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On fixed points of a generalized multidimensional affine recursion

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Abstract Let G be a multiplicative subsemigroup of the general linear group $\mathrm{Gl}(\mathbb{R}^d)$ which consists of matrices with positive entries such that every column and every row contains a strictly positive element. Given a G-valued random matrix A, we consider the following generalized multidimensional affine equation

$$R \stackrel{\mathcal{D}}{=} \sum_{i=1}^{N} A_i R_i + B,$$

where $N \geq 2$ is a fixed natural number, $A_1, \ldots, A_N$ are independent copies of A, $B \in \mathbb{R}^d$ is a random vector with positive entries, and $R_1, \ldots, R_N$ are independent copies of $R \in \mathbb{R}^d$, which have also positive entries. Moreover, all of them are mutually independent and $\stackrel{\mathcal{D}}{=}$ stands for the equality in distribution. We will show with the aid of spectral theory developed by Guivarc'h and Le Page (Simplicité de spectres de Lyapounov et propriété d'isolation spectrale pour une famille d'opérateurs de transfert sur l'espace projectif. Random Walks and Geometry, Walter de Gruyter GmbH & Co. KG, Berlin, 2004; On matricial renewal theorems and tails of stationary measures for affine stochastic recursions, Preprint, 2011) and Kesten's renewal theorem (Kesten in Ann Probab 2:355–386, 1974), that under appropriate conditions, there exists $\chi > 0$ such that $\mathbb{P}(\{\langle R, u\rangle > t\}) \asymp t^{-\chi}$, as $t \to \infty$, for every unit vector $u \in \mathbb{S}^{d-1}$ with positive entries.

Keywords Heavy tailed random variables · Renewal theory · Stationary measures · Markov chains · Spectral theory

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1 Introduction and statement of the results

We consider the Euclidean space $\mathbb{R}^d$ endowed with the scalar product $\langle x, y\rangle = \sum_{i=1}^d x_i y_i$, the norm $|x| = \sqrt{\langle x, x\rangle}$, and its Borel σ-field $\mathcal{B}or(\mathbb{R}^d)$. We say that $\mathbb{R}^d \ni x = (x_1, \ldots, x_d) \geq 0$ is positive (resp. $\mathbb{R}^d \ni x = (x_1, \ldots, x_d) > 0$ is strictly positive) when $x_n \geq 0$, (resp. $x_n > 0$) for every $1 \leq n \leq d$. By $\mathbb{R}^d_+$ we denote the set of all positive vectors and we define the set $\mathbb{S}^+ = \mathbb{R}^d_+ \cap \mathbb{S}^{d-1}$ of all positive vectors on the unit sphere $\mathbb{S}^{d-1} = \{x \in \mathbb{R}^d : |x| = 1\}$ with the distance being the restriction of the Euclidean norm to $\mathbb{S}^+$. Given $x \in \mathbb{R}^d$ we denote its projection on $\mathbb{S}^{d-1}$ by $\overline{x} = \frac{x}{|x|}$.

Let $\mathrm{Gl}(\mathbb{R}^d)$ be the group of $d \times d$ invertible matrices on $\mathbb{R}^d$ with the operator norm $\|\cdot\|$ associated with the Euclidean norm $|\cdot|$ on $\mathbb{R}^d$, i.e. $\|a\| = \sup_{x \in \mathbb{S}^{d-1}} |ax|$ for every $a = (a(i, j))_{1 \leq i, j \leq d} \in \mathrm{Gl}(\mathbb{R}^d)$.

Suppose that G is a multiplicative subsemigroup of $\mathrm{Gl}(\mathbb{R}^d)$ which consists of matrices with positive entries such that every column and every row contains a strictly positive element. By G° we denote the multiplicative subsemigroup of G composed of matrices with strictly positive entries. It is easy to see that G provides a projective action on $\mathbb{S}^+$ which is given by

$$G \times \mathbb{S}^+ \ni (a, x) \mapsto a \cdot x = \frac{ax}{|ax|} \in \mathbb{S}^+.$$

Let A be a G-valued random matrix distributed according to a probability measure μ on G, and B be a random vector independent of A, taking its values in $\mathbb{R}^d_+$.

Let $A_1, \ldots, A_N$ and B_0 be independent random variables, where $N \geq 2$ is a fixed natural number, $A_1, \ldots, A_N$ are independent copies of A, and B_0 is an independent copy of B.

The aim of this paper is to find a random vector $R \in \mathbb{R}^d_+$, independent of A and B, which solves (in law $\overset{\mathcal{D}}{=}$) a generalized multidimensional affine equation, i.e.

$$R \overset{\mathcal{D}}{=} \sum_{i=1}^N A_i R_i + B_0, \tag{1.1}$$

where $R_1, \ldots, R_N$ are independent copies of $R \in \mathbb{R}^d_+$ and independent of $A, A_1, \ldots, A_N, B, B_0$, (see Theorem 1.7 stated below).

Furthermore, we would like to find possibly mild conditions, which allows us to establish an asymptotic tail formula for R. More precisely, we are interested in the existence $\chi > 0$, such that

$$\mathbb{P}(\{\langle R, u\rangle > t\}) \asymp t^{-\chi}, \quad \text{as} \quad t \to \infty, \tag{1.2}$$

for every $u \in \mathbb{S}^+$ (see Theorem 1.9 stated below).

The one dimensional version of Eq. (1.1) has been considered recently by Jelenković and Olvera-Cravioto [13–15] in the context of Google's PageRank algorithm. The authors solved Eq. (1.1) and justified formula (1.2) using the renewal theorem. It is worth emphasizing that the one dimensional version of Eq. (1.1) with $B = 0$, was studied by Liu in a series of articles (see for instance [18] and the references given there).

We are also motivated by the recent results of Buraczewski et al. [3], where the authors considered the multidimensional version of Eq. (1.1) with $B = 0$, and established formula (1.2) with the help of Kesten's renewal theorem [17] and the spectral method developed by Guivarc'h and Le Page [6,7]. Their approach sheds some new light on multidimensional problems and fits perfectly to our situation.

In order to avoid repetitions in the sequel, and shorten article we have decided to state all necessary definitions and notations in the introduction, and formulate our main results as general as it is possible.

Let $M^1(G)$ denotes the set of all probability measures on G endowed with the weak topology. We denote by suppμ the support of the measure $\mu \in M^1(G)$. If $E \subseteq G$, let $[E]$ be the subsemigroup of G generated by the set E. For $n \in \mathbb{N}$ let $S_n = A_n \cdot \dots \cdot A_1 \in G$, where $A_1, A_2, \ldots \in G$ is a sequence of independent copies of G-valued random matrix A distributed according to μ.

A subsemigroup [suppμ] of G is called *contractive* if $[\mathrm{supp}\mu] \cap G^\circ \neq \emptyset$. In other words,

$$\mathbb{P}\left(\bigcup_{n\in\mathbb{N}} \{S_n \in G^\circ\} \right) > 0. \tag{1.3}$$

The condition (1.3) was considered by Hennion [10], Hennion and Hervé [11] in the context of limit theorems for the products of positive random matrices.

An element $a \in \mathrm{Gl}(\mathbb{R}^d)$ is *proximal* if there exists a unique eigenvalue λ_a (the dominant eigenvalue) of a, such that $r(a) = \lim_{n\to\infty} \|a^n\|^{1/n} = |\lambda_a|$.

According to the Perron–Frobenius theorem [12] every $a \in G^\circ$ is proximal. Moreover, for every $a \in G^\circ$ and its adjoint $a^* \in G^\circ$ it is possible to choose $v_a, w_a \in \mathbb{R}^d_+$ such that $v_a > 0$, $w_a > 0$ and

$$av_a = \lambda_a v_a, \quad a^* w_a = \lambda_a w_a, \quad \langle v_a, w_a \rangle = 1, \quad |w_a| = 1.$$

The eigenvector v_a determined by these relations will be called the dominant eigenvector of $a \in G^\circ$. This means that we can write $\mathbb{R}^d = \mathbb{R} \cdot v_a \oplus v_a^\perp$, and the spectral radius of a restricted to $v_a^\perp = \{x \in \mathbb{R}^d : \langle x, v_a \rangle = 0\}$ is strictly less than $|\lambda_a|$. Furthermore, by the preceding relations we have

$$\lim_{n\to\infty} \frac{a^n}{r(a)^n} = v_a \otimes w_a, \tag{1.4}$$

where $v_a \otimes w_a$ is the matrix projector on $\mathbb{R} \cdot v_a$. Since $v_a \otimes w_a x = \langle x, w_a \rangle v_a$ for every $x \in \mathbb{R}^d$, (1.4) immediately yields

$$\lim_{n\to\infty} a^n \cdot x = \frac{v_a \otimes w_a x}{|v_a \otimes w_a x|} = \frac{v_a}{|v_a|} = \overline{v}_a \in \mathbb{S}^+, \quad \text{for every } x \in \mathbb{R}^d_+. \tag{1.5}$$

A subsemigroup $\Gamma \subseteq \mathrm{Gl}(\mathbb{R}^d)$ is *strongly irreducible* if there does not exist a finite number ($k \in \mathbb{N}$) of proper linear subspaces $V_1, \dots, V_k$ of $\mathbb{R}^d$ such that

$$\Gamma\left(\bigcup_{i=1}^{k} V_i\right) \subseteq \bigcup_{i=1}^{k} V_i. \tag{1.6}$$

If $E \subseteq \mathrm{Gl}(\mathbb{R}^d)$ we denote by E^{prox} the set of all proximal elements of E. A subsemigroup $\Gamma \subseteq \mathrm{Gl}(\mathbb{R}^d)$ is said to satisfy condition $(i-p)$ if Γ is strongly irreducible and $\Gamma^{\mathrm{prox}} \neq \emptyset$. This condition was widely investigated by Guivarc'h and Le Page [6,7], see also [3,8,9] and the references given there.

A subsemigroup $[\mathrm{supp}\mu] \subseteq G$, where $\mu \in M^1(G)$, is said to satisfy *condition* $(\mathcal{C})$ if $[\mathrm{supp}\mu]$ is contractive and strongly irreducible. Clearly, condition $(\mathcal{C})$ implies condition $(i-p)$ with $\Gamma = [\mathrm{supp}\mu]$.

For $s \geq 0$ we write

$$\kappa(s) = \kappa_\mu(s) = \lim_{n\to\infty}\left(\int_G \|a\|^s \mu^{*n}(da)\right)^{\frac{1}{n}},$$

where μ^{*n} is the nth convolution power of $\mu \in M^1(G)$. The limit above exists and it is equal to $\inf_{n\in\mathbb{N}}\left(\int_G \|a\|^s \mu^{*n}(da)\right)^{\frac{1}{n}}$, because $u_n(s) = \int_G \|a\|^s \mu^{*n}(da)$ is submultiplicative, i.e. $u_{m+n}(s) \leq u_m(s)u_n(s)$ for every $m, n \in \mathbb{N}$. Moreover,

$$I_\mu = \left\{s \in [0,\infty) : \kappa_\mu(s) < \infty\right\} = \left\{s \in [0,\infty) : \int_G \|a\|^s \mu(da) < \infty\right\}.$$

Let $s_\infty = \sup\left\{s \geq 0 : \kappa_\mu(s) < \infty\right\} \in \mathbb{R}_+ \cup \{\infty\}$, then by the Hölder inequality $I_\mu = [0, s_\infty)$ or $I_\mu = [0, s_\infty]$.

Our "existence" result is the following

Theorem 1.7 *Assume that A is a G-valued random matrix distributed according to a probability measure μ on G, and B is a random vector independent of A, taking its values in $\mathbb{R}^d_+$, such that $\mathbb{P}(\{B > 0\}) > 0$. Let $A_1, \dots, A_N$ and B_0 be independent random variables as in (1.1), where $N \geq 2$ is a fixed natural number, $A_1, \dots, A_N$ are independent copies of A, and B_0 is an independent copy of B. Suppose further that $[\mathrm{supp}\mu] \subseteq G$ satisfies condition $(\mathcal{C})$ and there exist $s_1 \in (0, 1/2]$, and $s_2 > s_1$ such that $\mathbb{E}(\|A\|^{s_1}) \leq \frac{1}{N}$, $\mathbb{E}(\|A\|^{s_2}) \leq \frac{1}{N}$, and $\mathbb{E}(|B|^{s_2}) < \infty$. Then there exists a unique vector $R \in \mathbb{R}^d_+$ and its independent copies $R_1, \dots, R_N$ independent of $A, A_1, \dots, A_N, B, B_0$ which solve (1.1) in law. Moreover, $\mathbb{E}(|R|^s) < \infty$ for every $s < s_2$.*

Remark 1.8 The uniqueness of the solution of (1.1) will be explained in details in Sect. 3—see the discussion after Lemma 3.8.

