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A Martingale approach to metastability

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Abstract We presented in Beltrán and Landim (J Stat Phys 140:1065–1114, 2010) Beltrán and Landim (J Stat Phys 149:598–618, 2012) an approach to derive the metastable behavior of continuous-time Markov chains. We assumed in these articles that the Markov chains visit points in the time scale in which it jumps among the metastable sets. We replace this condition here by assumptions on the mixing times and on the relaxation times of the chains reflected at the boundary of the metastable sets.

Keywords Metastability · Mixing times · Markov processes

Mathematics Subject Classification 60J27 · 82C26 · 60K35

In memoriam of Hermann Rost.

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1 Introduction

Cassandro et al. [13] proposed in a seminal paper a general method to derive the metastable behavior of continuous-time Markov chains with exponentially small jump rates, called the pathwise approach. In many different contexts these ideas permitted to prove that the exit time from a metastable set has an asymptotic exponential law; to provide estimates for the expectations of the exit times; to describe the typical escape trajectory from a metastable set; to compute the distribution of the exit (saddle) points from a metastable set; and to prove the convergence of the finite-dimensional distributions of the order parameter, the macroscopic variable which characterizes the state of the process, to the finite-dimensional distributions of a finite-state Markov chain. This approach has known a great success, and it is impossible to review here the main results. We refer to [24] for a recent account of this theory.

In Bovier et al. [9,10] proposed a new approach to prove the metastable behavior of continuous-time Markov chains, known as the potential theoretic approach. Motivated by the dynamics of mean field spin systems, the authors created tools, based on the potential theory of reversible Markov processes, to compute the expectation of the exit time from a metastable set and to prove that these exit times are asymptotically exponential. They also expressed the expectation of the exit time from a metastable set and the jump probabilities among the metastable sets in terms of eigenvalues and right-eigenvectors of the generator of the Markov chain.

Compared to the pathwise approach, the potential theoretic approach does not attempt to describe the typical exit path from a metastable set, but provides precise asymptotic formulas for the expectation of the exit time from a metastable set. This accuracy, not reached by the pathwise approach, whose estimates admit exponential errors in the parameter, permits to encompass in the theory dynamics which present logarithmic energy or entropy barriers such as [2,11,12]. Moreover, in the case of a transition from a metastable set to a stable set, it characterizes the asymptotic dynamics: the process remains at the metastable set an exponential time whose mean has been estimated sharply and then it jumps to the stable set.

As the pathwise approach, the potential theoretic approach has been successfully applied to a great number of models. We refer to the recently published paper [6] for references.

Inspired by the evolution of sticky zero-range processes [2,22], dynamics which have a finite number of stable sets with logarithmic energy barriers, we proposed in [1,5] a third approach to metastability, now called the martingale approach. This method was successfully applied to derive the asymptotic behavior of the condensate in sticky zero-range processes [2,22], to prove that in the ergodic time scale random walks among random traps [17,18] converge to K-processes, and to show that the evolution among the ground states of the Kawasaki dynamics for the two dimensional Ising lattice gas [4,16] on a large torus converges to a Brownian motion as the temperature vanishes.

To depict the asymptotic dynamics of the order parameter, one has to compute the expectation of the holding times of each metastable set and the jump probabilities amid the mestastable sets. The potential theoretic approach permits to compute the expectations of the holding times and yields a formula for the jump probabilities in terms of eigenvectors of the generator. This latter formula, although interesting from the theoretical point of view, since it establishes a link between the spectral properties of the generator and the metastable behavior of the process, is of little practical use because one is usually unable to compute the eigenvectors of the generator.

The martingale approach replaces the formula of the jump probabilities written through eigenvectors of the generator by one, [1, Remark 2.9 and Lemma 6.8], expressed only in terms of the capacities, capacities which can be estimated using the Dirichlet and the Thomson variational principles. We have, therefore, a precise description of the asymptotic dynamics of the order parameter: a sharp estimate of the holding times at each metastable set from the potential theoretical approach, and an explicit expression for the jump probabilities among the metastable sets from the aforementioned formula.

This informal description of the asymptotic dynamics of the order parameter among the metastable sets has been converted in [1,5] into a theorem which asserts that the order parameter converges to a Markov chain in a topology introduced in [18], weaker than the Skorohod one. The proof of this result relies on three hypotheses, formulated in terms of the stationary measure and of the capacities between sets, and it uses the martingale characterization of a Markovian dynamics and the notion of the trace of a Markov process on a subset of the configuration space.

In the martingale approach, the potential theory tools developed by Bovier et al. [9,10] to prove the metastability of Markov chains can be very useful in some models [2,22] or not needed at all, as in [17,18]. In these latter dynamics, the asymptotic jump probabilities among the metastable sets, which, as we said, can be expressed through capacities, are estimated by other means without reference to potential theory.

The proof of the convergence of the order parameter to a Markov chain presented in [1,5] requires that in each metastable set the time it takes for the process to visit a representative configuration of the metastable set is small compared to the time the process stays in the metastable set. We introduced in [1] a condition, expressed in terms of capacities, which guarantees that a representative point of the metastable set is visited before the process reaches another metastable set. This quite strong assumption, fulfilled by a large class of dynamics, fails in some cases, as in polymer models in the depinned phase [11,12] or in the dog graph [25]. The main goal of this article is to weaken this assumption.

More recently, Bianchi and Gaudillière [7] proposed still another approach based on the fact that the exit time from a set starting from the quasi-stationary measure associated to this set is an exponential random variable. The proof that the exit time from a metastable set is asymptotically exponential is thus reduced to the proof that the state of the process gets close to the quasi-stationary state before the process leaves the metastable set. To derive this property the authors obtained estimates on the mixing time towards the quasi-stationary state and on the asymptotic exit distribution with errors expressed in terms of the ratio between the spectral radius of the generator of the process killed when it leaves the metastable set and the spectral gap of the process reflected at the boundary of the metastable set, a ratio which has to be small if a metastable behavior is expected. They also introduced (κ, λ)-capacities, an object which plays an important role in this article.

After these historical remarks, we present the main results of this article. Consider a sequence of continuous-time Markov chains $\eta^N(t)$. To describe the asymptotic evolution of the dynamics among the metastable sets, let X_t^N be the functional of the process which indicates the current metastable set visited:

$$X_t^N = \sum_{x=1}^{\kappa} x \, \mathbf{1}\{\eta^N(t) \in \mathcal{E}_N^x\}.$$

In this formula, κ represents the number of metastable sets and $\mathcal{E}_N^x$, $1 \le x \le \kappa$, the metastable sets. The non-Markovian dynamics X_t^N is called the order process or the order in short.

The main result of [1,5] states that under certain conditions, which can be expressed only in terms of the stationary measure and of the capacities between the metastable sets, the order converges in some time scale and in some topology to a Markov process on $S = \{1, \dots, \kappa\}$.

The main drawback of the method [1,5] is that it requires the process to visit points. More precisely, we needed to assume that each metastable set $\mathcal{E}_N^x$ contains a configuration ξ_N^x which, once the process enters $\mathcal{E}_N^x$, is visited before the process reaches another metastable set:

$$\lim_{N\to\infty} \sup_{\eta\in\mathcal{E}_N^x} \mathbb{P}_\eta \Big[H_{\breve{\mathcal{E}}_N^x} < H_{\xi_N^x} \Big] = 0 \tag{1.1}$$

for all $x \in S$. Here, H_A, $A \subset E_N$, stands for the hitting time of A, $\breve{\mathcal{E}}_N^x = \cup_{y\neq x} \mathcal{E}_N^y$, and $\mathbb{P}_\eta$ represents the distribution of the process $\eta^N(t)$ starting from the configuration η. The configuration ξ_N^x is by no means special. It is shown in [1] that if this property holds for one configuration ξ in $\mathcal{E}_N^x$, it holds for any configuration in $\mathcal{E}_N^x$.

Property (1.1) is fulfilled by some dynamics, as sticky zero-range processes [2,22], trap models [17,18] or Markov processes on finite sets [3,4], but it is clearly not fulfilled in general.

The purpose of this paper is to replace condition (1.1) by assumptions on the relaxation time of the process reflected at the boundary of a metastable set. We propose two different set of hypotheses. The first set essentially requires only the spectral gap of the process to be much smaller than the spectral gaps of the reflected processes on each metastable set, and the average jump rates among the metastable sets to converge when properly renormalized. Under these conditions, Theorem 2.2 states that the finite-dimensional distributions of the order process converge to the finite-dimensional distributions of a finite state Markov chain, provided the initial distribution is not too far from the equilibrium measure.

On the other hand, if one is able to show that the mixing times of the reflected processes on each metastable set are much smaller than the relaxation time of the process, Theorem 2.4 and Lemma 2.6 affirm that the order process converges to a finite state Markov chain. Hence, the condition that the process visits points is replaced in this article by estimates on the mixing times of the reflected processes.

In Sect. 8, we apply these results to two models. We show that the polymer in the depinned phase considered by Caputo et al. in [11,12] satisfies the first set of conditions and that the dog graph introduced by Diaconis and Saloff-Coste [25] fulfills the second set of assumptions. Lacoin and Teixeira [21] proved that a polymer interface which interacts with an attractive substrate satisfies both set of conditions.

2 Notation and results

Fix a sequence $(E_N : N \geq 1)$ of countable state spaces. The elements of E_N are denoted by the Greek letters η, ξ. For each $N \geq 1$ consider matrix $R_N : E_N \times E_N \to \mathbb{R}$ such that $R_N(\eta, \xi) \geq 0$, $\eta \neq \xi$, $-\infty < R_N(\eta, \eta) < 0$, $\sum_\xi R_N(\eta, \xi) = 0$, $\eta \in E_N$. Denote by $\{\eta^N(t) : t \geq 0\}$ the right-continuous, continuous-time strong Markov process on E_N whose generator L_N is given by

$$(L_N f)(\eta) = \sum_{\xi \in E_N} R_N(\eta, \xi) \, \{f(\xi) - f(\eta)\}, \tag{2.1}$$

for bounded functions $f : E_N \to \mathbb{R}$. We assume that $\eta^N(t)$ is positive-recurrent and reversible. Denote by $\pi = \pi_N$ the unique invariant probability measure, by $\lambda_N(\eta)$, $\eta \in E_N$, the holding rates, $\lambda_N(\eta) = \sum_{\xi \neq \eta} R_N(\eta, \xi)$, and by $p_N(\eta, \xi)$, $\eta, \xi \in E_N$, the jump probabilities: $p_N(\eta, \xi) = \lambda_N(\eta)^{-1} R_N(\eta, \xi)$ for $\eta \neq \xi$, and $p_N(\eta, \eta) = 0$ for $\eta \in E_N$. We assume that $p_N(\eta, \xi)$ are the transition probabilities of a positive-recurrent discrete-time Markov chain. In particular the measure $M_N(\eta) := \pi_N(\eta)\lambda_N(\eta)$ is finite.

Throughout this article we omit the index N as much as possible. We write, for instance, $\eta(t)$, π for $\eta^N(t)$, π_N, respectively. Denote by $D(\mathbb{R}_+, E_N)$ the space of right-continuous trajectories with left limits endowed with the Skorohod topology. Let $\mathbb{P}_\eta = \mathbb{P}^N_\eta$, $\eta \in E_N$, be the probability measure on $D(\mathbb{R}_+, E_N)$ induced by the Markov process $\{\eta(t) : t \geq 0\}$ starting from η. Expectation with respect to $\mathbb{P}_\eta$ is denoted by $\mathbb{E}_\eta$.

For a subset $\mathcal{A}$ of E_N, denote by $H_\mathcal{A}$ the hitting time of $\mathcal{A}$ and by $H^+_\mathcal{A}$ the return time to $\mathcal{A}$:

$$\begin{aligned} H^+_\mathcal{A} &= \inf\{t > 0 : \eta(t) \in \mathcal{A},\ \eta(s) \neq \eta(0) \ \text{ for some } 0 < s < t\}, \\ H_\mathcal{A} &:= \inf\{t > 0 : \eta(t) \in \mathcal{A}\}, \end{aligned} \tag{2.2}$$

with the convention that $H_\mathcal{A} = \infty$, $H^+_\mathcal{A} = \infty$ if $\eta(s) \notin \mathcal{A}$ for all $s > 0$. We sometimes write $H(\mathcal{A})$ for $H_\mathcal{A}$. Denote by $\text{cap}_N(\mathcal{A}, \mathcal{B})$ the capacity between two disjoint subsets $\mathcal{A}, \mathcal{B}$ of E_N:

$$\text{cap}_N(\mathcal{A}, \mathcal{B}) = \sum_{\eta \in \mathcal{A}} \pi(\eta) \, \lambda(\eta) \, \mathbb{P}_\eta \left[H_\mathcal{B} < H^+_\mathcal{A} \right].$$

Denote by $L^2(\pi)$ the space of square summable functions $f : E_N \to \mathbb{R}$ endowed with the scalar product $\langle f, g \rangle_\pi = \sum_{\eta \in E_N} \pi(\eta) f(\eta) g(\eta)$. Let $\mathfrak{g} = \mathfrak{g}_N$ be the spectral gap of the generator L_N:

$$\mathfrak{g} = \inf_f \frac{\langle (-L_N) f, f\rangle_\pi}{\langle f, f\rangle_\pi},$$

where the infimum is carried over all functions f in $L^2(\pi)$ which are orthogonal to the constants: $\langle f, 1\rangle_\pi = 0$.

Fix a finite number of disjoint subsets $\mathcal{E}_N^1, \dots, \mathcal{E}_N^\kappa$, $\kappa \geq 2$, of E_N: $\mathcal{E}_N^x \cap \mathcal{E}_N^y = \varnothing$, $x \neq y$. The sets $\mathcal{E}_N^x$ have to be interpreted as wells for the Markov dynamics $\eta(t)$. Let $\mathcal{E}_N = \cup_{x\in S} \mathcal{E}_N^x$ and let $\Delta_N = E_N \setminus \mathcal{E}_N$ so that

$$E_N = \mathcal{E}_N^1 \cup \cdots \cup \mathcal{E}_N^\kappa \cup \Delta_N. \tag{2.3}$$

In contrast with the wells $\mathcal{E}_N^x$, Δ_N is a set of small measure which separates the wells.
A. Trace process. Denote by $\{\eta^{\mathcal{E}}(t) : t \geq 0\}$ the $\mathcal{E}_N$-valued Markov process obtained as the trace of $\{\eta^N(t) : t \geq 0\}$ on $\mathcal{E}_N$. This is a time-change of the original process in which the clock is stopped when the process reaches a configuration outside of $\mathcal{E}_N$ and it is restarted when the process returns to $\mathcal{E}_N$. We refer to [1, Section 6.1] for a precise definition. The rate at which the trace process jumps from η to $\xi \in \mathcal{E}_N$ is denoted by $R^{\mathcal{E}}(\eta, \xi)$ and its generator by $L_{\mathcal{E}}$:

$$(L_{\mathcal{E}} f)(\eta) = \sum_{\xi \in \mathcal{E}_N} R^{\mathcal{E}}(\eta, \xi) \, \{f(\xi) - f(\eta)\}, \quad \eta \in \mathcal{E}_N.$$

By [1, Proposition 6.3], the probability measure π conditioned to $\mathcal{E}_N$, $\pi_{\mathcal{E}}(\eta) = \pi(\eta)/\pi(\mathcal{E}_N)\, \mathbf{1}\{\eta \in \mathcal{E}_N\}$, is reversible for the trace process.

Let $\mathbb{P}_\eta^{\mathcal{E}}$, $\eta \in \mathcal{E}_N$, be the probability measure on $D(\mathbb{R}_+, \mathcal{E}_N)$ induced by the trace process $\{\eta^{\mathcal{E}}(t) : t \geq 0\}$ starting from η. Expectation with respect to $\mathbb{P}_\eta^{\mathcal{E}}$ is denoted by $\mathbb{E}_\eta^{\mathcal{E}}$. Denote by $\mathfrak{g}_{\mathcal{E}}$ the spectral gap of the trace process:

$$\mathfrak{g}_{\mathcal{E}} = \inf_f \frac{\langle (-L_{\mathcal{E}}) f, f\rangle_{\pi_{\mathcal{E}}}}{\langle f, f\rangle_{\pi_{\mathcal{E}}}},$$

where the infimum is carried over all functions f in $L^2(\pi_{\mathcal{E}})$ which are orthogonal to the constants: $\langle f, 1\rangle_{\pi_{\mathcal{E}}} = 0$.

Proposition 2.1 presents an estimate of the spectral gap of the trace process in terms of the spectral gap of the original process.

Proposition 2.1 *Let f be an eigenfunction associated to $\mathfrak{g}$ such that $E_\pi[f^2] = 1$, $E_\pi[f] = 0$. Then,*

$$\mathfrak{g}_{\mathcal{E}} \left\{ 1 - \frac{1}{\pi(\mathcal{E}_N)} E_\pi \left[f^2 \mathbf{1}\{\mathcal{E}_N^c\} \right] \right\} \;\leq\; \mathfrak{g} \;\leq\; \mathfrak{g}_{\mathcal{E}}.$$

In the examples we have in mind $\pi(\mathcal{E}_N)$ converges to 1. In particular, if we show that an eigenfunction associated to $\mathfrak{g}$ is bounded, $\mathfrak{g}_{\mathcal{E}}/\mathfrak{g}$ converges to 1. We provide in Lemma 6.1 an upper bound for $\mathfrak{g}_{\mathcal{E}}$ in terms of capacities.

Denote by $\Psi_N : \mathcal{E}_N \mapsto S = \{1, \dots, \kappa\}$, the projection given by

$$\Psi_N(\eta) = \sum_{x=1}^{\kappa} x\, \mathbf{1}\{\eta \in \mathcal{E}_N^x\}.$$

and by $\{X_t^N : t \geq 0\}$ the stochastic process on S defined by $X_t^N = \Psi_N(\eta^{\mathcal{E}}(t))$. Clearly, besides trivial cases, $\{X_t^N : t \geq 0\}$ is not Markovian. We refer to X_t^N as the *order* process or order for short.

B. Reflected process. Denote by $\{\eta^{\mathbf{r},x}(t) : t \geq 0\}$, $1 \leq x \leq \kappa$, the Markov process $\eta(t)$ reflected at $\mathcal{E}_N^x$. This is the process obtained from the Markov process $\eta(t)$ by forbidding all jumps from η to ξ if η or ξ do not belong to $\mathcal{E}_N^x$. The generator $L_{\mathbf{r},x}$ of this Markov process is given by

$$(L_{\mathbf{r},x} f)(\eta) = \sum_{\xi \in \mathcal{E}_N^x} R_N(\eta, \xi)\ \{f(\xi) - f(\eta)\}\,, \quad \eta \in \mathcal{E}_N^x.$$

Assume that the reflected process $\eta^{\mathbf{r},x}(t)$ is irreducible for each $1 \leq x \leq \kappa$. It is easy to show that the conditioned probability measure π_x defined by

$$\pi_x(\eta) = \frac{\pi(\eta)}{\pi(\mathcal{E}_N^x)}, \quad \eta \in \mathcal{E}_N^x, \tag{2.4}$$

is reversible for the reflected process. Let $\mathfrak{g}_{\mathbf{r},x}$ be the spectral gap of the reflected process:

$$\mathfrak{g}_{\mathbf{r},x} = \inf_f \frac{\langle (-L_{\mathbf{r},x}) f, f\rangle_{\pi_x}}{\langle f, f\rangle_{\pi_x}},$$

where the infimum is carried over all functions f in $L^2(\pi_x)$ which are orthogonal to the constants: $\langle f, 1\rangle_{\pi_x} = 0$.

