'_n \wedge y)$ for all $y \in \mathbf{T}$.

4. Construction of the process

Suppose that μ is a σ-finite Borel measure on $\mathbf{E}_+$ such that $0 < \mu(B) < \infty$ for every ball B in the metric δ. In particular, the support of μ is all of $\mathbf{E}_+$.

Remark 4.1. We note that the existence of such a measure μ is a more restrictive assumption on $\mathbf{T}$ than it might first appear. Let $\overline{\mu}$ be a finite measure on $\mathbf{E}_+$ that is equivalent to μ. Recall from (3.1) that $\mathbf{T}_t$, $t \in \mathbb{R}$, is the set of points in $\mathbf{T}$ with height t. As we remarked in §3, the set $\{\zeta \in \mathbf{E}_+ : \zeta|t = x\}$ is a ball in $\mathbf{E}_+$ for each $x \in \mathbf{T}_t$ and two such balls are disjoint. Because the $\overline{\mu}$ measure of each such ball is non-zero, the set $\mathbf{T}_t$ is necessarily countable and hence, by observations made in §3, both the complete metric spaces $\mathbf{T}$ and $\mathbf{E}_+$ are separable, and hence Lusin.

For $x \in \mathbf{T}$ and real numbers $b < c$ with $b < h(x)$, define a probability measure $\mu(x, b, c; \cdot)$ on $\mathbf{T}$ by

$$\mu(x, b, c; A) := \frac{\mu\{\xi \in \mathbf{E}_+ : \xi|c \in A,\ \xi|b = x|b\}}{\mu\{\xi \in \mathbf{E}_+ : \xi|b = x|b\}}.$$

Let (B_t, P^a) be a standard (real-valued) Brownian motion. Write $m_t := \inf_{0 \le s \le t} B_s$. Recall that the pair (m_t, B_t) has joint density

$$\phi_{a,t}(b, c) := \sqrt{\frac{2}{\pi}} \frac{c - 2b + a}{t^{3/2}} \exp\left(-\frac{(c - 2b + a)^2}{2t}\right), \quad b < a \wedge c,$$

under P^a (see, for example, Corollary 30 in Chapter 1 of [Fre83]).

Theorem 4.2. *There is a Markov semigroup $(P_t)_{t \ge 0}$ on $\mathbf{T}$ defined by*

$$P_t f(x) := P^{h(x)} [\mu(x, m_t, B_t; f)] .$$

Furthermore, there is a strong Markov process $(X_t, \mathbb{P}^x)$ on $\mathbf{T}$ with continuous sample paths and semigroup $(P_t)_{t \ge 0}$.

Proof. The proof of the semigroup property of $(P_t)_{t \ge 0}$ is immediate from the Markov property of Brownian motion and the readily checked observation that for $x, x' \in \mathbf{T}$, $b < c$, $b < h(x)$, and $b' < c \wedge c'$ we have

$$\int \mu(x', b', c'; A)\, \mu(x, b, c; dx') = \mu(x, b \wedge b', c'; A).$$

By Kolmogorov's extension theorem, there is a Markov process $(X_t, \mathbb{P}^x)$ on $\mathbf{T}$ with semigroup $(P_t)_{t \ge 0}$. In order to show that a version of X can be chosen with continuous sample paths, it suffices because $(\mathbf{T}, d)$ is complete and separable to check Kolmogorov's continuity criterion. Because of the Markov property of X, it further suffices to observe for $\alpha > 0$ that, by definition of $(P_t)_{t \ge 0}$,

$$\begin{aligned}
&\mathbb{P}^x \left[d(x, X_t)^\alpha\right] \\
&\quad = P^{h(x)} \left[\frac{\int [h(x) + h(\xi|B_t) - 2h(x \wedge (\xi|B_t))]^\alpha \mathbf{1}\{\xi|m_t = x|m_t\}\, \mu(d\xi)}{\mu\{\xi \in \mathbf{E}_+ : \xi|m_t = x|m_t\}} \right]
\end{aligned}$$

$$
\begin{aligned}
&\le P^{h(x)}\left[\frac{\int [h(x)+B_t-2m_t]^\alpha \mathbf{1}\{\xi|m_t = x|m_t\}\,\mu(d\xi)}{\mu\{\xi \in \mathbf{E}_+ : \xi|m_t = x|m_t\}}\right] \\
&\le C P^{h(x)}\left[\, |h(x)-m_t|^\alpha + |m_t - B_t|^\alpha \,\right] \\
&\le C' t^{\alpha/2}
\end{aligned}
$$

for some constants C, C' that depend on α but not on $x \in \mathbf{T}$.

The claim that X is strong Markov will follow if we can show that P_t maps $bC(\mathbf{T})$ into itself (see, for example, §§III.8, III.9 of [RW94], – it is assumed there that the underlying space is locally compact and the semigroup maps the space of continuous functions that vanish at infinity into itself, but this stronger assumption is only needed to establish the existence of a process with càdlàg sample paths and plays no rôle in the proof of the strong Markov property). By definition, for $f \in b\mathscr{B}(\mathbf{T})$ and $t > 0$

$$
\begin{aligned}
P_t f(x) = \int_{-\infty}^{h(x)} \int_b^\infty &\frac{\int f(\xi|c)\mathbf{1}\{\xi|b = x|b\}\mu(d\xi)}{\mu\{\xi \in \mathbf{E}_+ : \xi|b = x|b\}} \\
&\times \sqrt{\frac{2}{\pi}}\frac{c-2b+h(x)}{t^{3/2}} \exp\left(-\frac{(c-2b+h(x))^2}{2t}\right) dc\, db
\end{aligned}
$$

for $t > 0$. The right-hand side can be written as $\int_{-\infty}^\infty \int_{-\infty}^\infty F_{f,x}(b,c)\,dc\,db$ for a certain function $F_{f,x}$. Recall from (2.2) that $|h(x) - h(x')| \le d(x,x')$. Also, if $b < h(x)$, then $x'|b = x|b$ for x' such that $d(x,x') \le h(x) - b$. Therefore, $\lim_{x' \to x} F_{f,x'}(b,c) = F_{f,x}(b,c)$ except possibly at $b = h(x)$. Moreover, if $\sup_x |f(x)| \le C$, then $|F_{f,x}(b,c)| \le C F_{1,x}(b,c)$. Because

$$
\lim_{x' \to x} \int_{-\infty}^\infty \int_{-\infty}^\infty F_{1,x'}(b,c)\,dc\,db = \lim_{x' \to x} 1 = 1 = \int_{-\infty}^\infty \int_{-\infty}^\infty F_{1,x}(b,c)\,dc\,db,
$$

a standard generalisation of the dominated convergence theorem (see, for example, Proposition 18 in Chapter 11 of [Roy68]) shows that if $f \in b\mathscr{B}(\mathbf{T})$, then $P_t f \in bC(\mathbf{T})$ for $t > 0$. □

5. Symmetry and the Dirichlet form

Write λ for Lebesgue measure on $\mathbb{R}$. Consider the measure ν that is obtained by pushing forward the measure $\mu \otimes \lambda$ on $\mathbf{E}_+ \times \mathbb{R}$ with the map $(\xi, a) \mapsto \xi|a$. Note that for $x \in \mathbf{T}$ with $h(x) = h^*$ and $\epsilon > 0$ we have

$$
\begin{aligned}
&\nu\{y \in \mathbf{T} : d(x,y) \le \epsilon\} \\
&\quad \le \nu\{y \in \mathbf{T} : y|(h^* - \epsilon) = x|(h^* - \epsilon),\ h^* - \epsilon \le h(y) \le h^* + \epsilon\} \\
&\quad \le 2\epsilon\mu\{\xi \in \mathbf{E}_+ : \xi|(h^* - \epsilon) = x|(h^* - \epsilon)\}.
\end{aligned}
$$

That is, ν assigns finite mass to balls in $\mathbf{T}$ and, in particular, is Radon.

We begin by showing that each operator P_t, $t > 0$, can be continuously extended from $b\mathscr{B}(\mathbf{T}) \cap L^2(\mathbf{T}, \nu)$ to $L^2(\mathbf{T}, \nu)$ and that the resulting semigroup is a strongly continuous, self-adjoint, Markovian semigroup on $L^2(\mathbf{T}, \nu)$.

