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Ray Hölder-continuity for fractional Sobolev spaces in infinite dimensions and applications

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Abstract. We prove Hölder-continuity on rays in the direction of vectors in the (generalized) Cameron-Martin space for functions in Sobolev spaces in L^p of fractional order $\alpha \in (\frac{1}{p}, 1)$ over infinite dimensional linear spaces. The underlying measures are required to satisfy some easy standard structural assumptions only. Apart from Wiener measure they include Gibbs measures on a lattice and Euclidean interacting quantum fields in infinite volume. A number of applications, e.g., to the two-dimensional polymer measure, are presented. In particular, irreducibility of the Dirichlet form associated with the latter measure is proved without restrictions on the coupling constant.

1. Introduction

Let $H_p^\alpha(\mathbb{R}^n)$ denote the usual (α, p)-Sobolev space over $\mathbb{R}^n$ constructed in terms of Fourier transforms of tempered distributions. The well-known Sobolev embedding theorem states that if $\alpha p > n$, then every element in $H_p^\alpha(\mathbb{R}^n)$ admits a continuous version ([42, Chap. 2] or [1]).

The analogue of $H_p^\alpha(\mathbb{R}^n)$ over an abstract (infinite dimensional) Wiener space is the Malliavin-Watanabe space $\mathscr{D}_p^\alpha$ which is defined in terms of the Ornstein-Uhlenbeck operator and which is a fundamental object in Malliavin Calculus. Since the main Wiener functionals of interest in Malliavin Calculus are Itô functionals obtained as stochastic integrals and solutions of stochastic differential equations with smooth coefficients, it was the spaces $\mathscr{D}_p^\alpha$ with *integer* α which were mainly studied ([19, 27, 44]).

Recently, however, due to rising interest in refining certain results in Malliavin Calculus and also due to the fact that many important Wiener functionals such as local times and self-intersection local time are not smooth Wiener functionals

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but only belong to some fractional Sobolev spaces, the latter have received much attention ([18, 25, 33, 45]).

Because of the infinite dimensionality of the underlying space, Sobolev spaces over an abstract Wiener space lack many properties of Sobolev spaces over finite dimensional spaces. For example, it is well-known that even smooth Wiener functionals (that is, Wiener functionals belonging to all the Sobolev spaces) may be discontinuous. Thus analogues of Sobolev-type embedding theorems do not hold. Nevertheless, we know that for every fixed direction in the Cameron-Martin space every element in $\mathscr{D}_p^1$ has a modification which is almost surely absolutely continuous in this direction, provided $p > 1$ ([21, 41]).

The corresponding result for fractional Sobolev spaces on Wiener space to be expected is the following (see also Remark 2.2 below for the higher dimensional analogue):

Suppose $\alpha p > 1$. For every fixed direction in the Cameron-Martin space H, any element in $\mathscr{D}_p^\alpha$ admits a modification which is Hölder continuous along this direction.

We give a proof of this result which (apart from Kolmogorov's continuity criterion) is based on operator semi-group theory. It turns out that this proof works in much more general cases than just the abstract Wiener space case. In fact, a large class of probability measures on linear spaces satisfying some easy standard structural conditions (see condition (C) in Subsection 2.1) can be taken to replace Wiener measure. We describe the framework and the general result in Section 2 (cf. Theorem 2.1), where we also present some general examples in the case $p = 2$ (see Subsection 2.3).

In Section 3 we give a number of concrete examples showing that the above class of probability measures apart from Wiener measure, in particular, includes Gibbs measures on a lattice as well as Euclidean quantum fields with polynomial (self-)interactions *in infinite volume*.

Section 4 is devoted to the proof of the general result, Theorem 2.1, while in Section 5 we discuss applications. We first prove general results on the invariance of closedness and irreducibility of classical Dirichlet forms under Doob transforms (cf. Proposition 5.1 and Corollary 5.3). As a special case we recover all main results in [2] on the stochastic quantization of the polymer measure μ_g in $\mathbb{R}^2$. We can even improve two of the main results in [2] in an essential way. First, we prove irreducibility of the corresponding Dirichlet form for all coupling constants $g \in (-g_0, +\infty)$ rather than only $(-g_0, g_0)$ as in [2] (cf. Remark 5.6 below). Second, we can prove that the stochastic process $(a_{th}^{\mu_g})_{t\in\mathbb{R}}$ given by the Radon-Nikodym derivatives

$$a_{th}^{\mu_g} := \frac{d\mu(\cdot + th)}{d\mu}, \quad t \in \mathbb{R} ,$$

has a version with continuous sample paths for *all h* in the classical Cameron-Martin space H rather than only those $h \in H$ with bounded derivatives as proved in [2].

As another application we prove that if A is a measurable set in the Wiener space with Wiener measure strictly between 0 and 1, its indicator $\mathbb{1}_A$ cannot be in

$\mathscr{D}_p^\alpha$ with $\alpha p > 1$ (cf. Proposition 5.5). This generalizes a result of D. Nualart for the special case $\alpha = 1$, $p = 2$ ([29]).

2. A general result for a class of probability measures on linear space

2.1. Framework

Let E be a locally convex topological vector space over $\mathbb{R}$ which is Souslin, i.e., the continuous image of a Polish space (e.g. E is a separable Banach space). Let E' denote its dual with dualization ${}_{E'}\langle\,,\,\rangle_E$ and let $\mathscr{B}(E)$ denote the Borel σ-algebra on E. Let μ be a positive measure on $(E, \mathscr{B}(E))$ such that $\mu(E) < \infty$ and let $\overline{\mathscr{B}(E)}^\mu$ denote the completion of $\mathscr{B}(E)$ w.r.t. μ. Let $L^p(E;\mu) := L^p(E;\mathscr{B}(E),\mu)$, $p \geq 1$, be the corresponding real L^p-spaces equipped with the usual norm $\|\ \|_p$.

Fix $p > 1$ and let $(T_t)_{t\geq 0}$ be a C^0 (i.e., strongly continuous) semi-group on $L^p(E;\mu)$ which is the restriction of a bounded analytic semi-group defined on the complexification of $L^p(E;\mu)$. Suppose $(T_t)_{t\geq 0}$ is sub-Markovian ($f \in L^p(E;\mu)$, $0 \leq f \leq 1 \Rightarrow 0 \leq T_t f \leq 1$ for all $t \geq 0$). In particular, $(T_t)_{t\geq 0}$ extends to a C^0-semi-group on all $L^{p'}(E;\mu)$ for all $p' \geq p$. Let $(L, D(L))$ be the generator of $(T_t)_{t\geq 0}$ on $L^p(E;\mu)$.

By [30, Section 2.6] for $\alpha > 0$ the fractional power $(1-L)^\alpha$ of $(1-L)$ is defined as the inverse of the bounded linear operator

$$(1-L)^{-\alpha} := \frac{1}{\Gamma(\alpha)} \int_0^\infty t^{\alpha-1} e^{-t} T_t \, dt \ , \tag{2.1}$$

where Γ denotes the classical Gamma-function. Let us define the corresponding Bessel-Sobolev spaces (cf. [12, 20]):

$$\mathscr{D}_p^\alpha := (1-L)^{-\alpha/2}(L^p(E;\mu)) \tag{2.2}$$

with norm

$$\|u\|_{p,\alpha} := \|(1-L)^{\alpha/2} u\|_p \ , \quad u \in \mathscr{D}_p^\alpha \ . \tag{2.3}$$

Clearly, $\mathscr{D}_{p'}^{\alpha'} \subset \mathscr{D}_p^\alpha$ if $\alpha' \geq \alpha$, $p' \geq p$. Below for $0 < \alpha < 1$ we want to give conditions on $k \in E$ ensuring that every $u \in \mathscr{D}_p^\alpha$ has a $\overline{\mathscr{B}(E)}^\mu$-measurable μ-version u_k such that

$$t \longmapsto u_k(z + tk) \ , \quad t \in \mathbb{R} \ ,$$

is Hölder continuous for all $z \in E$.