C. Enlarged process. Consider an irreducible, positive recurrent Markov process $\xi(t)$ on a countable set E which jumps from a state η to a state ξ at rate $R(\eta, \xi)$. Denote by π the unique stationary state of the process. Let $E^\star$ be a copy of E and denote by $\eta^\star \in E^\star$ the copy of $\eta \in E$. Following [7], for $\gamma > 0$ denote by $\xi^\gamma(t)$ the Markov process on $E \cup E^\star$ whose jump rates $R^\gamma(\eta, \xi)$ are given by

$$R^\gamma(\eta, \xi) = \begin{cases} R(\eta, \xi) & \text{if } \eta \text{ and } \xi \in E, \\ \gamma & \text{if } \xi = \eta^\star \text{ or if } \eta = \xi^\star, \\ 0 & \text{otherwise.} \end{cases}$$

Therefore, being at some state $\xi^\star$ in $E^\star$, the process may only jump to ξ and this happens at rate γ. In contrast, being at some state ξ in E, the process $\xi^\gamma(t)$ jumps with rate $R(\xi, \xi')$ to some state $\xi' \in E$, and jumps with rate γ to $\xi^\star$. We call the process $\xi^\gamma(t)$ the γ-*enlargement* of the process $\xi(t)$.

The enlarged process permits to formulate in mathematical terms the idea that a process reaches equilibrium inside a set before leaving this set. Let A, $B = A^c$ be a partition of E and let $\mathfrak{g}$ represent the spectral gap of the process reflected at A and θ the time it takes to reach B starting from A. Assume that $\mathfrak{g}^{-1} \ll \theta$ and choose γ so that $\mathfrak{g}^{-1} \ll \gamma^{-1} \ll \theta$. Starting from A the enlarged process reaches $A^\star$ before B with a probability close to 1 and its distribution at the hitting time of $A^\star$ is close to the quasi-stationary state for the process killed when it reaches B.

Let $\pi_\star$ be the probability measure on $E \cup E^\star$ defined by

$$\pi_\star(\eta) = (1/2)\,\pi(\eta), \quad \pi_\star(\eta^\star) = \pi_\star(\eta), \quad \eta \in E.$$

The probability measure $\pi_\star$ is invariant for the enlarged process $\xi^\gamma(t)$ and is reversible whenever π is reversible.

Let $\mathcal{E}_N^{\star,x}$, $1 \le x \le \kappa$, be a copy of the set $\mathcal{E}_N^x$ and let $\mathcal{E}_N^\star = \cup_{1\le x\le\kappa} \mathcal{E}_N^{\star,x}$, $\breve{\mathcal{E}}_N^{\star,x} = \cup_{y\neq x} \mathcal{E}_N^{\star,y}$. Fix a sequence $\gamma = \gamma_N$ and denote by $\eta^\star(t) = \eta^{\mathcal{E},\gamma}$ the γ-enlargement of the trace process $\eta^{\mathcal{E}}(t)$. Denote the generator of this Markov chain by $L_\star$, by $R_\star(\eta,\xi)$ the rate at which it jumps from η to ξ, and by $\lambda_\star(\eta)$ the holding rates, $\lambda_\star(\eta) = \sum_{\xi\in\mathcal{E}_N\cup\mathcal{E}_N^\star} R_\star(\eta,\xi)$.

Denote by $\mathbb{P}_\eta^{\star,\gamma}$, $\eta \in \mathcal{E}_N \cup \mathcal{E}_N^\star$, the probability measure on the path space $D(\mathbb{R}_+, \mathcal{E}_N \cup \mathcal{E}_N^\star)$ induced by the Markov process $\eta^\star(t)$ starting from η and recall the definition of the hitting time and the return time introduced in (2.2). For $x \neq y \in S$, let $r_N(x,y)$ be the average rate at which the enlarged process $\eta^\star(t)$ jumps from $\mathcal{E}_N^{\star,x}$ to $\mathcal{E}_N^{\star,y}$:

$$\begin{aligned} r_N(x,y) &= \frac{1}{\pi_\star(\mathcal{E}_N^{\star,x})} \sum_{\eta\in\mathcal{E}_N^{\star,x}} \pi_\star(\eta)\,\lambda_\star(\eta)\,\mathbb{P}_\eta^{\star,\gamma}\Big[H_{\mathcal{E}_N^{\star,y}} < H^+_{\breve{\mathcal{E}}_N^{\star,y}}\Big] \\ &= \frac{\gamma}{\pi_{\mathcal{E}}(\mathcal{E}_N^x)} \sum_{\eta\in\mathcal{E}_N^x} \pi_{\mathcal{E}}(\eta)\,\mathbb{P}_\eta^{\star,\gamma}\Big[H_{\mathcal{E}_N^{\star,y}} < H_{\breve{\mathcal{E}}_N^{\star,y}}\Big]. \end{aligned} \tag{2.5}$$

By [1, Proposition 6.2], $r_N(x,y)$ corresponds to the average rate at which the trace of the process $\eta^\star(t)$ on $\mathcal{E}_N^\star$ jumps from $\mathcal{E}_N^{\star,x}$ to $\mathcal{E}_N^{\star,y}$. This explains the terminology.

For two disjoint subsets $\mathcal{A}$, $\mathcal{B}$ of $\mathcal{E}_N \cup \mathcal{E}_N^\star$, denote by $\mathrm{cap}_\star(\mathcal{A},\mathcal{B})$ the capacity between $\mathcal{A}$ and $\mathcal{B}$:

$$\mathrm{cap}_\star(\mathcal{A},\mathcal{B}) = \sum_{\eta\in\mathcal{A}} \pi_\star(\eta)\,\lambda_\star(\eta)\,\mathbb{P}_\eta^{\star,\gamma}\big[H_{\mathcal{B}} < H^+_{\mathcal{A}}\big].$$

Let A, B be two disjoint subsets of S. Taking $\mathcal{A} = \cup_{x\in A}\mathcal{E}_N^{\star,x}$, $\mathcal{B} = \cup_{y\in B}\mathcal{E}_N^{\star,y}$ in the previous formula, since the enlarged process may only jump from $\eta^\star$ to η and since $\pi_\star(\eta^\star) = \pi_\star(\eta) = (1/2)\pi_{\mathcal{E}}(\eta)$,

$$\mathrm{cap}_\star\Bigg(\bigcup_{x\in A}\mathcal{E}_N^{\star,x}, \bigcup_{y\in B}\mathcal{E}_N^{\star,y}\Bigg) = \frac{\gamma}{2}\sum_{x\in A}\sum_{\eta\in\mathcal{E}_N^x} \pi_{\mathcal{E}}(\eta)\,\mathbb{P}_\eta^{\star,\gamma}\Bigg[H\Bigg(\bigcup_{y\in B}\mathcal{E}_N^{\star,y}\Bigg) < H\Bigg(\bigcup_{x\in A}\mathcal{E}_N^{\star,x}\Bigg)\Bigg]. \tag{2.6}$$

It follows from this identity and some simple algebra that

$$\pi_\star(\mathcal{E}_N^x)\sum_{y\neq x} r_N(x,y) = \frac{\gamma}{2}\sum_{\eta\in\mathcal{E}_N^x}\pi_{\mathcal{E}}(\eta)\,\mathbb{P}_\eta^{\star,\gamma}\left[H_{\breve{\mathcal{E}}_N^{\star,x}} < H_{\mathcal{E}_N^{\star,x}}\right] = \mathrm{cap}_\star\left(\mathcal{E}_N^{\star,x},\breve{\mathcal{E}}_N^{\star,x}\right). \tag{2.7}$$

D. L^2 theory. We show in this subsection that with very few assumptions one can prove the convergence of the finite-dimensional distributions of the order X_t^N. Let

$$\mathcal{M}_x = \min\left\{\pi_{\mathcal{E}}(\mathcal{E}_N^x),\ 1-\pi_{\mathcal{E}}(\mathcal{E}_N^x)\right\},\quad x\in S. \tag{2.8}$$

Theorem 2.2 *Suppose that there exist a non-negative sequence $\{\theta_N : N\geq 1\}$ and non-negative numbers $r(x,y)$, $x\neq y\in S$, such that*

$$\theta_N^{-1} \ll \min_{x\in S}\mathfrak{g}_{\mathbf{r},x}, \tag{L1}$$

$$\lim_{N\to\infty}\theta_N\, r_N(x,y) = r(x,y),\quad x\neq y\in S.$$

Fix $x_0\in S$. Let $\{\nu_N : N\geq 1\}$ be a sequence of probability measures concentrated on $\mathcal{E}_N^{x_0}$, $\nu_N(\mathcal{E}_N^{x_0}) = 1$, and such that

$$E_{\pi_{\mathcal{E}}}\left[\left(\frac{d\nu_N}{d\pi_{\mathcal{E}}}\right)^2\right] \leq \frac{C_0}{\max_{x\in S}\mathcal{M}_x} \tag{L2G}$$

for some finite constant C_0. Then, under $\mathbb{P}_{\nu_N}^{\mathcal{E}}$ the finite-dimensional distributions of the time-rescaled order $\mathbf{X}_t^N = X_{t\theta_N}^N$ converge to the finite-dimensional distributions of the Markov process on S which starts from x_0 and jumps from x to y at rate $r(x,y)$.

Let ν_N be the measure π_{x_0} defined in (2.4). In this case condition (L2G) becomes

$$\max_{x\in S}\mathcal{M}_x \leq C_0\,\pi_{\mathcal{E}}(\mathcal{E}_N^{x_0}). \tag{2.9}$$

Since

$$\min_{x\in S}\max_{z\neq x}\pi_{\mathcal{E}}(\mathcal{E}_N^z) \leq \max_{x\in S}\min\left\{\pi_{\mathcal{E}}(\mathcal{E}_N^x),\ 1-\pi_{\mathcal{E}}(\mathcal{E}_N^x)\right\} \leq \kappa\min_{x\in S}\max_{z\neq x}\pi_{\mathcal{E}}(\mathcal{E}_N^z),$$

condition (2.9) holds for all $x_0\in S$ if and only if

$$\min_{x\in S}\max_{z\neq x}\pi(\mathcal{E}_N^z) \leq C_0\min_{y\in S}\pi(\mathcal{E}_N^y) \tag{L2}$$

for some finite constant C_0. Condition (L2) is satisfied in two cases. Either if all wells $\mathcal{E}_N^y$ are stable sets (there exists a positive constant c_0 such that $\pi_{\mathcal{E}}(\mathcal{E}_N^y)\geq c_0$ for all $y\in S$, $N\geq 1$), or if there is only one stable set and all the other ones have comparable measures (there exists $x\in S$ and C_0 such that $\lim_N\pi_{\mathcal{E}}(\mathcal{E}_N^x) = 1$ and

$\pi_{\mathcal{E}}(\mathcal{E}_N^y) \le C_0 \pi_{\mathcal{E}}(\mathcal{E}_N^z)$ for all $y, z \neq x$). In particular, when there are only two wells, $|S| = 2$, assumption (L2G) is satisfied by the measures $\nu_N = \pi_x$, $x = 1, 2$.

Theorem 2.2 describes the asymptotic evolution of the trace of the Markov $\eta(t)$ on $\mathcal{E}_N$. The next lemma shows that in the time scale θ_N the time spent on the complement of $\mathcal{E}_N$ is negligible.

Lemma 2.3 *Assume that*

$$\lim_{N\to\infty} \frac{\pi(\Delta_N)}{\pi(\mathcal{E}_N^x)} = 0 \tag{L3}$$

for all $x \in S$. Let $\{\nu_N : N \ge 1\}$ be a sequence of probability measures concentrated on some well $\mathcal{E}_N^{x_0}$, $x_0 \in S$, and satisfying (L2G). *Then, for every $t > 0$,*

$$\lim_{N\to\infty} \mathbb{E}_{\nu_N}\left[\int_0^t \mathbf{1}\{\eta(s\theta_N) \in \Delta_N\}\, ds\right] = 0\,. \tag{2.10}$$

E. Mixing theory. If one is able to show that the process mixes inside each well before leaving the well, the assumptions on the initial state can be relaxed and the convergence of the order can be derived. Let $T^{\text{mix}}_{\mathbf{r},x}$, $x \in S$, be the mixing time of the reflected process $\eta^{\mathbf{r},x}(t)$.

Theorem 2.4 *Fix $x_0 \in S$. Suppose that there exist a non-negative sequence $\{\theta_N : N \ge 1\}$ and non-negative numbers $r(x, y)$, $x \neq y \in S$, satisfying conditions* (L1). *Assume that condition* (2.9) *is fulfilled and that there exists a sequence T_N, $T^{\text{mix}}_{\mathbf{r},x_0} \ll T_N \ll \theta_N$, such that*

$$\lim_{N\to\infty} \mathbb{P}^{\mathcal{E}}_{\pi_{x_0}}\left[H_{\breve{\mathcal{E}}_N^{x_0}} \le T_N\right] = 0. \tag{L4S}$$

Let $\{\nu_N : N \ge 1\}$ be a sequence of probability measures concentrated on $\mathcal{E}_N^{x_0}$, $\nu_N(\mathcal{E}_N^{x_0}) = 1$ and such that

$$\lim_{N\to\infty} \mathbb{P}^{\mathcal{E}}_{\nu_N}\left[H_{\breve{\mathcal{E}}_N^{x_0}} \le T'_N\right] = 0 \tag{L4}$$

for some sequence T'_N, $T^{\text{mix}}_{\mathbf{r},x_0} \ll T'_N \ll \theta_N$. Then, the finite-dimensional distributions of the time-rescaled order $\mathbf{X}^N_t = X^N_{t\theta_N}$ under $\mathbb{P}^{\mathcal{E}}_{\nu_N}$ converges to the finite-dimensional distributions of the Markov process on S which starts from x_0 and jumps from x to y at rate $r(x, y)$.

We assume in this latter theorem that the sequence of measures π_{x_0} fulfills all the hypotheses of Theorem 2.2. The unique advantage of Theorem 2.4 over Theorem 2.2 is that it replaces the condition (L2G) on the sequence ν_N by the somehow weaker condition (L4).

The next result asserts that condition (L4S) can be derived from one on the mean jump rate of the trace process.

Lemma 2.5 *Assume that there exist a non-negative sequence* $\{\theta_N : N \ge 1\}$ *such that* $T^{\rm mix}_{\mathbf{r},x_0} \ll \theta_N$ *and such that*

$$\limsup_{N\to\infty} \theta_N \, E_{\pi_{x_0}}[R^{\mathcal{E}}(\eta, \breve{\mathcal{E}}^{x_0})] \;<\; \infty. \tag{2.11}$$

Then, (L4S) *holds for any sequence* T_N *such that* $T^{\rm mix}_{\mathbf{r},x_0} \ll T_N \ll \theta_N$.

Assumption (2.11) is not difficult to be verified. By [1, Lemma 6.7],

$$E_{\pi_x}\Big[R^{\mathcal{E}}(\eta, \breve{\mathcal{E}}^x_N)\Big] = \frac{1}{\pi(\mathcal{E}^x_N)}\, {\rm cap}_N(\mathcal{E}^x_N, \breve{\mathcal{E}}^x_N)\;, \tag{2.12}$$

The Dirichlet principle [14,15] provides a variational formula for the capacity and a bound for the expression in (2.11). We show in (3.21) below that $\sum_{y\neq x} r_N(x,y) \le E_{\pi_x}[R^{\mathcal{E}}(\eta, \breve{\mathcal{E}}^x_N)]$.

A uniform version of assumption (L4) gives tightness of the speeded-up order. For a probability measure ν_N on $\mathcal{E}_N$, denote by $\mathbb{Q}_{\nu_N}$ the probability measure on the path space $D(\mathbb{R}_+, S)$ induced by the time-rescaled order $\mathbf{X}^N_t = \Psi_N(\eta^{\mathcal{E}}(t\theta_N))$ starting from ν_N.

Lemma 2.6 *Let* $\{\theta_N : N \ge 1\}$ *be a sequence such that* $\theta_N^{-1} \ll \min_{x\in S} \mathfrak{g}_{\mathbf{r},x}$ *and such that for all* $x \in S$,

$$\limsup_{N\to\infty} \theta_N \, E_{\pi_x}[R^{\mathcal{E}}(\eta, \breve{\mathcal{E}}^x_N)] \;<\; \infty. \tag{2.13}$$

Assume that there exists a sequence T_N *such that* $\max_{x\in S} T^{\rm mix}_{\mathbf{r},x} \ll T_N$ *and such that for all* $x \in S$,

$$\lim_{N\to\infty} \sup_{\eta\in\mathcal{E}^x_N} \mathbb{P}^{\mathcal{E}}_\eta\Big[H_{\breve{\mathcal{E}}^x_N} \le T_N\Big] = 0. \tag{L4U}$$

Let ν_N *be a sequence of probability measures on* $\mathcal{E}_N$. *Then, the sequence* $(\mathbb{Q}_{\nu_N} : N \ge 1)$ *is tight.*

In Sect. 4 we present a bound for the probability appearing in condition (L4U). Let $\mathcal{F}^x_N$, $x \in S$, be subsets of E_N containing $\mathcal{E}^x_N$, $\mathcal{E}^x_N \subset \mathcal{F}^x_N$. Denote by $T^{\rm mix}_{\mathbf{r},\mathcal{F}^x_N}$ the mixing time of the process $\eta(t)$ reflected at $\mathcal{F}^x_N$.

Lemma 2.7 *Fix* $x \in S$ *and suppose that there exist a set* $\mathcal{D}^x_N \subset \mathcal{E}^x_N$ *and a sequence* T_N, $T^{\rm mix}_{\mathbf{r},\mathcal{F}^x_N} \ll T_N \ll \theta_N$, *such that*

$$\lim_{N\to\infty} \max_{\eta\in\mathcal{D}^x_N} \mathbb{P}_\eta\Big[H_{(\mathcal{F}^x_N)^c} \le T_N\Big] = 0. \tag{L4E}$$

Then, (2.10) *holds for any* $t > 0$ *and any sequence of probability measures* ν_N *concentrated on* $\mathcal{D}^x_N$ *provided condition* (L3) *is in force.*

Even if we are not able to prove the pointwise versions (L4U) or (L4E) of the mixing condition, we can still show that the measures of the wells converge in the Cesaro sense.

Proposition 2.8 *Fix $x_0 \in S$. Assume that conditions (L1), (L2) and (2.11) are fulfilled. Let $\{\nu_N : N \ge 1\}$ be a sequence of probability measures concentrated on $\mathcal{E}_N^{x_0}$ and satisfying conditions (2.10) and (L4). Denote by $\{S_N(r) \,|\, r \ge 0\}$ the semigroup of the process $\eta(r)$. Then, for every $t > 0$ and $x \in S$,*

$$\lim_{N\to\infty} \int_0^t [\nu_N S_N(\theta_N r)]\ (\mathcal{E}_N^x)\, dr = \int_0^t [\delta_{x_0} S(r)](x)\, dr, \tag{2.14}$$

where $S(r)$ stands for the semigroup of the continuous-time Markov chain on S which jumps from y to z at rate $r(y,z)$, and where δ_{x_0} stands for the probability measure on S concentrated at x_0.