Observe that if $f \in b\mathscr{B}(\mathbf{T})$, then

$$\begin{aligned}
P_t f(x) &= \int_{\mathbf{E}_+} \int_{\mathbb{R}} \int_{-\infty}^{h(x)\wedge c} \frac{f(\xi|c)\mathbf{1}\{\xi|b = x|b\}}{\mu\{\xi \in \mathbf{E}_+ : \xi|b = x|b\}} \phi_{h(x),t}(b,c)\, db\, dc\, \mu(d\xi) \\
&= \int_{\mathbf{T}} f(y) \int_{-\infty}^{h(x)\wedge h(y)} \frac{\mathbf{1}\{x|b = y|b\}}{\mu\{\xi \in \mathbf{E}_+ : \xi|b = x|b\}} \\
&\quad \times \sqrt{\frac{2}{\pi}} \frac{h(x)+h(y)-2b}{t^{3/2}} \exp\left(-\frac{(h(x)+h(y)-2b)^2}{2t}\right) db\, \nu(dy) \\
&= \int_{\mathbf{T}} f(y) \int_{-\infty}^{h(x\wedge y)} \frac{1}{\mu\{\xi \in \mathbf{E}_+ : \xi|b = x|b\}} \\
&\quad \times \sqrt{\frac{2}{\pi}} \frac{h(x)+h(y)-2b}{t^{3/2}} \exp\left(-\frac{(h(x)+h(y)-2b)^2}{2t}\right) db\, \nu(dy)
\end{aligned}$$

for $t > 0$. Consequently, $P_t f(x) = \int_{\mathbf{T}} p_t(x,y) f(y)\, \nu(dy)$ for the jointly continuous, everywhere positive transition density

$$\begin{aligned}
p_t(x,y) := \int_{-\infty}^{h(x\wedge y)} &\frac{1}{\mu\{\xi \in \mathbf{E}_+ : \xi|b = x|b\}} \sqrt{\frac{2}{\pi}} \frac{h(x)+h(y)-2b}{t^{3/2}} \\
&\times \exp\left(-\frac{(h(x)+h(y)-2b)^2}{2t}\right) db.
\end{aligned} \tag{5.1}$$

Moreover, because $\mu\{\xi \in \mathbf{E}_+ : \xi|b = x|b\} = \mu\{\xi \in \mathbf{E}_+ : \xi|b = y|b\}$ when $b \le h(x \wedge y)$ (equivalently, when $x|b = y|b$), we have $p_t(x,y) = p_t(y,x)$. Therefore there exists a self-adjoint, Markovian semigroup on $L^2(\mathbf{T},\nu)$ that coincides with $(P_t)_{t\ge 0}$ on $b\mathscr{B}(\mathbf{T}) \cap L^2(\mathbf{T},\nu)$ (cf. §1.4 of [FOT94]). With the usual abuse of notation, we also denote this semigroup by $(P_t)_{t\ge 0}$.

Because ν is Radon, $bC(\mathbf{T}) \cap L^1(\mathbf{T},\nu)$ is dense in $L^2(\mathbf{T},\nu)$. It is immediate from the definition of $(P_t)_{t\ge 0}$ that $\lim_{t\downarrow 0} P_t f(x) = f(x)$ for all $f \in bC(\mathbf{T})$ and $x \in \mathbf{T}$. Therefore, by Lemma 1.4.3 of [FOT94], the semigroup $(P_t)_{t\ge 0}$ is strongly continuous on $L^2(\mathbf{T},\nu)$.

We now proceed to identify the Dirichlet form corresponding to $(P_t)_{t\ge 0}$.

Definition 5.1. Let $\mathscr{A}$ denote the class of functions $f \in bC(\mathbf{T})$ such that there exists $g \in \mathscr{B}(\mathbf{T})$ with the property that

$$f(\xi|b) - f(\xi|a) = \int_a^b g(\xi|u)\, du, \quad \xi \in \mathbf{E}_+,\ -\infty < a < b < \infty. \tag{5.2}$$

Note for $\xi \in \mathbf{E}_+$ that if $A \in \mathscr{B}(\mathbb{R})$ with $A \subseteq [a,b]$, where $-\infty < a < b < \infty$, then

$$\begin{aligned}
\mu\{\zeta \in \mathbf{E}_+ : \zeta|b = \xi|b\}\, \lambda(A) &\le \nu\{\xi|u : u \in A\} \\
&\le \mu\{\zeta \in \mathbf{E}_+ : \zeta|a = \xi|a\}\, \lambda(A).
\end{aligned}$$

Therefore, the function g in (5.2) is unique up to ν-null sets, and (with the usual convention of using function notation to denote equivalence classes of functions) we denote g by ∇f.

Definition 5.2. Write $\mathscr{D}$ for the class of functions $f \in \mathscr{A} \cap L^2(\mathbf{T}, \nu)$ such that $\nabla f \in L^2(\mathbf{T}, \nu)$.

Remark 5.3. By the observations made in Definition 5.1, the integral $\int_a^b \overline{g}(\xi|u)\,du$ is well-defined for any $\xi \in \mathbf{E}_+$ and $\overline{g} \in L^2(\mathbf{T}, \nu)$.

Theorem 5.4. *The Dirichlet form $\mathscr{E}$ corresponding to the strongly continuous, self-adjoint, Markovian semigroup $(P_t)_{t\geq 0}$ on $L^2(\mathbf{T}, \nu)$ has domain $\mathscr{D}$ and is given by*

$$\mathscr{E}(f, g) = \frac{1}{2}\int_{\mathbf{T}} \nabla f(x)\nabla g(x)\,\nu(dx), \;\; f, g \in \mathscr{D}. \tag{5.3}$$

Proof. A virtual reprise of the argument in Example 1.2.2 of [FOT94] shows that the form $\mathscr{E}'$ given by the right-hand side of (5.3) is a Dirichlet form.

Let $(G_\alpha)_{\alpha>0}$ denote the resolvent corresponding to $(P_t)_{t\geq 0}$: that is, $G_\alpha f = \int_0^\infty e^{-\alpha t} P_t f\,dt$ for $f \in L^2(\mathbf{T}, \nu)$. In order to show that $\mathscr{E} = \mathscr{E}'$, it suffices to show that $G_\alpha(L^2(\mathbf{T}, \nu)) \subseteq \mathscr{D}$ and $\mathscr{E}'_\alpha(G_\alpha f, g) := \mathscr{E}'(G_\alpha f, g) + \alpha(f, g) = (f, g)$ for $f \in L^2(\mathbf{T}, \nu)$ and $g \in \mathscr{D}$, where we write $(\cdot, \cdot)$ for the $L^2(\mathbf{T}, \nu)$ inner product. (cf. the proof of Theorem 1.3.1 in [FOT94]) By a simple approximation argument, it further suffices to check that $G_\alpha(b\mathscr{B}(\mathbf{T}) \cap L^2(\mathbf{T}, \nu)) \subseteq \mathscr{D}$ and $\mathscr{E}'_\alpha(G_\alpha f, g) = (f, g)$ for $f \in b\mathscr{B}(\mathbf{T}) \cap L^2(\mathbf{T}, \nu)$ and $g \in \mathscr{D}$.

Observe that

$$\int_0^\infty e^{-\alpha t}\phi_{a,t}(b, c)\,dt = 2\exp\left(-\sqrt{2\alpha}(c - 2b + a)\right), \quad b < a \wedge c,$$

(see Equations 3.71.13 and 6.23.15 of [Wat44]). Therefore, for $f \in b\mathscr{B}(\mathbf{T}) \cap L^2(\mathbf{T}, \nu)$ we have

$$G_\alpha f(x) = 2\int_{-\infty}^{h(x)}\int_b^\infty \mu(x, b, c; f)e^{-\sqrt{2\alpha}(c-2b+h(x))}\,dc\,db. \tag{5.4}$$

Thus, $G_\alpha f \in \mathscr{A}$ with

$$\nabla(G_\alpha f)(x) = 2\int_{h(x)}^\infty \mu(x, h(x), c; f)e^{-\sqrt{2\alpha}(c-h(x))}\,dc - \sqrt{2\alpha}G_\alpha f(x). \tag{5.5}$$

In order to show that $G_\alpha f \in \mathscr{D}$ is remains to show that the first term on the right-hand side of (5.5) is in $L^2(\mathbf{T}, \nu)$. By the Cauchy–Schwarz inequality and recalling the definition of $\mathbf{T}_t$ from (3.1),

$$\begin{aligned}
&\int_{\mathbf{T}}\left[\int_{h(x)}^\infty \mu(x, h(x), c; f)e^{-\sqrt{2\alpha}(c-h(x))}\,dc\right]^2 \nu(dx) \\
&= \int_{-\infty}^\infty \sum_{x\in\mathbf{T}_a}\left[\int_a^\infty \frac{\int_{\mathbf{E}_+} f(\xi|c)\mathbf{1}\{\xi|a = x\}\mu(d\xi)}{\mu\{\xi : \xi|a = x\}}e^{-\sqrt{2\alpha}(c-a)}\,dc\right]^2 \\
&\quad \times\mu\{\xi : \xi|a = x\}\,da
\end{aligned}$$

$$
\begin{aligned}
&\leq \frac{1}{2\sqrt{2\alpha}} \int_{-\infty}^{\infty} \sum_{x \in \mathbf{T}_a} \left[\int_a^{\infty} \left[\frac{\int_{\mathbf{E}_+} f(\xi|c) \mathbf{1}\{\xi|a = x\} \mu(d\xi)}{\mu\{\xi : \xi|a = x\}} \right]^2 e^{-\sqrt{2\alpha}(c-a)}\, dc \right] \\
&\quad \times \mu\{\xi : \xi|a = x\}\, da \\
&\leq \frac{1}{2\sqrt{2\alpha}} \int_{-\infty}^{\infty} \sum_{x \in \mathbf{T}_a} \left[\int_a^{\infty} \frac{\int_{\mathbf{E}_+} f^2(\xi|c) \mathbf{1}\{\xi|a = x\} \mu(d\xi)}{\mu\{\xi : \xi|a = x\}} e^{-\sqrt{2\alpha}(c-a)}\, dc \right] \\
&\quad \times \mu\{\xi : \xi|a = x\}\, da \\
&= \frac{1}{2\sqrt{2\alpha}} \int_{-\infty}^{\infty} \int_a^{\infty} \left[\int_{\mathbf{E}_+} f^2(\xi|c) \mu(d\xi) \right] e^{-\sqrt{2\alpha}(c-a)}\, dc\, da \\
&= \frac{1}{4\alpha} \int_{-\infty}^{\infty} \int_{\mathbf{E}_+} f^2(\xi|c)\, dc\, \mu(d\xi) = \frac{1}{4\alpha} \int_{\mathbf{T}} f^2(x)\, \nu(dx) < \infty,
\end{aligned}
$$

as required.

