Probab. Theory Relat. Fields (2016) 164:1079–1083
DOI 10.1007/s00440-015-0660-6

ERRATUM

Erratum to: Asymptotic behaviour of first passage time distributions for Lévy processes

R. A. Doney[1] · **V. Rivero**[2]

Published online: 30 September 2015

Erratum to: Probab Theory Relat Fields (2013) 157:1–45 DOI 10.1007/s00440-012-0448-x

In this Erratum, we correct an error in our paper "Asymptotic behaviour of first passage time distributions for Lévy processes" published in *Probab Theory Relat Fields*, 157(1–2):1–45, 2013.

Let X be a real valued Lévy process with law $\mathbb{P}$, characteristic exponent Ψ and characteristic triplet (a, σ, Π). We assume that X is in the domain of attraction of a stable distribution without centering, that is there exists a deterministic function $c : (0, \infty) \to (0, \infty)$ such that

$$\frac{X_t}{c(t)} \xrightarrow{\mathcal{D}} Y_1, \quad \text{as } \ t \to \infty, \tag{1}$$

with Y_1 a strictly stable random variable of parameter $0 < \alpha \leq 2$, and positivity parameter $\rho = \mathbb{P}(Y_1 > 0)$. In this case we will use the notation $X \in D(\alpha, \rho)$, and put $\overline{\rho} = 1 - \rho$. Hereafter $(Y_t, t \geq 0)$ will denote an α-stable Lévy process with positivity parameter $\rho = \mathbb{P}(Y_1 > 0)$. We write f for the density of Y_1, and Ψ_α for its characteristic exponent.

The online version of the original article can be found under doi:10.1007/s00440-012-0448-x.

✉ V. Rivero
rivero@cimat.mx

R. A. Doney
ron.doney@manchester.ac.uk

1 Department of Mathematics, University of Manchester, Manchester M13 9PL, UK

2 Centro de Investigacion en Matematicas A.C., Calle Jalisco s/n, 36240 Guanajuato, Mexico

In the original paper we provided sharp estimates for the local behaviour of the distribution of the first passage time of X below 0, i.e. $T_0 = \inf\{t > 0 : X_t < 0\}$, under $\mathbb{P}_x(\cdot)$, for $x > 0$, both in the event of creeping and non-creeping. The proof of our results has been based in the validity of the following result, Proposition 13 in the original paper.

Proposition 1 *Assume that* $X \in D(\alpha, \rho)$. *Then uniformly in* $0 < \Delta < \Delta_0$, *with* $\Delta_0 > 0$, *and* $x \in \mathbb{R}$,

$$c(t)\mathbb{P}(X_t \in (x, x+\Delta]) = \Delta\left(f\left(\frac{x}{c(t)}\right) + o(1)\right) \quad as \quad t \to \infty. \tag{2}$$

Consequently, given any $\Delta_0 > 0$ *there are constants* k_0 *and* t_0 *such that*

$$c(t)\mathbb{P}(X_t \in (x, x+\Delta]) \leq k_0\Delta, \quad for\ all \quad t \geq t_0 \quad and \quad \Delta \in (0, \Delta_0]. \tag{3}$$

We claimed that this result can easily be proved by repeating the argument used for non-lattice random walks in [3], with very minor changes. This fact is indeed true to some extent, but the uniformity in Δ is only true in general on intervals $[a, b]$ with $0 < a < b < \infty$; and further assumptions are needed to obtain the uniformity on $0 < \Delta < \Delta_0$, for $\Delta_0 > 0$. Hence for our results in original paper to be valid we require two extra assumptions, namely (H2) and (H3) below.

(H1) $X \in D(\alpha, \rho)$.
(H2) There exists a $t_0 > 0$ such that

$$\int_{|\lambda|>1} e^{-t_0 \Re\Psi(\lambda)} d\lambda < \infty.$$

(H3) X is strongly non-lattice,

$$m = \liminf_{|\lambda|\to\infty} \Re\Psi(\lambda) > 0.$$

Observe that the assumption (H2) implies that the law of X_t has a density for all $t > t_0$, see e.g. Proposition 2.5 in [2].

Under these assumptions we have the following Lemma which replaces the Proposition 13 in the original paper.