Section 3 contains a detailed proof of Theorem 1.7, which is similar in spirit to that of [13]. However, the multidimensional framework, we consider, provides some difficulties which do not appear in the one dimensional case. Namely, the method developed in [13], which gives finiteness of appropriate moments for the solution of (1.1), breaks down in higher dimensions. This problem will be dealt with the help of condition $(\mathcal{C})$ and additionally for technical reasons we have to assume that there is $s_1 \leq \frac{1}{2}$ such that $\mathbb{E}(\|A\|^{s_1}) \leq \frac{1}{N}$.

The last condition allows us to give elementary proof of Theorem 1.7, which follows the ideas introduced in [13]. If we did not assume that $s_1 \leq \frac{1}{2}$, it would generate many obstacles difficult to surmount. In particular straightforward proof of Lemma 3.12 which we propose in Sect. 3 might not work at all. On the one hand, the existence of $s_1 \leq \frac{1}{2}$ such that $\mathbb{E}(\|A\|^{s_1}) \leq \frac{1}{N}$ can be relaxed in the one dimensional case (see [13–15]). On the other hand, it can be also relaxed in the multidimensional settings, but this requires more sophisticated techniques giving the existence of the solution of (1.1). This approach will be discussed in the forthcoming article of Buraczewski et al. [4].

Let λ_d be the Lebesgue measure on $\mathbb{R}^d$. If ν is a probability measure on $\mathbb{R}^d$, then by $\nu = \nu_a + \nu_s$ we denote its Lebesgue decomposition with respect to λ_d where ν_a is the absolutely continuous part with respect to λ_d, i.e. $\nu_a \ll \lambda_d$, and ν_s is the singular part with respect to λ_d, i.e. $\nu_s \perp \lambda_d$. We have also $\nu_a \perp \nu_s$. Since ν is positive then its total variation $\|\nu\| = \nu(\mathbb{R}^d) = 1$. We say that the measure ν is singular if $\|\nu_s\| = 1$, otherwise ν is nonsingular, i.e. $\|\nu_s\| < 1$.

Now we can state our main "tail" result.

Theorem 1.9 *Fix a natural number $N \geq 2$, a G-valued random matrix A distributed according to μ, and a random vector B with law η, independent of A, taking its values in $\mathbb{R}^d_+$, such that $\mathbb{P}(\{B > 0\}) > 0$.*

- *Assume that $[\mathrm{supp}\mu] \subseteq G$ satisfies condition $(\mathcal{C})$, and there is $s_1 \in (0, 1/2]$, such that $\mathbb{E}(\|A\|^{s_1}) < \frac{1}{N}$. Moreover, we assume $s_\infty > s_1$ and $\lim_{s\to s_\infty} \kappa(s) > \frac{1}{N}$. Then there exists $\chi \geq s_1$ such that $N\kappa(\chi) = 1$.*
- *Furthermore, if $\mathbb{E}(\|A\|^\chi \log^+ \|A\|) < \infty$, $E(|B|^{\chi+\varepsilon}) < \infty$ for some $\varepsilon > 0$, and either*
  - (i) *η is nonsingular, i.e. $\|\eta_s\| < 1$, or*
  - (ii) *η is singular, i.e. $\|\eta_s\| = 1$, and $\mathbb{P}(\{\langle B, u\rangle = r\}) = 0$ for every $(u, r) \in \mathbb{S}^+ \times \mathbb{R}_+$.*

Then there exists a positive function $e_*^\chi : \mathbb{S}^+ \mapsto (0, \infty)$ and a constant $C_\chi \geq 0$ such that

$$\lim_{t\to\infty} t^\chi \mathbb{P}(\{\langle R, u\rangle > t\}) = C_\chi e_*^\chi(u) \geq 0, \tag{1.10}$$

for every $u \in \mathbb{S}^+$, where $R \in \mathbb{R}^d_+$ is the stationary solution of Eq. (1.1) as in Theorem 1.7. Moreover, if $\chi \geq 1$ then $C_\chi > 0$, and the limit in (1.10) is strictly positive.

Now we give an example of singular measure η, i.e. $\|\eta_s\| = 1$, on the plane ($d = 2$), such that $\eta(\{x \in \mathbb{R}^2 : \langle x, u\rangle = r\}) = 0$ for every $(u, r) \in \mathbb{S}^+ \times \mathbb{R}_+$, and $\eta(\{x \in$

$\mathbb{R}^2 : x > 0\}) > 0$. Define $S = \{(\cos\alpha, \sin\alpha) : 0 < \alpha < \pi/2\} \subseteq \mathbb{S}^+$ and let η be the normalized one dimensional Lebesgue measure on S, i.e. supp$\eta = \mathbb{S}^+$ and $\eta(S) = 1$. It is not hard to see that η is singular with respect to two dimensional Lebesgue measure λ_2. Obviously $\eta(\{x \in \mathbb{R}^2 : x > 0\}) = \eta(S) = 1$, and notice that $\{x \in \mathbb{R}^2 : \langle x, u\rangle = r\}$ intersects S at most two points, hence finally $\eta(\{x \in \mathbb{R}^2 : \langle x, u\rangle = r\}) = 0$.

As we mentioned before, the proof is based on concepts of [3] with considerable complications determined by the structure of Eq. (1.1). The most important tool which allows us to establish relation (1.10) is Kesten's renewal theorem [17]. We need to check that its assumptions are satisfied (see Sect. 4). This is the most difficult part of the paper and requires the spectral theory of transfer operators developed by Guivarc'h and Le Page [3,6,7], which is summarized in Sect. 2. But we touch only a few aspects of their theory and restrict our attention to the results which will be used in Sects. 3 and 4. Guivarc'h and Le Page approach significantly simplifies and clarifies proofs developed by Kesten [16], and what is most important for us, it is applicable to our situation.

The positivity of the limit constant $C_\chi > 0$ in (1.10) if $\chi \geq 1$, is a very delicate issue. This relies strongly on the positivity of matrices and the fact that $\chi \geq 1$. In the case when $\chi < 1$ the positivity of $C_\chi > 0$ seems to be a very difficult problem and is unavailable in our situation at the moment. However, in the one dimensional case and the case of group of similarities (instead of group G), a very careful study (requiring complex analysis methods) of the formula defining the limit constant $C_\chi \geq 0$ allows us to conclude that the constant C_χ is nonzero. A detailed exposition of these, and related problems, are discussed in [4].

Remark 1.11 We would like to emphasize that there are some possible extensions of Theorems 1.7 and 1.9 which relax the assumption that $N \geq 2$ is constant and allows us to consider an integer-valued random variable $N \geq 2$ with appropriate moment conditions (see [3,13–15,18]). But this is not the main issue of this paper and therefore, for simplicity, we decided to assume that $N \geq 2$ is constant.

2 Transfer operators

Let $\mathcal{C}(\mathbb{S}^+)$ be the space of continuous functions on $\mathbb{S}^+$ with the supremum norm $|\cdot|_\infty$. $\mathcal{H}_\varepsilon = \{\phi \in \mathcal{C}(\mathbb{S}^+) : \|\phi\|_\varepsilon = |\phi|_\infty + [\phi]_\varepsilon < \infty\}$, $\varepsilon \in (0, 1]$ is the space of all ε-Hölder functions on $\mathbb{S}^+$ with

$$[\phi]_\varepsilon = \sup_{x \neq y} \frac{|\phi(x) - \phi(y)|}{|x - y|^\varepsilon}.$$

Given a closed subset V of $\mathbb{S}^+$, $M^1(V)$ denotes the set of all probability measures on V, endowed with the weak topology. We say that $U \subseteq \mathbb{S}^+$ is a subspace of $\mathbb{S}^+$, if $U = V \cap \mathbb{S}^+$ for some subspace $V \subseteq \mathbb{R}^d$. A measure $\nu \in M^1(\mathbb{S}^+)$ is said to be proper if $\nu(U) = 0$ for every subspace $U \subsetneq \mathbb{S}^+$. Here and subsequently, $\Lambda(\Gamma) = \{\overline{v}_a \in \mathbb{S}^+ : v_a$ is the dominant eigenvector of $a \in \Gamma^{\text{prox}}\}$, where Γ is a subsemigroup of G such that $\Gamma^{\text{prox}} \neq \emptyset$.

The following Proposition 2.1 due to Guivarc'h and Raugi [8] (see also [9]) contains the relevant properties of $(i-p)$ semigroups which will be used in the sequel.

Proposition 2.1 *Let $\mu \in M^1(G)$ and $\Gamma = [supp\mu]$ satisfies condition $(i-p)$. Then there exists a unique proper μ-stationary measure $\nu \in M^1(\mathbb{S}^+)$ such that $supp\nu = \Lambda(\Gamma)$. Furthermore, $\Lambda(\Gamma)$ is the unique Γ-minimal subset of $\mathbb{S}^+$ (i.e. if $Z \subseteq \mathbb{S}^+$ is closed and $\Gamma \cdot Z \subseteq Z$, then $\Lambda(\Gamma) \subseteq Z$), and the subgroup of $\mathbb{R}^*_+$ generated by the set $\{|\lambda_a| : a \in \Gamma^{\text{prox}}\}$ is dense in $\mathbb{R}^*_+$.*

Let $\mu \in M^1(G)$. For $s \in I_\mu, x \in \mathbb{S}^+$ and a measurable function ϕ on $\mathbb{S}^+$ we consider the following transfer operators

$$\begin{aligned} P^s\phi(x) &= \int_G |ax|^s \phi(a\cdot x)\mu(da), \\ P^s_*\phi(x) &= \int_G |a^*x|^s \phi(a^*\cdot x)\mu(da) = \int_G |ax|^s \phi(a\cdot x)\mu_*(da), \end{aligned} \tag{2.2}$$

where $\mu_* \in M^1(G)$ and $\mu_*(U) = \mu(\{a \in G : a^* \in U\})$ for every $U \in \mathcal{B}or(G)$.

The main purpose of this section is to summarize a number of properties of operators P^s, P^s_*, see Theorem 2.3 below.

Theorem 2.3 *Assume that $\mu \in M^1(G)$, $s \in I_\mu$ and $\Gamma = [supp\mu]$ satisfies condition $(i-p)$. Then*

- *there exists a unique probability measure $\nu^s \in M^1(\mathbb{S}^+)$, $(\nu^s_* \in M^1(\mathbb{S}^+))$ such that*
  - (i) $P^s\nu^s = \kappa(s)\nu^s$, $(P^s_*\nu^s_* = \kappa(s)\nu^s_*)$.
  - (ii) $supp\nu^s = \Lambda([supp\mu])$, $(supp\nu^s_* = \Lambda([supp\mu_*]))$ *and it is not contained in any proper subspace of* $\mathbb{S}^+$.
  - (iii) $I_\mu \ni s \mapsto \nu^s \in M^1(\mathbb{S}^+)$, $(I_\mu \ni s \mapsto \nu^s_* \in M^1(\mathbb{S}^+))$ *is continuous in the weak topology.*
- $I_\mu \ni s \mapsto \kappa(s)$ *is strictly* log-*convex function.*
- *there exists a unique $\underline{s}$-Hölder continuous function $e^s : \mathbb{S}^+ \mapsto (0,\infty)$, $(e^s_* : \mathbb{S}^+ \mapsto (0,\infty))$ with $\underline{s} = \min\{s,1\}$ such that*
  - (i) $P^s e^s = \kappa(s)e^s$, $(P^s_* e^s_* = \kappa(s)e^s_*)$.
  - (ii) e^s, (e^s_*) *is given by the formula*

$$e^s(x) = \int_{\mathbb{S}^+} \langle x, y\rangle^s\, \nu^s_*(dy), \quad \left(e^s_*(x) = \int_{\mathbb{S}^+} \langle x, y\rangle^s\, \nu^s(dy)\right), \quad \textit{for} \quad x \in \mathbb{S}^+.$$

  - (iii) $I_\mu \ni s \mapsto e^s \in \mathcal{C}(\mathbb{S}^+)$, $(I_\mu \ni s \mapsto e^s_* \in \mathcal{C}(\mathbb{S}^+))$ *is continuous in the uniform topology.*
- *Moreover, there exists a unique stationary measure $\pi^s \in M^1(\mathbb{S}^+)$, $(\pi^s_* \in M^1(\mathbb{S}^+))$ for operator $Q^s f = \frac{P^s(e^s f)}{\kappa(s)e^s}$, $\left(Q^s_* f = \frac{P^s_*(e^s_* f)}{\kappa(s)e^s_*}\right)$ where $f \in \mathcal{C}(\mathbb{S}^+)$, such that*
  - (i) $\pi^s = \frac{e^s\nu^s}{\nu^s(e^s)}$, $\left(\pi^s_* = \frac{e^s_*\nu^s_*}{\nu^s_*(e^s_*)}\right)$.