We need some preparations.

For $k \in E$ define

$$\tau_k(z) := z - k \ , \quad z \in E \ . \tag{2.4}$$

Fix $k \in E$ satisfying the following condition:

(C) μ is k–quasi-invariant, i.e., $\mu \circ \tau_{sk}^{-1} \cong \mu$ for all $s \in \mathbb{R}$, and the Radon-Nikodymderivatives

$$a^{\mu}_{sk} := \frac{d(\mu \circ \tau^{-1}_{sk})}{d\mu}\ , \quad s \in \mathbb{R}\ ,$$

have the following properties:

(C1) $a^{\mu}_{sk} \in \cap_{q\geq 1} L^q(E;\mu)$, for all $s \in \mathbb{R}$, and for all $q \in [1,\infty)$ the function $s \longmapsto \|a^{\mu}_{sk}\|_q$ is locally bounded on $\mathbb{R}$.

(C2) For all compact $C \subset \mathbb{R}$

$$\int_C \frac{1}{a^{\mu}_{sk}(z)} ds < \infty \quad \text{for} \quad \mu\text{-a.e.} z \in E\ .$$

Here ds denotes Lebesgue measure on $\mathbb{R}$. We recall that choosing appropriate versions by [4, Prop. 2.4] we may always assume that $a^{\mu}_{sk}(z)$ is jointly measurable in s and z and that (C.2) holds for all $z \in E$ (rather than only μ-a.e. $z \in E$).

For examples of measures μ satisfying condition (C) we appeal to the reader's patience and refer to Section 3 below.

As in [21] we define the measure

$$\sigma^{\mu}_k(A) := \int_{\mathbb{R}} \mu \circ \tau^{-1}_{sk}(A) ds\ , \quad A \in \mathscr{B}(E)\ . \tag{2.5}$$

Note that obviously for $l \in E'$ (:= dual of E) with ${}_{E'}\langle l,k\rangle_E = 1$ we have that $\sigma^{\mu}_k \circ l^{-1} = ds$, that $\sigma^{\mu}_k \circ \tau^{-1}_{sk} = \sigma^{\mu}_k$ for all $s \in \mathbb{R}$, and that σ^{μ}_k is equivalent to μ, i.e., there exists $\rho^{\mu}_k : E \to (0,\infty)$ such that

$$\mu = \rho^{\mu}_k \cdot \sigma^{\mu}_k\ . \tag{2.6}$$

Again by [4, Prop. 2.4] choosing appropriate versions we may always assume to have the following relations between a^{μ}_{sk} and ρ^{μ}_k:

$$a^{\mu}_{sk}(z) = \frac{\rho^{\mu}_k(z+sk)}{\rho^{\mu}_k(z)} \quad \text{for all } z \in E,\ \ s \in \mathbb{R}\ , \tag{2.7}$$

$$\rho^{\mu}_k(z) = \left(\int_{\mathbb{R}} a^{\mu}_{sk}(z) ds\right)^{-1} \quad \text{for all } z \in E\ . \tag{2.8}$$

In particular, by (C2) and the remarks following it we have that

$$\int_C \rho^{\mu}_k(z+sk)^{-1} ds < \infty \quad \text{for all compact } C \subset \mathbb{R} \text{ and all } z \in E\ . \tag{2.9}$$

2.2. The general result

Before we state our general result we recall the definition of one-component Dirichlet forms from [6].

Let $l \in E'$ such that ${}_{E'}\langle l,k\rangle_E = 1$ and define

$$P_k(z) := {}_{E'}\langle l,z\rangle_E \cdot k\ , z \in E,\ \text{and}\ \pi_k := Id_E - P_k\ . \tag{2.10}$$

Clearly, $\pi_k(E)$ is a closed linear subspace of E and

$$E = \pi_k(E) \oplus k\mathbb{R} \ . \tag{2.11}$$

Then by (2.6) and an elementary calculation we obtain that for all $f : E \to \mathbb{R}$, $\mathscr{B}(\mathbb{R})$-measurable, bounded,

$$\int_E f(z)\mu(dz) = \int_{\pi_k(E)} \int_{\mathbb{R}} f(x+sk)\rho_k^{\mu}(x+sk)\, ds\, \mu_k(dx) \ , \tag{2.12}$$

where

$$\mu_k := \mu \circ \pi_k^{-1} \ , \tag{2.13}$$

and correspondingly,

$$L^2(E;\mu) = \int_{\pi_k(E)}^{\oplus} L^2(\mathbb{R}; \rho_k^{\mu}(x+sk)ds)\, \mu_k(dx) \tag{2.14}$$

in the sense that each $f \in L^2(E;\mu)$ corresponds to a "field of vectors" $(f_x)_{x\in\pi_k(E)}$ where $f_x := f(x + \cdot k)$, $x \in \pi_k(E)$ (cf. [6] for details and references). Define $D(\mathscr{E}_{\mu,k})$ to be the set of all $u \in L^2(E;\mu)$ such that for the corresponding element $(u_x)_{x\in\pi_k(E)}$ in $\int_{\pi_k(E)}^{\oplus} L^2(\mathbb{R}; \rho_k^{\mu}(x+sk)ds)\mu_k(dx)$ for μ_k-a.e. $x \in \pi_k(E)$ there exists a locally absolutely continuous ds-version $\tilde{u}_x$ of u_x on $\mathbb{R}$ such that $(\frac{d\tilde{u}_x}{ds})_{x\in\pi_k(E)} \in \int_{\pi_k(E)}^{\oplus} L^2(\mathbb{R}; \rho_k^{\mu}(x+sk)ds)\mu_k(dx)$. For $u \in D(\mathscr{E}_{\mu,k})$ we define $\frac{\partial^{\mu} u}{\partial k}$ as the element in $L^2(E;\mu)$ corresponding to $(\frac{d\tilde{u}_x}{ds})_{x\in\pi_k(E)}$ according to (2.14). Note that, since $\rho_k^{\mu} > 0$,

$$u \longmapsto \frac{\partial^{\mu} u}{\partial k}$$

is a well-defined operator on $L^2(E;\mu)$. Define

$$\mathscr{E}_{\mu,k}(u,v) := \int \frac{\partial^{\mu} u}{\partial k}\frac{\partial^{\mu} v}{\partial k} d\mu \ ; \quad u, v \in D(\mathscr{E}_{\mu,k}) \ . \tag{2.15}$$

$(\mathscr{E}_{\mu,k}, \mathscr{D}(\mathscr{E}_{\mu,k}))$ is called *one-component Dirichlet form* on $L^2(E;\mu)$ if it is closed (which is e.g. the case if (C.2) holds, cf. [6, Theorem 2.2]).

Now we are prepared to state our general result:

Theorem 2.1. *Let $p > 1$ and let E, μ, $(T_t)_{t\geq 0}$ be as above and let $k \in E$ satisfy condition (C1). Assume that the following condition holds:*

(A) There exist $T, C \in (0,\infty)$ such that for all $t \in (0,T]$, $T_t(L^{\infty}(E;\mu)) \subset D(\mathscr{E}_{\mu,k})$ and

$$\|\frac{\partial^{\mu} T_t f}{\partial k}\|_p \leq C\|T_t f\|_{p,1} \quad \textit{for all } f \in L^{\infty}(E;\mu).$$

Let $\alpha \in (\frac{1}{p}, 1)$. Then every $u \in \mathscr{D}_p^{\alpha}$ has a $\overline{\mathscr{B}(E)}^{\mu}$-measurable μ-version u_k such that for all $z \in E$,

$$t \longmapsto u_k(z+tk), \quad t \in \mathbb{R} \ ,$$

is Hölder-continuous of order β for all $\beta \in (0, \alpha - \frac{1}{p})$.