From (5.5) we have for $g \in \mathscr{D}$ that

$$
\begin{aligned}
&\mathscr{E}'(G_\alpha f, g) \\
&\quad = \int_{-\infty}^{\infty} \int_{\mathbf{E}+} \left[\int_a^{\infty} \mu(\xi|a, a, c; f) e^{-\sqrt{2\alpha}(c-a)}\, dc \right] \nabla g(\xi|a)\, \mu(d\xi)\, da \\
&\quad\quad - \frac{1}{2}\sqrt{2\alpha} \int_{-\infty}^{\infty} \int_{\mathbf{E}+} G_\alpha f(x) \nabla g(\xi|a), \mu(d\xi)\, da.
\end{aligned} \tag{5.6}
$$

Consider the first term on the right-hand side of (5.6). Note that it can be written as

$$
\begin{aligned}
&\int_{-\infty}^{\infty} \sum_{x \in \mathbf{T}_a} \left[\int_a^{\infty} \frac{\int_{\mathbf{E}_+} f(\xi|c) \mathbf{1}\{\xi|a = x\} \mu(d\xi)}{\mu\{\xi : \xi|a = x\}} e^{-\sqrt{2\alpha}(c-a)}\, dc \right] \\
&\quad \times \nabla g(x) \mu\{\xi : \xi|a = x\}\, da \\
&\quad\quad = \int_{-\infty}^{\infty} \int_{\mathbf{E}+} \left[\int_a^{\infty} f(x|c) e^{-\sqrt{2\alpha}(c-a)}\, dc \right] \nabla g(\xi|a)\, \mu(d\xi)\, da.
\end{aligned} \tag{5.7}
$$

Substitute (5.7) into (5.6), integrate by parts, and use (5.5) to get that

$$
\begin{aligned}
\mathscr{E}'(G_\alpha f, g) = &\int_{\mathbf{E}_+} \int_{-\infty}^{\infty} f(\xi|a) g(\xi|a)\, da\, \mu(dx) \\
&- \sqrt{2\alpha} \int_{\mathbf{E}_+} \int_{-\infty}^{\infty} \left[\int_a^{\infty} f(x|c) e^{-\sqrt{2\alpha}(c-a)}\, dc \right] g(\xi|a)\, da\, \mu(d\xi) \\
&+ \sqrt{2\alpha} \int_{\mathbf{E}_+} \int_{-\infty}^{\infty} \left[\int_a^{\infty} \mu(\xi|a, a, c; f) e^{-\sqrt{2\alpha}(c-a)}\, dc \right] \\
&\times g(\xi|a)\, da\, \mu(d\xi) - \alpha \int_{\mathbf{E}_+} \int_{-\infty}^{\infty} G_\alpha f(\xi|a) g(\xi|a)\, da\, \mu(dx).
\end{aligned}
$$

Argue as in (5.7) to see that the second and third terms on the right-hand side cancel and so

$$\mathscr{E}'(G_\alpha f, g) = (f, g) - \alpha(G_\alpha f, g),$$

as required. □

Remark 5.5. We wish to apply to X the theory of symmetric processes and their associated Dirichlet forms developed in [FOT94]. Because $\mathbf{T}$ is not generally locally compact, we cannot do so directly. Rather, we have to proceed via the embedding results outlined in §7.3 of [FOT94]. We quickly check the relevant conditions for these results to apply.

As usual, set $\mathscr{E}_1 := \mathscr{E} + (\cdot, \cdot)$ with domain $\mathscr{D}$. We begin by showing that conditions (C.1) – (C.3) in §7.3 of [FOT94] hold. That is, there is a countably generated subalgebra $\mathscr{C} \subseteq bC(\mathbf{T}) \cap \mathscr{D}$ such that $\mathscr{C}$ is $\mathscr{E}_1$–dense in $\mathscr{D}$, $\mathscr{C}$ separates points of $\mathbf{T}$, and for each $x \in \mathbf{T}$ there exists $f \in \mathscr{C}$ with $f(x) > 0$. Let $\mathscr{C}_0$ be a countable subset of $bC(\mathbf{T}) \cap L^2(\mathbf{T}, \nu)$ that separates points of $\mathbf{T}$ and is such that for every $x \in \mathbf{T}$ there exists $f \in \mathscr{C}_0$ with $f(x) > 0$. Let $\mathscr{C}$ be the algebra generated by the countable collection $\bigcup_\alpha G_\alpha \mathscr{C}_0$, where the union is over the positive rationals. It is clear that $\mathscr{C}$ is $\mathscr{E}_1$-dense in $\mathscr{D}$. We observed in the proof of Theorem 4.2 that $P_t : bC(\mathbf{T}) \to bC(\mathbf{T})$ for all $t \geq 0$ and $\lim_{t \downarrow 0} P_t f(x) = f(x)$ for all $f \in bC(\mathbf{T})$. Thus, $G_\alpha : bC(\mathbf{T}) \to bC(\mathbf{T})$ for all $\alpha > 0$ and $\lim_{\alpha \to \infty} \alpha G_\alpha f(x) = f(x)$ for all $f \in bC(\mathbf{T})$. Therefore, $\mathscr{C}$ separates points of $\mathbf{T}$ and for every $x \in \mathbf{T}$ there exists $f \in \mathscr{C}$ with $f(x) > 0$.

It remains to check that the tightness condition (7.3.2) of [FOT94] holds. That is, for all $\epsilon > 0$ there exists a compact set K such that $\mathrm{Cap}(\mathbf{T} \backslash K) < \epsilon$ where Cap denotes the capacity associated with $\mathscr{E}_1$. However, it follows from the sample path continuity of X and Theorem IV.1.15 of [MR92] that, in the terminology of that result, the process X is ν-tight. Conditions IV.3.1 (i)–(iii) of [MR92] then hold by Theorem IV.5.1 of [MR92], and this suffices by Theorem III.2.11 of [MR92] to establish condition (7.3.2) of [FOT94].

6. Recurrence, transience, and regularity of points

The Green operator G associated with the semigroup $(P_t)_{t \geq 0}$ is defined by $Gf(x) := \int_0^\infty P_t f(x)\, dt = \sup_{\alpha > 0} G_\alpha f(x)$ for $f \in p\mathscr{B}(\mathbf{T})$. In the terminology of [FOT94], we say that X is *transient* is $Gf < \infty$, ν-a.e., for any $f \in L^1_+(\mathbf{T}, \nu)$, whereas X is *recurrent* if $Gf \in \{0, \infty\}$, ν-a.e., for any $f \in L^1_+(\mathbf{T}, \nu)$.

As we observed in §5, X has symmetric transition densities $p_t(x, y)$ with respect to ν such that $p_t(x, y) > 0$ for all $x, y \in \mathbf{T}$. Consequently, in the terminology of [FOT94], X is *irreducible*. Therefore, by Lemma 1.6.4 of [FOT94], X is either transient or recurrent, and if X is recurrent, then $Gf = \infty$ for any $f \in L^1_+(\mathbf{T}, \nu)$ that is not ν-a.e. 0.

Taking limits as $\alpha \downarrow 0$ in (5.4), we see that

$$Gf(x) = \int_{\mathbf{T}} g(x, y) f(y)\, \nu(dy),$$

where

$$\begin{aligned} g(x,y) :&= 2\int_{-\infty}^{h(x\wedge y)} \frac{1}{\mu\{\xi : \xi|b = x|b\}}\, db \\ &= 2\int_{-\infty}^{h(x\wedge y)} \frac{1}{\mu\{\xi : \xi|b = y|b\}}\, db. \end{aligned} \tag{6.1}$$

Note that the integrals

$$\int_{-\infty}^{a} \frac{1}{\mu\{\xi : \xi|b = \zeta|b\}}\, db, \;\; a \in \mathbb{R}, \; \zeta \in \mathbf{E}_+, \tag{6.2}$$

are either simultaneously finite or infinite. The following is now obvious.

Theorem 6.1. *If the integrals in* (*6.2*) *are finite* (*resp. infinite*)*, then* $g(x,y) < \infty$ (*resp.* $g(x,y) = \infty$) *for all* $x, y \in \mathbf{T}$ *and* X *is transient* (*resp. recurrent*)*.*

Remark 6.2. For $B \in \mathscr{B}(\mathbf{T})$ write $\sigma_B := \inf\{t > 0 : X_t \in B\}$. We note from Theorem 4.6.6 and Problem 4.6.3 of [FOT94] that if $\mathbb{P}^x\{\sigma_B < \infty\} > 0$ for some $x \in \mathbf{T}$, then $\mathbb{P}^x\{\sigma_B < \infty\} > 0$ for all $x \in \mathbf{T}$. Moreover, if X is recurrent, then $\mathbb{P}^x\{\sigma_B < \infty\} > 0$ for some $x \in \mathbf{T}$ implies that $\mathbb{P}^x\{\forall N \in \mathbb{N}, \exists t > N : X_t \in B\} = 1$ for all $x \in \mathbf{T}$.