(ii) $(Q^s)^n f$, $((Q^s_*)^n f)$ *converges uniformly to* $\pi^s(f)$, $(\pi^s_*(f))$ *for any* $f \in \mathcal{C}(\mathbb{S}^+)$.

(iii) $supp\pi^s = \Lambda([supp\mu])$, $(supp\pi^s_* = \Lambda([supp\mu_*]))$.

This result was proved by Guivarc'h and Le Page and its, quite long and far from being obvious, proof can be found in [6,7]. Notice that in view of the cocycle property $\sigma^s(x, a_2a_1) = \sigma^s(x, a_1)\sigma^s(a_1 \cdot x, a_2)$, $(\sigma^s_*(x, a_2a_1) = \sigma^s_*(x, a_1)\sigma^s_*(a_1 \cdot x, a_2))$, a_1, $a_2 \in G$, $x \in \mathbb{S}^+$ of

$$\sigma^s(x, a) = |ax|^s \frac{e^s(a \cdot x)}{e^s(x)}, \quad \left(\sigma^s_*(x, a) = |ax|^s \frac{e^s_*(a \cdot x)}{e^s_*(x)}\right), \tag{2.4}$$

the Markov operators Q^s and Q^s_* defined in Theorem 2.3 can be rewritten in the following form

$$(Q^s)^n\phi(x) = \int_G \phi(a \cdot x) q^s_n(x, a)\mu^{*n}(da), \tag{2.5}$$

$$(Q^s_*)^n\phi(x) = \int_G \phi(a \cdot x) q^{s,*}_n(x, a)\mu^{*n}_*(da), \tag{2.6}$$

where

$$\begin{aligned} q^s_n(x, a) &= \frac{1}{\kappa^n(s)} \frac{e^s(a \cdot x)}{e^s(x)} |ax|^s = \frac{\sigma^s(x, a)}{\kappa^n(s)}, \\ q^{s,*}_n(x, a) &= \frac{1}{\kappa^n(s)} \frac{e^s_*(a \cdot x)}{e^s_*(x)} |ax|^s = \frac{\sigma^s_*(x, a)}{\kappa^n(s)}, \end{aligned} \tag{2.7}$$

$n \in \mathbb{N}$, $x \in \mathbb{S}^+$, $a \in G$ and ϕ is an arbitrary measurable function on $\mathbb{S}^+$.

3 Construction of the solution

Recall that A stands for a G-valued random matrix distributed according to the measure $\mu \in M^1(G)$, and B for a random vector taking its values in $\mathbb{R}^d_+$, independent of A. In this section we construct a solution of Eq. (1.1). The idea of the construction goes back to [13]. It is not difficult to imagine that we have to study a sequence of random variables that are obtained by iterating (1.1). Let $N \geq 2$ be a fixed natural number and $R^*_{0,1}, \dots, R^*_{0,N}$ be independent and identically distributed (i.i.d.) copies of the initial random variable $R^*_0 \in \mathbb{R}^d_+$. Throughout the paper we will assume that $\mathbb{E}(|R^*_0|^{s_2}) < \infty$, for $s_2 > 0$ as in Theorem 1.7. We consider the sequence $(R^*_n)_{n\geq 0}$ such that

$$R^*_{n+1} = \sum_{k=1}^{N} A_{n+1,k} R^*_{n,k} + B_{n+1}, \quad \text{for every } n \geq 0, \tag{3.1}$$

where $A_{n+1,1},\dots,A_{n+1,N}$, B_{n+1} and $R^*_{n,1},\dots,R^*_{n,N}$, $n\ge 0$ are independent. Moreover, for $n\ge 1$ $R^*_{n,1},\dots,R^*_{n,N}$ are i.i.d. copies of R^*_n obtained at the previous iteration. For $n\ge 0$ $A_{n+1,1},\dots,A_{n+1,N}$ are i.i.d. copies of A and B_{n+1} is an independent copy of B.

We will look more closely at the sequence $(R^*_n)_{n\ge 0}$. Let $\mathcal{A}=\{A_{i_1,\dots,i_n}:(i_1,\dots,i_n)\in\{1,\dots,N\}^n,\ n\in\mathbb{N}\}$ be the set consisting of i.i.d. copies of A, and $\mathcal{B}=\{B_{i_1,\dots,i_n}:(i_1,\dots,i_n)\in\{1,\dots,N\}^n,\ n\in\mathbb{N}\}\cup\{B_0\}$ the set consisting of i.i.d. copies of B independent of $\mathcal{A}$. Additionally we assume that $A_0=\mathrm{Id}$ a.s. and the initial random variable R^*_0 is always independent of A, B, $\mathcal{A}$ and $\mathcal{B}$.

Now let $W_0=A_0B_0=B_0$ a.s.,

$$W_n=\sum_{(i_1,\dots,i_n)\in\{1,\dots,N\}^n} A_{i_1}A_{i_1,i_2}\cdot\dots\cdot A_{i_1,\dots,i_n}B_{i_1,\dots,i_n},\quad n\ge 1,\tag{3.2}$$

and for $n\ge 0$

$$R^{(n)}=\sum_{i=0}^{n}W_i,\tag{3.3}$$

be the partial sum of the sequence $(W_n)_{n\ge 0}$. Since $R^{(n+1)}-R^{(n)}\ge 0$ is a positive vector for every $n\in\mathbb{N}$ then

$$R=\lim_{n\to\infty}R^{(n)}=\sum_{i=0}^{\infty}W_i,\tag{3.4}$$

exists a.s. and is a candidate for a solution of (1.1). Indeed, it is not hard to see that W_n satisfies

$$\begin{aligned}W_n&=\sum_{(i_1,\dots,i_n)\in\{1,\dots,N\}^n} A_{i_1}A_{i_1,i_2}\cdot\dots\cdot A_{i_1,\dots,i_n}B_{i_1,\dots,i_n}\\&=\sum_{k=1}^{N}A_k\left(\sum_{(k,i_2,\dots,i_n)\in\{1,\dots,N\}^n}A_{k,i_2}\cdot\dots\cdot A_{k,i_2,\dots,i_n}B_{k,i_2,\dots,i_n}\right)=\sum_{k=1}^{N}A_kW_{n-1,k},\end{aligned}\tag{3.5}$$

where A_k and $W_{n-1,k}$ are independent of each other and $W_{n-1,1},\dots,W_{n-1,N}$ have the same distribution as W_{n-1}. In view of the above calculations, $R^{(n)}$ satisfies the recursion

$$R^{(n)}=\sum_{k=1}^{N}A_kR_k^{(n-1)}+B_0,\tag{3.6}$$

for every $n \in \mathbb{N}$, where $R_1^{(n-1)}, \ldots, R_N^{(n-1)}$ are independent copies of $R^{(n-1)}$. This allows us to conclude that R is a solution of (1.1) in law provided that R is finite a.s., but this will be shown in the proof of Theorem 1.7 below.

To obtain a solution with an initial condition, let $R^*_{0,(i_1,\ldots,i_n)}$, $(i_1,\ldots,i_n) \in \{1,\ldots,N\}^n$, $n \in \mathbb{N}$, be i.i.d. copies of the initial random variable $R_0^* \in \mathbb{R}_+^d$ independent of the families $\mathcal{A}$ and $\mathcal{B}$. For $n \geq 1$, we define similarly as in (3.2)

$$W_n(R_0^*) = \sum_{(i_1,\ldots,i_n)\in\{1,\ldots,N\}^n} A_{i_1} A_{i_1,i_2} \cdot \cdots \cdot A_{i_1,\ldots,i_n} R^*_{0,(i_1,\ldots,i_n)}. \tag{3.7}$$

Moreover, as in (3.5), we obtain $W_n(R_0^*) \stackrel{\mathcal{D}}{=} \sum_{k=1}^N A_k W_{n-1,k}(R_0^*)$, where A_k and $W_{n-1,k}(R_0^*)$ are independent of each other and $W_{n-1,1}(R_0^*), \ldots, W_{n-1,N}(R_0^*)$ have the same distribution as $W_{n-1}(R_0^*)$. Now we have the following

Lemma 3.8 *Assume now that $(R_n^*)_{n\geq 0}$ and $(R^{(n)})_{n\geq 0}$ are the sequences defined in (3.1) and (3.3), respectively, then for every $n \in \mathbb{N}$ we have*

$$R_n^* \stackrel{\mathcal{D}}{=} R^{(n-1)} + W_n(R_0^*). \tag{3.9}$$

Proof Observe that for $n = 1$, (3.9) follows from definition. For more details we refer to [13]. □

In view of formula (3.9) we will be able to show (in the proof of Theorem 1.7 below) that every sequence $(R_n^*)_{n\geq 0}$ obtained from the iterations described at the beginning of Sect. 3 (see (3.1)) converges in law to the random variable R defined in (3.4), provided that $\mathbb{E}(|R_0^*|^{s_2}) < \infty$. The uniqueness of the solution of (1.1) will be understood exactly in the sense described above. Therefore, one may think that the solution of (1.1) does not depend on the choice of the initial random variable R_0^*.

Now we have the simple, but very useful

Lemma 3.10 *Under the assumptions of Theorem 2.3 there exists $c_s > 0$ such that for every $n \in \mathbb{N}$ we have*

$$c_s \int_G \|a\|^s \mu^n(da) \leq \kappa^n(s) \leq \int_G \|a\|^s \mu^n(da). \tag{3.11}$$

Proof We refer to [6]. □

To take the limit in (3.4) we need an estimate for $\mathbb{E}(|W_n|^s)$. Suppose for a moment that $s \le 1$. Then, in view of inequality (3.11), we have

$$\begin{aligned}\mathbb{E}\left(|W_n|^s\right) &\le \mathbb{E}\left(\sum_{(i_1,\dots,i_n)\in\{1,\dots,N\}^n} \|A_{i_1}A_{i_1,i_2}\cdot\dots\cdot A_{i_1,\dots,i_n}\|^s |B_{i_1,\dots,i_n}|^s\right)\\ &\le \sum_{(i_1,\dots,i_n)\in\{1,\dots,N\}^n} \mathbb{E}\left(\|A_{i_1}A_{i_1,i_2}\cdot\dots\cdot A_{i_1,\dots,i_n}\|^s\right)\mathbb{E}\left(|B|^s\right)\\ &= N^n\int_G \|a\|^s\mu^{*n}(da)\mathbb{E}\left(|B|^s\right) \le \frac{1}{c_s}\mathbb{E}\left(|B|^s\right)N^n\kappa^n(s).\end{aligned}$$

We would like to show that for an appropriate $s > 0$, not necessarily less or equal 1, the quantity $\mathbb{E}(|W_n|^s)$ decays exponentially. This is contained in Lemma 3.12. For the sake of computations we have to assume that there exists $s_1 \in (0, 1/2]$ such that $\mathbb{E}(\|A\|^{s_1}) \le \frac{1}{N}$.