Remark 2.2. Theorem 2.1 has its obvious generalization to obtain μ-versions $u_{\underline{k}}$ of $u \in \mathscr{D}_p^\alpha$ for $\underline{k} := (k_1, \dots, k_n) \in E^n$ so that for all $z \in E$,

$$(t_1, \dots, t_n) \longmapsto u_{\underline{k}}(z + t_1 k_1 + \dots + t_n k_n), \quad (t_1, \dots, t_n) \in \mathbb{R}^n$$

is Hölder-continuous of order β for all $\beta \in (0, \alpha - \frac{n}{p})$ provided $\alpha \in (\frac{n}{p}, 1), n \in \mathbb{N}$.

The proof of Theorem 2.1 will be given in Section 4 below. For *concrete* examples where condition (A) holds we again refer to Section 3. For the case $p = 2$ a whole class of examples will be discussed in the next subsection. They are provided by the classical gradient Dirichlet forms introduced in [6].

2.3. Classical gradient Dirichlet forms

Let E, μ be as in the previous subsection and assume there exists a real separable Hilbert space $(H, \langle\ ,\ \rangle)$ densely and continuously embedded into E. H should be thought of as a "tangent space" to E at each $z \in E$. Let K be a dense linear subspace of H such that Condition (C) in Subsection 2.1 holds for all $k \in K$. Define

$$\begin{aligned} D(\mathscr{E}_\mu) : = \Big\{ u \in \bigcap_{k \in K} D(\mathscr{E}_{\mu,k}) \mid & \text{ there exists a } \mathscr{B}(E)/\mathscr{B}(H)\text{-measurable} \\ & \text{function } \nabla u : E \longmapsto H \text{ such that for each } k \in K, \\ & \langle \nabla u(z), k \rangle_H = \frac{\partial^\mu u}{\partial k}(z) \text{ for } \mu\text{-a.e. } z \in E \text{ and} \\ & \int_E \langle \nabla u(z), \nabla u(z) \rangle_H \, \mu(dz) < \infty \Big\} \ , \end{aligned} \tag{2.16}$$

and for $u, v \in D(\mathscr{E}_\mu)$

$$\mathscr{E}_\mu(u, v) := \int_E \langle \nabla u(z), \nabla v(z) \rangle_H \mu(dz) \ . \tag{2.17}$$

Then by [6, Theorem 3.10], $(\mathscr{E}_\mu, D(\mathscr{E}_\mu))$ is a *symmetric Dirichlet form* on $L^2(E; \mu)$ in the sense of e.g. [28]. Most importantly,

$$(\mathscr{E}_\mu, D(\mathscr{E}_\mu)) \text{ is closed } , \tag{2.18}$$

i.e., $D(\mathscr{E}_\mu)$ equipped with the Hilbertian norm $(\mathscr{E}_\mu(\cdot, \cdot) + \| \cdot \|_2^2)^{1/2}$ is complete. The conditions in [6, Theorem 3.10] are, in fact, satisfied, since every $k \in K$ satisfies condition (C2). We also note that any finitely based bounded smooth cylinder function

$$E \ni z \longmapsto f({}_{E'}\langle l_1, z \rangle_E, \dots, {}_{E'}\langle l_m, z \rangle_E) \ ,$$

$l_1, \dots, l_m \in E'$, $f \in C_b^\infty(\mathbb{R}^m)$, belongs to $D(\mathscr{E}_\mu)$, so $(\mathscr{E}_\mu, D(\mathscr{E}_\mu))$ is densely defined.

In [6] $(\mathscr{E}_\mu, D(\mathscr{E}_\mu))$ was called *classical (gradient) Dirichlet form*. We do not recall the general definition of a Dirichlet form (since we do not really use it below) but instead refer to [11], [9], [28], [13]. We only need the fact that $(\mathscr{E}_\mu, D(\mathscr{E}_\mu))$ has associated to it a unique negative definite self-adjoint linear operator $(L_\mu, D(L_\mu))$ satisfying

$$\mathscr{E}_\mu(u,v) = \left(\sqrt{-L_\mu}u, \sqrt{-L_\mu}v\right)_{L^2(E;\mu)}$$
$$\text{for all } u, v \in D(\mathscr{E}_\mu) = D\left(\sqrt{-L_\mu}\right) \quad . \tag{2.19}$$

Let $T_t := e^{tL_\mu}$, $t \geq 0$, denote the corresponding strongly continuous semi-group on $L^2(E;\mu)$. Then $(T_t)_{t\geq 0}$ satisfies all assumptions imposed on the C^0-semi-group in Subsection 2.1 (cf. e.g. [28, Chap. I, Section 1.2]). Furthermore, as follows directly from spectral theory we then have

$$D(\mathscr{E}_\mu) = \mathscr{D}_2^1, \quad T_t(L^2(E;\mu)) \subset D(\mathscr{E}_\mu) \subset D(\mathscr{E}_{\mu,k}) \quad \text{for all } k \in K \quad ,$$

where $\mathscr{D}_2^1$ is as defined in Subsection 2.1 with L_μ taking the rôle of L. In particular, for all $k \in K$ condition (A) in Theorem 2.1 holds with $p = 2$ for the semi-group $(T_t)_{t\geq 0}$, since

$$\mathscr{E}_{\mu,k}(u,u) \leq \text{ const. } \mathscr{E}_\mu(u,u) \quad \text{for all } u \in D(\mathscr{E}_\mu) \quad .$$

Hence Theorem 2.1 applies for all $k \in K$, $p = 2$.

Remark 2.3. In fact the assumption that $(\mathscr{E}_\mu, D(\mathscr{E}_\mu))$ is of gradient type made above is not necessary. Everything works for the more general (but mostly less interesting) classical Dirichlet forms discussed in [6, Theorem 3.8].

3. Examples

Apart from Gaussian cases (see Subsections 3.1 and 3.3) we particularly want to present in detail the Euclidean $P(\Phi)_2$-quantum field in infinite volume as a "really non-Gaussian" measure (i.e., not absolutely continuous w.r.t. a Gaussian measure) to which Theorem 2.1 applies (cf. Subsection 3.4 below). Other non-Gaussian cases, i.e., Gibbs measures on a lattice are only briefly touched upon in Subsection 3.2.

3.1. Abstract Wiener spaces

Let μ be a Gaussian mean-zero measure on $(E, \mathscr{B}(E))$ whose *reproducing kernel Hilbert space* or *Cameron-Martin space* is H, i.e., (E, H, μ) is an *abstract Wiener space* if E is a separable Banach space ([15]). Then by the classical Cameron-Martin Theorem condition (C) in Subsection 2.1 holds for all $k \in H(\subset E)$. (cf. e.g., [28, Chap. I, Lemma 3.12]).

Let $(T_t)_{t\geq 0}$ be the C^0-semi-group defined in Subsection 2.3 for E, H, μ as above and $K = H$. It is well-known that $(T_t)_{t\geq 0}$ is given by *Mehler's formula,* i.e., for $t \geq 0$, $f \in L^2(E;\mu)$

$$T_t f(z) = \int_E f(e^{-t}z + \sqrt{1-e^{-2t}}z')\mu(dz') \tag{3.1}$$

for μ-a.e. $z \in E$ (cf. [7, Sect. 5] resp. [8, Sect. 6] which even cover more general "tangent spaces" H resp. non-symmetric cases). Then condition (A) in Theorem 2.1 holds for all $p > 1$. This follows from *Meyer's equivalence*. We refer e.g. to [41] for details. So, Theorem 2.1 applies in this case for all $k \in H$ and all $p > 1$, and thus we have proved what was asserted in the introduction of this paper.