Given $y \in \mathbf{T}$, write σ_y for $\sigma_{\{y\}}$. Set $C = \{z \in \mathbf{T} : y \le z\}$. Pick $x \le y$ with $x \ne y$. By definition of $(P_t)_{t\ge 0}$, $\mathbb{P}^x\{X_t \in C\} > 0$ for all $t > 0$. In particular, $\mathbb{P}^x\{\sigma_C < \infty\} > 0$. It follows from Axioms I and II that if $\gamma : \mathbb{R}_+ \mapsto \mathbf{T}$ is any continuous map with $\{x, z\} \subset \gamma(\mathbb{R}_+)$ for some $z \in C$, then $y \in \gamma(\mathbb{R}_+)$ also. Therefore, by the sample path continuity of X, $\mathbb{P}^x\{\sigma_y < \infty\} > 0$ for this particular choice of x. However, Remark 6.2 then gives that $\mathbb{P}^x\{\sigma_y < \infty\} > 0$ for all $x \in \mathbf{T}$. By Theorem 4.1.3 of [FOT94] we have that points are regular for themselves. That is, $\mathbb{P}^x\{\sigma_x = 0\} = 1$ for all $x \in \mathbf{T}$.

7. Examples

In this section we exhibit a parametric family of $\mathbb{R}$-trees $(\mathbf{T}, d)$ with measures μ on the corresponding collection of ends $\mathbf{E}_+$ such that associated process X is either recurrent or transient depending on the parameter values.

Fix a prime number p and constants $r_-, r_+ \ge 1$. Let $\mathbb{Q}$ denote the rational numbers. Define an equivalence relation $\sim$ on $\mathbb{Q} \times \mathbb{R}$ as follows. Given $a, b \in \mathbb{Q}$ with $a \ne b$ write $a - b = p^{v(a,b)}(m/n)$ for some $v(a,b), m, n \in \mathbb{Z}$ with m and n not divisible by p. For $v(a,b) \ge 0$ put $w(a,b) = \sum_{i=0}^{v(a,b)} r_+^i$, and for $v(a,b) < 0$ put $w(a,b) := 1 - \sum_{i=0}^{-v(a,b)} r_-^i$. Set $w(a,a) := +\infty$. Given $(a,s), (b,t) \in \mathbb{Q} \times \mathbb{R}$ declare that $(a,s) \sim (b,t)$ if and only if $s = t \le w(a,b)$. Note that

$$v(a,c) \ge v(a,b) \wedge v(b,c) \tag{7.1}$$

so that

$$w(a,c) \ge w(a,b) \wedge w(b,c) \tag{7.2}$$

and $\sim$ is certainly transitive (reflexivity and symmetry are obvious).

Let $\mathbf{T}$ denote the collection of equivalence classes for this equivalence relation. Define a partial order $\le$ on $\mathbf{T}$ as follows. Suppose that $x, y \in \mathbf{T}$ are equivalence classes with representatives (a, s) and (b, t). Say that $x \le y$ if and only if $s \le w(a, b) \wedge t$. It follows from (7.2) that $\le$ is indeed a partial order. A pair $x, y \in \mathbf{T}$ with representatives (a, s) and (b, t) has a unique greatest common lower bound $x \wedge y$ in this order given by the equivalence class of $(a, s \wedge t \wedge w(a, b))$, which is also the equivalence class of $(b, s \wedge t \wedge w(a, b))$.

For $x \in \mathbf{T}$ with representative (a, s), put $h(x) := s$. Define a metric d on $\mathbf{T}$ by setting $d(x, y) := h(x) + h(y) - 2h(x \wedge y)$. We leave it to the reader to check that $(\mathbf{T}, d)$ is an $\mathbb{R}$–tree satisfying Axioms I–IV, and that the definitions of $x \le y$, $x \wedge y$ and $h(x)$ fit into the general framework of §2, with the set $\mathbf{E}_+$ corresponding to $\mathbb{Q} \times \mathbb{R}$-valued paths $s \mapsto (a(s), s)$ such that $s \le w(a(s), a(t)) \wedge t$.

Note that there is a natural Abelian group structure on $\mathbf{E}_+$: if ξ and ζ correspond to paths $s \mapsto (a(s), s)$ and $s \mapsto (b(s), s)$, then define $\xi + \zeta$ to correspond to the path $s \mapsto (a(s) + b(s), s)$. We mention in passing that there is a bi-continuous group isomorphism between $\mathbf{E}_+$ and the additive group of the p-adic integers $\mathbb{Q}_p$. (This map is, however, not an isometry if $\mathbf{E}_+$ is equipped with the δ metric and $\mathbb{Q}_p$ is equipped with the usual p-adic metric.)

Define a Borel measure μ on $\mathbf{E}_+$ as follows. Write $\cdots \le w_{-1} \le w_0 = 1 \le w_1 \le w_2 \le \cdots$ for the possible values of $w(\cdot, \cdot)$. That is, $w_k = \sum_{i=0}^{k} r_+^i$ if $k \ge 0$, whereas $w_k = 1 - \sum_{i=0}^{-k} r_-^i$ if $k < 0$. By construction, closed balls in $\mathbf{E}_+$ all have diameters of the form 2^{-w_k} for some $k \in \mathbb{Z}$ and such a ball is the disjoint union of p balls of diameter $2^{-w_{k+1}}$. We can therefore uniquely define μ by requiring that each closed ball of diameter 2^{-w_k} has mass p^{-k}. The measure μ is nothing but the (unique up to constants) Haar measure on the locally compact Abelian group $\mathbf{E}_+$.

Applying Theorem 6.1, we see that X will be transient if and only if $\sum_{k=0}^{\infty} p^{-k} r_-^k < \infty$, that is, if and only if $r_- < p$. As one might have expected, transience and recurrence are unaffected by the value of r_+: Theorem 6.1 shows that transience and recurrence are features of the structure of $\mathbf{T}$ “near” †, whereas r_+ only dictates the structure of the $\mathbf{T}$ “near” points of $\mathbf{E}_+$.

8. Triviality of the tail σ-field

Theorem 8.1. *For all $x \in \mathbf{T}$ the tail σ-field $\bigcap_{s \ge 0} \sigma\{X_t : t \ge s\}$ is $\mathbb{P}^x$-trivial* (*that is, consists of sets with $\mathbb{P}^x$-measure* 0 *or* 1).

Proof. Fix $x \in \mathbf{T}$. By the continuity of the sample paths of X, $\sigma_{x|a} = \inf\{t > 0 : h(X_t) = a\}$. Because $h(X)$ is a Brownian motion, this stopping time is $\mathbb{P}^x$-a.s. finite. Put $T_0 := 0$ and $T_k := \sigma_{x|(h(x)-k)}$ for $k = 1, 2, \ldots$ By the strong Markov property we get that $\mathbb{P}^x\{T_1 < T_2 < \cdots < \infty\} = 1$. Set $X_k(t) := X((T_k + t) \wedge T_{k+1})$ for $k = 0, 1, \ldots$ Note that the tail σ-field in the statement of the result can also be written as $\bigcap_{k \ge 0} \sigma\{(T_\ell, X_\ell) : \ell \ge k\}$.

By the strong Markov property, the pairs $((T_{k+1} - T_k, X_k))_{k=0}^{\infty}$ are independent. Moreover, by the spatial homogeneity of Brownian motion, the random variables $(T_{k+1} - T_k)_{k=0}^{\infty}$ are identically distributed. The result now follows from Lemma 8.2 below. □

Lemma 8.2. *Let $((Y_n, Z_n))_{n\in\mathbb{N}}$ be a sequence of independent $\mathbb{R} \times \mathbf{U}$-valued random variables, where $(\mathbf{U}, \mathscr{U})$ is a measurable space. Suppose further that that the random variables $Y_n, n \in \mathbb{N}$, have a common distribution. Put $W_n := Y_1 + \cdots + Y_n$. Then the tail σ-field $\bigcap_{m=1}^{\infty} \sigma\{(W_n, Z_n) : n \geq m\}$ is trivial.*

Proof. Consider a real-valued random variable V that is measurable with respect to the tail σ-field in the statement. For each $m \in \mathbb{N}$ we have by conditioning on $\sigma\{W_n : n \geq m\}$ and using Kolmogorov's zero-one law that there is a $\sigma\{W_n : n \geq m\}$-measurable random variable V'_m such that $V'_m = V$ almost surely. Consequently, there is a random variable V' measurable with repect to $\bigcap_{m=1}^{\infty} \sigma\{W_n : n \geq m\}$ such that $V' = V$ almost surely, and the proof is completed by an application of the Hewitt–Savage zero-one law. □

Definition 8.3. A function $f \in \mathscr{B}(\mathbf{T} \times \mathbb{R}_+)$ (resp. $f \in \mathscr{B}(\mathbf{T})$) is said to be *space-time harmonic* (resp. *harmonic*) if $0 \leq f < \infty$ and $P_s f(\cdot, t) = f(\cdot, s + t)$ (resp. $P_s f = f$) for all $s, t \geq 0$.

Remark 8.4. There does not seem to be a generally agreed upon convention for the use of the term "harmonic". It is often used for the analogous definition without the requirement that the function is non-negative, and $P_t f(x) = \mathbb{P}^x[f(X_t)]$ is sometimes replaced by $\mathbb{P}^x[f(X_\tau)]$ for suitable stopping times τ. Also, the terms *invariant* and *regular* are sometimes used.