Lemma 3.12 *Assume that* $[\mathrm{supp}\mu] \subseteq G$ *satisfies condition* $(\mathcal{C})$*, and there exist* $s_1 \in (0, 1/2]$*, and* $s_2 > 1$ *such that* $\mathbb{E}(\|A\|^{s_1}) \le \frac{1}{N}$, $\mathbb{E}(\|A\|^{s_2}) \le \frac{1}{N}$*, and* $\mathbb{E}(|B|^{s_2}) < \infty$*. Then for every* $s \in (s_1, s_2)$*, there exist finite constants* $K_s > 0$ *and* $\eta < 1$ *such that for every* $n \in \mathbb{N}$

$$\mathbb{E}\left(|W_n|^s\right) \le K_s\eta^n. \tag{3.13}$$

Proof By Theorem 2.3 $\kappa(s)$ is strictly log-convex so $N\kappa(s) < 1$, for every $s \in (s_1, s_2)$ and for $s \le 1$, (3.13) follows from the calculation above. From now we assume that $s \in (1, s_2)$ and it is fixed. Let $S_{i_1,\dots,i_n} = A_{i_1}A_{i_1,i_2}\cdot\dots\cdot A_{i_1,\dots,i_n}$ for $(i_1,\dots,i_n) \in \{1,\dots,N\}^n$ and $n \in \mathbb{N}$. We order the set of indices writing $\{1,\dots,N\}^n = \{\mathbf{i}_1,\dots,\mathbf{i}_{N^n}\}$ and we choose $p \in \mathbb{N}$ and $p \ge 2$, such that $p-1 < s \le p$. Then $s_1 \le 1/2 < s/p \le 1$ (here is the first time where we have used that $s_1 \le \frac{1}{2}$, this allows us to make the specific choice of p, which in turn guarantees that $ks/p \in (s_1, s_2)$ for every $k \in \{1, 2, \dots, p\}$ and the inequalities $N\kappa(ks/p) < 1$, hold for every $k \in \{1, 2, \dots, p\}$). Moreover

$$\begin{aligned}&\mathbb{E}\left(|W_n|^s\right)\\ &\le \mathbb{E}\left(\left(\sum_{(i_1,\dots,i_n)\in\{1,\dots,N\}^n} |S_{i_1,\dots,i_n}B_{i_1,\dots,i_n}|^{s/p}\right)^p\right)\\ &= \mathbb{E}\left(\sum_{j_{\mathbf{i}_1}+\dots+j_{\mathbf{i}_{N^n}}=p}\binom{p}{j_{\mathbf{i}_1},\dots,j_{\mathbf{i}_{N^n}}}|S_{\mathbf{i}_1}B_{\mathbf{i}_1}|^{sj_{\mathbf{i}_1}/p}\cdot\dots\cdot|S_{\mathbf{i}_{N^n}}B_{\mathbf{i}_{N^n}}|^{sj_{\mathbf{i}_{N^n}}/p}\right)\\ &\le \sum_{j_{\mathbf{i}_1}+\dots+j_{\mathbf{i}_{N^n}}=p}\binom{p}{j_{\mathbf{i}_1},\dots,j_{\mathbf{i}_{N^n}}}\mathbb{E}\left(\left(\|S_{\mathbf{i}_1}\||B_{\mathbf{i}_1}|\right)^{sj_{\mathbf{i}_1}/p}\right)\cdot\dots\cdot\mathbb{E}\left(\left(\|S_{\mathbf{i}_{N^n}}\||B_{\mathbf{i}_{N^n}}|\right)^{sj_{\mathbf{i}_{N^n}}/p}\right).\end{aligned}$$

Notice that $\mathbb{E}\left(|B_{\mathbf{i}_1}|^{sj_{\mathbf{i}_1}/p}\right)\cdot\dots\cdot\mathbb{E}\left(|B_{\mathbf{i}_{N^n}}|^{sj_{\mathbf{i}_{N^n}}/p}\right) = \|B\|_{sj_{\mathbf{i}_1}/p}^{sj_{\mathbf{i}_1}/p}\cdot\dots\cdot\|B\|_{sj_{\mathbf{i}_{N^n}}/p}^{sj_{\mathbf{i}_{N^n}}/p} \le \|B\|_s^s$, since $\|B\|_r = \mathbb{E}(|B|^r)^{1/r}$ is increasing and $\|B\|_0 = 1$. This implies that

$$\begin{aligned}
&\sum_{j_{\mathbf{i}_1}+\dots+j_{\mathbf{i}_{N^n}}=p}\binom{p}{j_{\mathbf{i}_1},\dots,j_{\mathbf{i}_{N^n}}}\mathbb{E}\left((\|S_{\mathbf{i}_1}\||B_{\mathbf{i}_1}|)^{sj_{\mathbf{i}_1}/p}\right)\cdot\dots\cdot\mathbb{E}\left((\|S_{\mathbf{i}_{N^n}}\||B_{\mathbf{i}_{N^n}}|)^{sj_{\mathbf{i}_{N^n}}/p}\right)\\
&\le \mathbb{E}\left(|B|^s\right)\sum_{j_{\mathbf{i}_1}+\dots+j_{\mathbf{i}_{N^n}}=p}\binom{p}{j_{\mathbf{i}_1},\dots,j_{\mathbf{i}_{N^n}}}\mathbb{E}\left(\|S_{\mathbf{i}_1}\|^{sj_{\mathbf{i}_1}/p}\right)\cdot\dots\cdot\mathbb{E}\left(\|S_{\mathbf{i}_{N^n}}\|^{sj_{\mathbf{i}_{N^n}}/p}\right)\\
&= \mathbb{E}\left(|B|^s\right)\sum_{j_{\mathbf{i}_1}+\dots+j_{\mathbf{i}_{N^n}}=p}\binom{p}{j_{\mathbf{i}_1},\dots,j_{\mathbf{i}_{N^n}}}\int_G\|a\|^{sj_{\mathbf{i}_1}/p}\mu^{*n}(da)\cdot\dots\cdot\int_G\|a\|^{sj_{\mathbf{i}_{N^n}}/p}\mu^{*n}(da).
\end{aligned}$$

Observe that by the inequality (3.11), there exist constants $c_{sj_{\mathbf{i}_1}/p}, c_{sj_{\mathbf{i}_2}/p},\dots, c_{sj_{\mathbf{i}_{N^n}}/p}\in(0,1]$, such that for all $n\in\mathbb{N}$

$$\begin{aligned}
\int_G\|a\|^{sj_{\mathbf{i}_1}/p}\mu^{*n}(da) &\le c_{sj_{\mathbf{i}_1}/p}^{-1}\kappa^n(sj_{\mathbf{i}_1}/p),\\
\int_G\|a\|^{sj_{\mathbf{i}_2}/p}\mu^{*n}(da) &\le c_{sj_{\mathbf{i}_2}/p}^{-1}\kappa^n(sj_{\mathbf{i}_2}/p),\\
&\vdots\\
\int_G\|a\|^{sj_{\mathbf{i}_{N^n}}/p}\mu^{*n}(da) &\le c_{sj_{\mathbf{i}_{N^n}}/p}^{-1}\kappa^n(sj_{\mathbf{i}_{N^n}}/p).
\end{aligned}$$

Since $j_{\mathbf{i}_1}, j_{\mathbf{i}_2},\dots, j_{\mathbf{i}_{N^n}}\in\{0,1,\dots,p\}$, the constants above do not depend on $n\in\mathbb{N}$ and we may define $c_{p,s} = \max\{c_0^{-1}, c_{s/p}^{-1}, c_{2s/p}^{-1},\dots, c_{(p-1)s/p}^{-1}, c_s^{-1}\}$ that dominates all of them.

When $N^n\le p$, we have

$$\int_G\|a\|^{sj_{\mathbf{i}_1}/p}\mu^{*n}(da)\cdot\dots\cdot\int_G\|a\|^{sj_{\mathbf{i}_{N^n}}/p}\mu^{*n}(da)\le c_{p,s}^p\kappa^n(sj_{\mathbf{i}_1}/p)\cdot\dots\cdot\kappa^n(sj_{\mathbf{i}_{N^n}}/p). \tag{3.14}$$

Therefore,

$$\begin{aligned}
&\sum_{j_{\mathbf{i}_1}+\dots+j_{\mathbf{i}_{N^n}}=p}\binom{p}{j_{\mathbf{i}_1},\dots,j_{\mathbf{i}_{N^n}}}\int_G\|a\|^{sj_{\mathbf{i}_1}/p}\mu^{*n}(da)\cdot\dots\cdot\int_G\|a\|^{sj_{\mathbf{i}_{N^n}}/p}\mu^{*n}(da)\\
&\le c_{p,s}^p\sum_{j_{\mathbf{i}_1}+\dots+j_{\mathbf{i}_{N^n}}=p}\binom{p}{j_{\mathbf{i}_1},\dots,j_{\mathbf{i}_{N^n}}}\kappa^n(sj_{\mathbf{i}_1}/p)\cdot\dots\cdot\kappa^n(sj_{\mathbf{i}_{N^n}}/p)\\
&\le c_{p,s}^p\cdot\max\{\kappa(s/p),\kappa(2s/p),\dots,\kappa((p-1)s/p),\kappa(s)\}^n
\end{aligned}$$

$$
\begin{aligned}
&\cdot \sum_{j_{\mathbf{i}_1}+\cdots+j_{\mathbf{i}_{N^n}}=p} \binom{p}{j_{\mathbf{i}_1},\ldots,j_{\mathbf{i}_{N^n}}} \\
&\le c_{p,s}^p N^{pn} \cdot \max\{\kappa(s/p),\kappa(2s/p),\ldots,\kappa((p-1)s/p),\kappa(s)\}^n \\
&\le c_{p,s}^p p^{p-1} N^n \cdot \max\{\kappa(s/p),\kappa(2s/p),\ldots,\kappa((p-1)s/p),\kappa(s)\}^n,
\end{aligned}
$$

since $ks/p \in (s_1,s_2)$ for every $k \in \{1,\ldots,p\}$. This yields (3.13) with $K_s = c_{p,s}^p p^{p-1}\mathbb{E}(|B|^s) < \infty$ and $\eta = N \cdot \max\{\kappa(s/p),\ldots,\kappa(s)\} < 1$. As we said before the assumption $s_1 \le 1/2$ is indispensable, because it guarantees that $N\cdot\kappa(ks/p) < 1$ for every $k \in \{1,2,\ldots,p\}$.

When $N^n > p$, (3.14) also holds with the universal constant $c_{p,s}^p$ which does not depend on $n \in \mathbb{N}$, but we have to estimate

$$
\sum_{j_{\mathbf{i}_1}+\cdots+j_{\mathbf{i}_{N^n}}=p} \binom{p}{j_{\mathbf{i}_1},\ldots,j_{\mathbf{i}_{N^n}}} \kappa^n(sj_{\mathbf{i}_1}/p)\cdot\cdots\cdot\kappa^n(sj_{\mathbf{i}_{N^n}}/p),
$$

in a more subtle way. Before we do that we need to introduce a portion of necessary definitions.

For every $r \le k$, and $j_1 \le \cdots \le j_k$, let

$$
L(j_1,\ldots,j_k) = \binom{k}{l_1,l_2,\ldots,l_r},
$$

when $j_1 = \cdots = j_{l_1} < j_{l_1+1} = \cdots = j_{l_2+l_1} < j_{l_2+l_1+1} = \cdots = j_{l_3+l_2+l_1} < \cdots < j_{l_{r-1}+\cdots+l_1+1} = \cdots = j_{l_r+\cdots+l_1}$ and $l_1+l_2+\cdots+l_r = k$. Then it is not difficult to see that for every $k \le p$

$$
\begin{gathered}
L(j_1,\ldots,j_k) \le k!,\\
\binom{p}{j_1,\ldots,j_k} \le p!,\\
\binom{N^n}{k} L(j_1,\ldots,j_k) \le \frac{N^n!}{(N^n-k)!} \le N^{kn}.
\end{gathered}
$$

Let now $\eta = \max\{\eta_1,\eta_2,\ldots,\eta_p\} < 1$, where

$$
\begin{aligned}
&\eta_k = \max\{(N\kappa(sj_1/p))\cdot\cdots\cdot(N\kappa(sj_k/p)) : j_1+\cdots+j_k = p, \text{ and} \\
&j_1 \le \cdots \le j_k\} < 1.
\end{aligned}
$$