3.2. Gibbs measures for lattice systems

Condition (C2) for all k in a dense subspace of E has been verified for a large class of Gibbs measures on lattices with finite and infinite dimensional single spin spaces in [4, Subsections 4.1 resp. 4.2]. In these cases E is a subspace of $\mathbb{R}^{\mathbb{Z}^d}$ resp. $C([0,1],\mathbb{R}^{\mathbb{N}})^{\mathbb{Z}^d}$.

Condition (C.1) can easily be checked for many concrete interactions. So, by Subsection 2.3 for $(T_t)_{t\geq 0}$ as defined there with $H = l_2(\mathbb{Z}^d)$ and $K :=$ linear span of the canonical basis of $l_2(\mathbb{Z}^d)$, Theorem 2.1 applies with $p = 2$. (cf. [4, Subsection 4.1]).

3.3. The free Euclidean field

Let $E := \mathscr{D}' := \mathscr{D}'(\mathbb{R}^d)$ (i.e., the space of Schwartz distributions on $\mathbb{R}^d$) and let $\mu := \mu_0$ be the *free Euclidean field* of mass $m > 0$ on $\mathbb{R}^d$, i.e., the mean-zero Gaussian measure on $\mathscr{B}(\mathscr{D}')$ with covariance

$$\begin{aligned}&\int {}_{\mathscr{D}}\langle l_1, z\rangle_{\mathscr{D}'}\, {}_{\mathscr{D}}\langle l_2, z\rangle_{\mathscr{D}'}\, \mu_0(dz)\\ &= \Big(l_1, (-\Delta + m^2)^{-1} l_2\Big)_{L^2(\mathbb{R}^d;dx)}\ , l_1, l_2 \in \mathscr{D} := C_0^\infty(\mathbb{R}^d)\ , \qquad (3.2)\end{aligned}$$

where Δ denotes the Laplacian and dx Lebesgue measure on $\mathbb{R}^d$. Since the Cameron-Martin space (i.e., the reproducing kernel Hilbert space) for μ_0 is $H^{-1,2}(\mathbb{R}^d)$ (i.e., the dual of the classical Sobolev space of order 1 in $L^2(\mathbb{R}^d; dx)$), as in Subsection 3.1 it follows by the Cameron-Martin Theorem that condition (C), in particular, holds for all $k \in K := \mathscr{D}$. Let $(T_t)_{t\geq 0}$ be the C^0-semi-group defined in Subsection 2.3 for E, K, μ as above but with

$$H := L^2(\mathbb{R}^d; dx)\ .$$

(So in contrast to Subsection 3.1., H is *not* the Cameron-Martin space of μ_0.) It is well-known (cf. [7, Sect. 5]) that $(T_t)_{t\geq 0}$ is given by the following *(generalized) Mehler formula*:

$$T_t f(z) = \int f(e^{-t(-\Delta+m^2)} z + \sqrt{1 - e^{-2t(-\Delta+m^2)}} z')\mu_0(dz')\ , \quad \text{for } \mu_0\text{-a.e. } z \in E\ , \qquad (3.3)$$

for all $f \in L^2(E; \mu_0)$, $t \geq 0$. In particular, it follows by [38, Theorem 3.1] that Meyer's equivalence also holds in this case at least for finitely based smooth functions whose derivatives of all orders are polynomially bounded. (We should mention that this theorem really applies though only stated in [38] for separable Banach spaces E. But as follows from [7, Sect. 5] (see, in particular, [5, Example 5.6] for a similar case), $\mathscr{D}'$ above can always be replaced by a properly constructed separable Banach space). So, for all $p > 1$, condition (A) holds at least for such nice functions u. But since such functions are dense in $L^p(E; \mu_0)$ and since by analyticity T_t is continuous from $L^p(E; \mu_0)$ to $\mathscr{D}_p^1$ for all $t > 0$, it follows that (A) holds for all $u \in L^p(E; \mu_0)$. So, Theorem 2.1 applies for all $k \in \mathscr{D}$ and all $p > 1$ in this case for $(T_t)_{t\geq 0}$ as in (3.3).

Remark 3.1. Above we considered the case of the free field only for simplicity and since we need it in the next subsection. Everything above, of course, extends for properly chosen spaces K in full generality to the situation considered in [38].

3.4. Euclidean $P(\Phi)_2$-quantum fields in infinite volume

Let $d = 2$. As in the previous subsection let $E := \mathscr{D}' = \mathscr{D}'(\mathbb{R}^2)$, $H := L^2(\mathbb{R}^2; dx)$, $K := \mathscr{D} = \mathscr{D}(\mathbb{R}^2)$ and μ_0 the free Euclidean field of mass $m > 0$ on $\mathbb{R}^2$.

Clearly, for $\varphi_1, \dots, \varphi_j \in \mathscr{D}$, $\prod_{i=1}^j {}_{\mathscr{D}}\langle \varphi_i, \cdot\rangle_{\mathscr{D}'} \in L^2(\mathscr{D}'; \mu_0)$. Define for $n \in \mathbb{N}$, $P^{(n)} := P^{(\leq n)} - P^{(\leq n-1)}$ with $P^{(\leq n)}$ being the closed linear span of the monomials $\prod_{i=1}^j {}_{\mathscr{D}}\langle \varphi_i, \cdot\rangle_{\mathscr{D}'}$, $j \leq n$ in $L^2(\mathscr{D}'; \mu_0)$. Now if $\varepsilon \in (0, 1]$, $h \in L^{1+\varepsilon}(\mathbb{R}^2; \lambda^2)$ and $n \in \mathbb{N}$, define $: z^n : (h)$ to be the unique element in $P^{(n)}$ such that

$$\int_{\mathscr{D}'} : z^n : (h) \ : \prod_{i=1}^n {}_{\mathscr{D}}\langle \varphi_i, \cdot\rangle_{\mathscr{D}'} : \ d\mu_0$$
$$= n! \int_{\mathbb{R}^2} \prod_{i=1}^n \left(\int_{\mathbb{R}^2} (-\Delta + m^2)^{-1}(x - y_i)\, \varphi_i(y_i)\lambda^2(dy_i) \right) h(x)\, \lambda^2(dx) \ ,$$

where $: \prod_{i=1}^n {}_{\mathscr{D}}\langle \varphi_i, \cdot\rangle_{\mathscr{D}'} :$ is the orthogonal projection of $\prod_{i=1}^n {}_{\mathscr{D}}\langle \varphi_i, \cdot\rangle_{\mathscr{D}'}$ in $L^2(\mathscr{D}'; \mu_0)$ onto $P^{(n)}$ (see [39, p.12] for an explicit definition of the "Wick product" $: \prod_{i=1}^n {}_{\mathscr{D}}\langle \varphi_i, \cdot\rangle_{\mathscr{D}'} :$ and [39, §V.1] for the existence of $: z^n : (h)$). Clearly, for $h_1, \ h_2 \in L^{1+\varepsilon}(\mathbb{R}^2; \lambda^2)$, $\alpha, \ \beta \in \mathbb{R}$

$$: z^n : (\alpha h_1 + \beta h_2) = \alpha \ : z^n : (h_1) + \beta \ : z^n : (h_2) \quad \mu_0 - a.e \ . \tag{3.4}$$

Let $P : \mathbb{R} \to \mathbb{R}$ be defined by

$$P(s) := \sum_{n=0}^{2N} b_n\, s^n, \ s \in \mathbb{R} \ , \tag{3.5}$$

$b_n \in \mathbb{R}$, $N \in \mathbb{N}$ and $b_{2N} > 0$. Define for $U \subset \mathbb{R}^2$, U open, bounded,

$$a_U(z) := \sum_{n=0}^{2N} b_n : z^n : (1\!\!1_U) \ , \tag{3.6}$$

where as usual $\mathbb{1}_U$ means indicator function of U. We have that

$$e^{-a_U} \in L^p(\mathscr{D}'; \mu_0) \quad \text{for all } p \in [1, \infty) \tag{3.7}$$

(cf. [39, §5.2] and [14]).