The following is a standard consequence of the triviality of the tail σ-field and irreducibility of the process, but we include a proof for completeness.

Corollary 8.5. *There are no non-constant bounded space-time harmonic functions (and hence,* a fortiori*, no non-constant bounded harmonic functions).*

Proof. Suppose that f is a bounded space-time harmonic function. For each $x \in \mathbf{T}$ and $s \geq 0$ the process $(f(X_t, s + t))_{t\geq 0}$ is a bounded $\mathbb{P}^x$-martingale. Therefore $\lim_{t\to\infty} f(X_t, s + t)$ exists $\mathbb{P}^x$-a.s. and $f(x, s) = \mathbb{P}^x[\lim_{t\to\infty} f(X_t, s + t)] = \lim_{t\to\infty} f(X_t, s + t)$, $\mathbb{P}^x$-a.s., by the triviality of the tail. By the Markov property and the fact that X has everywhere positive transition densities with respect to ν we get that $f(s, x) = f(t, y)$ for ν-a.e. y for each $t > s$, and it is clear from this that f is a constant. □

Remark 8.6. The conclusion of Corollary 8.5 for harmonic functions has the following alternative probabilistic proof. By the arguments in the proof of Theorem 8.1 we have that if $n \in \mathbb{Z}$ is such that $n < h(x)$, then $\mathbb{P}^x\{\sigma_{x|n} < \sigma_{x|(n-1)} < \sigma_{x|(n-2)} < \cdots < \infty\} = 1$. Suppose that f is a bounded harmonic function. Then $f(x) = \mathbb{P}^x[\lim_{t\to\infty} f(X_t)] = \lim_{k\to\infty} f(x|(-k))$. Now note for each pair $x, y \in \mathbf{T}$ that $x|(-k) = y|(-k)$ for $k \in \mathbb{N}$ sufficiently large.

9. Martin compactification and excessive functions

Suppose in this section that X is transient. Recall that $f \in \mathscr{B}(\mathbf{T})$ is *excessive* for $(P_t)_{t\geq 0}$ if $0 \leq f < \infty$, $P_t f \leq f$, and $\lim_{t\downarrow 0} P_t f = f$ pointwise. Recall the definition of harmonic function from §8. In this section we will obtain an integral representation for the excessive and harmonic functions.

Fix $x_0 \in \mathbf{T}$ and define $k : \mathbf{T} \times \mathbf{T} \to \mathbb{R}$, the corresponding *Martin kernel*, by

$$\begin{aligned} k(x,y) := \frac{g(x,y)}{g(x_0,y)} &= \frac{\int_{-\infty}^{h(x\wedge y)} \mu\{\xi : \xi|b = y|b\}^{-1} db}{\int_{-\infty}^{h(x_0\wedge y)} \mu\{\xi : \xi|b = y|b\}^{-1} db} \\ &= \frac{\int_{-\infty}^{h(x\wedge y)} \mu\{\xi : \xi|b = x|b\}^{-1} db}{\int_{-\infty}^{h(x_0\wedge y)} \mu\{\xi : \xi|b = x_0|b\}^{-1} db}. \end{aligned} \tag{9.1}$$

Note that the function k is continuous in both arguments and

$$0 < \mathbb{P}^x\{\sigma_{x_0} < \infty\} \le k(x,y) = \frac{\mathbb{P}^x\{\sigma_y < \infty\}}{\mathbb{P}^{x_0}\{\sigma_y < \infty\}} \le \mathbb{P}^{x_0}\{\sigma_x < \infty\}^{-1} < \infty.$$

We can follow the standard approach to constructing a Martin compactification when there are well-behaved potential kernel densities (e.g. [KW65, Mey70]). That is, we choose a countable, dense subset $T \subset \mathbf{T}$ and compactify $\mathbf{T}$ using the sort of Stone–Čech-like procedure described in §3 to obtain a metrisable compactification $\mathbf{T}^M$ such that a sequence $(y_n)_{n\in\mathbb{N}} \subset \mathbf{T}$ converges if and only if $\lim_n k(x, y_n)$ exists for all $x \in T$. Recall the compactification $\overline{\mathbf{T}}$ of §3.

Proposition 9.1. *The compact metric spaces* $\overline{\mathbf{T}}$ *and* $\mathbf{T}^M$ *are homeomorphic, so that* $\mathbf{T}^M$ *can be identified with* $\mathbf{T} \cup \mathbf{E}$*. If we define*

$$k(x,\eta) := \frac{\int_{-\infty}^{h(x\wedge y)} \mu\{\xi : \xi|b = \eta|b\}^{-1} db}{\int_{-\infty}^{h(x_0\wedge y)} \mu\{\xi : \xi|b = \eta|b\}^{-1} db}, \quad x \in \mathbf{T}, \quad \eta \in \mathbf{T} \cup \mathbf{E}_+,$$

and $k(x, \dagger) = 1$*, then* $k(x,\cdot)$ *is continuous on* $\mathbf{T} \cup \mathbf{E}$*. Moreover,*

$$\sup_{x\in B} \sup_{\eta\in\mathbf{T}\cup\mathbf{E}} k(x,\eta) < \infty$$

for all balls $B \subset \mathbf{T}$.

Proof. The rest of the proof will be almost immediate once we show for a sequence $(y_n)_{n\in\mathbb{N}} \subset \mathbf{T}$ that $\lim_n k(x,y_n)$ exists for all $x \in T$ if and only if $\lim_n h(x \wedge y_n)$ exists (in the extended sense) for all $x \in T$.

It is clear that if $\lim_n h(x \wedge y_n)$ exists for all $x \in T$, then $\lim_n k(x, y_n)$ exists for all $x \in T$.

Suppose, on the other hand, that $\lim_n k(x,y_n)$ exists for all $x \in T$ but $\lim_n h(x' \wedge y_n)$ does not exist for some $x' \in T$. Then we can find $\epsilon > 0$ and $a < h(x') - \epsilon$ such that $x'' := x'|a \in T$, $\liminf_n h(x' \wedge y_n) \le a - \epsilon$, and $\limsup_n h(x' \wedge y_n) \ge a + \epsilon$. This implies that for any $N \in \mathbb{N}$ there exists $p, q \ge N$ such that $h(x'' \wedge y_p) = h(x' \wedge y_p)$ and $h(x'' \wedge y_q) = a < a + \epsilon/2 < h(x' \wedge y_q)$. We thus obtain the contradiction

$$\liminf_n \frac{k(x', y_n)}{k(x'', y_n)} = \liminf_n \frac{g(x', y_n)}{g(x'', y_n)} = 1,$$

while

$$\limsup_n \frac{k(x', y_n)}{k(x'', y_n)} = \limsup_n \frac{g(x', y_n)}{g(x'', y_n)}$$
$$\geq \frac{\int_{-\infty}^{a+\epsilon/2} \mu\{\xi : \xi|b = x'|b\}^{-1}\, db}{\int_{-\infty}^{a} \mu\{\xi : \xi|b = x'|b\}^{-1}\, db} > 1. \qquad \square$$

The following theorem essentially follows from results in [Mey70], with most of the work that is particular to our setting being the argument that the points of $\mathbf{E}_+$ are, in the terminology of [Mey70], *minimal*. Unfortunately, the standing assumption in [Mey70] is that the state-space is locally compact. The requirement for this hypothesis can be circumvented using the special features of our process, but checking this requires a fairly close reading of much of [Mey70]. Later, more probabilistic or measure-theoretic, approaches to the Martin boundary such as [Dyn72, GM73, Gar76, Jeu78] do not require local compactness, but are rather less concrete and less pleasant to compute with. We therefore sketch the relevant arguments.

Definition 9.2. An excessive function f is said to be a *potential* if $\lim_{t\to\infty} P_t f = 0$. (The term *purely excessive function* is also sometimes used.)

Theorem 9.3. *If u is an excessive function, then there is a unique finite measure γ on $\overline{\mathbf{T}} = \mathbf{T} \cup \mathbf{E}$ such that $u(x) = \int_{\mathbf{T}\cup\mathbf{E}} k(x, \eta)\, \gamma(d\eta)$, $x \in \mathbf{T}$. Furthermore, u is harmonic* (*resp. a potential*) *if and only if $\gamma(\mathbf{T}) = 0$* (*resp.* $\gamma(\mathbf{E}) = 0$).