F. Two valleys. We suppose from now on that there are only two valleys, $\mathcal{E}_N^1 = \mathcal{A}_N$ and $\mathcal{E}_N^2 = \mathcal{B}_N$. It is possible in this case to establish a relation between the spectral gap of the trace process and the capacities of the enlarged process, and to re-state Theorems 2.2 and 2.4 in a simpler form. Assume that the sets $\mathcal{E}_N^x$, $x = 1, 2$, have an asymptotic measure:

$$\lim_{N\to\infty} \pi_{\mathcal{E}}(\mathcal{E}_N^x) = m(x),$$

and suppose, to fix ideas, that $m(1) \le m(2)$.

Theorem 2.9 *Assume that $\mathfrak{g}_{\mathcal{E}} \ll \min_{x=1,2} \mathfrak{g}_{\mathbf{r},x}$ and consider a sequence γ_N such that $\mathfrak{g}_{\mathcal{E}} \ll \gamma_N \ll \min_{x=1,2} \mathfrak{g}_{\mathbf{r},x}$. Then,*

$$\lim_{N\to\infty} \frac{\mathrm{cap}_\star(\mathcal{A}_N^\star, \mathcal{B}_N^\star)}{\mathfrak{g}_{\mathcal{E}}\, \pi_{\mathcal{E}}(\mathcal{A}_N)\, \pi_{\mathcal{E}}(\mathcal{B}_N)} = \frac{1}{2}\,\cdot$$

This result follows from [7, Theorem 2.12]. Under the assumptions of Proposition 2.1 we may replace in this statement the spectral gap of the trace process by the spectral gap of the original process. Moreover, in view of (2.7),

$$\lim_{N\to\infty} \mathfrak{g}_{\mathcal{E}}^{-1}\, r_N(x,y) = m(y). \tag{2.15}$$

When there are only two valleys, the right hand side of Eq. (L2G) is equal to $C_0 \min\{\pi_{\mathcal{E}}(\mathcal{E}_N^1), \pi_{\mathcal{E}}(\mathcal{E}_N^2)\}^{-1}$. Condition (L2G) then becomes

$$E_{\pi_{\mathcal{E}}}\left[\left(\frac{d\nu_N}{d\pi_{\mathcal{E}}}\right)^2\right] \le \max_{x=1,2} \frac{C_0}{\pi_{\mathcal{E}}(\mathcal{E}_N^x)} \tag{2.16}$$

for some finite constant C_0. The measures $\nu_N = \pi_1, \pi_2$ clearly fulfill this condition. We summarize in the next lemma the observations just made.

Lemma 2.10 *Suppose that there are only two wells, $S = \{1, 2\}$, and set $\theta_N = \mathfrak{g}_{\mathcal{E}}^{-1}$. Then, condition (L1) is reduced to the condition that*

$$\mathfrak{g}_{\mathcal{E}} \ll \min\{\mathfrak{g}_{\mathbf{r},1},\ \mathfrak{g}_{\mathbf{r},2}\}, \tag{L1B}$$

the asymptotic rates $r(x, y)$ are given by $r(x, y) = m(y)$, and condition (L2) is always in force.

In the case of two wells, there are two different asymptotic behaviors. Assume first that $m(1) > 0$. In this case $\mathcal{E}_N^1$ and $\mathcal{E}_N^2$ are stable sets and $\mathbf{X}_t^N$ jumps asymptotically from x to $3-x$ at rate $m(3-x)$. If $m(1) = 0$ and if ν_N is a sequence of measures concentrated on $\mathcal{E}_N^1$, $\mathcal{E}_N^1$ is a metastable set, $\mathcal{E}_N^2$ a stable set, and $\mathbf{X}_t^N$ jumps asymptotically from 1 to 2 at rate 1, remaining forever at 2 after the jump.

Remark 2.11 The average rates $r_N(x, y)$ introduced in (2.5) are different from those which appeared in [1], but can still be expressed in terms of the star-capacities:

$$\begin{aligned}&\pi_\star(\mathcal{E}_N^x)\, r_N(x, y)\\ &\quad= \frac{1}{2}\left\{\mathrm{cap}_\star(\mathcal{E}_N^{\star,x}, \breve{\mathcal{E}}_N^{\star,x}) + \mathrm{cap}_\star(\mathcal{E}_N^{\star,y}, \breve{\mathcal{E}}_N^{\star,y}) - \mathrm{cap}_\star(\mathcal{E}_N^{\star,x} \cup \mathcal{E}_N^{\star,x}, \cup_{z\neq x,y}\mathcal{E}_N^{\star,z})\right\}.\end{aligned}$$

To prove this identity observe that by (2.6) the right hand side is equal to

$$\frac{\gamma}{4}\sum_{\eta\in\mathcal{E}_N^x} \pi_{\mathcal{E}}(\eta)\, \mathbb{P}_\eta^{\star,\gamma}\left[H_{\mathcal{E}_N^{\star,y}} < H_{\breve{\mathcal{E}}_N^{\star,y}}\right] + \frac{\gamma}{4}\sum_{\eta\in\mathcal{E}_N^y} \pi_{\mathcal{E}}(\eta)\, \mathbb{P}_\eta^{\star,\gamma}\left[H_{\mathcal{E}_N^{\star,x}} < H_{\breve{\mathcal{E}}_N^{\star,x}}\right].$$

By (2.5), the first term is equal to $(1/2)\pi_\star(\mathcal{E}_N^x) r_N(x, y)$. By definition of the enlarged process, the second term can be written as

$$\frac{\gamma}{2}\sum_{\eta\in\mathcal{E}_N^{\star,y}} \pi_\star(\eta)\, \mathbb{P}_\eta^{\star,\gamma}\left[H_{\mathcal{E}_N^{\star,x}} = H^+_{\mathcal{E}_N^\star}\right].$$

By reversibility, $\pi_\star(\eta)\, \mathbb{P}_\eta^{\star,\gamma}[H_\xi = H^+_{\mathcal{E}_N^\star}] = \pi_\star(\xi)\, \mathbb{P}_\xi^{\star,\gamma}[H_\eta = H^+_{\mathcal{E}_N^\star}]$, $\eta \in \mathcal{E}_N^{\star,y}$, $\xi \in \mathcal{E}_N^{\star,x}$. This concludes the proof of the remark.

We conclude this section pointing out an interesting difference between Markov processes exhibiting a metastable behavior and a Markov processes exhibiting the cutoff phenomena [23]. On the level of trajectories, after remaining a long time in a metastable set, the first ones perform a sudden transition from one metastable set to another, while on the level of distributions, as stated in Proposition 2.8 below, in the relevant time scale these processes relax smoothly to the equilibrium state. In contrast, processes exhibiting the cutoff phenomena do not perform sudden transitions on the path level, but do so on the distribution level, moving quickly in a certain time scale from far to equilibrium to close to equilibrium.

3 Convergence of the finite-dimensional distributions

We prove in this section the main results of the article. We start this section with an important estimate which allows the replacement of the time integral of a function $f : \mathcal{E}_N \to \mathbb{R}$ by the time integral of the conditional expectation of f with respect to the σ-algebra generated by the partition $\mathcal{E}_N^1, \dots, \mathcal{E}_N^\kappa$.

Local Ergodicity. Denote by $\|f\|_{-1}$ the $\mathcal{H}_{-1}$ norm associated to the generator $L_{\mathcal{E}}$ of a function $f : \mathcal{E}_N \to \mathbb{R}$ which has mean zero with respect to $\pi_{\mathcal{E}}$:

$$\|f\|_{-1}^2 = \sup_h \big\{2\langle f, h\rangle_{\pi_{\mathcal{E}}} - \langle h, (-L_{\mathcal{E}})h\rangle_{\pi_{\mathcal{E}}}\big\},$$

where the supremum is carried over all functions $h : \mathcal{E}_N \to \mathbb{R}$ with finite support. By [19, Lemma 2.4], for every function $f : \mathcal{E}_N \to \mathbb{R}$ which has mean zero with respect to $\pi_{\mathcal{E}}$, and every $T > 0$,

$$\mathbb{E}^{\mathcal{E}}_{\pi_{\mathcal{E}}}\left[\sup_{0\le t\le T}\left(\int_0^t f(\eta^{\mathcal{E}}(s))\,ds\right)^2\right] \le 24\,T\,\|f\|_{-1}^2. \tag{3.1}$$

Similarly, for a function $f : \mathcal{E}_x^N \to \mathbb{R}$ which has mean zero with respect to π_x, denote by $\|f\|_{x,-1}$ the $\mathcal{H}_{-1}$ norm of f with respect to the generator $L_{\mathbf{r},x}$ of the reflected process at $\mathcal{E}_N^x$:

$$\|f\|_{x,-1}^2 = \sup_h \big\{2\langle f, h\rangle_{\pi_x} - \langle h, (-L_{\mathbf{r},x})h\rangle_{\pi_x}\big\}, \tag{3.2}$$

where the supremum is carried over all functions $h : \mathcal{E}_N^x \to \mathbb{R}$ with finite support. It is clear that

$$\sum_{x\in S}\pi_{\mathcal{E}}(\mathcal{E}_N^x)\,\langle h, (-L_{\mathbf{r},x})h\rangle_{\pi_x} \le \langle h, (-L_{\mathcal{E}})h\rangle_{\pi_{\mathcal{E}}}$$

for any function $h : \mathcal{E}_N \to \mathbb{R}$ with finite support. Note that the generator of the trace process $L_{\mathcal{E}}$ may have jumps from the boundary of a set $\mathcal{E}_x^N$ to its boundary which do not exist in the original process. There are therefore two types of contributions which appear on the right hand side but do not on the left hand side. These ones, and jumps from one set $\mathcal{E}_N^x$ to another. It follows from the previous inequality that for every function $f : \mathcal{E}_N \to \mathbb{R}$ which has mean zero with respect to each measure π_x,

$$\|f\|_{-1}^2 \le \sum_{x\in S}\pi_{\mathcal{E}}(\mathcal{E}_N^x)\|f\|_{x,-1}^2. \tag{3.3}$$

Proposition 3.1 *Let $\{\nu_N : N \ge 1\}$ be a sequence of probability measures on $\mathcal{E}_N$. Then, for every function $f : \mathcal{E}_N \to \mathbb{R}$ which has mean zero with respect to each measure π_x and for every $T > 0$,*

$$\Big(\mathbb{E}^{\mathcal{E}}_{\nu_N} \Big[\sup_{t\le T} \Big| \int_0^t f(\eta^{\mathcal{E}}(s))\, ds \Big| \Big] \Big)^2 \;\le\; 24\, T\, E_{\pi_{\mathcal{E}}} \Big[\Big(\frac{\nu_N}{\pi_{\mathcal{E}}} \Big)^2 \Big] \sum_{x\in S} \pi_{\mathcal{E}}(\mathcal{E}^x_N) \| f\|^2_{x,-1} \,.$$

Proof By Schwarz inequality, the expression on the left hand side of the previous displayed equation is bounded above by

$$E_{\pi_{\mathcal{E}}} \Big[\Big(\frac{\nu_N}{\pi_{\mathcal{E}}} \Big)^2 \Big] \mathbb{E}^{\mathcal{E}}_{\pi_{\mathcal{E}}} \Big[\sup_{t\le T} \Big(\int_0^t f(\eta^{\mathcal{E}}(s))\, ds \Big)^2 \Big] .$$

By (3.1) and by (3.3), the second expectation is bounded by

$$24\, T \sum_{x\in S} \pi_{\mathcal{E}}(\mathcal{E}^x_N)\, \| f\|^2_{x,-1} \,,$$

which concludes the proof of the proposition.

By the spectral gap, for any function $f : \mathcal{E}^x_N \to \mathbb{R}$ which has mean zero with respect to π_x, $\| f\|^2_{x,-1} \;\le\; \mathfrak{g}_{\mathbf{r},x}^{-1} \langle f, f\rangle_{\pi_x}$. The next result follows from this observation and the previous proposition.

Corollary 3.2 *Let* $\{\nu_N : N \ge 1\}$ *be a sequence of probability measures on* $\mathcal{E}_N$*. Then, for every function* $f : \mathcal{E}_N \to \mathbb{R}$ *which has mean zero with respect to each measure* π_x *and for every* $T > 0$*,*

$$\Big(\mathbb{E}^{\mathcal{E}}_{\nu_N} \Big[\sup_{t\le T} \Big| \int_0^t f(\eta^{\mathcal{E}}(s))\, ds \Big| \Big] \Big)^2 \le 24T\, E_{\pi_{\mathcal{E}}} \Big[\Big(\frac{\nu_N}{\pi_{\mathcal{E}}} \Big)^2 \Big] \sum_{x\in S} \pi_{\mathcal{E}}(\mathcal{E}^x_N)\, \mathfrak{g}_{\mathbf{r},x}^{-1} \langle f, f\rangle_{\pi_x} .$$

We have seen in (2.7) that $\mathrm{cap}_\star(\mathcal{E}^{\star,x}_N, \breve{\mathcal{E}}^{\star,x}_N) = \pi_\star(\mathcal{E}^x_N) \sum_{y\neq x} r_N(x,y)$. For similar reasons, $\mathrm{cap}_\star\Big(\mathcal{E}^{\star,x}_N, \breve{\mathcal{E}}^{\star,x}_N\Big) \;=\; \sum_{y\neq x} \pi_\star(\mathcal{E}^y_N)\, r_N(y,x)$. If $\theta_N r_N(x,y)$ converges, as postulated in assumption (L1), we obtain from these identities that

$$\mathrm{cap}_\star(\mathcal{E}^{\star,x}_N, \breve{\mathcal{E}}^{\star,x}_N) \;\le\; C_0\, \theta_N^{-1}\, \mathcal{M}_x \tag{3.4}$$

for some finite constant C_0, where $\mathcal{M}_x$ has been introduced in (2.8).

The equilibrium potentials. Fix a sequence $\gamma = \gamma_N$ such that $\theta_N^{-1} \ll \gamma \ll \min_{x\in S} \mathfrak{g}_{\mathbf{r},x}$ and recall that we denote by $\eta^\star(t)$ the γ-enlargement of the trace process $\eta^{\mathcal{E}}(t)$. Denote by V_x, $x \in S$, the equilibrium potential between the sets $\mathcal{E}^{\star,x}_N$ and $\breve{\mathcal{E}}^{\star,x}_N$, $V_x(\eta) = \mathbb{P}^{\star,\gamma}_\eta [H_{\mathcal{E}^{\star,x}_N} < H_{\breve{\mathcal{E}}^{\star,x}_N}]$. Since $L_\star V_x = 0$ on $\mathcal{E}_N$, we deduce that

$$(L_{\mathcal{E}} V_x)(\eta) = -\gamma\, [1 - V_x(\eta)], \quad \eta \in \mathcal{E}_N^x,$$
$$(L_{\mathcal{E}} V_x)(\eta) = \gamma\, V_x(\eta), \quad \eta \in \breve{\mathcal{E}}_N^x. \tag{3.5}$$

Moreover, since $\pi_\star(\eta) = (1/2)\pi_{\mathcal{E}}(\eta)$, $\eta \in \mathcal{E}_N$,

$$\begin{aligned}&\mathrm{cap}_\star(\mathcal{E}_N^{\star,x}, \breve{\mathcal{E}}_N^{\star,x})\\ &\quad = \frac{1}{2}\left\{\gamma \sum_{\eta\in\mathcal{E}_N^x} \pi_{\mathcal{E}}(\eta)[1 - V_x(\eta)]^2 + \langle(-L_{\mathcal{E}})V_x, V_x\rangle_{\pi_{\mathcal{E}}} + \gamma \sum_{\eta\in\breve{\mathcal{E}}_N^x} \pi_{\mathcal{E}}(\eta)\, V_x(\eta)^2\right\}.\end{aligned} \tag{3.6}$$

By assumption (L1), for all $x \neq y \in S$, $r_N(x, y) \le C_0\theta_N^{-1}$ for some finite constant C_0 and for all N large enough. Hence, by (3.4) and by (3.6), for all $x \in S$

$$\gamma \sum_{\eta\in\mathcal{E}_N^x} \pi_{\mathcal{E}}(\eta)\, [1 - V_x(\eta)]^2 + \langle(-L_{\mathcal{E}})V_x, V_x\rangle_{\pi_{\mathcal{E}}} + \gamma \sum_{\eta\in\breve{\mathcal{E}}_N^x} \pi_{\mathcal{E}}(\eta) V_x(\eta)^2 \;\le\; \frac{C_0\, \mathcal{M}_x}{\theta_N}. \tag{3.7}$$

Uniqueness of limit points. Recall the definition of the measure $\mathbb{Q}_{\nu_N}$ introduced just before Lemma 2.6, and let $\mathfrak{L}$ be the generator of the S-valued Markov process given by

$$(\mathfrak{L}F)(x) = \sum_{y\in S} r(x, y)[F(y) - F(x)].$$

Proposition 3.3 *Assume that the hypotheses of Theorem 2.2 are in force. Then, the sequence $\mathbb{Q}_{\nu_N}$ has at most one limit point, the probability measure on $D(\mathbb{R}_+, S)$ induced by the Markov process with generator $\mathfrak{L}$ starting from x_0.*

Proof To prove the uniqueness of limit points, we use the martingale characterization of Markov processes. Fix a function $F : S \to \mathbb{R}$ and a limit point $\mathbb{Q}_*$ of the sequence $\mathbb{Q}_{\nu_N}$. We claim that

$$M_t^F \;:=\; F(X_t) \;-\; F(X_0) \;-\; \int_0^t (\mathfrak{L}F)(X_s)\, ds \tag{3.8}$$

is a martingale under $\mathbb{Q}_*$.