Define for $U \subset \mathbb{R}^2$, U open,

$$\mathscr{D}(U) := \{\varphi \in \mathscr{D} \mid \operatorname{supp} \varphi \subset U\} \ ,$$

and let $\sigma(U)$ be the sub-σ-algebra of $\mathscr{B}(\mathscr{D}')$ generated by all $\varphi \in \mathscr{D}(U)$. For $A \subset \mathbb{R}^2$ define

$$\sigma(A) := \bigcap_{\substack{A \subset U \\ U \text{ open}}} \sigma(U) \ .$$

Definition 3.2. *([16, 17])* A probability measure ν on $(\mathscr{D}', \mathscr{B}(\mathscr{D}'))$ is called a *Guerra–Rosen–Simon (=GRS)–Gibbs state* with coupling constant $\lambda \geq 0$ if for any $U \subset \mathbb{R}^2$, U open and bounded,

(i) $\nu_{|\sigma(U)}$ (i.e., the restriction of ν to $\sigma(U)$) is absolutely continuous with respect to $\mu_{0|\sigma(U)}$.
(ii) $E_\nu[f \mid \sigma(U^c)] = E_\nu[f \mid \sigma(\partial U)]$ ν-a.e. for any $\sigma(U)$-measurable $f : \mathscr{D}' \to \mathbb{R}_+$ (where $U^c := \mathbb{R}^2 \setminus U$ and ∂U means topological boundary of U).
(iii) For every $\sigma(U)$-measurable $f : \mathscr{D}' \to \mathbb{R}_+$

$$E_\nu[f \mid \sigma(\partial U)] = \frac{E_{\mu_0}[f e^{-\lambda a_U} \mid \sigma(\partial U)]}{E_{\mu_0}[e^{-\lambda a_U} \mid \sigma(\partial U)]} \quad \nu - a.e \ .$$

Let $\mathscr{G}_\lambda$ denote the set of all GRS-Gibbs states with coupling constant $\lambda \geq 0$.

From now on we fix $\lambda \geq 0$.

Remark 3.3. (i) In [35] a (local) specification $(\pi_U^\lambda)_{U \in \mathbb{L}}$ was constructed such that the associated Gibbs states are exactly the GRS-Gibbs states above and a representation formula of arbitrary GRS-Gibbs states in terms of extremal Gibbs states was derived. We refer to [35] for the precise definition of $(\pi_U^\lambda)_{U \in \mathbb{L}}$. We emphasize that it is entirely useless to construct some abstract specification so that the associated Gibbs states are exactly the GRS-Gibbs states. The point of [35] is that the corresponding kernels are given by *explicit* formulae.

(ii) By [35, Theorems 5.4 and 5.6] in Definition 3.2 (i) “absolutely continuous” can be replaced by ”equivalent”.

(iii) We have that $\mathscr{G}_\lambda \neq \emptyset$ and, in general, $\#\mathscr{G}_\lambda > 1$ as shown in the above quoted literature (cf. e.g. [16, 17, 10, 35, 34]).

For $k \in \mathscr{D}$, $t \in \mathbb{R}$, define for $z \in \mathscr{D}'$

$$a_{tk}(z) := a_{tk}^0(z) \cdot a_{tk}^\lambda(z) \tag{3.8}$$

where

$$a^0_{tk}(z) := \exp\left[-t\, {}_{\mathscr{D}}\langle(-\Delta+m^2)\,k, z\rangle_{\mathscr{D}'} - \frac{1}{2}t^2\left((-\Delta+m^2)\,k, k\right)_{L^2(\mathbb{R}^2;dx)}\right] \tag{3.9}$$

and

$$a^\lambda_{tk}(z) := \exp\left[-\lambda\sum_{n=0}^{2N} b_n \sum_{i=0}^{n-1}\binom{n}{i}t^{n-i}\ :z^i:\ (k^{n-i})\right]\ . \tag{3.10}$$

The reader should forgive us concerning the abuse of notation concerning (3.6) and (3.8).

Define the convex set

$$\mathscr{G}_{\lambda,0} := \{\mu \in \mathscr{G}_\lambda \,|\, \text{(C1) holds for all } k \in \mathscr{D}\}\ .$$

Remark 3.4. By [14, Lemma 12.5.2] we know that $\mathscr{G}_{\lambda,0} \neq \emptyset$ at least if the polynomial P is of type "even plus linear".

By (the easy half of) [4, Theorem 4.11] we know that every $\mu \in \mathscr{G}_\lambda$ is k-quasi-invariant for every $k \in \mathscr{D}$ with corresponding Radon-Nikodym derivative given by (3.8). In particular, (C2) holds. So, by Subsection 2.3 for $(T_t)_{t\geq 0}$ as defined there, Theorem 2.1 applies to all $\mu \in \mathscr{G}_{\lambda,0}$ and all $k \in \mathscr{D}$ with $p = 2$.

4. Proof of Theorem 2.1

Consider the situation of the theorem and fix α, p as there. We need two lemmas:

Lemma 4.1. *There exists $C_\alpha \in (0,\infty)$ such that for all $t > 0$ and all $u \in \mathscr{D}^\alpha_p$*

(i) $\|T_t u\|_{p,1} \leq C_\alpha\, t^{-\frac{1}{2}(1-\alpha)}\, e^{-\frac{1}{2}t}\, \|u\|_{p,\alpha}$,

(ii) $\|T_t u - u\|_p \leq C_\alpha\, t^{\alpha/2}\, \|u\|_{p,\alpha}$.

Proof. (i):

$$\begin{aligned}\|T_t u\|_{p,1} &= \|(1-L)^{1/2}T_t u\|_p \\ &= \|(1-L)^{\frac{1}{2}(1-\alpha)}T_t(1-L)^{\alpha/2}u\|_p \\ &\leq C_\alpha\, t^{-\frac{1}{2}(1-\alpha)}\, \|u\|_{p,\alpha}\end{aligned} \tag{4.1}$$

where the last inequality follows by [30, Theorem 6.13 (c)].

(ii): The assertion follows by [30, Theorem 6.13 (d)]. Q.E.D.

Lemma 4.2. *Suppose $p' \in (1,p)$, $f \in L^\infty(E;\mu)$, $t \in (0,T]$, $M \in \mathbb{N}$, and t_1, $t_2 \in [-M,M]$, $t_1 < t_2$. Then*

$$\|T_t f(\cdot + t_2 k) - T_t f(\cdot + t_1 k)\|_{p'} \leq C \sup_{|s|\leq M} \|a^\mu_{sk}\|^{1/p'}_{\frac{p}{p-p'}}\, |t_2 - t_1|\, \|T_t f\|_{p,1}\ , \tag{4.2}$$

(where C, T are as in condition (A)).