Proof. From Theorem IX.T64 in [Mey66] there exists a sequence $(f_n)_{n\in\mathbb{N}}$ of non-negative functions such that $Gf_1(x) \leq Gf_2(x) \leq \cdots \leq G_n f(x) \uparrow u(x)$ as $n \to \infty$ for all $x \in \mathbf{T}$. Define a measure γ_n by $\gamma_n(dy) := g(x_0, y) f_n(y)\, \nu(dy)$, so that $Gf_n(x) = \int_{\mathbf{T}} k(x, y)\, \gamma_n(dy)$. Note that $\gamma_n(\mathbf{T}) = Gf_n(x_0) \leq u(x_0) < \infty$. We can think of $(\gamma_n)_{n\in\mathbb{N}}$ as a sequence of finite measures on the compact space $\overline{\mathbf{T}}$ with bounded total mass. Therefore, there exists a subsequence $(n_\ell)_{\ell\in\mathbb{N}}$ such that $\gamma = \lim_\ell \gamma_{n_\ell}$ exists in the topology of weak convergence of finite measures on $\overline{\mathbf{T}}$. By Proposition 9.1, each of the functions $k(x, \cdot)$ is bounded and continuous, and so

$$\begin{aligned}\int_{\mathbf{T}\cup\mathbf{E}} k(x, \eta)\, \gamma(d\eta) &= \lim_\ell \int_{\mathbf{T}\cup\mathbf{E}} k(x, \eta)\, \gamma_{n_\ell}(d\eta) \\ &= \lim_\ell \int_{\mathbf{T}} k(x, y)\, \gamma_{n_\ell}(dy) \\ &= \lim_\ell Gf_{n_\ell}(x) = u(x).\end{aligned}$$

Write k_η for the excessive function $k(\cdot, \eta)$, $\eta \in \mathbf{T} \cup \mathbf{E}$. Each of the functions k_y, $y \in \mathbf{T}$, is clearly a potential. A direct calculation using (5.4), which we omit, shows that if $\xi \in \mathbf{E}$, then $\alpha G_\alpha k_\xi = k_\xi$ for all $\alpha > 0$, and this implies that k_ξ is harmonic.

This completes the proof of the theorem except for the uniqueness claim. From Proposition 9.1, all excessive functions are bounded on balls and hence ν-integrable on balls. We can therefore equip the cone of excessive functions with the metrisable $L^1_{\mathrm{loc}}(\mathbf{T}, \nu)$ topology. Consider the convex set of excessive functions u such that $u(x_0) = 1$. Any measure appearing in the representation of such a function u

is necessarily a probability measure. Given a sequence $(u_n)_{n\in\mathbb{N}}$ of such functions, we can, by the weak compactness argument described above, find a subsequence $(u_{n_\ell})_{\ell\in\mathbb{N}}$ that converges bounded pointwise, and hence also in $L^1_{\mathrm{loc}}(\mathbf{T},\nu)$, to some limit u. Therefore the excessive functions are a cone over a compact metrisable base. Moreover, this cone is a lattice in the associated intrinsic (that is, strong) order (see §XV.4 of [Mey67]).

Uniqueness will now follow from the standard Choquet uniqueness theorem (see, for example, Theorem XI.T29 of [Mey66]) provided we can show for all $\eta \in \mathbf{T}\cup\mathbf{E}$ that if $k_\eta = \int k_{\eta'}\,\gamma(d\eta')$ for some finite measure γ, then γ is necessarily the point mass at η.

Consider first the case of representing k_ξ for some $\xi \in \mathbf{E}_+$. For $x \in \mathbf{T}$ and $a > h(x \wedge \xi)$

$$\begin{aligned}
k_\xi(x) &\ge \mathbb{P}^x[k_\xi(X_{\sigma_{\xi|a}})] \\
&= \frac{g(x,\xi|a)}{g(\xi|a,\xi|a)} k(\xi|a,\xi) \\
&= \frac{\int_{-\infty}^{h(x\wedge(\xi|a))} \mu\{\zeta : \zeta|b = (\xi|a)|b\}^{-1}\,db}{\int_{-\infty}^{h(\xi|a)} \mu\{\zeta : \zeta|b = (\xi|a)|b\}^{-1}\,db} \\
&\quad \times \frac{\int_{-\infty}^{h((\xi|a)\wedge\xi)} \mu\{\zeta : \zeta = \xi|b\}^{-1}\,db}{\int_{-\infty}^{h(x_0\wedge\xi)} \mu\{\zeta : \zeta|b = \xi|b\}^{-1}\,db} \\
&= \frac{\int_{-\infty}^{h(x\wedge\xi)} \mu\{\zeta : \zeta|b = \xi|b\}^{-1}\,db}{\int_{-\infty}^{a} \mu\{\zeta : \zeta|b = \xi|b\}^{-1}\,db} \\
&\quad \times \frac{\int_{-\infty}^{a} \mu\{\zeta : \zeta = \xi|b\}^{-1}\,db}{\int_{-\infty}^{h(x_0\wedge\xi)} \mu\{\zeta : \zeta|b = \xi|b\}^{-1}\,db} \\
&= k_\xi(x).
\end{aligned}$$

Thus $k_\xi(x) = \mathbb{P}^x[k_\xi(X_{\sigma_{\xi|a}})]$ for all a sufficiently large. On the other hand, a similar argument shows for $\xi' \in \mathbf{E}_+\backslash\{\xi\}$ that

$$k_{\xi'}(x) \ge \mathbb{P}^x[k_{\xi'}(X_{\sigma_{\xi|a}})]$$

and

$$\mathbb{P}^x[k_{\xi'}(X_{\sigma_{\xi|a}})] = \frac{\int_{-\infty}^{h(\xi\wedge\xi')} \mu\{\zeta : \zeta|b = \xi|b\}^{-1}\,db}{\int_{-\infty}^{a} \mu\{\zeta : \zeta|b = \xi|b\}^{-1}\,db}\, k_{\xi'}(x),$$

for sufficiently large a, where the right-hand side converges to 0 as $a \to 0$. Similarly, $\lim_{a\to\infty} \mathbb{P}^x[k_\dagger(X_{\sigma_{\xi|a}})] = 0$. This clearly shows that if $k_\xi = \int_{\mathbf{E}} k_{\xi'}\,\gamma(d\xi')$, then γ cannot assign any mass to $\mathbf{E}\backslash\{\xi\}$. Uniqueness for the representation of $k_\dagger$ is handled similarly.

Uniqueness for the representation of k_y, $y \in \mathbf{T}$, is an immediate consequence of the principle of masses (see Proposition 1.1 of [GG83]). □

Remark 9.4. Theorem 9.3 can be used as follows to give an analytic proof (in the transient case) of the conclusion of Corollary 8.5 that bounded harmonic functions are necessarily constant.

First extend the definition of the Green kernel g to $\mathbf{T} \cup \mathbf{E}$ by setting

$$\begin{aligned} g(\eta, \rho) &:= 2 \int_{-\infty}^{h(\eta \wedge \rho)} \mu\{\zeta : \zeta|b = \eta|b\}^{-1} \, db \\ &= 2 \int_{-\infty}^{h(\eta \wedge \rho)} \mu\{\zeta : \zeta|b = \rho|b\}^{-1} \, db. \end{aligned}$$

By Theorem 9.3, non-constant bounded harmonic functions exist if and only if there is a non-trivial finite measure γ concentrated on $\mathbf{E}_+$ such that

$$\sup_{x \in \mathbf{T}} \int_{\mathbf{E}_+} k(x, \zeta) \, \gamma(d\zeta) < \infty. \tag{9.2}$$

Note that for any ball $B \subset \mathbf{E}_+$ of the form $B = \{\zeta \in \mathbf{E}_+ : \zeta|h(x^*) = x^*\}$ for $h(x^*) \geq h(x_0)$ we have $g(x_0, \zeta) = g(x_0, x^*)$. Thus, by possibly replacing the measure γ in (9.2) by its trace on a ball, we have that non-constant bounded harmonic functions exist if and only if there is a probability measure (that we also denote by γ) concentrated on a ball $B \subset \mathbf{E}_+$ such that

$$\sup_{x \in \mathbf{T}} \int_{B} g(x, \zeta) \, \gamma(d\zeta) < \infty. \tag{9.3}$$

Observe that $g(\xi|t, \zeta)$ increases monotonically to $g(\xi, \zeta)$ as $t \to \infty$ and so, by monotone convergence, (9.3) holds if and only if

$$\sup_{\xi \in \mathbf{E}_+} \int_{B} g(\xi, \zeta) \, \gamma(d\zeta) < \infty. \tag{9.4}$$

It is further clear that if (9.4) holds, then

$$\int_B \int_B g(\xi, \zeta) \, \gamma(d\xi) \, \gamma(d\zeta) < \infty. \tag{9.5}$$

Suppose that (9.5) holds. For $b \in \mathbb{R}$ write $\mathbf{T}_b^\gamma$ for the subset of $\mathbf{T}_b$ consisting of $x \in \mathbf{T}_b$ such that $\gamma\{\xi \in B : \eta|b = x\} > 0$. In other words, $\mathbf{T}_b^\gamma$ is the collection of points of the form $\eta|b$ for some η in the closed support of γ. Note that $\sum_{x \in \mathbf{T}_b^\gamma} \mu\{\eta : \eta|b = x\} \leq \mu(B)$ if 2^{-b} is at most the diameter of B. Applying Jensen's inequality, we obtain the contradiction

$$\begin{aligned} &\int_B \int_B g(\xi, \zeta) \, \gamma(d\xi) \, \gamma(d\zeta) \\ &\quad = 2 \int_{-\infty}^{\infty} \int_B \int_B \frac{\mathbf{1}\{\xi|b = \zeta|b\}}{\mu\{\eta : \eta|b = \xi|b\}} \, \gamma(d\xi) \, \gamma(d\zeta) \, db \\ &\quad = 2 \int_{-\infty}^{\infty} \int_B \frac{\gamma\{\eta : \eta|b = \xi|b\}}{\mu\{\eta : \eta|b = \xi|b\}} \, \gamma(d\xi) \, db \end{aligned}$$

$$
\begin{aligned}
&\geq 2\int_{-\infty}^{\infty}\left[\int_B \frac{\mu\{\eta : \eta|b = \xi|b\}}{\gamma\{\eta : \eta|b = \xi|b\}}\,\gamma(d\xi)\right]^{-1} db \\
&= 2\int_{-\infty}^{\infty}\left[\sum_{x\in\mathbf{T}_b^\gamma} \frac{\mu\{\eta : \eta|b = x\}}{\gamma\{\eta : \eta|b = x\}}\,\gamma\{\eta : \eta|b = x\}\right]^{-1} db \\
&= \infty.
\end{aligned}
$$

10. Entrance laws

Recall that a *probability entrance law* for the semigroup $(P_t)_{t\geq 0}$ is a family $(\gamma_t)_{t>0}$ of probability measures on $\mathbf{T}$ such that $\gamma_s P_t = \gamma_{s+t}$ for all $s, t > 0$. Given such a probability entrance law, we can construct on some probability space $(\Omega, \mathscr{F}, \mathbb{P})$ a continuous process that, with a slight abuse of notation, we denote $X = (X_t)_{t>0}$ such that X_t has law γ_t and X is a time-homogeneous Markov process with transition semigroup $(P_t)_{t\geq 0}$.