Fix $0 \le s < t$ and a bounded function $U : D(\mathbb{R}_+, S) \mapsto \mathbb{R}$ depending only on $\{X_r : 0 \le r \le s\}$ and continuous for the Skorohod topology. We shall prove that

$$\mathbb{E}_{\mathbb{Q}_*}\left[M_t^F U\right] = \mathbb{E}_{\mathbb{Q}_*}\left[M_s^F U\right]. \tag{3.9}$$

Let $G(\eta) = \sum_{x\in S} F(x)V_x(\eta)$, $\eta \in \mathcal{E}_N$. By the Markov property of the trace process $\eta^{\mathcal{E}}(t)$,

$$M_t^N = G(\eta^{\mathcal{E}}(t\theta_N)) \;-\; G(\eta^{\mathcal{E}}(0)) \;-\; \int_0^{t\theta_N} (L_{\mathcal{E}}G)(\eta^{\mathcal{E}}(s))\, ds$$

is a martingale. Let $U^N := U(X_\cdot^N)$. As $\{M_t^N : t \geq 0\}$ is a martingale,

$$\mathbb{E}_{\nu_N}^{\mathcal{E}}\left[M_t^N U^N\right] = \mathbb{E}_{\nu_N}^{\mathcal{E}}\left[M_s^N U^N\right]$$

so that

$$\mathbb{E}_{\nu_N}^{\mathcal{E}}\left[U^N\left\{G(\eta^{\mathcal{E}}(t\theta_N)) \;-\; G(\eta^{\mathcal{E}}(s\theta_N)) \;-\; \int_{s\theta_N}^{t\theta_N} (L_{\mathcal{E}}G)(\eta^{\mathcal{E}}(r))\, dr\right\}\right] = 0\,. \tag{3.10}$$

Claim A *For all $x \in S$,*

$$\lim_{N\to\infty}\sup_{t\geq 0} \mathbb{E}_{\nu_N}^{\mathcal{E}}\left[|\mathbf{1}_{\mathcal{E}_N^x}(\eta^{\mathcal{E}}(t\theta_N)) - V_x(\eta^{\mathcal{E}}(t\theta_N))|\right] = 0.$$

Indeed, denote by $S_{\mathcal{E}}(t), t \geq 0$, the semigroup associated to the trace process $\eta^{\mathcal{E}}(t)$, and by h_t the Radon-Nikodym derivative $d\nu_N S_{\mathcal{E}}(t)/d\pi_{\mathcal{E}}$. It is well known that $E_{\pi_{\mathcal{E}}}[h_t^2] \leq E_{\pi_{\mathcal{E}}}[h_0^2]$. Hence, by Schwarz inequality, the square of the expectation appearing in the previous displayed formula is bounded above by

$$E_{\pi_{\mathcal{E}}}\left[\left(\frac{d\nu_N}{d\pi_{\mathcal{E}}}\right)^2\right] E_{\pi_{\mathcal{E}}}\left[|\mathbf{1}_{\mathcal{E}_N^x} - V_x|^2\right].$$

To conclude the proof of the claim it remains to recall the definition of the sequence γ, the estimate (3.7) and the assumption on the sequence of probability measures ν_N.

It follows from Claim A that

$$\lim_{N\to\infty}\sup_{t\geq 0} \mathbb{E}_{\nu_N}^{\mathcal{E}}\left[|(F\circ\Psi)(\eta^{\mathcal{E}}(t\theta_N)) - G(\eta^{\mathcal{E}}(t\theta_N))|\right] = 0.$$

Therefore, by (3.10),

$$\lim_{N\to\infty} \mathbb{E}_{\nu_N}^{\mathcal{E}}\left[U^N\left\{\Delta_{s,t}F \;-\; \int_{s\theta_N}^{t\theta_N} (L_{\mathcal{E}}G)(\eta^{\mathcal{E}}(r))\, dr\right\}\right] = 0\,. \tag{3.11}$$

where $\Delta_{s,t}F = (F\circ\Psi)(\eta^{\mathcal{E}}(t\theta_N)) \;-\; (F\circ\Psi)(\eta^{\mathcal{E}}(s\theta_N)) = F(X_{t\theta_N}^N) - F(X_{s\theta_N}^N)$.

Claim B *Denote by* $\mathcal{P}$ the $\sigma-$*algebra generated by the partition* $\mathcal{E}_N^z$, $z \in S$. *For all* $T > 0$, $x \in S$,

$$\lim_{N\to\infty} \mathbb{E}^{\mathcal{E}}_{\nu_N} \left[\sup_{t \le T\theta_N} \Big| \int_0^t \Big\{ (L_{\mathcal{E}} V_x)(\eta^{\mathcal{E}}(s)) - E\left[L_{\mathcal{E}} V_x \,\middle|\, \mathcal{P}\right](\eta^{\mathcal{E}}(s)) \Big\}\, ds \Big| \right] = 0.$$

By the assumption on the sequence ν_N and by Proposition 3.1, the square of the expectation appearing in the previous formula is bounded by

$$\frac{C_0\, T\, \theta_N}{\max_{z\in S} \mathcal{M}_z} \sum_{y\in S} \pi_{\mathcal{E}}(\mathcal{E}_N^y)\, \|\overline{L_{\mathcal{E}} V_x}\|^2_{y,-1} \tag{3.12}$$

for some finite constant C_0, where $\overline{G}$ stands for $G - E_{\pi_y}[G]$. By (3.5), on the set $\mathcal{E}_N^x$, $L_{\mathcal{E}} V_x = -\gamma[1 - V_x(\eta)]$. Hence, by the spectral gap o the reflected process and by (3.7),

$$\|\overline{L_{\mathcal{E}} V_x}\|^2_{x,-1} = \gamma^2\, \|\overline{1 - V_x}\|^2_{x,-1} \;\le\; \frac{\gamma^2}{\mathfrak{g}_{\mathbf{r},x}} \|1 - V_x\|^2_{\pi_x} \;\le\; \frac{C_0\, \gamma\, \mathcal{M}_x}{\pi_{\mathcal{E}}(\mathcal{E}_N^x)\, \mathfrak{g}_{\mathbf{r},x}\, \theta_N}$$

for some finite constant C_0. Similarly, since $L_{\mathcal{E}} V_x = \gamma V_x(\eta)$ on the set $\mathcal{E}_N^y$, $y \ne x$,

$$\|\overline{L_{\mathcal{E}} V_x}\|^2_{y,-1} \;\le\; \frac{C_0\, \gamma\, \mathcal{M}_x}{\pi_{\mathcal{E}}(\mathcal{E}_N^y)\, \mathfrak{g}_{\mathbf{r},y}\, \theta_N}\,.$$

Therefore, the sum appearing in (3.12) is bounded by $C_0 T\, |S|\, \gamma\, \max_{z\in S} \mathfrak{g}_{\mathbf{r},z}^{-1}$ which vanishes as $N \uparrow \infty$ by definition of γ, proving Claim B.

It follows from (3.11) and Claim B that

$$\lim_{N\to\infty} \mathbb{E}^{\mathcal{E}}_{\nu_N} \left[U^N \left\{ \Delta_{s,t} F \;-\; \int_s^t \theta_N\, E\left[L_{\mathcal{E}} G \,\middle|\, \mathcal{P}\right](\eta^{\mathcal{E}}(r\theta_N))\, dr \right\} \right] = 0\,. \tag{3.13}$$

We affirm that

$$E\left[L_{\mathcal{E}} G \,\middle|\, \mathcal{P}\right](\eta) = \sum_{x\in S} \mathbf{1}\{\eta \in \mathcal{E}_N^x\} \sum_{y\in S} r_N(x,y)[F(y) - F(x)]. \tag{3.14}$$

Indeed, by (3.5),

$$E\left[L_{\mathcal{E}} V_x \,\middle|\, \mathcal{P}\right] = \begin{cases} -\gamma \sum_{\eta\in\mathcal{E}_N^x} \dfrac{\pi_{\mathcal{E}}(\eta)}{\pi_{\mathcal{E}}(\mathcal{E}_N^x)} \mathbb{P}^{\star,\gamma}_\eta \left[H_{\breve{\mathcal{E}}_N^{\star,x}} < H_{\mathcal{E}_N^{\star,x}} \right], & \eta \in \mathcal{E}_N^x, \\ \gamma \sum_{\eta\in\mathcal{E}_N^y} \dfrac{\pi_{\mathcal{E}}(\eta)}{\pi_{\mathcal{E}}(\mathcal{E}_N^y)} \mathbb{P}^{\star,\gamma}_\eta \left[H_{\mathcal{E}_N^{\star,x}} < H_{\breve{\mathcal{E}}_N^{\star,x}} \right], & \eta \in \mathcal{E}_N^y,\; y \ne x. \end{cases}$$

By (2.7), on the set $\mathcal{E}_N^x$, $E[L_{\mathcal{E}} V_x \,|\, \mathcal{P}] = -\sum_{y \neq x} r_N(x, y)$, and by (2.5), on the set $\mathcal{E}_N^y$, $E[L_{\mathcal{E}} V_x \,|\, \mathcal{P}] = r_N(y, x)$. To conclude the proof of (3.14) it remains to recall the definition of G.

By (3.13), (3.14) and by definition of X_t^N,

$$\lim_{N\to\infty} \mathbb{E}^{\mathcal{E}}_{\nu_N} \Big[U^N \Big\{ \Delta_{s,t} F \;-\; \int_s^t \sum_{y\in S} \theta_N \, r_N(X^N_{r\theta_N}, y) \, [F(y) - F(X^N_{r\theta_N})] \, dr \Big\} \Big] = 0\,.$$

Since $\Delta_{s,t} F = F(X^N_{t\theta_N}) - F(X^N_{s\theta_N})$, since U has been assumed to be continuous for the Skorohod topology and since $\mathbb{Q}_*$ is a limit point of the sequence $\mathbb{Q}_{\nu_N}$, by assumption (L1)

$$E_{\mathbb{Q}_*} \Big[U \Big\{ F(X_t) - F(X_s) \;-\; \int_s^t \sum_{y\in S} r(X_r, y) \, [F(y) - F(X_r)] \, dr \Big\} \Big] = 0,$$

proving (3.8) and the proposition. □

Proof of Theorem 2.2 The proof is similar to the one of Proposition 3.3. We prove the convergence of the one-dimensional distributions. The extension to higher dimensional distributions is clear. Fix a function $F : S \to \mathbb{R}$. We claim that for every $T \geq 0$,

$$\limsup_{N\to\infty} \sup_{0\le s<t\le T} \Big| \mathbb{E}_{\nu_N} \Big[F(X^N_{t\theta_N}) - F(X^N_{s\theta_N}) - \int_s^t (\mathfrak{L} F)(X^N_{r\theta_N}) dr \Big] \Big| = 0. \tag{3.15}$$

Recall the definition of the function $G : \mathcal{E}_N \to \mathbb{R}$ introduced in the proof of the previous proposition. By Claim A and since $G(\eta^{\mathcal{E}}(t\theta_N)) - \int_0^t \theta_N (L_{\mathcal{E}} G)(\eta^{\mathcal{E}}(s\theta_N))\, ds$ is a martingale, to prove (3.15), it is enough to show that

$$\limsup_{N\to\infty} \sup_{0\le t\le T} \Big| \mathbb{E}_{\nu_N} \Big[\int_0^t \theta_N (L_{\mathcal{E}} G)(\eta^{\mathcal{E}}(r\theta_N)) \, dr - \int_0^t (\mathfrak{L} F)(X^N_{r\theta_N}) \, dr \Big] \Big| = 0.$$

By Claim B, by the identity (3.14) and by the definition of X_t^N, the proof of (3.15) is further reduced to the proof that

$$\limsup_{N\to\infty} \sup_{0\le t\le T} \Big| \mathbb{E}_{\nu_N} \Big[\int_0^t (\mathfrak{L}_N F)(X^N_{r\theta_N}) \, dr - \int_0^t (\mathfrak{L} F)(X^N_{r\theta_N}) \, dr \Big] \Big| = 0,$$

where $(\mathfrak{L}_N F)(x) = \sum_{y\in S} \theta_N \, r_N(x, y) \, [F(y) - F(x)]$. To conclude the proof of (3.15), it remains to recall assumption (L1).

It follows from (3.15) that the sequence $f_N(t) = \mathbb{E}_{\nu_N}[F(X^N_{t\theta_N})]$ is equicontinuous in any compact interval $[0, T]$. Moreover, if F is an eigenfunction of the operator $\mathfrak{L}$

associated to an eigenvalue λ, all limit points $f(t)$ of the subsequence $f_N(t)$ are such that

$$f(t) - F(x_0) = \int_0^t \lambda f(r)\, dr, \quad 0 \le t \le T,$$

which yields uniqueness of limit points. □

Proof of Lemma 2.5 Let T_N be a sequence such that $T^{\text{mix}}_{\mathbf{r},x_0} \ll T_N \ll \theta_N$. The probability $\mathbb{P}^{\mathcal{E}}_{\pi_{x_0}}[H_{\breve{\mathcal{E}}^{x_0}_N} \le T_N]$ can be written as

$$\sum_{\eta \in \mathcal{E}^{x_0}_N} \big\{\pi_{x_0}(\eta) - \pi^*_{x_0}(\eta)\big\}\, \mathbb{P}^{\mathcal{E}}_{\eta}\Big[H_{\breve{\mathcal{E}}^{x_0}_N} \le T_N\Big] + \mathbb{P}^{\mathcal{E}}_{\pi^*_{x_0}}\Big[H_{\breve{\mathcal{E}}^{x_0}_N} \le T_N\Big], \tag{3.16}$$

where $\pi^*_{x_0}$ is the quasi-stationary measure associated to the trace process $\eta^{\mathcal{E}}(t)$ killed when it hits $\breve{\mathcal{E}}^{x_0}_N$. The first term is less than or equal to

$$\sum_{\eta \in \mathcal{E}^{x_0}_N} \pi_{x_0}(\eta) \left| \frac{\pi^*_{x_0}(\eta)}{\pi_{x_0}(\eta)} - 1 \right| \le \left\{ \sum_{\eta \in \mathcal{E}^{x_0}_N} \pi_{x_0}(\eta) \left(\frac{\pi^*_{x_0}(\eta)}{\pi_{x_0}(\eta)} - 1 \right)^2 \right\}^{1/2} .$$

By Proposition 2.1, (17) and Lemma 2.2 in [7], the expression inside the square root on the right hand side of the previous formula is bounded by $\varepsilon_{x_0}/[1 - \varepsilon_{x_0}]$, where $\varepsilon_{x_0} = E_{\pi_{x_0}}[R^{\mathcal{E}}(\eta, \breve{\mathcal{E}}^{x_0}_N)]/\mathfrak{g}_{\mathbf{r},x_0}$. By (2.11), $\varepsilon_{x_0} \le C_0(\theta_N \mathfrak{g}_{\mathbf{r},x_0})^{-1}$ for some finite constant C_0 and by hypothesis, $\theta_N^{-1} \ll (T^{\text{mix}}_{\mathbf{r},x_0})^{-1} \le C_0 \mathfrak{g}_{\mathbf{r},x_0}$. This shows that the first term in (3.16) vanishes as $N \uparrow \infty$.

On the other hand, since $\pi^*_{x_0}$ is the quasi-stationary state, under $\mathbb{P}_{\pi^*_{x_0}}$, the hitting time of $\breve{\mathcal{E}}^{x_0}_N$, denoted by $H_{\breve{\mathcal{E}}^{x_0}}$, has an exponential distribution whose parameter we represent by $\phi^*_{x_0}$. By [7, Lemma 2.2], $\phi^*_{x_0}$ is bounded by $E_{\pi_{x_0}}[R^{\mathcal{E}}(\eta, \breve{\mathcal{E}}^{x})] \le C_0/\theta_N$, for some finite constant C_0. Hence,

$$\mathbb{P}^{\mathcal{E}}_{\pi^*_{x_0}}\big[H_{\breve{\mathcal{E}}^{x_0}} \le T_N\big] = 1 - e^{-\phi^*_{x_0} T_N} \le 1 - e^{-C_0(T_N/\theta_N)},$$

an expression which vanishes as $N \uparrow \infty$. □

Proof of Theorem 2.4 Recall that $T^{\text{mix}}_{\mathbf{r},x}$, $x \in S$, stands for the mixing time of the reflected process $\eta^{\mathbf{r},x}(t)$. We prove that the one-dimensional distributions converge. The extension to higher dimensional distributions is straightforward. Since the sequence of measures π_{x_0} satisfies the assumptions of Theorem 2.2, in view of its assertions it is enough to show that for each function $F : S \to \mathbb{R}$,

$$\lim_{N\to\infty} \Big| \mathbb{E}^{\mathcal{E}}_{\nu_N}\Big[F(X^N_{t\theta_N})\Big] - \mathbb{E}^{\mathcal{E}}_{\pi_{x_0}}\Big[F(X^N_{t\theta_N})\Big] \Big| = 0. \tag{3.17}$$

Let T_N be a sequence satisfying the assumptions of the theorem. We may write $\mathbb{E}^{\mathcal{E}}_{\nu_N}[F(X^N_{t\theta_N})]$ as

$$\mathbb{E}^{\mathcal{E}}_{\nu_N}\left[\mathbf{1}\left\{H_{\breve{\mathcal{E}}^{x_0}_N} > T_N\right\} F(X^N_{t\theta_N})\right] + \mathbb{E}^{\mathcal{E}}_{\nu_N}\left[\mathbf{1}\left\{H_{\breve{\mathcal{E}}^{x_0}_N} \le T_N\right\} F(X^N_{t\theta_N})\right].$$

The second term is absolutely bounded by $C_0\mathbb{P}^{\mathcal{E}}_{\nu_N}[H_{\breve{\mathcal{E}}^{x_0}_N} \le T_N]$ for some finite constant C_0 independent of N and which may change from line to line. By hypothesis, this latter probability vanishes as $N \uparrow \infty$. By the Markov property, the first term in the previous displayed equation is equal to

$$\mathbb{E}^{\mathcal{E}}_{\nu_N}\left[\mathbf{1}\{H_{\breve{\mathcal{E}}^{x_0}_N} > T_N\}\, \mathbb{E}^{\mathcal{E}}_{\eta(T_N)}\left[F(X^N_{t\theta_N - T_N})\right]\right].$$

On the set $\{H_{\breve{\mathcal{E}}^{x_0}_N} > T_N\}$ we may couple the trace process with the reflected process in such a way that $\eta^{\mathcal{E}}(t) = \eta^{\mathbf{r},x_0}(t)$ for $t \le T_N$. The previous expectation is thus equal to

$$\mathbb{E}^{\mathcal{E}}_{\nu_N}\left[\mathbb{E}^{\mathcal{E}}_{\eta^{\mathbf{r},x_0}(T_N)}\left[F(X^N_{t\theta_N - T_N})\right]\right] - \mathbb{E}^{\mathcal{E}}_{\nu_N}\left[\mathbf{1}\{H_{\breve{\mathcal{E}}^{x_0}_N} \le T_N\}\, \mathbb{E}^{\mathcal{E}}_{\eta^{\mathbf{r},x_0}(T_N)}\left[F\left(X^N_{t\theta_N - T_N}\right)\right]\right].$$

As before, the second term vanishes as $N \uparrow \infty$. The first expectation is equal to

$$\mathbb{E}^{\mathcal{E}}_{\pi_{x_0}}\left[F(X^N_{t\theta_N - T_N})\right] + R_N(t),$$

where $R_N(t)$ is absolutely bounded by $C_0\|\nu_N S^{\mathbf{r},x_0}(T_N) - \pi_{x_0}\|_{\mathrm{TV}}$. In this formula, $\|\mu - \nu\|_{\mathrm{TV}}$ stands for the total variation distance between μ and ν and $S^{\mathbf{r},x}(t)$ represents the semi-group of the reflected process. By definition of the mixing time, this last expression is less than or equal to $(1/2)^{(T_N/T^{\mathrm{mix}}_{\mathbf{r},x})}$, which vanishes as $N \uparrow \infty$ by assumption.