Proof. Set $G := T_t f$. Then by condition (A) we have $G \in D(\mathscr{E}_{\mu,k})$ and, therefore, using (2.12) we obtain (with $\tilde{G}_x$ as in the definition of $D(\mathscr{E}_{\mu,k})$) that

$$\begin{aligned}
&\|G(\cdot + t_2 k) - G(\cdot + t_1 k)\|_{p'} \\
&= \left[\int_{\pi_k(E)} \int_{\mathbb{R}} |\tilde{G}_x(s+t_2) - \tilde{G}_x(s+t_1)|^{p'} \rho_k^\mu(x+sk)\, ds\, \mu_k(dx)\right]^{1/p'} \\
&= \left[\int_{\pi_k(E)} \int_{\mathbb{R}} \left|\int_{t_1}^{t_2} \frac{d\tilde{G}_x}{d\tau}(s+\tau)\, d\tau\right|^{p'} \rho_k^\mu(x+sk)\, ds\, \mu_k(dx)\right]^{1/p'} \\
&\le |t_2 - t_1|^{\frac{p'-1}{p'}} \left[\int_{\pi_k(E)} \int_{t_1}^{t_2} \int_{\mathbb{R}} \left|\frac{d\tilde{G}_x}{ds}(s)\right|^{p'} \right. \\
&\qquad\qquad \left. \rho_k^\mu(x+(s-\tau)k)\, ds\, d\tau\, \mu_k(dx)\right]^{1/p'} \\
&\underset{(2.7)}{=} |t_2 - t_1|^{\frac{p'-1}{p'}} \left[\int_{t_1}^{t_2} \int_{\pi_k(E)} \int_{\mathbb{R}} \left|\frac{d\tilde{G}_x}{ds}(s)\right|^{p'} \right. \\
&\qquad\qquad \left. a_{-\tau k}^\mu(x+sk) \rho_k^\mu(x+sk)\, ds\, \mu_k(dx)\, d\tau\right]^{1/p'} \\
&\underset{(2.12)}{=} |t_2 - t_1|^{\frac{p'-1}{p'}} \left[\int_{t_1}^{t_2} \int \left|\frac{\partial^\mu G}{\partial k}\right|^{p'} a_{-\tau k}^\mu\, d\mu\, d\tau\right]^{1/p'} \\
&\le |t_2 - t_1| \sup_{|\tau| \le M} \|a_{\tau k}^\mu\|_{\frac{p}{p-p'}}^{1/p'} \left\|\frac{\partial^\mu G}{\partial k}\right\|_p \\
&\underset{(A)}{\le} C\, |t_2 - t_1| \sup_{|\tau| \le M} \|a_{\tau k}^\mu\|_{\frac{p}{p-p'}}^{1/p'} \|G\|_{p,1} \quad .
\end{aligned}$$

Q.E.D.

Now we are prepared to prove Theorem 2.1.

Proof of Theorem 2.1. Let $p' \in (\frac{1}{\alpha}, p)$, $M \in \mathbb{N}$, and $s, t \in [-M, M]$ such that $|t - s| \le T$ where T is as in condition (A). Choose the unique $n \in \mathbb{N}$ such that

$$2^{-n} T < |t - s| \le 2^{-n+1} T \quad . \tag{4.3}$$

Set $t_n := (2^{-n} T)^2$. We note that if $u_m := \inf(\sup(u, -m), m)$, $m \in \mathbb{N}$, then by the analyticity of $(T_t)_{t \ge 0}$,

$$\lim_{m \to \infty} \|T_{t_n} u_m - T_{t_n} u\|_{p,1} = 0 \quad .$$

Hence by Lemmas 4.1, 4.2 and (C1)

$$\begin{aligned}
&\|u(\cdot+tk)-u(\cdot+sk)\|_{p'}\\
&\quad\le \|u(\cdot+tk)-T_{t_n}u(\cdot+sk)\|_{p'}+\|T_{t_n}u(\cdot+sk)-u(\cdot+sk)\|_{p'}\\
&\qquad+\limsup_{m\to\infty}\|T_{t_n}u(\cdot+tk)-T_{t_n}u_m(\cdot+tk)\|_{p'}\\
&\qquad+\limsup_{m\to\infty}\|T_{t_n}u_m(\cdot+sk)-T_{t_n}u(\cdot+sk)\|_{p'}\\
&\qquad+\limsup_{m\to\infty}\|T_{t_n}u_m(\cdot+tk)-T_{t_n}u_m(\cdot+sk)\|_{p'}\\
&\quad\le \sup_{|\tau|\le M}\|a_{\tau k}\|_{\frac{p}{p-p'}}^{1/p'}\left(2C_\alpha t_n^{\alpha/2}+C|t-s|\limsup_{m\to\infty}\|T_{t_n}u_m\|_{p,1}\right)\\
&\quad\le C_\alpha\sup_{|\tau|\le M}\|a_{\tau k}\|_{\frac{p}{p-p'}}^{1/p'}\left(2t_n^{\alpha/2}+C|t-s|t_n^{-\frac{1}{2}(1-\alpha)}\right)\|u\|_{p,\alpha}\\
&\quad\le 2C_\alpha\sup_{|\tau|\le M}\|a_{\tau k}\|_{\frac{p}{p-p'}}^{1/p'}\left(1+C2^{-\alpha}\right)|t-s|^\alpha\|u\|_{p,\alpha}\ .
\end{aligned}$$

Since $\alpha p'>1$, it follows by Kolmogorov's theorem (cf. e.g. [40, Corollary 2.1.4]) that there exists a $\mathscr{B}(\mathbb{R})\otimes\overline{\mathscr{B}(E)}^\mu$-measurable function $g:\mathbb{R}\times E\longmapsto\mathbb{R}$ such that for all $z\in E$, $g(\cdot,z)$ is Hölder-continuous of order β for all $\beta\in(0,\alpha-\frac{1}{p})$ and for all $t\in\mathbb{R}$,

$$u(\cdot+tk)=g(t,\cdot)\quad \mu-\text{a.e.}\ . \tag{4.4}$$

Now we fix a $\mathscr{B}(E)$-measurable μ-version of u and define for $z\in E$

$$T_z:=\{t\in\mathbb{R}\mid g(t,z)=u(z+tk)\}\ . \tag{4.5}$$

Since by (4.4) and Fubini's theorem T_z has full Lebesgue measure for μ-a.e. $z\in E$, we can find $D_k\in\overline{\mathscr{B}(E)}^\mu$ with $\mu(D_k)=1$ and

$$ds(\mathbb{R}\backslash T_z)=0\quad\text{and}\quad\mathbb{Q}\subset T_z\quad\text{for all } z\in D_k\ . \tag{4.6}$$

Now for $z\in E$ we define

$$u_k(z):=\begin{cases} g(t,z_0) & \text{if } z=tk+z_0\in\mathbb{R}\cdot k+D_k\\ 0 & \text{if } z\in E\backslash(\mathbb{R}\cdot k+D_k)\ .\end{cases} \tag{4.7}$$

Of course, representations $z=tk+z_0$ of $z\in\mathbb{R}\cdot k+D_k$ with $t\in\mathbb{R}$, $z_0\in D_k$ are not unique. So, we first have to show that u_k is well-defined. Suppose

$$t_1k+z_1=t_2k+z_2 \tag{4.8}$$

with $t_1,t_2\in\mathbb{R}$, $z_1,z_2\in D_k$. Then for all $t\in(T_{z_1}-t_1)\cap(T_{z_2}-t_2)$,

$$\begin{aligned}
g(t+t_1,z_1)&=u(z_1+(t+t_1)k)\\
&=u(z_2+(t+t_2)k)\\
&=g(t+t_2,z_2).
\end{aligned}$$

Since $(T_{z_1}-t_1)\cap(T_{z_2}-t_2)$ has full Lebesgue-measure and since $g(\cdot,z_i)$, $i=1,2$, are continuous, it follows that

$$g(t+t_1,z_1)=g(t+t_2,z_2)\text{ for all } t\in\mathbb{R}\ .$$

In particular, for $t = 0$ we obtain

$$g(t_1, z_1) = g(t_2, z_2) \quad ,$$

so u_k is well-defined.