In this section we show that the only probability entrance laws are the trivial ones.

Theorem 10.1. *If $(\gamma_t)_{t>0}$ is a probability entrance law for $(P_t)_{t\geq 0}$, then $\gamma_t = \gamma_0 P_t$, $t > 0$, for some probability measure γ_0 on $\mathbf{T}$.*

Proof. Construct a Ray–Knight compactification $(\mathbf{T}^R, \rho)$, say, as in §17 of [Sha88]. Write $(\overline{P}_t)_{t\geq 0}$ and $(\overline{G}_\alpha)_{\alpha>0}$ for the corresponding extended semigroup and resolvent.

Construct X with one-dimensional distributions $(\gamma_t)_{t>0}$ and semigroup $(P_t)_{t\geq 0}$ as described above. By Theorem 40.4 of [Sha88], $\lim_{t\downarrow 0} X_t$ exists in the Ray topology, and if γ_0 denotes the law of this limit, then $\gamma_0\overline{P}_t$ is concentrated on $\mathbf{T}$ for all $t > 0$ and γ_t is the restriction of $\gamma_0\overline{P}_t$ to $\mathbf{T}$. We therefore need to establish that γ_0 is concentrated on $\mathbf{T}$. Moreover, it suffices to consider the case when γ_0 is a point mass at some $x_0 \in \mathbf{T}^R$, so that $\lim_{t\downarrow 0} X_t = x_0$ in the Ray topology. Note by Theorem 4.10 of [Sha88] that the germ σ-field $\mathscr{F}_{0+} := \bigcap_\epsilon \sigma\{X_t : 0 \leq t \leq \epsilon\}$ is trivial under $\mathbb{P}$ in this case.

By construction of $(P_t)_{t\geq 0}$, the family obtained by pushing forward each γ_t by the map h is an entrance law for standard Brownian motion on $\mathbb{R}$. Because Brownian motion is a Feller–Dynkin process, the only entrance laws for it are the trivial ones $(\rho Q_t)_{t>0}$, where $(Q_t)_{t\geq 0}$ is the semigroup of Brownian motion and ρ is a probability measure on $\mathbb{R}$. Thus, by the triviality $\mathscr{F}_{0+}$, there is a constant $h_0 \in \mathbb{R}$ such that $\lim_{t\downarrow 0} h(X_t) = h_0$, $\mathbb{P}$-a.s.

As usual, regard functions on $\mathbf{T}$ as functions on $\mathbf{T}^R$ by extending them to be 0 on $\mathbf{T}^R\backslash\mathbf{T}$. For every $f \in b\mathscr{B}(\mathbf{T})$ we have by Theorem 40.4 of [Sha88] that $\lim_{t\downarrow 0} G_\alpha f(X_t) = \lim_{t\downarrow 0} \overline{G}_\alpha f(X_t) = \overline{G}_\alpha f(x)$.

From (5.4),

$$
G_\alpha f(x) = \int_{\mathbf{T}} g_\alpha(x, y) f(y)\,\nu(dy),
$$

where

$$\begin{aligned} g_\alpha(x,y) &:= 2\int_{-\infty}^{h(x\wedge y)} \frac{\exp(-\sqrt{2\alpha}(h(x)+h(y)-2b))}{\mu\{\xi : \xi|b = x|b\}}\,db \\ &= 2\int_{-\infty}^{h(x\wedge y)} \frac{\exp(-\sqrt{2\alpha}(h(x)+h(y)-2b))}{\mu\{\xi : \xi|b = y|b\}}\,db. \end{aligned} \tag{10.1}$$

It follows straightforwardly that $\lim_{t\downarrow 0} h(X_t \wedge y)$ exists for all $y \in \mathbf{T}$, $\mathbb{P}$-a.s., and so, by the discussion in §3 and the triviality of $\mathscr{F}_{0+}$, there exists $\eta \in \mathbf{T} \cup \mathbf{E}$ such that $h(\eta) \le h_0$ and $\lim_{t\downarrow 0} h(X_t \wedge y) = h(\eta \wedge y)$, $\mathbb{P}$-a.s. Note, in particular, that we actually have $\eta \in \mathbf{T} \cup \{\dagger\}$ because $h(\eta) < \infty$. Moreover, we conclude that

$$\begin{aligned} &\int_0^\infty e^{-\alpha t}\gamma_t(f)\,dt = \overline{G}_\alpha f(x_0) \\ &\quad = 2\int_{\mathbf{T}}\left[\int_{-\infty}^{h(\eta\wedge y)} \frac{\exp(-\sqrt{2\alpha}(h_0+h(y)-2b))}{\mu\{\xi : \xi|b = y|b\}}\,db\right]\nu(dy) \end{aligned}$$

for all $f \in b\mathscr{B}(\mathbf{T})$.

We cannot have $\eta = \dagger$, because this would imply that γ_t is the null measure for all $t > 0$. If $\eta \in \mathbf{T}$ and $h_0 = h(\eta)$, then we have $\gamma_t = \delta_\eta P_t$.

We therefore need only rule out the possibility that $\eta \in \mathbf{T}$ but $h(\eta) < h_0$. In this case we have

$$\int_0^\infty e^{-\alpha t}\gamma_t(f)\,dt = \exp\left(-\sqrt{2\alpha}(h_0 - h(\eta))\right)\int_0^\infty e^{-\alpha t}\delta_\eta P_t(f)\,dt$$

and so, by comparison of Laplace transforms, $\gamma_t = \int_0^t \delta_\eta P_{t-s}\,\kappa(ds)$, where κ is a certain stable-$\frac{1}{2}$ distribution. In particular, γ_t has total mass $\kappa([0,t]) < 1$ and is not a probability distribution. □

11. Local times and semimartingale decompositions

Our aim in this section is to give a semimartingale decomposition for the process $H_\xi(t) := h(X_t \wedge \xi)$, $t \ge 0$, for $\xi \in \mathbf{E}_+$. From the intuitive description of X in the Introduction, we expect H_ξ to remain constant over time intervals when X_t is not in the ray $R_\xi := \{x \in \mathbf{T} : x \le \xi\}$. During time intervals when X_t is in R_ξ we expect H_ξ to evolve as a standard Brownian motion except at branch points of $\mathbf{T}$ where it receives negative "kicks" from a local time additive functional in the same manner that skew Brownian motion receives kicks at 0, with the magnitude of the kicks related to how much μ-mass is being lost to the rays that are branching off from R_ξ. To make this description precise, we first need to introduce appropriate local time processes.

We showed in §6 that $\mathbb{P}^x\{\sigma_y < \infty\}$ for any $x, y \in \mathbf{T}$. By Theorems 4.2.1 and 2.2.3 of [FOT94], the point mass δ_y at any $y \in \mathbf{T}$ belongs to the set of measures S_{00}. (See (2.2.10) of [FOT94] for a definition of S_{00}. Another way of seeing that δ_y is in S_{00} is just to observe that $\sup_x g_\alpha(x,y) < \infty$ for all $\alpha > 0$.) By Theorem 5.1.6 of [FOT94] there exists for each $y \in \mathbf{T}$ a strict sense positive continuous additive functional L^y with Revuz measure δ_y. As usual, we call L^y the *local time* at y.

Definition 11.1. Given $\xi \in \mathbf{E}_+$, write m_ξ for the Radon measure on $\mathbf{T}$ that is supported on the ray R_ξ and for each $a \in \mathbb{R}$ assigns mass $\mu\{\zeta \in \mathbf{E}_+ : \zeta|a = \xi|a\}$ to the set $\{\xi|b : b \geq a\} = \{x \in R_\xi : h(x) \geq a\}$.

Remark 11.2. Note that m_ξ is a discrete measure that is concentrated on the countable set of points of the form $\xi \wedge \zeta$ for some $\zeta \in \mathbf{E}_+ \backslash \{\xi\}$ (that is, on the points where other rays branch from R_ξ).