This implies that

$$
\sum_{j_{\mathbf{i}_1}+\cdots+j_{\mathbf{i}_{N^n}}=p} \binom{p}{j_{\mathbf{i}_1},\ldots,j_{\mathbf{i}_{N^n}}} \kappa^n(sj_{\mathbf{i}_1}/p)\cdot\cdots\cdot\kappa^n(sj_{\mathbf{i}_{N^n}}/p) = N^n\kappa^n(s)
$$

$$
\begin{aligned}
&+\binom{N^n}{2}\sum_{\substack{j_1+j_2=p\\ j_1\le j_2}}\binom{p}{j_1,\,j_2}L(j_1,j_2)\kappa^n(sj_1/p)\kappa^n(sj_2/p)\\
&+\binom{N^n}{3}\sum_{\substack{j_1+j_2+j_3=p\\ j_1\le j_2\le j_3}}\binom{p}{j_1,\,j_2,\,j_3}L(j_1,j_2,j_3)\kappa^n(sj_1/p)\kappa^n(sj_2/p)\kappa^n(sj_3/p)\\
&\ \vdots\\
&+\binom{N^n}{k}\sum_{\substack{j_1+\dots+j_k=p\\ j_1\le\dots\le j_k}}\binom{p}{j_1,\dots,j_k}L(j_1,\dots,j_k)\kappa^n(sj_1/p)\cdot\dots\cdot\kappa^n(sj_k/p)\\
&\ \vdots\\
&+\binom{N^n}{p}\sum_{\substack{j_1+\dots+j_p=p\\ j_1\le\dots\le j_p}}\binom{p}{j_1,\dots,j_p}L(j_1,\dots,j_p)\kappa^n(sj_1/p)\cdot\dots\cdot\kappa^n(sj_p/p)\\
&\le N^n\kappa^n(s)+\sum_{\substack{j_1+j_2=p\\ j_1\le j_2}}p!N^{2n}\kappa^n(sj_1/p)\kappa^n(sj_2/p)\\
&\quad+\dots+\sum_{\substack{j_1+\dots+j_k=p\\ j_1\le\dots\le j_k}}p!N^{kn}\kappa^n(sj_1/p)\cdot\dots\cdot\kappa^n(sj_k/p)\\
&\quad+\dots+\sum_{\substack{j_1+\dots+j_p=p\\ j_1\le\dots\le j_p}}p!N^{pn}\kappa^n(sj_1/p)\cdot\dots\cdot\kappa^n(sj_p/p)\le\eta^n\cdot p!\sum_{k=1}^{p}\sum_{\substack{j_1+\dots+j_k=p\\ j_1\le\dots\le j_k}}1\\
&\le\eta^n\cdot p!\sum_{k=1}^{p}\binom{p-1}{k-1}\le 2^{p-1}p!\cdot\eta^n.
\end{aligned}
$$

Hence in this case (3.13) follows with $K_s = 2^{p-1}p!c_{p,s}^p\mathbb{E}(|B|^s) < \infty$ and $\eta < 1$. □

Proof of Theorem 1.7 First of all we show that $\mathbb{E}(|R|^s) < \infty$ for every $s < s_2$, where R was defined in (3.4). This shows that R is finite a.s. moreover, in view of formula (3.6) R solves Eq. (1.1) in law. Its uniqueness will be a consequence of Lemma 3.8 as we mentioned above.

By Lemma 3.12 there exist $\eta < 1$ and $K_s < \infty$ such that for every $n \in \mathbb{N}$ we have $\mathbb{E}(|W_n|^s) \le K_s\eta^n$. Observe now

$$
\mathbb{E}(|R|^s)=\mathbb{E}(\liminf_{n\to\infty}|R^{(n)}|^s)\le\liminf_{n\to\infty}\mathbb{E}(|R^{(n)}|^s)\le\liminf_{n\to\infty}\mathbb{E}\left(\sum_{k=0}^{n}|W_k|\right)^s.
$$

When $0 < s \le 1$, we have

$$
\liminf_{n\to\infty}\mathbb{E}\left(\sum_{k=0}^{n}|W_k|\right)^s\le\liminf_{n\to\infty}\mathbb{E}\left(\sum_{k=0}^{n}|W_k|^s\right)\le\liminf_{n\to\infty}K_s\sum_{k=0}^{n}\eta^k=\frac{K_s}{1-\eta}<\infty.
$$

When $s > 1$, we have

$$\liminf_{n\to\infty} \mathbb{E}\left(\sum_{k=0}^{n} |W_k|\right)^s \le \liminf_{n\to\infty}\left(\sum_{k=0}^{n} \mathbb{E}\left(|W_k|^s\right)^{1/s}\right)^s$$
$$\le \liminf_{n\to\infty} K_s\left(\sum_{k=0}^{n} \eta^{k/s}\right)^s = \frac{K_s}{(1-\eta^{1/s})^s} < \infty.$$

It immediately implies that $\mathbb{E}(|R|^s) < \infty$, which in turn gives $|R| < \infty$ a.s.

Now we want to show that R is the unique solution of (1.1). It is enough to show that R_n^*, with an arbitrary initial random variable $R_0^* \in \mathbb{R}_+^d$ converges weakly to R as $n \to \infty$. Recall that the initial random variable R_0^* has finite s_2th moment, i.e. $\mathbb{E}(|R_0^*|^{s_2}) < \infty$. We show that $\mathbb{E}(f(R_n^*)) \underset{n\to\infty}{\longrightarrow} \mathbb{E}(f(R))$ for an arbitrary uniformly continuous function f defined on $\mathbb{R}^d$. Fix $\varepsilon > 0$, and choose $\delta > 0$ such that

$$|x-y| < \delta \implies |f(x) - f(y)| < \varepsilon.$$

By (3.9) we know that $R_n^* \overset{\mathcal{D}}{=} R^{(n-1)} + W_n(R_0^*)$ for every $n \in \mathbb{N}$, hence

$$\left|\mathbb{E}(f(R_n^*) - f(R))\right| \le \left|\mathbb{E}(f(R^{(n-1)} + W_n(R_0^*)) - f(R^{(n-1)}))\right|$$
$$+ \left|\mathbb{E}(f(R^{(n-1)}) - f(R))\right|.$$

It is enough to show that $\left|\mathbb{E}(f(R^{(n-1)} + W_n(R_0^*)) - f(R^{(n-1)}))\right| \underset{n\to\infty}{\longrightarrow} 0$. Fix $s < s_2$ and observe that, in view of inequality (3.13) with $W_n(R_0^*)$ instead of W_n, we have

$$\begin{aligned}
&\left|\mathbb{E}(f(R^{(n-1)} + W_n(R_0^*)) - f(R^{(n-1)}))\right| \\
&\le \mathbb{E}(|\mathbf{1}_{\{|W_n(R_0^*)|\le\delta\}}(f(R^{(n-1)} + W_n(R_0^*)) - f(R^{(n-1)}))|) \\
&+ \mathbb{E}(|\mathbf{1}_{\{|W_n(R_0^*)|>\delta\}}(f(R^{(n-1)} + W_n(R_0^*)) - f(R^{(n-1)}))|) \\
&\le \varepsilon\mathbb{P}(\{|W_n(R_0^*)| \le \delta\}) + 2M_f\mathbb{P}(\{|W_n(R_0^*)| > \delta\}) \\
&\le \varepsilon + 2M_f\mathbb{P}(\{|W_n(R_0^*)| > \delta\}) \le \varepsilon + 2M_f\frac{\mathbb{E}(|W_n(R_0^*)|^s)}{\delta^s} \\
&\le \varepsilon + \frac{2M_f K_s}{\delta^s}\eta^n \underset{n\to\infty}{\longrightarrow} \varepsilon,
\end{aligned}$$

for some $\eta < 1$ and $K_s < \infty$ (see Lemma 3.12). Since $\varepsilon > 0$ is arbitrary we have shown that $\mathbb{E}(f(R_n^*)) \underset{n\to\infty}{\longrightarrow} \mathbb{E}(f(R))$, and Theorem 1.7 follows. □

4 Application of Kesten's renewal theorem

In order to prove Theorem 1.9, as mentioned in the introduction, we will use Kesten's renewal theorem [17] which allows us to describe the desired tail asymptotic (1.10).

Before we state Kesten's theorem we have to introduce necessary definitions and to prove a number of auxiliary results. They are contained in the three lemmas of Sect. 4.1 and they will be used later on to check that the assumptions of Kesten's renewal theorem are satisfied in our settings. The material presented in this section is adapted from [3,6,7,16].

4.1 Some general results

At first we define the probability space $\Omega = G^{\mathbb{N}}$. $\mathcal{B}or(X)$ stands for the Borel σ-field of the space X. For any sequence $\omega = (a_1, a_2, \dots) \in \Omega$ we write

$$S_n(\omega) = a_n \cdot \dots \cdot a_1 \in G, \quad \text{for } n \in \mathbb{N} \text{ and } S_0(\omega) = \mathrm{Id} \in G.$$

Let $\theta : \Omega \mapsto \Omega$ be the shift on Ω, i.e.

$$\theta((a_1, a_2, \dots)) = (a_2, a_3, \dots), \quad \text{for every } \omega = (a_1, a_2, \dots) \in \Omega.$$

As in Sect. 2 (see (2.4) and (2.7)), for every $n \in \mathbb{N}$, we define the kernel

$$q_n^s(x, \omega) = \prod_{k=1}^{n} q_1^s(S_{k-1}(\omega) \cdot x, a_k), \quad \text{for every } x \in \mathbb{S}^+ \text{ and } \omega = (a_1, a_2, \dots) \in \Omega.$$

The cocycle property gives a very useful relation, i.e. for every $m, n \in \mathbb{N}$, $x \in \mathbb{S}^+$ and $\omega \in \Omega$ we have

$$q_{m+n}^s(x, \omega) = q_n^s(x, S_n(\omega)) q_m^s(S_n(\omega) \cdot x, S_m(\theta^n(\omega))). \tag{4.2}$$

The Kolmogorov's consistency theorem guarantees the existence of the probability measure $\mathbb{Q}_x^s$ on Ω being the unique extension of measures $q_k^s(x, a)\mu^{*k}(da)$. Next we define the probability measure

$$\mathbb{Q}^s = \int_{\mathbb{S}^+} \mathbb{Q}_x^s \pi^s(dx), \quad \text{on } \Omega,$$

where π^s is the unique Q^s stationary measure on $\mathbb{S}^+$ (see Theorem 2.3). By $\mathbb{E}_x^s$ we denote the expectation corresponding to $\mathbb{Q}_x^s$. We extend the probability space Ω to ${}^a\Omega = \mathbb{S}^+ \times \Omega$. Let ${}^a\theta : {}^a\Omega \mapsto {}^a\Omega$ be the shift defined by

$${}^a\theta(x, \omega) = (a_1 \cdot x, \theta(\omega)), \quad \text{for every } x \in \mathbb{S}^+ \text{ and } \omega = (a_1, a_2, \dots) \in \Omega.$$

We now define the probability measure ${}^aQ^s$ on ${}^a\Omega$ as follows

$${}^aQ^s = \int_{\mathbb{S}^+} \delta_x \otimes \mathbb{Q}_x^s \pi^s(dx).$$

In the same way, starting with μ_* instead of μ, and with kernels $q_n^{s,*}$ instead of q_n^s (see (2.7) for the definition) we introduce the measure $\mathbb{Q}_x^{s,*}$, and $\mathbb{E}_x^{s,*}$ denotes its expectation. Moreover, the probabilities $\mathbb{Q}^{s,*}$ and ${}^aQ^{s,*}$ are defined similarly, i.e.

$$\mathbb{Q}^{s,*} = \int_{\mathbb{S}^+} \mathbb{Q}_x^{s,*} \pi_*^s(dx), \quad \text{and} \quad {}^aQ^{s,*} = \int_{\mathbb{S}^+} \delta_x \otimes \mathbb{Q}_x^{s,*} \pi_*^s(dx),$$

where π_*^s is the unique Q_*^s stationary measure on $\mathbb{S}^+$ (see Theorem 2.3). Let $\omega^* = (a_1^*, a_2^*, \ldots) \in \Omega$ for every $\omega = (a_1, a_2, \ldots) \in \Omega$. Then $S_n(\omega^*) = a_n^* \cdot \cdots \cdot a_1^* \in G$.

Remark 4.3 The properties of the stationary measures π^s and π_*^s developed in Sect. 2 imply that $(\Omega, \mathcal{B}or(\Omega), \mathbb{Q}^s, \theta)$, $(\Omega, \mathcal{B}or(\Omega), \mathbb{Q}^{s,*}, \theta)$, $({}^a\Omega, \mathcal{B}or({}^a\Omega), {}^aQ^s, {}^a\theta)$ and $({}^a\Omega, \mathcal{B}or({}^a\Omega), {}^aQ^{s,*}, {}^a\theta)$ are ergodic.

From now we will work with the measures $\mathbb{Q}_x^{s,*}$, π_*^s, $\mathbb{Q}^{s,*}$ and ${}^aQ^{s,*}$. Clearly, all the results stated below remain valid for the measures $\mathbb{Q}_x^s$, π^s, $\mathbb{Q}^s$ and ${}^aQ^s$.