Lemma 2 *Assume* (*H1–3*) *hold. We have that*

$$c(t)\mathbb{P}(X_t \in (x, x+\Delta]) = \Delta\left(f\left(\frac{x}{c(t)}\right) + o(1)\right), \quad as \quad t \to \infty,$$

uniformly in $x \in \mathbb{R}$, *and* $0 \leq \Delta < \Delta_0$. *Consequently* (3) *holds.*

Proof We would like to estimate

$$c(t)\mathbb{P}(X_t \in (x, x+\Delta])$$

uniformly in $x \in \mathbb{R}$, and uniformly in $|\Delta| < \Delta_0$ for Δ_0 fixed. For $\Delta > 0$, the function

$$g_{\Delta,t}(x) = \frac{1}{\Delta}\mathbb{P}(X_t \in (x, x+\Delta]), \quad x \in \mathbb{R},$$

is a probability density function, that of $X_t + U_\Delta$, with U_Δ an independent r.v. with uniform distribution over $(-\Delta, 0)$. Its characteristic function is given by

$$\widehat{g}_{\Delta,t}(\lambda) := \int_{\mathbb{R}} e^{i\lambda x} g_{\Delta,t}(x)dx = \mathbb{E}\left(e^{i\lambda X_t}\right)\frac{(1-e^{-i\lambda\Delta})}{i\lambda\Delta} = e^{-t\Psi(\lambda)}\frac{(1-e^{-i\lambda\Delta})}{i\lambda\Delta}.$$

The integrability assumption (H2) implies that for $t > t_0$, and $\Delta > 0$

$$\int_{-\infty}^{\infty} |\widehat{g}_{\Delta,t}(\lambda)|d\lambda < \infty.$$

By the Fourier inversion theorem we have that for $\Delta > 0, t > t_0$

$$\begin{aligned}
&\Delta c(t) g_{\Delta,t}(x) - \Delta f(x/c(t)) \\
&\quad = \frac{\Delta}{2\pi}\int_{-\infty}^{\infty} d\lambda e^{-ix\lambda/c(t)}\left[e^{-t\Psi(\lambda/c(t))} - e^{-\Psi_\alpha(\lambda)}\right]\left(\frac{1-e^{-i\lambda\Delta/c(t)}}{i\lambda\Delta/c(t)}\right) \\
&\qquad + \frac{\Delta}{2\pi}\int_{-\infty}^{\infty} d\lambda e^{-ix\lambda/c(t)} e^{-\Psi_\alpha(\lambda)}\left(\frac{1-e^{-i\lambda\Delta/c(t)}}{i\lambda\Delta/c(t)} - 1\right).
\end{aligned}$$

To estimate this expression we will use among other things the inequalities

$$\left|\frac{e^{iu}-1}{iu}\right| \le 1, \quad \left|\frac{e^{iu}-1-iu}{iu}\right| \le \frac{|u|}{2}, \quad u \in \mathbb{R},$$

see [2] Lemma 8.6. Using the latter, the second term in the above expression can be estimated by

$$\begin{aligned}
&\frac{\Delta}{2\pi}\left|\int_{-\infty}^{\infty} d\lambda e^{-ix\lambda/c(t)} e^{-\Psi_\alpha(\lambda)}\left(\frac{1-e^{-i\lambda\Delta/c(t)}}{i\lambda\Delta/c(t)} - 1\right)\right| \\
&\quad \le \frac{\Delta}{4\pi}\left(\frac{\Delta_0}{c(t)}\right)\int_{-\infty}^{\infty} d\lambda e^{-\Re\Psi_\alpha(\lambda)}|\lambda|.
\end{aligned}$$

Since $\Re\Psi_\alpha(\lambda) = |\lambda|^\alpha c_\alpha$, with $c_\alpha \in (0, \infty)$ a constant, the latter integral is finite, and hence its product with $\Delta/c(t)$ tends to zero uniformly in x and Δ, as long as Δ remains bounded.