Repeating the same arguments presented above with the measure ν_N replaced by the local equilibrium π_{x_0} we conclude the proof of (3.17). □

Proof of Lemma 2.3 Let ν_N be a sequence of probability measures satisfying (L2G). By Schwarz inequality, the square of the expectation appearing in the statement of the lemma is bounded above by

$$\frac{1}{\pi(\mathcal{E}_N)}\, E_{\pi_{\mathcal{E}}}\left[\left(\frac{d\nu_N}{d\pi_{\mathcal{E}}}\right)^2\right] \mathbb{E}_{\pi}\left[\left(\int_0^t \mathbf{1}\{\eta(s\theta_N) \in \Delta_N\}\, ds\right)^2\right]$$

By assumption (L2G), the first expectation is bounded by $C_0 \min_{x\in S} \mathcal{M}_x^{-1}$. Since $\mathcal{M}_x \ge \min_y \pi_{\mathcal{E}}(\mathcal{E}^y_N)$, $\min_{x\in S}\mathcal{M}_x^{-1} \le \max_{y\in S} \pi_{\mathcal{E}}(\mathcal{E}^y_N)^{-1}$. On the other hand, by

Schwarz inequality, the second expectation is less than or equal to

$$t\,\mathbb{E}_{\pi}\left[\int_0^t \mathbf{1}\{\eta(s\theta_N) \in \Delta_N\}\,ds\right] = t^2\pi(\Delta_N),$$

which concludes the proof. □

Proof of Lemma 2.7 The proof of this result is similar to the previous one with obvious modifications. Consider a sequence of initial states η^N in $\mathcal{D}_N^x$. By the Markov property, the expectation appearing in (2.10) with $\nu_N = \delta_{\eta^N}$ is bounded above by

$$\mathbb{E}_{\eta^N}\left[\mathbf{1}\left\{H_{(\mathcal{F}_N^x)^c} > T_N\right\}\,\mathbb{E}_{\eta(T_N)}\left[\int_0^t \mathbf{1}\{\eta(s\theta_N) \in \Delta_N\}\,ds\right]\right] + T_N/\theta_N \; + \; t\,\mathbb{P}_{\eta^N}\left[H_{(\mathcal{F}_N^x)^c} \le T_N\right],$$

where we replaced $t - T_N$ by t in the time integral. By assumption, the second and the third term vanish as $N \uparrow \infty$. On the set $\{H_{(\mathcal{F}_N^x)^c} > T_N\}$ we may replace $\eta(T_N)$ by $\eta^{\mathbf{r},\mathcal{F}^x}(T_N)$, where $\eta^{\mathbf{r},\mathcal{F}^x}(t)$ stands for the process $\eta(t)$ reflected at $\mathcal{F}_N^x$. After this replacement, we may remove the indicator and estimate the expectation by

$$t\,\|\delta_{\eta^N} S^{\mathbf{r},\mathcal{F}^x}(T_N) - \pi_{\mathcal{F}^x}\|_{\mathrm{TV}} \; + \; \mathbb{E}_{\pi_{\mathcal{F}^x}}\left[\int_0^t \mathbf{1}\{\eta(s\theta_N) \in \Delta_N\}\,ds\right],$$

where $S^{\mathbf{r},\mathcal{F}^x}(t)$ represents the semigroup of the reflected process $\eta^{\mathbf{r},\mathcal{F}^x}(t)$ and $\pi_{\mathcal{F}^x}$ the measure π conditioned to $\mathcal{F}_N^x$. The first term vanishes by definition of T_N, while the second one is bounded by $t\pi(\Delta_N)/\pi(\mathcal{F}_N^x)$, which vanishes in view of condition (L3). □

Proof of Proposition 2.8 The proof of this proposition relies on a comparison between the original process and the trace process presented below in Eqs. (3.18) and (3.19). Let $\{T_{\mathcal{E}}(t)\,|\,t \ge 0\}$ be the time spent on the set $\mathcal{E}_N$ by the process $\eta(s)$ in the time interval $[0, t]$,

$$T_{\mathcal{E}}(t) = \int_0^t \mathbf{1}\{\eta(s) \in \mathcal{E}_N\}\,ds.$$

Denote by $S_{\mathcal{E}}(t)$ the generalized inverse of $T_{\mathcal{E}}(t)$, $S_{\mathcal{E}}(t) = \sup\{s \ge 0\,|\,T_{\mathcal{E}}(s) \le t\}$, and recall that the trace process is defined as $\eta^{\mathcal{E}}(t) = \eta(S_{\mathcal{E}}(t))$.

By definition of the trace process, for every $t \ge 0$,

$$\int_0^t \mathbf{1}\{\eta(s) \in \mathcal{E}_N^x\}\,ds \;\le\; \int_0^t \mathbf{1}\{\eta^{\mathcal{E}}(s) \in \mathcal{E}_N^x\}\,ds. \tag{3.18}$$

On the other hand,

$$\int_0^t \mathbf{1}\{\eta^{\mathcal{E}}(r) \in \mathcal{E}_N^x\}\, dr = \int_0^t \mathbf{1}\{\eta(S_{\mathcal{E}}(r)) \in \mathcal{E}_N^x\}\, dr.$$

By a change of variables, the previous integral is equal to

$$\int_0^{S_{\mathcal{E}}(t)} \mathbf{1}\{\eta(r) \in \mathcal{E}_N^x\}\, dr.$$

Let $T_\Delta(t)$, $t \geq 0$, be the time spent by the process $\eta(s)$ on the set Δ_N in the time interval $[0, t]$, $T_\Delta(t) = \int_0^t \mathbf{1}\{\eta(s) \in \Delta_N\}\, ds$. Since $T_{\mathcal{E}}(t) + T_\Delta(t) = t$, on the set $T_\Delta(t_0) < \delta$ and for $t \leq t_0 - \delta$, $T_{\mathcal{E}}(t+\delta) > t$, so that $S_{\mathcal{E}}(t) \leq t + \delta$. Putting together all previous estimates we get that on the set $T_\Delta(t_0) < \delta$ and for $t \leq t_0 - \delta$,

$$\int_0^t \mathbf{1}\{\eta^{\mathcal{E}}(s) \in \mathcal{E}_N^x\}\, ds \;\leq\; \int_0^{t+\delta} \mathbf{1}\{\eta(s) \in \mathcal{E}_N^x\}\, ds. \tag{3.19}$$

We turn now to the proof of the proposition. We may rewrite the time integral appearing on the left hand side of (2.14) as

$$\mathbb{E}_{\nu_N}\left[\int_0^t \mathbf{1}\{\eta(r\theta_N) \in \mathcal{E}_N^x\}\, dr\right]. \tag{3.20}$$

By (3.18), this expectation is bounded above by

$$\mathbb{E}_{\nu_N}\left[\int_0^t \mathbf{1}\{\eta^{\mathcal{E}}(r\theta_N) \in \mathcal{E}_N^x\}\, dr\right] = \mathbb{E}_{\nu_N}\left[\int_0^t \mathbf{1}\{X^N_{r\theta_N} = x\}\, dr\right].$$

By Theorem 2.4, the right hand side converges as $N \uparrow \infty$ to the right hand side of (2.14).

Fix $\delta > 0$. The expectation (3.20) is bounded below by

$$\mathbb{E}_{\nu_N}\left[\mathbf{1}\{T_\Delta(t\theta_N) < \delta\theta_N\} \int_0^t \mathbf{1}\{\eta(r\theta_N) \in \mathcal{E}_N^x\}\, dr\right].$$

By (3.19), this expression is bounded below by

$$\mathbb{E}_{\nu_N}\left[\mathbf{1}\{T_\Delta(t\theta_N) < \delta\theta_N\} \int_0^{t-\delta} \mathbf{1}\{\eta^{\mathcal{E}}(r\theta_N) \in \mathcal{E}_N^x\}\, dr\right]$$
$$\geq \mathbb{E}_{\nu_N}\left[\int_0^{t-\delta} \mathbf{1}\{\eta^{\mathcal{E}}(r\theta_N) \in \mathcal{E}_N^x\}\, dr\right] - t\,\mathbb{P}_{\nu_N}\left[T_\Delta(t\theta_N) \geq \delta\theta_N\right].$$

By (2.10), the second term vanishes as $N \uparrow \infty$, while by Theorem 2.4 the second one converges to the right hand side of (2.14) as $N \uparrow \infty$ and then $\delta \downarrow 0$. □

The jump rates. Recall the definition (2.5) of the rates $r_N(x, y)$. For all $x \in S$,

$$\sum_{y \neq x} r_N(x, y) \leq E_{\pi_x}\left[R^{\mathcal{E}}(\eta, \breve{\mathcal{E}}_N^x)\right]. \tag{3.21}$$

Indeed, by (2.7) and by the Dirichlet principle,

$$\pi_\star(\mathcal{E}_N^x) \sum_{y \neq x} r_N(x, y) = \mathrm{cap}_\star(\mathcal{E}_N^{\star,x}, \breve{\mathcal{E}}_N^{\star,x}) = \inf_f \langle (-L_\star f), f \rangle_{\pi_\star},$$

where the infimum is carried over all functions $f : \mathcal{E}_N \cup \mathcal{E}_N^\star \to \mathbb{R}$ equal to 1 on $\mathcal{E}_N^{\star,x}$ and equal to 0 on $\breve{\mathcal{E}}_N^{\star,x}$. Taking $f = \mathbf{1}\{\mathcal{E}_N^x \cup \mathcal{E}_N^{\star,x}\}$ and computing the Dirichlet form of this function we get (3.21).

4 On assumptions (L4) and (L4U)

We present in this section two estimates of $\mathbb{P}_\eta^{\mathcal{E}}[H_{\breve{\mathcal{E}}_N^x} \leq T_N]$. We start with a bound of this probability in terms of an equilibrium potential. Denote by $W_{x,\gamma}^\star$, $x \in S$, $\gamma > 0$, the equilibrium potential between $\breve{\mathcal{E}}_N^x \cup \breve{\mathcal{E}}_N^{\star,x}$ and $\mathcal{E}_N^{\star,x}$ for the γ-enlargement of the trace process $\eta^{\mathcal{E}}(t)$:

$$W_{x,\gamma}^\star(\eta) = \mathbb{P}_\eta^{\star,\gamma}\left[H_{\breve{\mathcal{E}}_N^x \cup \breve{\mathcal{E}}_N^{\star,x}} < H_{\mathcal{E}_N^{\star,x}}\right] = \mathbb{P}_\eta^{\star,\gamma}\left[H_{\breve{\mathcal{E}}_N^x} < H_{\mathcal{E}_N^{\star,x}}\right].$$

Lemma 4.1 *Fix $x \in S$. Then, for all $\eta \in \mathcal{E}_N^x$, $\gamma > 0$ and $A > 0$,*

$$\mathbb{P}_\eta^{\mathcal{E}}\left[H_{\breve{\mathcal{E}}_N^x} \leq \gamma^{-1}\right] \leq e\, W_{x,\gamma}^\star(\eta)\,,$$
$$W_{x,\gamma}^\star(\eta) - \frac{e^{-A}}{1 - e^{-A}} \leq \mathbb{P}_\eta^{\mathcal{E}}\left[H_{\breve{\mathcal{E}}_N^x} \leq A\gamma^{-1}\right].$$

Proof Fix $x \in S$. By definition of the equilibrium potential,

$$\begin{cases} (L_\star W^\star_{x,\gamma})(\eta) = 0 & \eta \in \mathcal{E}^x_N, \\ W^\star_{x,\gamma}(\eta) = 1 & \eta \in \breve{\mathcal{E}}^x_N \cup \breve{\mathcal{E}}^{\star,x}_N, \\ W^\star_{x,\gamma}(\eta) = 0 & \eta \in \mathcal{E}^{\star,x}_N. \end{cases}$$

By definition of the generator $L_\star$ and since the equilibrium potential $W^\star_{x,\gamma}$ vanishes on the set $\mathcal{E}^{\star,x}_N$, on the set $\mathcal{E}^x_N$, we have that

$$(L_{\mathcal{E}} W^\star_{x,\gamma})(\eta) = \gamma W^\star_{x,\gamma}(\eta), \quad \eta \in \mathcal{E}^x_N.$$

Since $W^\star_{x,\gamma}$ is equal to 1 on the set $\breve{\mathcal{E}}^x_N$, we conclude that

$$W^\star_{x,\gamma}(\eta) = \mathbb{E}^{\mathcal{E}}_\eta[\exp\{-\gamma H_{\breve{\mathcal{E}}^x_N}\}], \quad \eta \in \mathcal{E}_N.$$

On the other hand, by Tchebychev inequality and by the previous identity,

$$\mathbb{P}^{\mathcal{E}}_\eta\left[H_{\breve{\mathcal{E}}^x_N} \le \gamma^{-1}\right] = \mathbb{P}^{\mathcal{E}}_\eta\left[e^{-\gamma H_{\breve{\mathcal{E}}^x_N}} \ge e^{-1}\right] \le e\,\mathbb{E}^{\mathcal{E}}_\eta\left[e^{-\gamma H_{\breve{\mathcal{E}}^x_N}}\right] = e\, W^\star_{x,\gamma}(\eta).$$

Conversely, fix $A > 0$ and let $T(\gamma)$ be an exponential time of parameter γ independent of the trace process $\eta^{\mathcal{E}}(s)$. It is clear that for $\eta \in \mathcal{E}^x_N$,

$$W^\star_{x,\gamma}(\eta) = \mathbb{P}^{\star,\gamma}_\eta\left[H_{\breve{\mathcal{E}}^x_N} < H_{\mathcal{E}^{\star,x}_N}\right] = \mathbb{P}^{\mathcal{E}}_\eta\left[H_{\breve{\mathcal{E}}^x_N} < T(\gamma)\right].$$

By definition of $T(\gamma)$, the last probability is equal to

$$\int_0^\infty \mathbb{P}^{\mathcal{E}}_\eta\left[H_{\breve{\mathcal{E}}^x_N} < t\right] \gamma e^{-\gamma t}\, dt \le \mathbb{P}^{\mathcal{E}}_\eta\left[H_{\breve{\mathcal{E}}^x_N} \le A\gamma^{-1}\right](1 - e^{-A}) + e^{-A}.$$

An elementary computation permits to conclude the proof of the lemma. □

The second assertion of the previous lemma shows that we do not lose much in the first one.

Corollary 4.2 *Let ν_N be a probability measure concentrated on the set $\mathcal{E}^x_N$. Then, for all $\gamma > 0$,*

$$\mathbb{P}^{\mathcal{E}}_{\nu_N}\left[H_{\breve{\mathcal{E}}^x_N} \le \gamma^{-1}\right]^2 \le \frac{2\,e^2}{\gamma}\, E_{\pi_{\mathcal{E}}}\left[\left(\frac{\nu_N}{\pi_{\mathcal{E}}}\right)^2\right] \mathrm{cap}_\star(\mathcal{E}^{\star,x}_N, \breve{\mathcal{E}}^x_N).$$

Proof Recall that we denote by $\eta^\star$ the copy of the state η. By definition of the enlarged process and by Schwarz inequality,

$$\mathbb{P}^{\star,\gamma}_{\nu_N}\Big[H_{\breve{\mathcal{E}}^x_N} < H_{\mathcal{E}^{\star,x}_N}\Big] = \sum_{\eta\in\mathcal{E}^x_N} \nu_N(\eta)\,\mathbb{P}^{\star,\gamma}_{\eta^\star}\Big[H_{\breve{\mathcal{E}}^x_N} < H^+_{\mathcal{E}^{\star,x}_N}\Big]$$
$$\le \Big\{E_{\pi_{\mathcal{E}}}\Big[\Big(\frac{\nu_N}{\pi_{\mathcal{E}}}\Big)^2\Big] \sum_{\eta\in\mathcal{E}^x_N} \pi_{\mathcal{E}}(\eta)\,\mathbb{P}^{\star,\gamma}_{\eta^\star}\Big[H_{\breve{\mathcal{E}}^x_N} < H^+_{\mathcal{E}^{\star,x}_N}\Big]\Big\}^{1/2}.$$

In the previous sum we may replace $\pi_{\mathcal{E}}(\eta)$ by $2\,\pi_\star(\eta^\star)$. After the replacement, the sum becomes $2\gamma^{-1}\mathrm{cap}_\star(\mathcal{E}^{\star,x}_N, \breve{\mathcal{E}}^x_N)$. This estimate together with Lemma 4.1 concludes the proof of the corollary. □

Comments on assumption (L4U). We present in this subsection two strategies to prove that the equilibrium potential $W^\star_{x,\gamma}(\eta)$ vanishes. We apply the first technique in Example A of Sect. 8.

A. Monotonicity. It is always possible to couple two trace processes $\eta^{\mathcal{E}}(t)$ starting from different initial states in such a way that both reach the set $\mathcal{E}^\star_N$ at the same time. Assume that the equilibrium potential $W^\star_{x,\gamma}$ satisfies some property $\mathcal{P}$. For example, suppose that the equilibrium potential is monotone with respect to some partial order defined on $\mathcal{E}_N$. By the Dirichlet principle,

$$\mathrm{cap}_\star(\mathcal{E}^{\star,x}_N, \breve{\mathcal{E}}^x_N \cup \breve{\mathcal{E}}^{\star,x}_N) = \langle W^\star_{x,\gamma}, (-L_\star) W^\star_{x,\gamma}\rangle_{\pi_\star} = \inf_f \langle f, (-L_\star) f\rangle_{\pi_\star},$$

where the supremum is carried over all functions f vanishing at $\mathcal{E}^{\star,x}_N$, equal to 1 on $\breve{\mathcal{E}}^x_N \cup \breve{\mathcal{E}}^{\star,x}_N$ and satisfying condition $\mathcal{P}$. Fix a configuration $\eta \in \mathcal{E}^x_N$ and denote by $R_N(\varepsilon)$, $\varepsilon > 0$, the right hand side of the previous formula when we impose the further restriction that $f(\eta) \ge \varepsilon$.

To prove that $W^\star_{x,\gamma}(\eta)$ vanishes as $N \uparrow \infty$, it is enough to show that for every $\varepsilon > 0$, $\mathrm{cap}_\star(\mathcal{E}^{\star,x}_N, \breve{\mathcal{E}}^x_N \cup \breve{\mathcal{E}}^{\star,x}_N) \ll R_N(\varepsilon)$. Indeed, suppose by contradiction that $W^\star_{x,\gamma}(\eta)$ does not vanish as $N \uparrow \infty$. There exists in this case $\varepsilon > 0$ and a subsequence N_j, still denoted by N, for which $W^\star_{x,\gamma}(\eta) \ge \varepsilon$ for all N. Therefore,

$$R_N(\varepsilon) \;\le\; \langle W^\star_{x,\gamma}, (-L_\star) W^\star_{x,\gamma}\rangle_{\pi_\star} = \mathrm{cap}_\star(\mathcal{E}^{\star,x}_N, \breve{\mathcal{E}}^x_N \cup \breve{\mathcal{E}}^{\star,x}_N),$$

proving our claim.

B. Capacities. To present the second form of estimating the equilibrium potential, we start with a general result which expresses the equilibrium potential as a ratio between capacities. Consider a reversible Markov chain $\eta(t)$ on some countable state space E. Denote by $\mathbf{P}_\xi$, $\xi \in E$, the probability measure on the path space $D(\mathbb{R}_+, E)$ induced by the Markov process $\eta(t)$ starting from ξ, and by $\mathrm{cap}(A, B)$ the capacity between two disjoint subsets, A, B, of E. Next result is Lemma 1.15 in [20].