It remains to show that u_k has the desired properties. Since $0 \in \mathbb{Q} \subset T_z$ for all $z \in D_k$, we have that

$$u_k(z) = g(0, z) = u(z) \quad \text{for all } z \in D_k \quad ,$$

so

$$u_k = u \ \mu - \text{a.e.} \quad .$$

Furthermore, let $z \in E$. If $z \in \mathbb{R} \cdot k + D_k$, i.e., $z = tk + z_0$, $t \in \mathbb{R}$, $z_0 \in D_k$, then

$$u_k(z + sk) = g(s + t, z_0) \quad \text{for all } s \in \mathbb{R} \quad . \tag{4.9}$$

But, if $z \in E \backslash (\mathbb{R} \cdot k + D_k)$, then $sk + z \in E \backslash (\mathbb{R} \cdot k + D_k)$ for all $s \in \mathbb{R}$ (because if $sk + z = t_1 k + z_1$, $t_1 \in \mathbb{R}$, $z_1 \in D_k$, then

$$z = (t_1 - s)k + z_1 \in \mathbb{R} \cdot k + D_k) \quad .$$

Hence

$$u_k(z + sk) = 0 \quad \text{for all } s \in \mathbb{R} \quad . \tag{4.10}$$

(4.9) and (4.10) imply that for all $z \in E$

$$s \longmapsto u_k(z + sk), \qquad s \in \mathbb{R} \quad ,$$

has the desired Hölder-continuity property and the proof is completed. Q.E.D.

5. Applications

The general results presented in Subsections 5.1 and 5.2 below are motivated by the applications to the polymer measure over $\mathbb{R}^2$ discussed in Subsection 5.3.

5.1. Invariance of closedness under Doob transforms

We consider the situation of Subsection 2.3. So let E, H, K, μ, $(\mathscr{E}_\mu, D(\mathscr{E}_\mu))$, $(L_\mu, D(L_\mu))$ and $(T_t)_{t \geq 0}$ be as defined there. In particular, condition (C) holds for all $k \in K$. Let $\mathscr{D}_p^\alpha$ be the Sobolev space corresponding to $L := L_\mu$. Let $p \in (1, \infty)$ and assume throughout this and the next subsection that:

$$\text{Condition (A) holds for all } k \in K \quad . \tag{5.0}$$

By the results in Subsection 2.3 condition (5.0) always holds for $p = 2$.

Proposition 5.1. *Let $\alpha \in (\frac{1}{p}, 1)$, $u \in \mathscr{D}_p^\alpha$ and $f \in C(\mathbb{R})$ such that $\varphi := f(u) > 0$ μ-a.e. and $\varphi \in L^1(E;\mu)$. Then $(\mathscr{E}_{\varphi\cdot\mu}, D(\mathscr{E}_{\varphi\cdot\mu}))$ (defined as in Subsection 2.3 with $\varphi \cdot \mu$ replacing μ) is closed on $L^2(E;\varphi\cdot\mu)$ (cf. (2.18) in Subsection 2.3).*

Remark 5.2. (i) The Dirichlet form $(\mathscr{E}_{\varphi\cdot\mu}, D(\mathscr{E}_{\varphi\cdot\mu}))$ is sometimes called the *Doob-transform* of $(\mathscr{E}_\mu, D(\mathscr{E}_\mu))$. So, by Proposition 5.1 we have "invariance of closedness" under such transforms.

(ii) Of course, Proposition 5.1 has its natural analogue for the one-component Dirichlet forms $(\mathscr{E}_{\mu,k}, D(\mathscr{E}_{\mu,k}))$ introduced in Subsection 2.2 as well as for the more general classical Dirichlet forms mentioned in Remark 2.3.

(iii) Proposition 5.1 is well-known for $\alpha \geq 1$ in particular cases (cf. e.g. [28] or [36]).

(iv) We note that (C2) might not hold for $k \in K$ and $\varphi \cdot \mu$ replacing μ. But the mere definition of $(\mathscr{E}_\mu, D(\mathscr{E}_\mu))$ in Subsection 2.3 does not require this. So, $(\mathscr{E}_{\varphi\cdot\mu}, D(\mathscr{E}_{\varphi\cdot\mu}))$ is really defined in the same way as $(\mathscr{E}_\mu, D(\mathscr{E}_\mu))$.

Proof of Proposition 5.1. Applying (2.12) we get for all $k \in K$ and all $g : E \longmapsto \mathbb{R}$, $\mathscr{B}(\mathbb{R})$-measurable, bounded,

$$\int_E g(z)\varphi(z)\mu(dz) = \int_{\pi_k(E)} \int_{\mathbb{R}} g(x+sk)\varphi_k(x+sk)\rho_k^\mu(x+sk)\,ds\,\mu_k(dx) \ , \tag{5.1}$$

where $\varphi_k := f(u_k)$ and u_k is as in Theorem 2.1. Now the assertion follows directly from [6, Theorem 3.10], since every $k \in K$ satisfies (C.2) (for μ).

Q.E.D.

5.2. Invariance of irreducibility under Doob-transforms

We consider the same situation as in the previous subsection. In particular, condition (5.0) is still in force. We recall that a Dirichlet form $(\mathscr{E}, D(\mathscr{E}))$ on $L^2(E;\mu))$ is called *irreducible*, if

$$v \in D(\mathscr{E}) \ , \quad \mathscr{E}(v, v) = 0 \Rightarrow v = \text{const} \ . \tag{5.2}$$

It is well-known that it is enough to check (5.2) for bounded v (cf. [28, Chap. I, Proposition 4.17].

Corollary 5.3. *Let $\alpha \in (\frac{1}{p}, 1)$, $u \in \mathscr{D}_p^\alpha$, and $f \in C(\mathbb{R})$, $f > 0$, such that $\varphi := f(u) \in L^1(E;\mu)$. Then, if $(\mathscr{E}_\mu, D(\mathscr{E}_\mu))$ is irreducible on $L^2(E;\mu)$, so is $(\mathscr{E}_{\varphi\cdot\mu}, D(\mathscr{E}_{\varphi\cdot\mu}))$ on $L^2(E;\varphi\cdot\mu)$).*

Proof. Let $v \in D(\mathscr{E}_{\varphi\cdot\mu}) \cap L^\infty(E;\varphi\cdot\mu)$ such that $\mathscr{E}_{\varphi\cdot\mu}(v,v) = 0$ and let $k \in K$. Then by (5.1) (with φ_k as defined there)

$$0 = \int_E \left(\frac{\partial^{\varphi\cdot\mu} v}{\partial k}\right)^2 \varphi_k\,d\mu = \int_{\pi_k(E)} \int_{\mathbb{R}} \left(\frac{d\tilde{v}_x}{ds}(s)\right)^2 \varphi_k(x+sk)\,\rho_k^\mu(x+sk)\,ds\,\mu_k(dx) \ .$$

Since $\varphi_k(x+sk) > 0$ for all $x \in \pi_k(E)$, $s \in \mathbb{R}$, it is easily seen that $v \in D(\mathscr{E}_\mu)$ and

$$0 = \int_{\pi_k(E)} \int_{\mathbb{R}} \left(\frac{d\tilde{v}_x}{ds}(s) \right)^2 \rho_k^\mu(x+sk)\, ds\, \mu_k(dx) = \int_E \left(\frac{\partial^\mu v}{\partial k} \right)^2 d\mu \quad .$$

Consequently, $v \in D(\mathscr{E}_\mu)$ and $\mathscr{E}_\mu(v, v) = 0$, so $v = \text{const.}$ μ-a.e., hence $\varphi \cdot \mu$-a.e..

Q.E.D.

We emphasize that Corollary 5.3 is really a consequence of the explicit description of the domains $D(\mathscr{E}_\mu)$ and $D(\mathscr{E}_{\varphi\cdot\mu})$ which are otherwise quite unrelated.