Because $X \in D(\alpha, \rho)$ we have that

$$\lim_{t\to\infty} |\exp\{-t\Psi(\lambda/c(t))\} - e^{-\Psi_\alpha(\lambda)}| = 0,$$

uniformly over closed intervals $[-A, A]$, and also $\left|\frac{1-e^{-i\lambda\Delta/c(t)}}{i\lambda\Delta/c(t)}\right| \leq 1$. It follows that for any $A > 0$

$$\Delta\left|\int_{-A}^{A} d\lambda e^{-ix\lambda/c(t)}\left[e^{-t\Psi(\lambda/c(t))} - e^{-\Psi_\alpha(\lambda)}\right]\left(\frac{1-e^{-i\lambda\Delta/c(t)}}{i\lambda\Delta/c(t)}\right)\right| \to 0, \quad t \to \infty,$$

uniformly in x and $\Delta < \Delta_0$. To finish it will be sufficient to prove that

$$\Delta\int_{(-A,A)^c} d\lambda e^{-t\Re\Psi(\lambda/c(t))} \to 0,$$

uniformly in x, and $\Delta < \Delta_0$. We proceed as follows. Because the function $\lambda \mapsto \Re\Psi(\lambda)$ is regularly varying at 0, the Potter bounds, [1] Theorem 1.5.6, ensure that for any $\alpha > \epsilon > 0$ there exists constant K and a B_1 such that

$$\Re\Psi(\lambda) \geq K\lambda^{\alpha-\epsilon}, \quad 0 \leq \lambda < B_1.$$

We apply this inequality to infer

$$\Delta\int_{A}^{B_1 c(t)} d\lambda e^{-t\Re\Psi(\lambda/c(t))} \leq \Delta\int_0^\infty d\lambda e^{-Kt\left(\frac{\lambda}{c(t)}\right)^{\alpha-\epsilon}}.$$

An application of the monotone convergence theorem shows that the latter term tends to 0, as $t \to \infty$, because $c(t) \in RV^\infty_{1/\alpha}$ and therefore $t/(c(t))^{\alpha-\epsilon} \in RV^\infty_{\epsilon/\alpha}$. The convergence is uniform in x, and in Δ on bounded intervals. By symmetry we also get the convergence

$$\Delta\int_{-B_1 c(t)}^{-A} d\lambda e^{-t\Re\Psi(\lambda/c(t))} \xrightarrow[t\to\infty]{} 0,$$

uniformly in x, and in Δ on bounded intervals. The assumption of having X strongly non-lattice, implies that, given $\epsilon > 0$ and small enough, there is a B_2 such that $\Re\Psi(\lambda) > m - \epsilon > 0$ for all $|\lambda| > B_2$. By the continuity of $\Re\Psi(\lambda)$ and the fact that this function does not take the value zero in $\mathbb{R}\backslash\{0\}$, since X is non-lattice, we can assume that $B_2 = B_1$, maybe at the price of replacing $m - \epsilon$ by $0 < \widetilde{m} \leq m - \epsilon$. We get that for $t > t_0 > 0$

$$\begin{aligned}\Delta\int_{(B_2 c(t),\infty)} d\lambda e^{-t\Re\Psi(\lambda/c(t))} &\leq \Delta c(t)\int_{B_2}^\infty \exp\{-t\Re\Psi(\lambda)\}d\lambda \\ &\leq \Delta c(t)\exp\{-(t-t_0)\widetilde{m}\}\int_{B_2}^\infty \exp\{-t_0\Re\Psi(\lambda)\}d\lambda.\end{aligned}$$

The rightmost term in the above inequality tends to zero as $t \to \infty$, because the function $c(\cdot)$ is regularly varying and hence its growth is at most polynomial. By symmetry we deduce the convergence

$$\Delta \int_{(-\infty,-B_2c(t))} d\lambda e^{-t\Re\Psi(\lambda/c(t))} \xrightarrow[t\to\infty]{} 0,$$

uniformly in x, and in Δ on bounded intervals. The result follows. □

References

1. Bingham, N.H., Goldie, C.M., Teugels, J.L.: Regular variation. Encyclopedia of Mathematics and Its Applications, vol. 27. Cambridge University Press, Cambridge (1989)
2. Sato, K.: Lévy processes and infinitely divisible distributions. In: Cambridge Studies in Advanced Mathematics, vol. 68. Cambridge University Press, Cambridge (2013)
3. Stone, C.: On local and ratio limit theorems. In: Proceedings of the 5th Berkeley Sympos. Math. Statist. and Probability (Berkeley, Calif., 1965/66), vol. II. Contributions to Probability Theory, Part 2, pp. 217–224. University of California Press, Berkeley (1967)