Lemma 4.3 *Let A, B be two disjoint subsets of E, $A \cap B = \varnothing$, and let $\eta \notin A \cup B$. Then,*

$$\mathbf{P}_\eta \left[H_B < H_A \right] = \frac{\mathrm{cap}(\eta, A \cup B) + \mathrm{cap}(B, A \cup \{\eta\}) - \mathrm{cap}(A, B \cup \{\eta\})}{2\, \mathrm{cap}(\eta, A \cup B)} \le \frac{\mathrm{cap}(\eta, B)}{\mathrm{cap}(\eta, A \cup B)} \,.$$

In some cases the estimate presented in the previous lemma has no content. On the one hand,

$$\mathrm{cap}_T(\eta, B) = \mu_T(\eta)\, R_T(\eta, B) \;+\; \mu_T(\eta)\, \lambda_T(\eta)\, \mathbf{P}^T_\eta \left[H_A < H_B < H^+_\eta \right].$$

The second term on the right hand side is the expression we added to the numerator to transform the identity presented in Lemma 4.3 into an inequality. On the other hand, since $H_A \wedge H_B < H^+_\eta$ $\mathbf{P}^T_\eta$-a.s.,

$$\mathbf{P}^T_\eta \left[H_A < H_B < H^+_\eta \right] = \mathbf{E}^T_\eta \left[\mathbf{1}\{H_A < H_B\}\, \mathbf{P}^T_{\eta^T(H_A)} \left[H_B < H_\eta \right] \right].$$

and

$$\begin{aligned} &\mu_T(\eta)\, R_T(\eta, B) \;+\; \mu_T(\eta)\, \lambda_T(\eta)\, \mathbf{P}^T_\eta \left[H_A < H_B \right] \\ &\quad = \mu_T(\eta)\, R_T(\eta, B) \;+\; \mu_T(\eta)\, R_T(\eta, A) = \mathrm{cap}_T(\eta, A \cup B), \end{aligned}$$

which is the expression which appears in the denominator in the proof of the lemma. Therefore, the statement of the lemma may have some interest only if $\mathbf{P}^T_{\eta^T(H_A)} \left[H_B < H_\eta \right] = \mathbf{P}_{\eta^T(H_A)} \left[H_B < H_\eta \right]$ is negligible, i.e., if the process starting from A reaches B before η with a vanishing probability.

We apply Lemma 4.3 to our context to obtain a bound on $\mathbb{P}^{\mathcal{E}}_\eta [H_{\breve{\mathcal{E}}^x_N} \le \gamma^{-1}]$. For $\gamma > 0$, consider the Markov process $\{\eta^{N,\star}(t) : t \ge 0\}$ on $E_N \cup \mathcal{E}^\star_N$ whose jump rates $R_{N,\star}(\eta, \xi) = R^\gamma_{N,\star}(\eta, \xi)$ are given by

$$R_{N,\star}(\eta, \xi) = \begin{cases} R_N(\eta, \xi) & \text{if } \eta \text{ and } \xi \in E_N, \\ \gamma & \text{if } \eta \in \mathcal{E}^\star_N \cup \mathcal{E}_N \text{ and if } [\xi = \eta^\star \text{or } \eta = \xi^\star], \\ 0 & \text{otherwise.} \end{cases}$$

Note that the process $\eta^\star(t)$ is the trace of the process $\eta^{N,\star}(t)$ on $\mathcal{E}^\star_N \cup \mathcal{E}_N$. Denote by $\mathrm{cap}_{N,\star}$ the capacity associated to the process $\eta^{N,\star}(t)$. Next result provides a bound for condition (L4U) in terms of capacities which can be estimated through the Dirichlet and the Thomson principles.

Corollary 4.4 *For every $x \in S$, $\eta \in \mathcal{E}_N^x$ and $\gamma > 0$,*

$$\mathbb{P}_\eta^{\mathcal{E}}\Big[H_{\breve{\mathcal{E}}_N^x} \le \gamma^{-1}\Big] \le \frac{e\,\mathrm{cap}_N(\eta, \breve{\mathcal{E}}_N^x)}{2\,\mathrm{cap}_{N,\star}(\eta, \mathcal{E}_N^{\star,x})} .$$

Proof By Lemmas 4.1 and 4.3,

$$\mathbb{P}_\eta^{\mathcal{E}}\Big[H_{\breve{\mathcal{E}}_N^x} \le \gamma^{-1}\Big] \le e\,\mathbb{P}_\eta^{\star,\gamma}\Big[H_{\breve{\mathcal{E}}_N^x} < H_{\mathcal{E}_N^{\star,x}}\Big] \le \frac{e\,\mathrm{cap}_\star(\eta, \breve{\mathcal{E}}_N^x)}{\mathrm{cap}_\star(\eta, \mathcal{E}_N^{\star,x})} .$$

It is clear from the Dirichlet principle and from the definition of the enlarged process that $\mathrm{cap}_\star(\eta, \breve{\mathcal{E}}_N^x) = (1/2)\,\mathrm{cap}_{\mathcal{E}}(\eta, \breve{\mathcal{E}}_N^x)$, where $\mathrm{cap}_{\mathcal{E}}$ stands for the capacity associated to the trace process $\eta^{\mathcal{E}}(t)$. By [1, Lemma 6.9], once more, $\mathrm{cap}_{\mathcal{E}}(\eta, \breve{\mathcal{E}}_N^x) = \pi(\mathcal{E}_N)^{-1}\mathrm{cap}_N(\eta, \breve{\mathcal{E}}_N^x)$ and $\mathrm{cap}_\star(\eta, \mathcal{E}_N^{\star,x}) = \pi(\mathcal{E}_N)^{-1}\mathrm{cap}_{N,\star}(\eta, \mathcal{E}_N^{\star,x})$. This concludes the proof of the lemma. □

5 Tightness

We prove in this section tightness of the process $\mathbf{X}_t^N$. By Aldous criterion (see Theorem 16.10 in [8]) we just need to show that for every $\epsilon > 0$ and $T > 0$

$$\lim_{\delta\downarrow 0}\lim_{N\to\infty}\sup_{a\le\delta}\sup_{\tau\in\mathfrak{T}_T}\mathbb{P}_{\nu_N}^{\mathcal{E}}\Big[\,|\mathbf{X}_{\tau+a}^N - \mathbf{X}_\tau^N| > \epsilon\,\Big] = 0, \tag{5.1}$$

where $\mathfrak{T}_T$ is the set of stopping times bounded by T.

In fact, in the present context of a finite state space, we do not need to consider all stopping times, but just the jump times. More precisely, the process $\mathbf{X}_t^N$ is tight provided

$$\lim_{\delta\to 0}\limsup_{N\to\infty}\sup_{i\ge 0}\mathbb{P}_{\nu_N}^{\mathcal{E}}\big[\tau_{i+1} - \tau_i \le \delta\big] = 0,$$

where $\tau_0 = 0$ and τ_i, $i \ge 1$, represent the jumping times of the process $\mathbf{X}_t^N$.

Proof of Lemma 2.6 We will prove that (5.1) holds. Fix $T > 0$, $\epsilon > 0$ and $\delta > 0$. By the strong Markov property, for every $0 < a \le \delta$ and stopping time $\tau \le T$,

$$\begin{aligned}\mathbb{P}_{\nu_N}^{\mathcal{E}}\Big[\,\big|\mathbf{X}_{\tau+a}^N - \mathbf{X}_\tau^N\big| > \epsilon\,\Big] &\le \mathbb{P}_{\nu_N}^{\mathcal{E}}\Big[\mathbb{P}_{\eta(\tau)}^{\mathcal{E}}\Big[\,\big|\mathbf{X}_a^N - \mathbf{X}_0^N\big| > \epsilon\,\Big]\Big]\\ &\le \sup_{\eta\in\mathcal{E}_N}\mathbb{P}_\eta^{\mathcal{E}}\Big[\,\big|\mathbf{X}_a^N - \mathbf{X}_0^N\big| > \epsilon\,\Big] \le \max_{x\in S}\sup_{\eta\in\mathcal{E}_N^x}\mathbb{P}_\eta^{\mathcal{E}}\big[H_{\breve{\mathcal{E}}^x} \le \delta\theta_N\big].\end{aligned}$$

To conclude the proof we need to show that the last term vanishes as $N \uparrow \infty$ and then $\delta \downarrow 0$. The arguments used are similar to the ones used in the proof of Theorem 2.4.

Let T_N be a sequence satisfying the assumptions (L4U). Fix $x \in S$ and $\eta \in \mathcal{E}_N^x$. The probability $\mathbb{P}_\eta^{\mathcal{E}}[H_{\breve{\mathcal{E}}^x} \le \delta\theta_N]$ is bounded above by

$$\mathbb{P}_\eta^{\mathcal{E}}\left[H_{\breve{\mathcal{E}}^x} \le T_N\right] + \mathbb{E}_\eta^{\mathcal{E}}\left[\mathbf{1}\left\{H_{\breve{\mathcal{E}}^x} > T_N\right\} \mathbb{P}_{\eta(T_N)}^{\mathcal{E}}\left[H_{\breve{\mathcal{E}}^x} \le \delta\theta_N\right]\right]. \tag{5.2}$$

The first term vanishes in view of assumption (L4U). On the set $\{H_{\breve{\mathcal{E}}^x} > T_N\}$, we may couple the process $\eta(t)$ with the reflected process $\eta^{\mathbf{r},x}(t)$ in a way that $\eta(t) = \eta^{\mathbf{r},x}(t)$ for $0 \le t \le T_N$. In particular, we may replace in the previous term $\mathbb{P}_{\eta(T_N)}^{\mathcal{E}}[H_{\breve{\mathcal{E}}^x} \le \delta\theta_N]$ by $\mathbb{P}_{\eta^{\mathbf{r},x}(T_N)}^{\mathcal{E}}[H_{\breve{\mathcal{E}}^x} \le \delta\theta_N]$. After this replacement we may bound the second term in (5.2) by

$$\sum_{\xi \in \mathcal{E}_N^x} \left\{\left(\delta_\eta S^{\mathbf{r},x}(T_N)\right)(\xi) - \pi_x(\xi)\right\} \mathbb{P}_\xi^{\mathcal{E}}\left[H_{\breve{\mathcal{E}}^x} \le \delta\theta_N\right] + \mathbb{P}_{\pi_x}^{\mathcal{E}}\left[H_{\breve{\mathcal{E}}^x} \le \delta\theta_N\right], \tag{5.3}$$

where $S^{\mathbf{r},x}(t)$ represents the semi-group of the reflected process. The first term of this sum is bounded by $\|\delta_\eta S^{\mathbf{r},x}(T_N) - \pi_x\|_{\mathrm{TV}}$, where $\|\mu - \nu\|_{\mathrm{TV}}$ stands for the total variation distance between μ and ν. By definition of the mixing time, this last expression is less than or equal to $(1/2)^{(T_N/T^{\mathrm{mix}}_{\mathbf{r},x})}$, which vanishes as $N \uparrow \infty$ by definition of the sequence T_N.

It remains to estimate the second term in (5.3). It can be written as

$$\sum_{\eta \in \mathcal{E}_N^x} \left\{\pi_x(\eta) - \pi_x^*(\eta)\right\} \mathbb{P}_\eta^{\mathcal{E}}\left[H_{\breve{\mathcal{E}}^x} \le \delta\theta_N\right] + \mathbb{P}_{\pi_x^*}^{\mathcal{E}}\left[H_{\breve{\mathcal{E}}^x} \le \delta\theta_N\right], \tag{5.4}$$

where π_x^* is the quasi-stationary measure associated to the trace process $\eta^{\mathcal{E}}(t)$ killed when it hits $\breve{\mathcal{E}}^x$. The first term is less than or equal to

$$\sum_{\eta \in \mathcal{E}_N^x} \pi_x(\eta) \left|\frac{\pi_x^*(\eta)}{\pi_x(\eta)} - 1\right| \le \left\{\sum_{\eta \in \mathcal{E}_N^x} \pi_x(\eta) \left(\frac{\pi_x^*(\eta)}{\pi_x(\eta)} - 1\right)^2\right\}^{1/2}.$$

By Proposition 2.1, (17) and Lemma 2.2 in [7], the expression inside the square root on the right hand side is bounded by $\varepsilon_x/[1-\varepsilon_x]$, where $\varepsilon_x = E_{\pi_x}[R^{\mathcal{E}}(\eta, \breve{\mathcal{E}}^x)]/\mathfrak{g}_{\mathbf{r},x}$. By (2.13), $\varepsilon_x \le C_0(\theta_N \mathfrak{g}_{\mathbf{r},x})^{-1}$ for some finite constant C_0 and by hypothesis, $\theta_N^{-1} \ll \mathfrak{g}_{\mathbf{r},x}$. This shows that the first term in (5.4) vanishes as $N \uparrow \infty$.

Finally, since π_x^* is the quasi-stationary state, under $\mathbb{P}_{\pi_x^*}$, the hitting time of $\breve{\mathcal{E}}_N^x$, $H_{\breve{\mathcal{E}}^x}$, has an exponential distribution whose parameter we denote by ϕ_x^*. By [7, Lemma 2.2], ϕ_x^* is bounded by $E_{\pi_x}[R^{\mathcal{E}}(\eta, \breve{\mathcal{E}}^x)] \le C_0/\theta_N$, for some finite constant C_0. Hence,

$$\mathbb{P}_{\pi_x^*}^{\mathcal{E}}\left[H_{\breve{\mathcal{E}}^x} \le \delta\theta_N\right] = 1 - e^{-\phi_x^*\delta\theta_N} \le 1 - e^{-C_0\delta},$$

an expression which vanishes as $\delta \downarrow 0$. This proves (5.1) and concludes the proof of the lemma. □

By a version of [23, Theorem 12.3] for continuous-time reversible Markov chains, $T^{\rm mix}_{\mathbf{r},x} \le \mathfrak{g}_{\mathbf{r},x}^{-1} \log(4/\min_{\eta\in\mathcal{E}_N^x} \pi_x(\eta))$. Hence, $\max_{x\in S} T^{\rm mix}_{\mathbf{r},x} \ll T_N$ if

$$\lim_{N\to\infty} \frac{1}{T_N\, \mathfrak{g}_{\mathbf{r},x}} \log \frac{1}{\min_{\eta\in\mathcal{E}_N^x} \pi_x(\eta)} = 0. \tag{5.5}$$

6 The spectral gap of the trace process

We prove in this section Proposition 2.1. We start with an elementary result which provides an upper bound for the spectral gap of the trace process in terms of capacities. Recall that $\eta(t)$ is a positive recurrent, reversible, continuous-time Markov chain on a countable state space E_N, whose embedded discrete-time chain is also positive recurrent. Let $\mathcal{E}_N$ a subset of E_N and denote by $\mathfrak{g}_{\mathcal{E}}$ the spectral gap of the trace of $\eta(t)$ on $\mathcal{E}_N$.

Lemma 6.1 *We have that*

$$\mathfrak{g}_{\mathcal{E}} \;\le\; \inf_{\mathcal{A}\subset\mathcal{E}_N} \frac{\pi(\mathcal{E}_N)\, \mathrm{cap}(\mathcal{A},\mathcal{B})}{\pi(\mathcal{A})\, \pi(\mathcal{B})},$$

where $\mathcal{B} = \mathcal{E}_N \setminus \mathcal{A}$.

Proof Fix a subset $\mathcal{A}$ of $\mathcal{E}_N$, and let $\mathcal{B} = \mathcal{E}_N\setminus\mathcal{A}$. By definition,

$$\mathfrak{g}_{\mathcal{E}} = \inf_f \frac{\langle f, (-L_{\mathcal{E}})f\rangle_{\pi_{\mathcal{E}}}}{\mathrm{Var}_{\pi_{\mathcal{E}}}(f)} \;\le\; \frac{\langle \mathbf{1}\{\mathcal{A}\}, (-L_{\mathcal{E}})\mathbf{1}\{\mathcal{A}\}\rangle_{\pi_{\mathcal{E}}}}{\mathrm{Var}_{\pi_{\mathcal{E}}}(\mathbf{1}\{\mathcal{A}\})},$$

where $\mathrm{Var}_{\pi_{\mathcal{E}}}(f)$ stands for the variance of f with respect to the measure $\pi_{\mathcal{E}}$. Since $\mathcal{E}_N = \mathcal{A}\cup\mathcal{B}$, $\mathbf{1}\{\mathcal{A}\}$ is the equilibrium potential between $\mathcal{A}$ and $\mathcal{B}$ so that $\langle \mathbf{1}\{\mathcal{A}\}, (-L_{\mathcal{E}})\mathbf{1}\{\mathcal{A}\}\rangle_{\pi_{\mathcal{E}}} = \mathrm{cap}_{\mathcal{E}}(\mathcal{A},\mathcal{B})$. Hence, by [1, Lemma 6.9],

$$\mathfrak{g}_{\mathcal{E}} \;\le\; \frac{\mathrm{cap}_{\mathcal{E}}(\mathcal{A},\mathcal{B})}{\pi_{\mathcal{E}}(\mathcal{A})\, \pi_{\mathcal{E}}(\mathcal{B})} = \frac{\pi(\mathcal{E}_N)\, \mathrm{cap}(\mathcal{A},\mathcal{B})}{\pi(\mathcal{A})\, \pi(\mathcal{B})}\,.$$

□

Proof of Proposition 2.1 Let $F : \mathcal{E}_N \to \mathbb{R}$ be a function in $L^2(\pi_{\mathcal{E}})$ and denote by $\hat F : E_N \to \mathbb{R}$ the harmonic extension of F to E_N, defined by

$$\hat F(\eta) = \begin{cases} F(\eta) & \text{if } \eta\in\mathcal{E}_N, \\ \mathbb{E}_\eta[F(\eta(H_{\mathcal{E}_N}))] & \text{if } \eta\notin\mathcal{E}_N. \end{cases}$$

We claim that

$$\langle (-L_N)\hat F,\ \hat F\rangle_\pi = \pi(\mathcal{E}_N)\, \langle (-L_{\mathcal{E}})F,\ F\rangle_{\pi_{\mathcal{E}}}\,. \tag{6.1}$$

Indeed, since $L_N \hat{F} = 0$ on $\mathcal{E}_N^c$ and since $\hat{F}$ and F coincide on $\mathcal{E}_N$, the Dirichlet form $\langle L_N \hat{F},\ \hat{F}\rangle_\pi$ is equal to

$$\sum_{\eta\in\mathcal{E}_N} \pi(\eta)\, F(\eta) \sum_{\xi\in E_N} R_N(\eta,\xi)\{\hat{F}(\xi) - F(\eta)\}. \tag{6.2}$$

We decompose the previous sum in two expressions, the first one including all terms for which ξ belongs to $\mathcal{E}_N$ and the second one including all terms for which ξ belongs to $E_N\backslash\mathcal{E}_N$. When ξ belongs to $\mathcal{E}_N$, we may replace $\hat{F}$ by F. The other expression, by definition of $\hat{F}$ is equal to

$$\sum_{\eta\in\mathcal{E}_N} \sum_{\xi\notin\mathcal{E}_N} \pi(\eta)\, F(\eta)\, R_N(\eta,\xi) \sum_{\zeta\in\mathcal{E}_N} \mathbb{P}_\xi \left[H_{\mathcal{E}_N} = H_\zeta\right] \{F(\zeta) - F(\eta)\}.$$

Since for $\eta \in \mathcal{E}_N$,

$$\mathbb{P}_\eta \left[H^+_{\mathcal{E}_N} = H_\zeta\right] = p_N(\eta,\zeta) \;+ \sum_{\xi\notin\mathcal{E}_N} p_N(\eta,\xi)\, \mathbb{P}_\xi \left[H_{\mathcal{E}_N} = H_\zeta\right],$$

and since by [1, Proposition 6.1] $R^{\mathcal{E}}(\eta,\zeta) = \lambda_N(\eta)\mathbb{P}_\eta\left[H^+_{\mathcal{E}_N} = H_\zeta\right]$ the previous sum is equal to

$$\sum_{\eta\in\mathcal{E}_N} \sum_{\zeta\in\mathcal{E}_N} \pi(\eta)\, F(\eta) \left\{R^{\mathcal{E}}(\eta,\zeta) - R_N(\eta,\zeta)\right\} \{F(\zeta) - F(\eta)\}.$$

Adding this sum to the first expression in our decomposition of (6.2) as the sum of two terms, we get that the left hand side of (6.1) is equal to

$$\sum_{\eta\in\mathcal{E}_N} \sum_{\zeta\in\mathcal{E}_N} \pi(\eta)\, F(\eta)\, R^{\mathcal{E}}(\eta,\zeta)\, \{F(\zeta) - F(\eta)\}.$$

To conclude the proof of Claim (6.1), it remains to recall that $\pi_{\mathcal{E}}(\eta) = \pi(\eta)/\pi(\mathcal{E}_N)$.