We begin with the following

Lemma 4.4 *Assume that $\mu \in M^1(G)$, $s \in I_\mu$ and $\Gamma = [supp \mu]$ satisfies condition $(i-p)$. Then there exists $c > 0$ such that $\mathbb{Q}_x^{s,*} \le c\mathbb{Q}^{s,*}$ for every $x \in \mathbb{S}^+$. Moreover the constant c does not depend on $x \in \mathbb{S}^+$.*

Proof We can repeat the argument from Sect. 3 in [3]. □

Lemma 4.5 *Assume that $\mu \in M^1(G)$, $s \in I_\mu$ and $\Gamma = [supp \mu]$ satisfies condition $(i-p)$. Then for every $x \in \mathbb{S}^+$ we have*

$$\mathbb{Q}_x^{s,*}\left(\{\omega \in \Omega : \exists\, C > 0\ \forall\, n \in \mathbb{N} \quad |S_n(\omega)x| \ge C\|S_n(\omega)\|\}\right) = 1, \quad \textit{and} \tag{4.6}$$

$$\mathbb{Q}^{s,*}\left(\{\omega \in \Omega : \exists\, C > 0\ \forall\, n \in \mathbb{N} \quad |S_n(\omega)x| \ge C\|S_n(\omega)\|\}\right) = 1. \tag{4.7}$$

Proof Observe that (4.7) implies (4.6). Indeed, let

$$Z_x = \{\omega \in \Omega : \exists\, C > 0\ \forall\, n \in \mathbb{N} \quad |S_n(\omega)x| \ge C\|S_n(\omega)\|\},$$

and let Z_x^c be the complement of Z_x. Then by Lemma 4.4

$$\mathbb{Q}_x^{s,*}(Z_x^c) \le c\mathbb{Q}^{s,*}(Z_x^c) = 0.$$

The proof of (4.7) is adapted from [16]. Condition (1.3) yields the existence of $n_0 \in \mathbb{N}$ and $0 < \tau < 1$ such that

$$p = \mathbb{P}^*\left(\{\omega \in \Omega : S_{n_0}(\omega)(i, j) > \tau, \ \text{for all } 1 \le i, j \le d\}\right) > 0, \tag{4.8}$$

where $\mathbb{P}^* = \mu_*^{\otimes \mathbb{N}}$. Let us introduce

$$T(\omega) = \min\{n \ge n_0 : S_{n_0}(\theta^{n-n_0}(\omega)) \in G^\circ\}.$$

First of all we need to show that

$$\mathbb{Q}_x^{s,*}(\{\omega \in \Omega : T(\omega) < \infty\}) = 1, \quad \text{for every } x \in \mathbb{S}^+ \quad \text{and} \tag{4.9}$$

$$\mathbb{Q}^{s,*}(\{\omega \in \Omega : T(\omega) < \infty\}) = 1. \tag{4.10}$$

Notice that (4.9) immediately gives (4.10), since the event $\{T < \infty\}$ does not depend on $x \in \mathbb{S}^+$ and $\mathbb{Q}^{s,*} = \int_{\mathbb{S}^+} \mathbb{Q}_x^{s,*} \pi_*^s(dx)$.

Assume for a moment that (4.10) holds and prove (4.7). If $x = (x_1, \dots, x_n) \in \mathbb{S}^+$ is such that $x > 0$ then for any $a \in G$ we have

$$\begin{aligned}|ax| &\geq d^{-1/2} \sum_{i=1}^{d} (a(i,1)x_1 + \cdots + a(i,n)x_n) \geq d^{-1/2} \min_{1 \leq i \leq d} x_i \sum_{i,j=1}^{d} a(i,j)\\ &\geq d^{-1/2} \min_{1 \leq i \leq d} x_i \cdot \sup_{|y|=1} \left(\sum_{i=1}^{d} (a(i,1)y_1 + \cdots + a(i,n)y_n)^2 \right)^{1/2}\\ &= d^{-1/2} \min_{1 \leq i \leq d} x_i \|a\|.\end{aligned}$$

We will use this inequality to show that (4.7) holds. Now fix an arbitrary $x \in \mathbb{S}^+$ and let $\Omega_1 = \{T < \infty\} \subseteq \Omega$. By assumption, $\mathbb{Q}^{s,*}(\Omega_1) = 1$. It is easy to see that $S_T(\omega^*)x > 0$ for $\omega^* \in \Omega_1$. Fix $\omega^* \in \Omega_1$ then for any $n \geq T(\omega^*)$ we have

$$\begin{aligned}|S_n(\omega^*)x| = |S_{n-T}(\theta^T(\omega^*))S_T(\omega^*)x| &\geq d^{-1/2} \min_{1 \leq i \leq d} (S_T(\omega^*)x)_i \|S_{n-T}(\theta^T(\omega^*))\|\\ &\geq d^{-1/2} \frac{\min_{1 \leq i \leq d}(S_T(\omega^*)x)_i}{\|S_T(\omega^*)\|} \|S_n(\omega^*)\|.\end{aligned}$$

It implies that $|S_n(\omega^*)x| \geq C_{T,x}(\omega^*)\|S_n(\omega^*)\|$ holds with the constant $C_{T,x}(\omega^*) > 0$ independent of $n \geq T(\omega^*)$, for every $\omega^* \in \Omega_1$. Recall that G is the multiplicative semigroup of $d \times d$ invertible matrices with positive entries such that every row and every column contains a strictly positive element. Now take $n \leq T(\omega^*)$ and notice that $C_{n,x}(\omega^*) = \frac{|S_n(\omega^*)x|}{\|S_n(\omega^*)\|} > 0$, for every $\omega^* \in \Omega_1$ by the definition of G and $x \in \mathbb{S}^+$. Therefore, we take $C(\omega^*) = \min\{C_{1,x}(\omega^*), \dots, C_{T,x}(\omega^*)\} > 0$, and (4.7) follows.

We need only to prove (4.9). In this purpose we define the events

$$E_k = \{\omega \in \Omega : S_{n_0}(\theta^k(\omega))(i,j) \geq \tau, \text{ for all } 1 \leq i,j \leq d\}, \quad k \in \mathbb{N}.$$

We show that there exists $\gamma \in [0,1)$ such that for all $l \in \mathbb{N}$

$$\mathbb{Q}_x^{s,*}(\{T > ln_0\}) \leq \mathbb{Q}_x^{s,*}(\{E_{jn_0} \text{ does not occur for any } 0 \leq j < l\}) \leq \gamma^l. \tag{4.11}$$

Then (4.11) with Borel–Cantelli lemma yield $\mathbb{Q}_x^{s,*}(\{T<\infty\})=1$. In fact it is enough to show that

$$\mathbb{Q}_x^{s,*}(E_0^c\cap\cdots\cap E_{(l-1)n_0}^c)\le\gamma\mathbb{Q}_x^{s,*}(E_0^c\cap\cdots\cap E_{(l-2)n_0}^c)\le\gamma^2\mathbb{Q}_x^{s,*}(E_0^c\cap\cdots\cap E_{(l-3)n_0}^c)$$
$$\le\cdots \text{ and inductively } \cdots\le\gamma^l. \tag{4.12}$$

Let $r_s=\frac{\inf_{x\in\mathbb{S}^+}e_*^s(x)}{\sup_{x\in\mathbb{S}^+}e_*^s(x)}$. Then

$$\begin{aligned}
&\mathbb{Q}_x^{s,*}(E_0^c\cap\cdots\cap E_{(l-2)n_0}^c\cap E_{(l-1)n_0})\\
&=\int_\Omega \mathbf{1}_{E_0^c\cap\cdots\cap E_{(l-2)n_0}^c\cap E_{(l-1)n_0}}(\omega^*)q_{ln_0}^{s,*}\big(x,S_{ln_0}(\omega^*)\big)\mu^{*ln_0}(d\omega)\\
&\ge\frac{r_s\tau^s}{d^{s/2}\kappa^{n_0}(s)}\int_\Omega \mathbf{1}_{E_0^c\cap\cdots\cap E_{(l-2)n_0}^c}(\omega^*)\mathbf{1}_{E_{(l-1)n_0}}(\omega^*)q_{(l-1)n_0}^{s,*}\big(x,S_{(l-1)n_0}(\omega^*)\big)\mu^{*ln_0}(d\omega)\\
&=\frac{r_s\tau^s}{d^{s/2}\kappa^{n_0}(s)}\mathbb{P}^*(E_{(l-1)n_0})\mathbb{Q}_x^{s,*}(E_0^c\cap\cdots\cap E_{(l-2)n_0}^c)\\
&=\frac{pr_s\tau^s}{d^{s/2}\kappa^{n_0}(s)}\mathbb{Q}_x^{s,*}(E_0^c\cap\cdots\cap E_{(l-2)n_0}^c),
\end{aligned}\tag{4.13}$$

($n_0\in\mathbb{N}$ and $\tau>0$ were defined in (4.8)) since by (4.2) we have the following lower bound

$$\begin{aligned}
&\mathbf{1}_{E_{(l-1)n_0}}(\omega^*)q_{ln_0}^{s,*}\big(x,S_{ln_0}(\omega^*)\big)\\
&=\mathbf{1}_{E_{(l-1)n_0}}(\omega^*)q_{(l-1)n_0}^{s,*}\big(x,S_{(l-1)n_0}(\omega^*)\big)q_{n_0}^{s,*}\Big(S_{(l-1)n_0}(\omega^*)\cdot x,S_{n_0}\big(\theta^{(l-1)n_0}(\omega^*)\big)\Big)\\
&\ge\frac{r_s}{\kappa^{n_0}(s)}\mathbf{1}_{E_{(l-1)n_0}}(\omega^*)q_{(l-1)n_0}^{s,*}\big(x,S_{(l-1)n_0}(\omega^*)\big)\big|S_{n_0}\big(\theta^{(l-1)n_0}(\omega^*)\big)\big(S_{(l-1)n_0}(\omega^*)\cdot x\big)\big|^s\\
&\ge\frac{r_s}{d^{s/2}\kappa^{n_0}(s)}\mathbf{1}_{E_{(l-1)n_0}}(\omega^*)q_{(l-1)n_0}^{s,*}\big(x,S_{(l-1)n_0}(\omega^*)\big)\\
&\times\left(\sum_{i=1}^d S_{n_0}\big(\theta^{(l-1)n_0}(\omega^*)\big)\big(S_{(l-1)n_0}(\omega^*)\cdot x\big)_i\right)^s\\
&\ge\frac{r_s\tau^s}{d^{s/2}\kappa^{n_0}(s)}\mathbf{1}_{E_{(l-1)n_0}}(\omega^*)q_{(l-1)n_0}^{s,*}\big(x,S_{(l-1)n_0}(\omega^*)\big).
\end{aligned}$$

Let $0<\gamma_s=\min\left\{1,\frac{pr_s\tau^s}{d^{s/2}\kappa^{n_0}(s)}\right\}$. For $\gamma=1-\gamma_s\in[0,1)$, by (4.13), we obtain that

$$\begin{aligned}
\mathbb{Q}_x^{s,*}(E_0^c\cap\cdots\cap E_{(l-2)n_0}^c\cap E_{(l-1)n_0}^c)&\le\gamma\mathbb{Q}_x^{s,*}(E_0^c\cap\cdots\cap E_{(l-2)n_0}^c\cap G_{(l-1)n_0})\\
&=\gamma\mathbb{Q}_x^{s,*}(E_0^c\cap\cdots\cap E_{(l-2)n_0}^c).
\end{aligned}$$

This finishes the proof of (4.12) and completes the proof of the lemma. □

Lemma 4.14 *Assume that $\mu \in M^1(G)$, $s \in I_\mu$ and $\Gamma = [supp \mu]$ satisfies condition $(i-p)$. Assume additionally that $\int_G \|a\|^s \log^+ \|a\| \mu(da) < \infty$. Then for any $x \in \mathbb{S}^+$*

$$\lim_{n\to\infty} \frac{1}{n} \log |S_n(\omega)x| = \lim_{n\to\infty} \frac{1}{n} \log \|S_n(\omega)\| = \alpha(s), \quad \mathbb{Q}_x^{s,*} \text{ and } \mathbb{Q}^{s,*} \text{ a.s.}, \tag{4.15}$$

where

$$\alpha(s) = \int_{\mathbb{S}^+} \int_G \log |ax| q_1^{s,*}(x,a) \mu_*(da) \pi_*^s(dx). \tag{4.16}$$