Theorem 11.3. *For each $\xi \in \mathbf{E}_+$ and $x \in \mathbf{T}$ the process H_ξ has a semimartingale decomposition*

$$H_\xi(t) = H_\xi(0) + M_\xi(t) - \frac{1}{2}\int_{R_\xi} L^y(t)\, m_\xi(dy), \quad t \geq 0,$$

under $\mathbb{P}^x$, where M_ξ is a continuous, square-integrable martingale with quadratic variation

$$\langle M_\xi \rangle(t) = \int_0^t \mathbf{1}\{X(s) \leq \xi\}\, ds, \quad t \geq 0.$$

Moreover, the martingales M_ξ and $M_{\xi'}$ for $\xi, \xi' \in \mathbf{E}_+$ have covariation

$$\langle M_\xi, M_{\xi'} \rangle_t = \int_0^t \mathbf{1}\{X(s) \leq \xi \wedge \xi'\}\, ds, \quad t \geq 0.$$

Proof. For $\xi \in \mathbf{E}_+$, $x \in \mathbf{T}$, and $A \in \mathbb{N}$, set $h_\xi(x) = h(x \wedge \xi)$ and $h_\xi^A(x) = (-A) \vee (h(x \wedge \xi) \wedge A)$.

It is clear that h_ξ^A is in the domain $\mathscr{D}$ of the Dirichlet form $\mathscr{E}$, with $\nabla h_\xi^A(x) = \mathbf{1}\{\xi|(-A) \leq x \leq \xi|A\}$. Given $f \in \mathscr{D}$, it follows from the product rule that

$$2\mathscr{E}(h_\xi^A f, h_\xi^A f) - \mathscr{E}((h_\xi^A)^2, f) = \int_{\mathbf{T}} f(x)\mathbf{1}\{\xi|(-A) \leq x \leq \xi|A\}\, \nu(dx).$$

In the terminology of §3.2 of [FOT94], the *energy measure* corresponding to h_ξ^A is $\nu_\xi^A(dx) := \mathbf{1}\{\xi|(-A) \leq x \leq \xi|A\}\, \nu(dx)$. A similar calculation shows that the joint energy measure corresponding to a pair of functions h_ξ^A and $h_{\xi'}^{A'}$ is $\mathbf{1}[\{\xi|(-A) \leq x \leq \xi|A\} \cap \{\xi'|(-A') \leq x \leq \xi'|A'\}]\, \nu(dx) = (\nu_\xi^A \wedge \nu_{\xi'}^{A'})(dx)$ in the usual lattice structure on measures.

An integration by parts establishes that for any $f \in \mathscr{D}$ we have

$$\mathscr{E}(h_\xi^A, f) = \frac{1}{2}\int_{\mathbf{T}} f(x)\, \tilde{m}_\xi^A(dx),$$

where

$$\tilde{m}_\xi^A := m_\xi^A - \mu\{\zeta : \zeta|(-A) = \xi|(-A)\}\delta_{\xi|(-A)} + \mu\{\zeta : \zeta|A = \xi|A\}\delta_{\xi|A}$$

with

$$m_\xi^A(dx) := \mathbf{1}\{\xi|(-A) \le x \le \xi|A\} m_\xi(dx).$$

Now ν_ξ^A is the Revuz measure of the strict sense positive continuous additive functional $\int_0^t \mathbf{1}\{\xi|(-A) \le X(s) \le \xi|A\}\,ds$ and $\nu_\xi^A \wedge \nu_{\xi'}^{A'}$ is the Revuz measure of the strict sense positive continuous additive functional $\int_0^t \mathbf{1}[\{\xi|(-A) \le X(s) \le \xi|A\} \cap \{\xi'|(-A') \le X(s) \le \xi'|A'\}]\,ds$. A straightforward calculation shows that $\sup_x \int g_\alpha(x,y)\, m_\xi^A(dy) < \infty$, and so $m_\xi^A \in S_{00}$ is the Revuz measure of the strict sense positive continuous additive functional $\int_{R_\xi} L^y(t)\, m_\xi^A(dy)$ (because the integral is just a sum, we do not need to address the measurability of $y \mapsto L^y(t)$).

Put $H_\xi^A(t) := h_\xi^A(X(t))$, $t \ge 0$. Theorem 5.2.5 of [FOT94] applies to give that

$$H_\xi^A(t) = H_\xi^A(0) + M_\xi^A(t) - \frac{1}{2}\int_{R_\xi} L^y(t)\, \tilde{m}_\xi^A(dy), \quad t \ge 0,$$

under $\mathbb{P}^x$ for each $x \in \mathbf{T}$, where M_ξ^A is a continuous, square-integrable martingale with quadratic variation

$$\langle M_\xi^A \rangle(t) = \int_0^t \mathbf{1}\{\xi|(-A) \le X(s) \le \xi|A\}\,ds.$$

Moreover, the martingales M_ξ^A and $M_{\xi'}^{A'}$ for $\xi, \xi' \in \mathbf{E}_+$ have covariation

$$\begin{aligned}&\langle M_\xi^A, M_{\xi'}^{A'} \rangle(t) \\ &\quad= \int_0^t \mathbf{1}\left[\{\xi|(-A) \le X(s) \le \xi|A\} \cap \{\xi'|(-A') \le X(s) \le \xi'|A'\}\right] ds.\end{aligned}$$

In particular,

$$\begin{aligned}&\langle M_\xi^B - M_\xi^A \rangle(t) \\ &\quad= \int_0^t \mathbf{1}\left[\{\xi|(-B) \le X(s) \le \xi|B\} \backslash \{\xi|(-A) \le X(s) \le \xi|A\}\right] ds \end{aligned}\tag{11.1}$$

for $A < B$.

For each $t \ge 0$ we have that $H_\xi^A(s) = H_\xi(s)$ and $\int_{R_\xi} L^y(s)\, \tilde{m}_\xi^A(dy) = \int_{R_\xi} L^y(s)\, m_\xi(dy)$ for all $0 \le s \le t$ when $A > \sup\{|H_\xi(s)| : 0 \le s \le t\}$, $\mathbb{P}^x$-a.s. Therefore there exists a continuous process M_ξ such that $M_\xi^A(s) = M_\xi(s)$ for all $0 \le s \le t$ when $A > \sup\{|H_\xi(s)| : 0 \le s \le t\}$, $\mathbb{P}^x$-a.s. It follows from (11.1) that $\lim_{A \to \infty} \mathbb{P}^x[\sup_{0 \le s \le t} |M_\xi^A(s) - M_\xi(s)|^2] = 0$. By standard arguments, the processes M_ξ are continuous, square-integrable martingales with the stated quadratic variation and covariation properties. $\square$

Remark 11.4. There is more that can be said about the process H_ξ. For instance, given $x \in \mathbf{T}$ and $\xi \in \mathbf{E}_+$ with $x \in R_\xi$ and $a > h(x)$, we can explicitly calculate the Laplace transform of $\inf\{t > 0 : H_\xi(t) = a\} = \sigma_{\xi|a}$ under $\mathbb{P}^x$. We have

$$\mathbb{P}^x[\exp(-\alpha\sigma_{\xi|a})] = g_\alpha(x, \xi|a) \,/\, g_\alpha(\xi|a, \xi|a),$$

where g_α is given explicitly by (10.1). When X is transient, the distribution of $\sigma_{\xi|a}$ has an atom at ∞ and we have

$$\mathbb{P}^x\left\{\sup_{0\le t<\infty} H_\xi(t) \ge a\right\} = \mathbb{P}^x\{\sigma_{\xi|a} < \infty\} = g(x, \xi|a) \,/\, g(\xi|a, \xi|a).$$

By the strong Markov property, the càdlàg process $(\sigma_{\xi|a})_{a\ge h(x)}$ has independent (although, of course, non-stationary) increments under $\mathbb{P}^x$, with the usual appropriate definition of this notion for non-decreasing $\mathbb{R} \cup \{+\infty\}$-valued processes.

Remark 11.5. The stochastic calculus can be used to further analyse X. As a typical example, when X is transient consider the harmonic functions $k_\xi = k(\cdot, \xi)$, $\xi \in \mathbf{E}_+$, introduced in §9 and the corresponding harmonic transformed laws $\mathbb{P}^x_{k_\xi}$, $x \in \mathbf{T}$. That is, $\mathbb{P}^x_{k_\xi}$, $x \in \mathbf{T}$, is the collection of laws of a Markov process X^ξ such that $\mathbb{P}^x_{k_\xi}[f(X^\xi_t)] = k_\xi(x)^{-1}\mathbb{P}^x[k_\xi(X_t)f(X_t)]$, $f \in b\mathscr{B}(\mathbf{T})$. Recall that $(h(X_t))_{t\ge 0}$ is a standard Brownian motion under $\mathbb{P}^x$. Arguing as in the proof of Theorem 11.3 and using Girsanov's theorem, we have under $\mathbb{P}^x_{k_\xi}$ that

$$h(X^\xi_t) = h(X^\xi_0) + W_t + D_t,$$

where W is a standard Brownian motion and

$$D_t = \int_0^t \left[\frac{\mathbf{1}\{X_s \le \xi\}}{\mu\{\zeta : X_s \le \zeta\}}\right] \Big/ \left[\int_{-\infty}^{h(X_s)} \frac{1}{\mu\{\zeta : X_s|b \le \zeta\}}\, db\right] ds.$$

In other words, when X^ξ_t is not on the ray R_ξ the height process $h(X^\xi_t)$ evolves as a standard Brownian motion, but when X^ξ_t is on the ray R_ξ the height experiences an added positive drift. We leave the details to the reader.

Acknowledgements. The author thanks Pat Fitzsimmons, Hans Föllmer, Jim Pitman, Tom Salisbury and Jason Schweinsberg for helpful discussions. The research was conducted while the author was visiting the Fields Institute for Research in the Mathematical Sciences in Toronto, Canada.

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