Fix a function $F : \mathcal{E}_N \to \mathbb{R}$. We claim that

$$\inf_g \langle(-L_N)g,\ g\rangle_\pi = \langle(-L_N)\hat{F},\ \hat{F}\rangle_\pi, \tag{6.3}$$

where the infimum is carried over all functions $g : E_N \to \mathbb{R}$ which are equal to F on $\mathcal{E}_N$. Indeed, it is simple to show that any function f which solves the variational problem on the left hand side of (6.3) is harmonic on $\mathcal{E}_N^c$ and coincides with F on $\mathcal{E}_N$, $L_N f = 0$ on $\mathcal{E}_N^c$ and $f = F$ on $\mathcal{E}_N$. The unique solution to this problem is $\hat{F}$, which proves (6.3).

Fix an eigenfunction F associated to $\mathfrak{g}_{\mathcal{E}}$ such that $E_{\pi_{\mathcal{E}}}[F^2] = 1$, $E_{\pi_{\mathcal{E}}}[F] = 0$. By (6.1) we have that

$$\mathfrak{g}_{\mathcal{E}} = \langle (-L_{\mathcal{E}})F,\, F\rangle_{\pi_{\mathcal{E}}} = \frac{1}{\pi(\mathcal{E}_N)} \langle (-L_N)\hat{F},\, \hat{F}\rangle_{\pi}.$$

By the spectral gap, the Dirichlet form on the right hand side is bounded below by $\mathfrak{g}$ times the variance of $\hat{F}$. This latter variance, in view of the definition of $\hat{F}$ and the properties of F, is equal to

$$\pi(\mathcal{E}_N) \;+\; \sum_{\eta\notin\mathcal{E}_N} \pi(\eta)\hat{F}(\eta)^2 - \Bigg(\sum_{\eta\notin\mathcal{E}_N} \pi(\eta)\hat{F}(\eta)\Bigg)^2 \;\geq\; \pi(\mathcal{E}_N).$$

This proves that $\mathfrak{g} \leq \mathfrak{g}_{\mathcal{E}}$.

Fix an eigenfunction f associated to $\mathfrak{g}$ such that $E_\pi[f^2] = 1$, $E_\pi[f] = 0$. Let $F : \mathcal{E}_N \to \mathbb{R}$ be the restriction to $\mathcal{E}_N$ of f: $F(\eta) = f(\eta)\mathbf{1}\{\eta \in \mathcal{E}_N\}$. By definition of $\mathfrak{g}$,

$$\mathfrak{g} = \langle (-L_N)f,\, f\rangle_{\pi} \;\geq\; \inf_g \langle (-L_N)g,\, g\rangle_{\pi},$$

where the infimum is carried over all functions g which coincide with F on $\mathcal{E}_N$. By (6.3), by (6.1) and by definition of the spectral gap $\mathfrak{g}_{\mathcal{E}}$, the right hand side of the previous term is equal to

$$\langle (-L_N)\hat{F},\, \hat{F}\rangle_{\pi} = \pi(\mathcal{E}_N)\langle (-L_{\mathcal{E}})F,\, F\rangle_{\pi_{\mathcal{E}}} \;\geq\; \mathfrak{g}_{\mathcal{E}}\, \pi(\mathcal{E}_N) \left\{ E_{\pi_{\mathcal{E}}}[F^2] - E_{\pi_{\mathcal{E}}}[F]^2 \right\}.$$

Since $F = f\mathbf{1}\{\mathcal{E}_N\}$, up to this point we proved that

$$\mathfrak{g}_{\mathcal{E}}\, \pi(\mathcal{E}_N) \left\{ E_{\pi_{\mathcal{E}}}[f^2 1\{\mathcal{E}_N\}] - E_{\pi_{\mathcal{E}}}[f 1\{\mathcal{E}_N\}]^2 \right\} \;\leq\; \mathfrak{g}$$

Since the eigenfunction f associated to $\mathfrak{g}$ is such that $E_\pi[f^2] = 1$, $E_\pi[f] = 0$, we may rewrite the previous inequality as

$$\mathfrak{g}_{\mathcal{E}} \left\{ 1 - \left[E_\pi\left[f^2\mathbf{1}\{\mathcal{E}_N^c\} \right] \;+\; \frac{1}{\pi(\mathcal{E}_N)} E_\pi\left[f\mathbf{1}\{\mathcal{E}_N^c\} \right]^2 \right] \right\} \;\leq\; \mathfrak{g}.$$

By Schwarz inequality, $E_\pi[f\mathbf{1}\{\mathcal{E}_N^c\}]^2 \leq E_\pi[f^2\mathbf{1}\{\mathcal{E}_N^c\}]\pi(\mathcal{E}_N^c)$ so that

$$\mathfrak{g}_{\mathcal{E}} \left\{ 1 - \frac{1}{\pi(\mathcal{E}_N)} E_\pi\left[f^2\mathbf{1}\{\mathcal{E}_N^c\} \right] \right\} \;\leq\; \mathfrak{g},$$

which proves the proposition. □

7 Proof of Theorem 2.9

We assume in this section that the state space E_N has been divided in three disjoint sets $\mathcal{E}_N^1 = \mathcal{A}$, $\mathcal{E}_N^2 = \mathcal{B}$ and $\Delta_N = E_N \setminus \mathcal{E}_N$, where $\mathcal{E}_N = \mathcal{A} \cup \mathcal{B}$. Recall that $\eta^{\mathcal{E}}(t)$ represents the trace of the process $\eta(t)$ on the set $\mathcal{E}_N$ and $\eta^\star(t)$ the γ-enlargement of the process $\eta^{\mathcal{E}}(t)$ to the set $\mathcal{E}_N \cup \mathcal{E}_N^\star$, where $\gamma = \gamma_N$ is a sequence of positive numbers and $\mathcal{E}_N^\star = \mathcal{A}^\star \cup \mathcal{B}^\star$, $\mathcal{A}^\star$, $\mathcal{B}^\star$ being copies of the sets $\mathcal{A}$, $\mathcal{B}$, respectively. Denote by $\mathfrak{g}_{\mathcal{A}}$, $\mathfrak{g}_{\mathcal{B}}$ the spectral gap of the process $\eta(t)$ reflected at $\mathcal{A}$, $\mathcal{B}$, respectively.

Let $\widehat{\text{cap}}_\star(\mathcal{A}^\star, \mathcal{B}^\star)$ be the normalized capacity between $\mathcal{A}^\star$ and $\mathcal{B}^\star$:

$$\widehat{\text{cap}}_\star(\mathcal{A}^\star, \mathcal{B}^\star) = \frac{\text{cap}_\star(\mathcal{A}^\star, \mathcal{B}^\star)}{\pi_{\mathcal{E}}(\mathcal{A})\, \pi_{\mathcal{E}}(\mathcal{B})} \cdot$$

By [7, Theorem 2.12],

$$\left(1 - \frac{2\,\widehat{\text{cap}}_\star(\mathcal{A}^\star, \mathcal{B}^\star)}{\gamma}\right)^2 \le \frac{2\,\widehat{\text{cap}}_\star(\mathcal{A}^\star, \mathcal{B}^\star)}{\mathfrak{g}_{\mathcal{E}}} \le 1 + \frac{\gamma + 2\,\widehat{\text{cap}}_\star(\mathcal{A}^\star, \mathcal{B}^\star)}{\min\{\mathfrak{g}_{\mathcal{A}}, \mathfrak{g}_{\mathcal{B}}\}} \cdot \tag{7.1}$$

The factor 2, which is not present in [7], appears because we consider the capacity with respect to the probability measure $\pi_\star$, while [7] defines the capacity with $\pi_{\mathcal{E}}$ as reference measure.

Theorem 2.9 is a simple consequence of (7.1). For sake of completeness, we present a proof of the lower bound of (7.1). Let V be the equilibrium potential between $\mathcal{A}^\star$ and $\mathcal{B}^\star$: $V(\eta) = \mathbb{P}_\eta^{\star,\gamma}[H_{\mathcal{A}^\star} < H_{\mathcal{B}^\star}]$. We sometimes consider below V as a function on $\mathcal{E}_N$. By definition of the spectral gap,

$$\mathfrak{g}_{\mathcal{E}} \;\le\; \frac{\langle V, (-L_{\mathcal{E}})V\rangle_{\pi_{\mathcal{E}}}}{\text{Var}_{\pi_{\mathcal{E}}}(V)},$$

where $\text{Var}_{\pi_{\mathcal{E}}}(V)$ stands for the variance of V with respect to the measure $\pi_{\mathcal{E}}$. We estimate the numerator and the denominator separately.

Since the capacity between $\mathcal{A}^\star$ and $\mathcal{B}^\star$ is equal to the Dirichlet form of the equilibrium potential,

$$(1/2)\langle V, (-L_{\mathcal{E}})V\rangle_{\pi_{\mathcal{E}}} \;\le\; \text{cap}_\star(\mathcal{A}^\star, \mathcal{B}^\star)\ .$$

A martingale decomposition of the variance of V gives that

$$\text{Var}_{\pi_{\mathcal{E}}}(V) \;\ge\; \pi_{\mathcal{E}}(\mathcal{A})\, \pi_{\mathcal{E}}(\mathcal{B})\, \big(E_{\pi_{\mathcal{A}}}[V_{\mathcal{A}}] - E_{\pi_{\mathcal{B}}}[V_{\mathcal{B}}]\big)^2\,,$$

where $V_{\mathcal{A}} = V\mathbf{1}\{\mathcal{A}\}$, $V_{\mathcal{B}} = V\mathbf{1}\{\mathcal{B}\}$. Since $\text{cap}_\star(\mathcal{A}^\star, \mathcal{B}^\star) = \langle V, (-L_\star)V\rangle_{\pi_\star}$, since $(L_\star V)(\eta^\star) = \gamma[V(\eta) - 1]$, where $\eta^\star$ is the state in $\mathcal{E}_N^\star$ corresponding to the state $\eta \in \mathcal{E}_N$, and since $\pi_\star(\eta^\star) = (1/2)\pi_{\mathcal{E}}(\eta)$, $2\text{cap}_\star(\mathcal{A}^\star, \mathcal{B}^\star) = \gamma\pi_{\mathcal{E}}(\mathcal{A}) - \gamma \sum_{\eta\in\mathcal{A}} \pi_{\mathcal{E}}(\eta)V(\eta)$. Therefore,

$$E_{\pi_{\mathcal{A}}}[V_{\mathcal{A}}] = 1 \;-\; \frac{2\,\text{cap}_\star(\mathcal{A}^\star, \mathcal{B}^\star)}{\gamma\, \pi_{\mathcal{E}}(\mathcal{A})}\ .$$

Repeating the previous argument with $1 - V$ in place of V we obtain that

$$E_{\pi_{\mathcal{B}}}[V_{\mathcal{B}}] = \frac{2\,\mathrm{cap}_\star(\mathcal{A}^\star, \mathcal{B}^\star)}{\gamma\, \pi_{\mathcal{E}}(\mathcal{B})}\ .$$

Putting together the previous estimates, we conclude the proof of the lower bound of (7.1). □

8 Applications

We present in this section two applications of Theorems 2.2 and 2.4. Both processes do not visit points in the time scale where tunneling occurs, in the sense that the probability that the process visits a specific configuration, in a time interval whose length is of the order of the tunneling time, vanishes. In particular, these models do not satisfy the hypotheses of the theory developed in [1,5]. Furthermore, these models have logarithmic energy or entropy barriers, restraining the application of large deviations methods. On the other hand, both dynamics are monotone with respect to a partial order, allowing the use of coupling techniques. The first model, which has only entropy barriers, was suggested by A. Gaudillière to the authors as a model for testing metastability techniques. We prove for to this model the mixing conditions introduced in Section 2. E. The second one has been examined in details in [11,12]. We apply to this model the L^2-theory presented in Sect. 2. D.

8.1 The dog graph [25]

For $N \ge 1$ and $d \ge 2$, let $Q_N = \{0, \dots, N\}^d$ be a d-dimensional cube of length N, let $\breve{Q}_N$ be the reflection of Q_N through the origin, $\breve{Q}_N = \{\eta \in \mathbb{Z}^d : -\eta \in Q_N\}$, and let $V_N = Q_N \cup \breve{Q}_N$. Denote by E_N the set of edges formed by pairs of nearest-neighbor sites of V_N, $E_N = \{(\eta, \xi) \in V_N \times V_N : |\eta - \xi| = 1\}$. The graph $G_N = (V_N, E_N)$ is called the dog graph [25].

Let $\{\eta(t) : t \ge 0\}$ be the continuous-time Markov chain on G_N which jumps from η to ξ at rate 1 if $(\eta, \xi) \in E_N$. The uniform measure on V_N, denoted by π, is the unique stationary state. Diaconis and Saloff Coste [25, Example 3.2.5] proved that there exist constants $0 < c(d) < C(d) < \infty$ such that for all $N \ge 1$,

$$\frac{c(2)}{N^2\ \log N} \ \le\ \mathfrak{g}\ \le\ \frac{C(2)}{N^2\ \log N} \quad \text{in } d = 2 \text{ and } \quad \frac{c(d)}{N^d}\ \le\ \mathfrak{g}\ \le\ \frac{C(d)}{N^d} \tag{8.1}$$

in dimension $d \ge 3$.

Fix a sequence α_N, $(\log N)^{-1/2} \ll \alpha_N \ll 1$, and let $\mathcal{B}_N = \{\eta = (\eta_1, \dots, \eta_d) \in V_N : \min_j \eta_j \ge \alpha_N N\}$, $\mathcal{A}_N = -\mathcal{B}_N = \{\eta \in V_N : -\eta \in \mathcal{B}_N\}$. Denote by $\mathfrak{g}_{\mathcal{A}}$ and $T^{\text{mix}}_{\mathbf{r},\mathcal{A}}$ (resp. $\mathfrak{g}_{\mathcal{B}}$ and $T^{\text{mix}}_{\mathbf{r},\mathcal{B}}$) the spectral gap and the mixing time of the continuous-time random walk $\eta(t)$ reflected at $\mathcal{A}_N$ (resp. $\mathcal{B}_N$). It is well known that there exist finite constants $0 < c(d) < C(d) < \infty$ such that for all $N \ge 1$,

$$\frac{c(d)}{N^2}\ \le\ \mathfrak{g}_{\mathcal{A}}\ \le\ \frac{C(d)}{N^2},\quad c(d)\, N^2\ \le\ T^{\text{mix}}_{\mathbf{r},\mathcal{A}}\ \le\ C(d) N^2, \tag{8.2}$$

with similar inequalities if $\mathcal{B}$ replaces $\mathcal{A}$.

Condition (2.11). Let $\mathcal{E}_N = \mathcal{A}_N \cup \mathcal{B}_N$, and recall the notation introduced in Sect. 2. We claim that condition (2.11) is fulfilled for $\theta_N = N^2 \log N$ in dimension 2 and for $\theta_N = N^d$ in dimension $d \geq 3$. Indeed, if $\pi_{\mathcal{A}_N}, \pi_{\mathcal{B}_N}, \pi_{\mathcal{E}}$ represent the uniform measure π conditioned to $\mathcal{A}_N$, $\mathcal{B}_N$, $\mathcal{E}_N$, respectively, by (2.12),

$$E_{\pi_{\mathcal{A}_N}}[R^{\mathcal{E}_N}(\eta, \mathcal{B}_N)] = \frac{1}{\pi(\mathcal{A}_N)} \operatorname{cap}_N(\mathcal{A}_N, \mathcal{B}_N).$$

By the Dirichlet principle, the capacity is bounded by the Dirichlet form of any function which vanishes on $\mathcal{A}_N$ and is equal to 1 on $\mathcal{B}_N$. In dimension $d \geq 3$ we simply choose the indicator of the set Q_N. In dimension 2, let $D_k = \{\eta \in Q_N : \eta_1 + \eta_2 = k\}$, $k \geq 0$. Fix $1 \leq L \leq N$ and consider the function $f_L : Q_N \to \mathbb{R}_+$ defined by $f(0) = 0$,

$$f_L(\eta) = \frac{1}{\Phi(L)} \sum_{j=1}^{k} \frac{1}{j} \quad \eta \in D_k, \quad 1 \leq k \leq L, \tag{8.3}$$

where $\Phi(L) = \sum_{1 \leq j \leq L} j^{-1}$, and $f_L(\eta) = 1$ otherwise. It is easy to see that the Dirichlet form of f_L is bounded by $C_0(N^2 \log L)^{-1}$ for some finite constant C_0. Choosing $L = N^a$, for some $0 < a < 1$, we conclude that there exists a finite constant C_0 such that

$$\operatorname{cap}_N(\mathcal{A}_N, \mathcal{B}_N) \leq \frac{C_0}{N^2 \log N}, \quad d = 2, \quad \operatorname{cap}_N(\mathcal{A}_N, \mathcal{B}_N) \leq \frac{C_0}{N^d}, \quad d \geq 3. \tag{8.4}$$

Condition (2.11) follows from this estimate and the definition of the sequence θ_N.

Condition (L1B) in Lemma 2.10. By Lemma 6.1 and by the previous estimate of the capacity, there exists a finite constant C_0 such that $\mathfrak{g}_{\mathcal{E}} \leq C_0[N^2 \log N]^{-1}$ in dimension 2 and $\mathfrak{g}_{\mathcal{E}} \leq C_0 N^{-d}$ in dimension $d \geq 3$. Condition (L1B) is thus fulfilled in view of (8.2).

Condition (L4) in Theorem 2.4. We claim that there exists a sequence T_N satisfying the conditions (L4) if ν_N is a sequence of measures concentrated on $\mathcal{A}_N$ and such that

$$\lim_{N\to\infty} \frac{1}{R_N} E_{\pi_{\mathcal{E}}} \left[\left(\frac{\nu_N}{\pi_{\mathcal{E}}} \right)^2 \right] = 0, \tag{8.5}$$

where $R_N = \log N$ in dimension $d = 2$, and $R_N = N^{d-2}$ in dimension $d \geq 3$. Let M_N be an increasing sequence, $M_N \gg 1$, for which (8.5) still holds if multiplied by M_N. Since $\operatorname{cap}_\star(\mathcal{A}_N^\star, \mathcal{B}_N) \leq \operatorname{cap}_{\mathcal{E}}(\mathcal{A}_N, \mathcal{B}_N)$, by Corollary 4.2, by [1, Lemma 6.9], by (8.4) and by (8.5) multiplied by M_N,

$$\lim_{N\to\infty} \mathbb{P}^{\mathcal{E}}_{\nu_N} \left[H_{\mathcal{B}_N} \leq N^2 M_N \right] = 0.$$

The strategy proposed in Sect. 6 permits to weaken assumption (8.5).