Proof We show that $f(x,\omega) = \log |S_1(\omega)x|$ is ${}^a Q^{s,*}$ integrable. Observe that there exists $0 < \delta < 1$ such that

$$0 < |ax| < \delta \implies |ax|^s \log |ax|^{-1} \le 1.$$

Then

$$\begin{aligned}
{}^a Q^{s,*}(|f|) &= \int_{\mathbb{S}^+} \int_{\Omega} |\log |S_1(\omega)y|| \delta_x(dy) \mathbb{Q}_x^{s,*}(d\omega) \pi_*^s(dx) \\
&= \int_{\mathbb{S}^+} \int_G |ax|^s |\log |ax|| \frac{e_*^s(a\cdot x)}{\kappa(s) e_*^s(x)} \mu_*(da) \pi_*^s(dx) \\
&\le C_s \int_{\mathbb{S}^+} \int_G |ax|^s |\log |ax|| \mu_*(da) \pi_*^s(dx) \\
&\le C_s \int_G \|a\|^s \log^+ \|a\| \mu(da) + C_s \int_{\mathbb{S}^+} \int_G |ax|^s \log^- |ax| \mu_*(da) \pi_*^s(dx) \\
&\le C_s \int_G \|a\|^s \log^+ \|a\| \mu(da) + C_s \mu_* \otimes \pi_*^s(\{(a,x) \in G \times \mathbb{S}^+ : 0 < |ax| < \delta\}) \\
&\quad + C_s \log\left(\frac{1}{\delta}\right) \mu_* \otimes \pi_*^s(\{(a,x) \in G \times \mathbb{S}^+ : \delta < |ax| \le 1\}) \\
&\le C_s \int_G \|a\|^s \log^+ \|a\| \mu(da) + C_s \left(1 + \log\left(\frac{1}{\delta}\right)\right) < \infty.
\end{aligned}$$

Hence in view of Remark 4.3, on the one hand, by the Birkhoff ergodic theorem (applied to ${}^a Q^{s,*}$ and ${}^a\theta$) we obtain

$$\begin{aligned}
{}^a Q^{s,*}\Bigg(\Bigg\{ (x,\omega) \in {}^a\Omega : \lim_{n\to\infty} \frac{1}{n} \log |S_n(\omega)x| \\
= \lim_{n\to\infty} \frac{1}{n} \cdot \sum_{k=0}^{n-1} f \circ {}^a\theta^k(x,\omega) = {}^a Q^{s,*}(f) = \alpha(s) \Bigg\}\Bigg) = 1.
\end{aligned}$$

On the other hand by the Kingman subadditive ergodic theorem (applied to $Q^{s,*}$ and θ) we have that for every $s \in I_\mu$ there exists $\alpha_s \in \mathbb{R}$ such that

$$\mathbb{Q}^{s,*}\left(\left\{\omega \in \Omega : \lim_{n\to\infty} \frac{1}{n} \log \|S_n(\omega)\| = \alpha_s\right\}\right) = 1.$$

Define $\Omega' = \{\omega \in \Omega : \exists\, C > 0\, \forall\, n \in \mathbb{N}\ |s_n(\omega)x| \geq C\|S_n(\omega)\|$ and $\lim_{n\to\infty} \frac{1}{n} \log \|S_n(\omega)\| = \alpha_s\}$, for every $x \in \mathbb{S}^+$. By Lemma 4.5 and calculations stated above we know that $\mathbb{Q}^{s,*}(\Omega') = 1$. Fix arbitrary $x \in \mathbb{S}^+$, take any $\omega^* \in \Omega'$ and notice that

$$0 < C_x(\omega^*) \leq \frac{|S_n(\omega^*)x|}{\|S_n(\omega^*)\|} \leq 1,$$

imply

$$\begin{aligned}
&\frac{1}{n}\log C_x(\omega^*) + \frac{1}{n}\log\|S_n(\omega^*)\| \leq \frac{1}{n}\log\frac{|S_n(\omega^*)x|}{\|S_n(\omega^*)\|} + \frac{1}{n}\log\|S_n(\omega^*)\| \\
&\leq \frac{1}{n}\log\|S_n(\omega^*)\|.
\end{aligned}$$

Since $\lim_{n\to\infty} \frac{1}{n} \log C_x(\omega^*) = 0$ we have

$$\mathbb{Q}^{s,*}\left(\left\{\omega \in \Omega : \lim_{n\to\infty} \frac{1}{n} \log |S_n(\omega)x| = \alpha_s\right\}\right) = 1.$$

And so, in view of Lemma 4.4,

$$\mathbb{Q}_x^{s,*}\left(\left\{\omega \in \Omega : \lim_{n\to\infty} \frac{1}{n} \log |S_n(\omega)x| = \alpha_s\right\}\right) = 1,$$

for all $x \in \mathbb{S}^+$ (by considering complements). Since ${}^a Q^{s,*} = \int_{\mathbb{S}^+} \delta_x \otimes \mathbb{Q}_x^{s,*} \pi_*^s(dx)$ we get $\alpha(s) = \alpha_s$ and Lemma 4.14 follows. □

4.2 Kesten's renewal theorem

For $x \in \mathbb{S}^+$ and $\omega \in \Omega$ define $X_0(\omega) = x$, and for $n \in \mathbb{N}$

$$X_n(\omega) = g_n(\omega) \cdot X_{n-1}(\omega) = S_n(\omega) \cdot x,$$

and

$$V_n(\omega) = \log |S_n(\omega)x| = \sum_{i=1}^{n} U_i(\omega), \quad \text{where} \quad U_i(\omega) = \log |g_i(\omega) X_{i-1}(\omega)|.$$

Let $F(dt|x, y)$ be the conditional law of U_1, given $X_0 = x$, $X_1 = y$, i.e.

$$\mathbb{Q}_x^{s,*}(X_1 \in A, U_1 \in B) = \int_A \int_B F(dt|x, y) Q_*^s(x, dy).$$

A function $g : \mathbb{S}^+ \times \mathbb{R} \to \mathbb{R}$ is called direct Riemann integrable ($d\mathcal{R}i$), if it is $\mathcal{B}or(\mathbb{S}^+) \times \mathcal{B}or(\mathbb{R})$ measurable and for every fixed $x \in \mathbb{S}^+$ and $0 < L < \infty$ the function $t \mapsto g(x, t)$ is Riemann integrable on $[-L, L]$, and satisfies

$$\sum_{k=0}^{\infty} \sum_{l=-\infty}^{\infty} (k+1) \sup\{|g(x, t)| : \ x \in C_{k+1} \setminus C_k, \quad \text{and} \quad t \in [l, l+1]\} < \infty, \tag{4.17}$$

where

$$C_k = \left\{ x \in \mathbb{S}^+ : \ \mathbb{Q}_x^{s,*}\left(\left\{ \frac{V_m}{m} \geq \frac{1}{k}, \ \text{for all } m \geq k \right\} \right) \geq \frac{1}{2} \right\}, \quad \text{for all } k \in \mathbb{N}. \tag{4.18}$$

For the reader's convenience we formulate Kesten's renewal theorem [17].

Theorem 4.19 *Assume the following conditions are satisfied:*

- ***Condition*** I.1 *There exists* $\pi_*^s \in M^1(\mathbb{S}^+)$ *such that* $\pi_*^s Q_*^s = \pi_*^s$ *and for every open set* $U \subseteq \mathbb{S}^+$ *with* $\pi_*^s(U) > 0$, $\mathbb{Q}_x^{s,*}(X_n \in U \text{ for some } n \in \mathbb{N}) = 1$ *for every* $x \in \mathbb{S}^+$.
- ***Condition*** I.2

$$\int_{\mathbb{S}^+} \int_{\mathbb{S}^+} \int_{\mathbb{R}} |t| F(dt|x, y) Q_*^s(x, dy) \pi_*^s(dx) < \infty,$$

and for all $x \in \mathbb{S}^+$,

$$\lim_{n\to\infty} \frac{V_n}{n} = \alpha(s) = \int t F(dt|x, y) Q_*^s(x, dy) \pi_*^s(dx) > 0 \quad \mathbb{Q}_x^{s,*} - a.e. \tag{4.20}$$

- ***Condition*** I.3 *There exists a sequence* $\{\zeta_i\} \subset \mathbb{R}$ *such that the group generated by* ζ_i *is dense in* $\mathbb{R}$ *and such that for each* ζ_i *and* $\lambda > 0$ *there exists* $y = y(\zeta_i, \lambda) \in \mathbb{S}^+$ *with the following property: for each* $\varepsilon > 0$, *there exists* $A \in \mathcal{B}or(\mathbb{S}^+)$ *with* $\pi_*^s(A) > 0$ *and* $m_1, m_2 \in \mathbb{N}$, $\tau \in \mathbb{R}$ *such that for any* $x \in A$

$$\mathbb{Q}_x^{s,*}\left\{|X_{m_1} - y| < \varepsilon, |V_{m_1} - \tau| \leq \lambda\right\} > 0, \tag{4.21}$$

$$\mathbb{Q}_x^{s,*}\left\{|X_{m_2} - y| < \varepsilon, |V_{m_2} - \tau - \zeta_i| \leq \lambda\right\} > 0. \tag{4.22}$$

- ***Condition*** I.4 *For each fixed $x \in \mathbb{S}^+$, $\varepsilon > 0$ there exists $r_0 = r_0(x, \varepsilon) > 0$ such that for all real valued functions f measurable with respect to $\mathcal{B}or\left((\mathbb{S}^+ \times \mathbb{R})^{\mathbb{N}}\right)$ and for all $y \in \mathbb{S}^+$ with $|x - y| < r_0$ one has:*

$$\mathbb{E}_x^{s,*} f(X_0, V_0, X_1, V_1, \dots) \le \mathbb{E}_y^{s,*} f^{\varepsilon}(X_0, V_0, X_1, V_1, \dots) + \varepsilon |f|_{\infty},$$
$$\mathbb{E}_y^{s,*} f(X_0, V_0, X_1, V_1, \dots) \le \mathbb{E}_x^{s,*} f^{\varepsilon}(X_0, V_0, X_1, V_1, \dots) + \varepsilon |f|_{\infty},$$

where $f^{\varepsilon}(x_0, v_0, x_1, v_1, \dots) = \sup\{f(y_0, u_0, y_1, u_1, \dots) : \forall\, i \in \mathbb{N}\ |x_i - y_i| + |v_i - u_i| < \varepsilon\}$.

If a function $g : \mathbb{S}^+ \times \mathbb{R} \mapsto \mathbb{R}$ is jointly continuous and $(d\mathcal{R}i)$, then for every $x \in \mathbb{S}^+$

$$\lim_{t \to \infty} \mathbb{E}_x^{s,*} \left(\sum_{n=0}^{\infty} g(X_n, t - V_n) \right) = \frac{1}{\alpha(s)} \int_{\mathbb{S}^+} \left(\int_{\mathbb{R}} g(y, x) dx \right) \pi_*^s(dy),$$

for $\alpha(s)$ defined in (4.20).

In the next four subsections we indicate how the material developed in Sects. 2 and 4.1, under the hypotheses of Theorem 1.9, may be used to check the assumptions of Theorem 4.19. From now we will work with the measures $\mathbb{Q}_x^{\chi,*}$ for $x \in \mathbb{S}^+$, where $\chi > 0$ solves equation $\kappa(\chi) = \frac{1}{N}$. Such $\chi > 0$ exists since $\kappa(s)$ is strictly log-convex and $\lim_{s \to s_\infty} \kappa(s) > \frac{1}{N}$, (see Theorems 1.9 and 2.3). We are going to prove that Conditions I.1–I.4 are satisfied for $s = \chi$.

4.3 Condition I.1.

Proof of Condition I.1. Theorem 2.3 with Breiman's strong law of large numbers [2] allow us to repeat the argument contained in Sect. 5 in [3]. □

4.4 Condition I.2.

Proof of Condition I.2. We know that $\int_G \|a\|^{\chi} \log^+ \|a\| \mu(da) < \infty$, hence

$$\int_{\mathbb{S}^+} \int_{\mathbb{S}^+} \int_{\mathbb{R}} |t| F(dt|x, y) Q_*^{\chi}(x, dy) \pi_*^{\chi}(dx)$$
$$= \int_{\mathbb{S}^+} \int_{\Omega} |\log |ax|| q_1^{\chi,*}(x, a) \mu_*(da) \pi_*^{\chi}(dx) < \infty,$$

by the arguments of Lemma 4.14 applied to $s = \chi$. The only point remaining concerns the positivity of $\alpha(\chi)$ defined in Lemma 4.14 (see also (4.20)).