Remark 5.4. For a characterization of irreducibility of Dirichlet forms of type $(\mathscr{E}_\mu, D(\mathscr{E}_\mu))$ we refer to [4, Theorem 3.3]. It is well-known that in the cases discussed in Subsections 3.1 and 3.3, i.e., the case of an abstract Wiener space resp. the free Euclidean field, we have irreducibility. The same is true for infinite volume Euclidean $P(\Phi)_2$-measures (cf. Subsection 3.4) if the coupling constant λ is small enough. We refer to [4, Remark 4.15 (iii)] for details. So, Corollary 5.3 applies in all these cases (but only for $p = 2$ in the latter situation).

By the previous remark the following generalizes a result due to D. Nualart who proved this for $\alpha = 1$, $p = 2$ in the abstract Wiener space case ([29, p.31, Remark 2]).

Proposition 5.5. *Let $\alpha \in (\frac{1}{p}, 1)$ and suppose that $(\mathscr{E}_\mu, D(\mathscr{E}_\mu))$ is irreducible. Let $1\!\!1_A \in \mathscr{B}(E)$ be such that $1\!\!1_A \in \mathscr{D}_p^\alpha$. Then $\mu(A) = 1$ or 0.*

Proof. Let $u := 1\!\!1_A$ and $k \in K$. Let u_k be as in Theorem 2.1. Then for μ-a.e. $z \in E$

$$u_k(z+tk) = 1\!\!1_A(z+tk) \in \{0, 1\} \quad \text{for all } t \in \mathbb{Q} \quad ,$$

hence by [4, Lemma 3.4]

$$u \in D(\mathscr{E}_{\mu,k}) \text{ and } \mathscr{E}_{\mu,k}(u, u) = 0.$$

Consequently, $u \in D(\mathscr{E}_\mu)$ and $\mathscr{E}_\mu(u, u) = 0$, so $u = 1\!\!1_A = \text{const.}$ μ-a.e. and the assertion follows.

Q.E.D.

5.3. The two-dimensional polymer measure

Let us first recall the (rigorous) definition of the two dimensional polymer measure μ_g. Let $E := C_0([0, 1], \mathbb{R}^2)$ be the set of all continuous paths in $\mathbb{R}^2$ indexed by $[0, 1]$ and starting at zero, equipped with the uniform topology. Let μ_0 denote Wiener measure on $(E, \mathscr{B}(E))$. Let H be the classical Cameron-Martin space, i.e.,

$$H := \{h \in E \mid h \text{ is absolutely continuous and } \|h\|_H^2 := \int_0^1 |\dot{h}(s)|^2 ds < \infty\} \quad .$$

Then (E, H, μ_0) is an abstract Wiener space. Let $\alpha(x, A)$ be the *self-intersection local time* at $x \in \mathbb{R}^2$ of $z \in E$ on the set $A \subset [0,1] \times [0,1]$, i.e.,

$$\alpha(x, A) = \int_A \delta_x(z_s - z_t) ds\, dt \ .$$

(For its precise definition, the reader is referred to [37] and the references therein.) It is well-known that $\alpha(0, \{(s,t) : 0 \le s < t \le 1\}) = \infty$, μ_0-a.e.. Therefore, one has to use renormalization. To this end we set

$$\alpha_{i,k} = \alpha\left(0, \left[\frac{2(i-1)}{2^k}, \frac{2i-1}{2^k}\right) \times \left[\frac{2i-1}{2^k}, \frac{2i}{2^k}\right)\right), \quad i = 1, \ldots, 2^{k-1}, \quad k = 1, 2, \ldots$$

Let E_{μ_0} denote the expectation with respect to μ_0, and set

$$\xi_n := \sum_{k=1}^{n} \sum_{i=1}^{2^{k-1}} (\alpha_{i,k} - E_{\mu_0}\alpha_{i,k}), \quad n \ge 1 \ .$$

Then one can prove (see [23], [43]) that $(\xi_n)_{n\ge 0}$ is almost surely convergent to a random variable $\xi \in L^2(E;\mu_0)$ and $\lim_{n\to\infty} E_{\mu_0}|\xi_n - \xi|^2 = 0$. The random variable ξ is usually called the *normalized self-intersection local time* of planar Brownian motion. One can prove that there is $g_0 \in (0, \infty)$ (see e.g. [24],[31]) such that

$$E_{\mu_0} \exp(-g\xi) \begin{cases} < \infty, & \forall g \in (-g_0, \infty), \\ = \infty, & \forall g \in (-\infty, -g_0) \ . \end{cases} \tag{5.3}$$

The two-dimensional polymer measure μ_g is defined by

$$\mu_g := (E_{\mu_0} \exp(-g\xi))^{-1} \exp(-g\xi)\mu_0 \ , \quad g \in (-g_0, \infty) \ .$$

Below we assume that $g \in (-g_0, \infty)$. It has been proved in [3] that $\xi \in \mathscr{D}_2^\alpha \backslash \mathscr{D}_2^1$ for all $\alpha < 1$. Hence by the result in Subsection 3.1 (cf. Remark 5.3) both Proposition 5.1 and Corollary 5.3 apply with $f(x) := \exp(-gx)$, $x \in \mathbb{R}$; $u := \xi$.

Let us close with commenting on the relation and, particularly, the progress w.r.t. the result in [2].

Remark 5.6. (i) Let $(\mathscr{E}_{\nu_g}, D(\mathscr{E}_{\nu_g}))$ on $L^2(E;\nu_g)$ be as defined in [2, Section 1]. Then by definition $D(\mathscr{E}_{\mu_g}) \supset D(\mathscr{E}_{\nu_g})$ and $\mathscr{E}_{\mu_g} = \mathscr{E}_{\nu_g}$ on $D(\mathscr{E}_{\nu_g}) \times D(\mathscr{E}_{\nu_g})$. Hence the closedness of $(\mathscr{E}_{\mu_g}, D(\mathscr{E}_{\mu_g}))$ on $L^2(E;\mu_g)$ ensured by Proposition 5.1 implies Theorem 1.1 in [2], which was essential there for constructing the stochastic quantization of the two-dimensional polymer measure (cf. [2, Theorem 1.2]).

(ii) The irreducibility of $(\mathscr{E}_{\mu_g}, D(\mathscr{E}_{\mu_g}))$ for all $g \in (-g_0, \infty)$ ensured by Corollary 5.3 generalizes [2, Theorem 1.5] where the same result was proved for $(\mathscr{E}_{\nu_g}, D(\mathscr{E}_{\nu_g}))$, but only for $g \in (-g_0, g_0)$ and by a completely different method.

(iii) Obviously, by the Cameron-Martin Theorem for all $k \in H$, $t \in \mathbb{R}$,

$$a_{tk}^{\mu_g}(z) = e^{-g(\xi(z+tk)-\xi(z))} e^{-t\int_0^1 \dot{k}(s)\, dz(s) - \frac{1}{2}t^2\|k\|_H^2}$$

for μ-a.e. $z \in E$, where $\int_0^1 \dot{k}(s)dz(s)$ is the Itô-integral w.r.t. the Brownian motion $(z(s))_{0\le s\le 1}$ under μ_0. Choosing ξ_k as in Theorem 2.1 we obtain a generalization of [2, Theorem 1.4] to *all* directions k in the Cameron-Martin space rather than only those with bounded derivatives as proved in [2]. Since this was a question posed by a referee to [2], we formulate this result as a theorem below.

Theorem 5.7. *Let for $h \in H$*

$$a_{th}^{\mu_g} := \frac{d(\mu_g \circ \tau_{th}^{-1})}{d\mu_g}, \quad t \in \mathbb{R} \ .$$

Then the process $(a_{th}^{\mu_g})_{t\in\mathbb{R}}$ has a version with continuous sample paths for all $h \in H$.

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