Lemma 8.1 *Let T_N be a sequence such that $T_N \ll \alpha_N^2\, N^2 \log N$ in dimension 2, and $T_N \ll \alpha_N^d\, N^d$ in dimension $d \ge 3$. Then,*

$$\lim_{N\to\infty} \max_{\eta\in\mathcal{A}_N} \mathbb{P}_\eta \big[H_{Q_N} \le T_N \big] = 0.$$

Proof In view of the definition of α_N, we may assume that $T_N \gg N^2$. We present the arguments in dimension 2, the case of higher dimension being similar. Fix a sequence $\eta^N \in \mathcal{A}_N$. Let $\gamma_N = T_N^{-1}$ and denote by $\eta^\star(t)$ the γ-enlargement of the process $\eta(t)$ on $V_N \cup V_N^\star$, as defined in Sect. 2. Here, $V_N^\star$ represents a copy of V_N, and the process $\eta^\star(t)$ jumps from η to $\eta^\star$ (and from $\eta^\star$ to η) at rate γ_N. Denote by $\mathbb{P}_\eta^\gamma$ the probability measure on the path space $D(\mathbb{R}_+, V_N \cup V_N^\star)$ induced by the Markov process $\eta^\star(t)$ starting from η.

Let W be the equilibrium potential $W(\eta) = \mathbb{P}_\eta^\gamma[H_0 < H_{\breve{Q}_N^\star}]$, where 0 represents the origin. In view of Lemma 4.1, it is enough to show that $W(\eta^N)$ vanishes as $N \uparrow \infty$. By the Dirichlet principle,

$$\langle W, (-L_\star) W\rangle_{\pi_\star} = \text{cap}_\star(0, \breve{Q}_N^\star) = \inf_f \langle f, (-L_\star) f\rangle_{\pi_\star}, \tag{8.6}$$

where the infimum is carried over all functions f which vanish on $\breve{Q}_N^\star$ and which are equal to 1 at the origin. Using the function f_L introduced in (8.3), we may show that the last term is bounded by $C_0 (N^2 \log N)^{-1}$ for some finite constant C_0. We used here the fact that $\gamma_N \ll (N \log N)^{-1}$.

Denote by $\prec$ the partial order of $\mathbb{Z}^d$ so that $\eta \prec \xi$ if $\eta_j \le \xi_j$ for $1 \le j \le d$. A coupling argument shows that the equilibrium potential W is monotone on $\breve{Q}_N$: $W(\eta) \le W(\xi)$ for $\eta \prec \xi$, $\eta, \xi \in \breve{Q}_N$. Suppose that $W(\eta^N)$ does not vanish as $N \uparrow \infty$. In this case there exists $\epsilon > 0$ and a subsequence N_j, still denoted by N, such that $W(\eta^N) \ge \epsilon$ for all N. Let $U_N = \{\xi \in \breve{Q}_N : \eta^N \prec \xi\}$. By monotonicity of the equilibrium potential, $W(\xi) \ge W(\eta^N) \ge \epsilon$ for all $\xi \in U_N$. Therefore,

$$\langle W, (-L_\star) W\rangle_{\pi_\star} \;\ge\; \gamma_N \sum_{\xi\in U_N} \pi_\star(\xi) W(\xi)^2 \;\ge\; c_0\, \gamma_N\, \epsilon^2\, \alpha_N^2$$

for some positive constant c_0. This contradicts the estimate (8.6) because $\gamma_N \gg (\alpha_N^2\, N^2 \log N)^{-1}$. □

Condition (L4U) The proof of Lemma 8.1 shows that condition (L4U) is in force.

Lemma 8.2 *let T_N be a sequence such that $T_N \ll \alpha_N^2\, N^2 \log N$ in dimension 2, and $T_N \ll \alpha_N^d\, N^d$ in dimension $d \ge 3$. Then,*

$$\lim_{N\to\infty} \max_{\eta\in\mathcal{A}_N} \mathbb{P}_\eta^{\mathcal{E}} \big[H_{\mathcal{B}_N} \le T_N \big] = 0.$$

Proof Consider the case of dimension 2. In view of the definition of α_N, we may assume that $T_N \gg N^2$. Let $\gamma_N = T_N^{-1}$, and fix a sequence $\eta^N \in \mathcal{A}_N$. By the proof of

Lemma 8.1, it is enough to show that $\mathbb{P}^{\star,\gamma}_{\eta^N}[H_{\mathcal{B}_N} < H_{\mathcal{A}^\star_N}]$ vanishes as $N \uparrow \infty$, where $\mathbb{P}^{\star,\gamma}_{\eta}$ has been introduced in Sect. 2 just after the definition of enlargements. Clearly,

$$\mathbb{P}^{\star,\gamma}_{\eta^N}\Big[H_{\mathcal{B}_N} < H_{\mathcal{A}^\star_N}\Big] = \mathbb{P}^{\gamma}_{\eta^N}\Big[H_{\mathcal{B}_N} < H_{\mathcal{A}^\star_N}\Big] \le \mathbb{P}^{\gamma}_{\eta^N}\Big[H_0 < H_{\mathcal{A}^\star_N}\Big],$$

where $\mathbb{P}^{\gamma}_{\eta}$ is the probability measure introduced in the proof of Lemma 8.1. Denote by $\eta^{\mathbf{r},\breve{Q}}(t)$ the process $\eta(t)$ reflected at $\breve{Q}_N$ and by $\eta^{\mathbf{r},\breve{Q},\gamma}(t)$ the γ-enlargement of the process $\eta^{\mathbf{r},\breve{Q}}(t)$ on $\breve{Q}_N \cup \breve{Q}^\star_N$. The last probability is clearly equal to $\mathbb{P}^{\mathbf{r},\breve{Q},\gamma}_{\eta^N}[H_0 < H_{\mathcal{A}^\star_N}]$, where $\mathbb{P}^{\mathbf{r},\breve{Q},\gamma}_{\eta}$ is the law of the process $\eta^{\mathbf{r},\breve{Q},\gamma}(t)$ starting from η.

Let $\mathcal{A}^j_N = \{\eta \in \breve{Q}_N : \eta_j > -\alpha_N N\}$, $j = 1, 2$, so that $\breve{Q}_N = \mathcal{A}_N \cup \mathcal{A}^1_N \cup \mathcal{A}^2_N$ and

$$\mathbb{P}^{\mathbf{r},\breve{Q},\gamma}_{\eta^N}\Big[H_0 < H_{\mathcal{A}^\star_N}\Big] \le \mathbb{P}^{\mathbf{r},\breve{Q},\gamma}_{\eta^N}\Big[H_0 < H_{\breve{Q}^\star_N}\Big] + \sum_{j=1}^{2} \mathbb{P}^{\mathbf{r},\breve{Q},\gamma}_{\eta^N}\Big[H_{\mathcal{A}^{j,\star}_N} < H_{\mathcal{A}^\star_N}\Big].$$

We have shown in the proof of Lemma 8.1 that the first term on the right hand side of the previous formula vanishes as $N \uparrow \infty$. The other two are one-dimensional problems.

Let $W(\eta)$ be the equilibrium potential $\mathbb{P}^{\mathbf{r},\breve{Q},\gamma}_{\eta}[H_{\mathcal{A}^{1,\star}_N} < H_{\mathcal{A}^\star_N}]$. We claim that

$$\lim_{N\to\infty} \max_{\eta\in\breve{Q}_N} W(\eta) = 0.$$

Let R_N be a sequence such that $N^2 \ll R_N \ll T_N$. With respect to the measure $\mathbb{P}^{\mathbf{r},\breve{Q},\gamma}_{\eta}$, $H_{\breve{Q}^\star_N}$ is a mean T_N exponential time. Hence, $\mathbb{P}^{\mathbf{r},\breve{Q},\gamma}_{\eta}[H_{\breve{Q}^\star_N} < R_N]$ vanishes as $N \uparrow \infty$. It is therefore enough to show that

$$\lim_{N\to\infty} \mathbb{P}^{\mathbf{r},\breve{Q},\gamma}_{\eta}\Big[H_{\mathcal{A}^{1,\star}_N} < H_{\mathcal{A}^\star_N},\ H_{\breve{Q}^\star_N} \ge R_N\Big] = 0.$$

By the Markov property, the previous probability is equal to

$$\mathbb{E}^{\mathbf{r},\breve{Q},\gamma}_{\eta}\Big[\mathbf{1}\{H_{\breve{Q}^\star_N} \ge R_N\}\, \mathbb{P}^{\mathbf{r},\breve{Q},\gamma}_{\eta^{\mathbf{r},\breve{Q}}(R_N)}\Big[H_{\mathcal{A}^{1,\star}_N} < H_{\mathcal{A}^\star_N}\Big]\Big],$$

where $\eta^{\mathbf{r},\breve{Q}}(t)$ is the process $\eta(t)$ reflected at $\breve{Q}_N$. We bound last expectation by removing the indicator of the set $H_{\breve{Q}^\star_N} \ge R_N$ and we estimate the remaining term by

$$\mathbb{P}^{\mathbf{r},\breve{Q},\gamma}_{\pi_{\breve{Q}}}\Big[H_{\mathcal{A}^{1,\star}_N} < H_{\mathcal{A}^\star_N}\Big] + \|\delta_\eta S^{\mathbf{r},\breve{Q}}(R_N) - \pi_{\breve{Q}}\|_{VT},$$

where $\pi_{\breve{Q}}$ is the uniform measure on $\breve{Q}$ and $S^{\mathbf{r},\breve{Q}}(t)$ the Markov semigroup of the process $\eta^{\mathbf{r},\breve{Q}}(t)$. As $R_N \gg N^2$, which is the mixing time of $\eta^{\mathbf{r},\breve{Q}}(t)$, the second term vanishes as $N \uparrow \infty$, while the first term is the expectation of the equilibrium potential W with respect to the measure $\pi_{\breve{Q}}$. If $L_{\mathbf{r},\breve{Q}}$ represents the generator of the Markov process $\eta^{\mathbf{r},\breve{Q}}(t)$, we have that $L_{\mathbf{r},\breve{Q}} W - \gamma W = -\gamma \mathbf{1}\{\mathcal{A}_N^1\}$. Taking the expectation with respect to $\pi_{\breve{Q}}$, we conclude that $E_{\pi_{\breve{Q}}}[W] = \pi_{\breve{Q}}(\mathcal{A}_N^1)$, which vanishes as $N \uparrow \infty$. This concludes the proof of the lemma.

In view of Lemma 2.10, we have just shown that all assumptions of Theorem 2.4 and Lemma 2.6 are in force. Moreover, by (8.2) and Lemma 8.1, the hypotheses of Lemma 2.7 are fulfilled for $\mathcal{D}_N = \mathcal{A}_N$, $\mathcal{F}_N = \breve{Q}_N$ and $N^2 \ll T_N \ll \alpha_N^2 (\log N) N^2$. Hence,

Proposition 8.3 *Consider the Markov process $\eta(t)$ on the dog graph. Assume that the initial state ν_N is concentrated on $\mathcal{A}_N$. Then, the time-rescaled order $\mathbf{X}_t^N = X^N_{\mathfrak{g}_{\mathcal{E}}^{-1} t}$ converges to the Markov process on $\{1, 2\}$ which starts from 1 and jumps from x to $3-x$ at rate $1/2$. Moreover, in the time scale $\mathfrak{g}_{\mathcal{E}}^{-1}$ the time spent by the original process $\eta(t)$ on the set $\Delta_N = V_N \setminus \mathcal{E}_N$ is negligible.*

As a last step, we replace in the previous statement the spectral gap $\mathfrak{g}_{\mathcal{E}}$ of the trace process by the spectral gap $\mathfrak{g}$ of the original process. Let T_N be a sequence such that $N^2 \ll T_N \ll \alpha_N^2 N^2 \log N$ in dimension 2, and $N^2 \ll T_N \ll \alpha_N^d N^d$ in dimension $d \geq 3$. It follows from Lemma 8.1 and from (8.2) that

$$\lim_{N\to\infty} \min_{\eta\in\mathcal{A}_N} \mathbb{P}_\eta \Big[\eta(T_N) \in \breve{Q}_N \Big] = 1,$$
$$\lim_{N\to\infty} \max_{\eta\in\mathcal{A}_N} \|\delta_\eta S(T_N) - \pi_{\breve{Q}_N}\|_{TV} = 0,$$

where $S(t)$ is the semigroup of the Markov process $\eta(t)$. These estimates are the two ingredients needed, together with monotonicity, in the proof of [11, Proposition 2.9], a result which states, among other things, that there exists an eigenfunction f_N of the generator L_N associated to the eigenvalue $\mathfrak{g}$ such that $E_\pi[f_N] = 0$, $E_\pi[f_N^2] = 1$, $\lim_N \|f_N\|_\infty = 1$. Here and below $\|h\|_\infty$ represents the sup norm of a function h. By this result and by Proposition 2.1, $\lim_N(\mathfrak{g}/\mathfrak{g}_{\mathcal{E}}) = 1$.

8.2 A polymer in the depinned phase [11,12]

Fix $N \geq 1$ and denote by E_N the set of all lattice paths starting at 0 and ending at 0 after $2N$ steps:

$$E_N = \{\eta \in \mathbb{Z}^{2N+1} : \eta_{-N} = \eta_N = 0,\ \eta_{j+1} - \eta_j = \pm 1,\ -N \leq j < N\}.$$

Fix $0 < \alpha < 1$ and consider the dynamics on E_N induced by the generator L_N defined by

$$(L_N f)(\eta) = \sum_{j=-N+1}^{N-1} c_{j,+}(\eta)[f(\eta^{j,+}) - f(\eta)] + \sum_{j=-N+1}^{N-1} c_{j,-}(\eta)[f(\eta^{j,-}) - f(\eta)],$$

for every function $f : E_N \to \mathbb{R}$. In this formula $\eta^{j,\pm}$ represents the configuration which is equal to η at every site $k \neq j$ and which is equal to $\eta_j \pm 2$ at site j. The jump rate $c_{j,+}(\eta)$ vanishes at configurations η which do not satisfy the condition $\eta_{j-1} = \eta_{j+1} = \eta_j + 1$, and it is given by

$$c_{j,+}(\eta) = \begin{cases} 1/2 & \text{if } \eta_{j-1} = \eta_{j+1} \neq \pm 1, \\ 1/[(1+\alpha)] & \text{if } \eta_{j-1} = \eta_{j+1} = 1, \\ \alpha/[(1+\alpha)] & \text{if } \eta_{j-1} = \eta_{j+1} = -1 \end{cases}$$

for configurations which fulfill the condition $\eta_{j-1} = \eta_{j+1} = \eta_j + 1$. Let $-\eta$ stand for the configuration η reflected around the horizontal axis, $(-\eta)_j = -\eta_j$, $-N \le j \le N$. The rates $c_{j,-}(\eta)$ are given by $c_{j,-}(\eta) = c_{j,+}(-\eta)$.

Denote by $\Sigma(\eta)$ the number of zeros in the path η, $\Sigma(\eta) = \sum_{-N \le j \le N} \mathbf{1}\{\eta_j = 0\}$. The probability measure π_N on E_N defined by $\pi_N(\eta) = Z_{2N}^{-1} \alpha^{\Sigma(\eta)}$, where Z_{2N} is a normalizing constant, is easily seen to be reversible for the dynamics generated by L_N.

By [12, Theorem 3.5], the spectral gap $\mathfrak{g}$ is bounded above by $C(\alpha)(\log N)^8/N^{5/2}$ for some finite constant $C(\alpha)$. Following [11], let $\mathcal{E}_N^1$ be the set of configurations in E_N such that $\eta_j > 0$ for all $-(N-\ell) < j < (N-\ell)$, where $\ell = \ell_N$ is a sequence such that $1 \ll \ell_N \ll N$, and let $\mathcal{E}_N^2 = \{\eta \in E_N : -\eta \in \mathcal{E}_N^1\}$, $\Delta_N = E_N \setminus (\mathcal{E}_N^1 \cup \mathcal{E}_N^2)$. By equation (2.27) in [11], $\pi(\mathcal{E}_N^1) = \pi(\mathcal{E}_N^1) = (1/2) + O(\ell^{-1/2})$. Moreover, taking $\ell_N = (\log N)^{1/4}$, by [11, Proposition 2.6], for every $\epsilon > 0$, there exists N_0 such that for all $N \ge N_0$, $\mathfrak{g}_{\mathbf{r},1} = \mathfrak{g}_{\mathbf{r},2} \ge N^{-(2+\epsilon)}$. In conclusion, choosing ϵ small enough and $\ell_N = (\log N)^{1/4}$,

$$\mathfrak{g} \ll \min\left\{\mathfrak{g}_{\mathbf{r},1},\ \mathfrak{g}_{\mathbf{r},2}\right\}$$

for all N large enough, which proves that condition (L1B) is in force.

By [11, Proposition 2.9], there exists an eigenfunction f of the generator L_N such that $E_\pi[f] = 0$, $E_\pi[f^2] = 1$, $L_N f = \mathfrak{g} f$ and $\|f\|_\infty = 1 + o_N(1)$ where $o_N(1)$ represents an expression which vanishes as $N \uparrow \infty$. Therefore, since $\pi(\Delta_N) \to 0$, by Proposition 2.1, $\mathfrak{g}/\mathfrak{g}_{\mathcal{E}}$ converges to 1 as $N \uparrow \infty$.

Let ν_N be a sequence of probability measures concentrated on $\mathcal{E}_N^1$ and satisfying condition (2.16). For example, one may define $\nu_N(\,\cdot\,)$ as $\pi(\,\cdot\,|\,\mathcal{F})$, where $\mathcal{F}$ is a subset of $\mathcal{E}_N^1$ such that $\liminf_{N\to\infty} \pi(\mathcal{F}) \ge c_0$ for some positive constant c_0. Define the trace process $\eta^{\mathcal{E}}(t)$ and the order X_t^N as in Section 2. By Proposition 2.1 and Lemma 2.10, and in view of the previous remarks, the time-rescaled process $\mathbf{X}_t^N = X_{t/\mathfrak{g}}^N$ converges to a Markov process on $\{1, 2\}$ which starts from 1 and jumps from x to $3-x$ at rate 1/2. Moreover, by Lemma 2.3, the time spent by the process $\eta(t)$ on the time scale $\mathfrak{g}^{-1}$ outside the set $\mathcal{E}_N$ is negligible.

The difference between this result, derived from a general statement, and Theorems 1.3 and 1.5 in [11] is that we require in Theorem 2.2 the initial state to be close to the stationary state of the reflected process in one of the wells, while [11] allows the process to start from any state in one of the wells. This strong assumption on the initial condition permits to consider larger wells and to have an explicit description of these wells. To prove tunneling for a process starting from a state, one needs to show that the mixing conditions (L4U) are in force.

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