Notice that if $\varepsilon > 0$ is sufficiently small, then for every $t \in (\chi - \varepsilon, \chi)$, we have $\kappa(t) < \kappa(\chi)$, since $\kappa(s)$ is strictly log-convex and $\lim_{s \to s_\infty} \kappa(s) > \frac{1}{N}$, (see Theorems 1.9 and 2.3). Fix $t \in (\chi - \varepsilon, \chi)$ such that $\chi/t \le 4/3$ and take $\gamma > 0$ such that $\kappa(t) e^{\gamma} < \kappa(\chi)$. In view of inequality (3.11), there is $C > 0$ such that

$$\int_G \|a\|^t \mu_*^{*n}(da) \le C\kappa^n(t) e^{\gamma n/3}, \quad \text{for every } n \in \mathbb{N},$$

since $1 \le e^{\gamma/3}$. Fix $x \in \mathbb{S}^+$. Then for $\delta = \gamma/3$ we have

$$\mu_*^{*n}(\{a \in G : |ax|^t > e^{-\delta n}\}) \le e^{\delta n} \int_G |ax|^t \mu_*^{*n}(da) \le C\kappa^n(t) e^{2\gamma n/3},$$

Now let $\rho = \gamma/6$. Throughout the proof will use the convention that $D > 0$ stands for a large positive constant whose value varies from occurrence to occurrence. Then

$$\begin{aligned}
&\mathbb{Q}_x^{\chi,*}(\{\omega \in \Omega : |S_n(\omega)x|^t < e^{\rho n}\}) = \int_G \mathbf{1}_{\{a\in G:|ax|^t<e^{\rho n}\}} q_n^{\chi,*}(x,a)\mu_*^{*n}(da) \\
&\le \frac{D}{\kappa^n(\chi)} \int_G \mathbf{1}_{\{a\in G:|ax|^t<e^{\rho n}\}} |ax|^\chi \mu_*^{*n}(da) \\
&\le \frac{D}{\kappa^n(\chi)} \int_G \mathbf{1}_{\{a\in G:|ax|^t<e^{-\delta n}\}} |ax|^\chi \mu_*^{*n}(da) + \frac{D}{\kappa^n(\chi)} \\
&\times \int_G \mathbf{1}_{\{a\in G:e^{-\delta n}\le|ax|^t<e^{\rho n}\}} |ax|^\chi \mu_*^{*n}(da) \\
&\le \frac{D\kappa^n(t)}{\kappa^n(\chi)} \cdot \frac{1}{\kappa^n(t)} e^{-\frac{\chi-t}{t}\delta n} \int_G |ax|^t \frac{e_*^t(a\cdot x)}{e_*^t(x)} \mu_*^{*n}(da) \\
&+ \frac{D}{\kappa^n(\chi)} \mu_*^{*n}(\{a \in G : |ax|^t > e^{-\delta n}\}) e^{\frac{\rho n \chi}{t}} \\
&\le De^{-\left(\gamma + \frac{\chi-t}{t}\delta\right)n} + De^{-\gamma n} e^{2\gamma n/3} e^{\rho n\chi/t} \\
&\le De^{-\left(\gamma + \frac{\chi-t}{t}\delta\right)n} + De^{-\gamma n/3 + 2\gamma n/9} \le De^{-\beta n},
\end{aligned}$$

for some $\beta > 0$. Thus

$$\sum_{n\in\mathbb{N}} \mathbb{Q}_x^{\chi,*}\left(\left\{\omega \in \Omega : \log|S_n(\omega)x| < \frac{\rho n}{t}\right\}\right) < \infty.$$

Therefore, by the Borel–Cantelli lemma we obtain that for every $x \in \mathbb{S}^+$

$$\mathbb{Q}_x^{\chi,*}\left(\left\{\omega \in \Omega : \liminf_{n\to\infty} \frac{\log|S_n(\omega)x|}{n} \ge \frac{\rho}{t} > 0\right\}\right) = 1.$$

This shows that $\alpha(\chi) > 0 \mathbb{Q}_x^{\chi,*}$ a.s. for every $x \in \mathbb{S}^+$ and finishes the proof of Condition I.2. □

4.5 Condition I.3.

Proof of Condition I.3. Proposition 2.1 and Theorem 2.3 allow us to use arguments from Section 5 in [3]. □

4.6 Condition I.4.

Proof of Condition I.4. The proof is a consequence of Lemma 4.5 and the argument given by Kesten [16]. □

4.7 Direct Riemann integrability

Now we derive an interesting criterium which significantly simplifies condition (4.17).

Lemma 4.23 *Assume that the hypotheses of Theorem 1.9 are satisfied. If h is any bounded and continuous function on $\mathbb{S}^+ \times \mathbb{R}$ which satisfies*

$$\sum_{l=-\infty}^{\infty} \sup \left\{ |h(x,t)| :\ x \in \mathbb{S}^+,\ \textit{and}\ t \in [l, l+1] \right\} < \infty, \tag{4.24}$$

then h is direct Riemann integrable i.e. it satisfies condition (4.17).

Proof We give only a sketch of the proof, for more details we refer to [3]. First of all we prove that $C_k = \mathbb{S}^+$, for some sufficiently large $k \in \mathbb{N}$, (C_k was defined in (4.18)). Then obviously (4.24) implies (4.17). There is a finite number N_1 of points such that $\mathbb{S}^+ \subseteq \bigcup_{i=1}^{N_1} B(x_i, 2)$, since $\mathbb{S}^+$ is compact. Let

$$\Omega' = \left\{ \lim_{n\to\infty} \frac{\log |S_n x_i|}{n} = \alpha(\chi) > 0,\ \text{and}\ \exists C > 0\ \forall n \in \mathbb{N}\ \ |S_n x_i| \geq C \|S_n\|, \right.$$
$$\left. \text{for all } 1 \leq i \leq N_1 \right\}.$$

Then $\mathbb{Q}^{\chi,*}(\Omega') = 1$, by Lemmas 4.5 and 4.14. Take any $y \in \mathbb{S}^+$, then there exists $1 \leq i \leq N_1$ such that $y \in B(x_i, 2)$. This implies the existence of $m_0 \in \mathbb{N}$ such that

$$\mathbb{Q}^{\chi,*}\left(\left\{ \omega \in \Omega : \frac{\log |S_n(\omega) y|}{n} > \alpha(\chi)/2,\ \text{for all } n \geq m_0 \right\} \right) \geq 1 - \frac{1}{2c},$$

with the constant $c > 0$ defined in Lemma 4.4. Taking any $1/k \leq \min\{\alpha(s)/2, 1/m_0\}$ Lemma 4.23 follows. □

5 Proof of the main theorem

In this section we give a detailed proof of Theorem 1.9. For that we consider the following smooth version of $\mathbb{P}(\{\langle R, u\rangle > t\})$

$$G(u,t) = \frac{1}{e^t e_*^\chi(u)} \int_0^{e^t} r^\chi \mathbb{P}(\{\langle R, u\rangle > r\})dr, \quad \text{where } (u,t) \in \mathbb{S}^+ \times \mathbb{R}, \tag{5.1}$$

where $R \in \mathbb{R}_+^d$ solves Eq. (1.1). Let $\mathbf{B}(\mathbb{S}^+ \times \mathbb{R})$ be the space of all bounded measurable functions on $\mathbb{S}^+ \times \mathbb{R}$. Define a linear operator $\Theta : \mathbf{B}(\mathbb{S}^+ \times \mathbb{R}) \mapsto \mathbf{B}(\mathbb{S}^+ \times \mathbb{R})$ given by the formula

$$\begin{aligned}\Theta f(u,t) &= \mathbb{E}_u^{\chi,*}(f(X_1, t - V_1)) \\ &= \frac{1}{\kappa(\chi)} \int_\Omega f(S_1(\omega^*)\cdot u, t - \log|S_1(\omega^*)u|) \frac{e_*^\chi(S_1(\omega^*)\cdot u)}{e_*^\chi(u)} |S_1(\omega^*)u|^\chi \mathbb{P}(d\omega).\end{aligned}$$

Observe that for every $n \in \mathbb{N}$

$$\Theta^n f(u,t) = \mathbb{E}_u^{\chi,*}(f(X_n, t - V_n)).$$

First we express $G(u,t)$ as a potential of a function $g(u,t)$ that turns out later on to be direct Riemann integrable. Recall that $A, A_1, A_2, \ldots \in G$ is a sequence of independent copies of G-valued random matrix A distributed according to μ and they are independent of R. For $n \in \mathbb{N}$ let $S_n = A_n \cdot \dots \cdot A_1 \in G$.

Lemma 5.2 *Assume that the hypotheses of Theorem 1.9 are satisfied. Let $G(u,t)$ be the function defined in (5.1), and*

$$G_0(u,t) = \frac{N}{e^t e_*^\chi(u)} \int_0^{e^t} r^\chi \mathbb{P}(\{\langle AR, u\rangle > r\})dr,$$

then

$$G_0(u,t) = \Theta G(u,t), \quad \textit{and} \tag{5.3}$$

$$\lim_{n\to\infty} \Theta^n G(u,t) = \lim_{n\to\infty} \mathbb{E}_u^{\chi,*}(G(X_n, t - V_n)) = 0. \tag{5.4}$$

Moreover,

$$G(u,t) = \sum_{n=0}^{\infty} \Theta^n g(u,t), \quad \textit{where} \tag{5.5}$$

$$g(u,t) = \frac{1}{e^t e_*^\chi(u)} \int_0^{e^t} r^\chi \left(\mathbb{P}(\{\langle R, u\rangle > r\}) - N\mathbb{P}(\{\langle AR, u\rangle > r\})\right) dr. \tag{5.6}$$

Proof First of all we show $G_0(u,t) = \Theta G(u,t)$. Indeed,

$$\begin{aligned}
G_0(u,t) &= \frac{N}{e^t e_*^\chi(u)} \int_0^{e^t} r^\chi \mathbb{P}(\{\langle R, A^*\cdot u\rangle |A^*u| > r\}) dr \\
&= \mathbb{E}\left(\frac{N}{e^t e_*^\chi(u)} \int_0^{e^t} r^\chi \mathbf{1}_{\left(\frac{r}{|A^*u|},\infty\right)}(\langle R, A^*\cdot u\rangle) dr \right) \\
&= \mathbb{E}\left(\frac{N}{\frac{e^t}{|A^*u|} e_*^\chi(u)} \int_0^{\frac{e^t}{|A^*u|}} r^\chi \mathbf{1}_{(r,\infty)}(\langle R, A^*\cdot u\rangle)|A^*u|^\chi dr \right) \\
&= \mathbb{E}\left(\frac{1}{\frac{e^t}{|A^*u|} e_*^\chi(A^*\cdot u)} \int_0^{\frac{e^t}{|A^*u|}} r^\chi \mathbf{1}_{(r,\infty)}(\langle R, A^*\cdot u\rangle) dr \frac{1}{\kappa(\chi)} \frac{e_*^\chi(A^*\cdot u)}{e_*^\chi(u)} |A^*u|^\chi \right) \\
&= \Theta G(u,t).
\end{aligned}$$

Now we have

$$\begin{aligned}
&\Theta^n G(u,t) \\
&= \mathbb{E}_u^{\chi,*}\left(G(X_n, t - V_n)\right) = \mathbb{E}^*\left(G(S_n\cdot u, t - \log|S_n u|) \frac{1}{\kappa^n(\chi)} \frac{e_*^\chi(S_n\cdot u)}{e_*^\chi(u)} |S_n u|^\chi \right) \\
&= N^n \mathbb{E}^*\left(\frac{|S_n u|}{e^t e_*^\chi(S_n\cdot u)} \int_0^{\frac{e^t}{|S_n u|}} r^\chi \mathbf{1}_{(r,\infty)}(\langle R, S_n\cdot u\rangle) \frac{e_*^\chi(S_n\cdot u)}{e_*^\chi(u)} |S_n u|^\chi dr \right) \\
&= N^n \mathbb{E}^*\left(\frac{|S_n u|^{\chi+1}}{e^t e_*^\chi(u)} \int_0^{\frac{e^t}{|S_n u|}} r^\chi \mathbf{1}_{(r,\infty)}(\langle R, S_n\cdot u\rangle) dr \right) \\
&= N^n \mathbb{E}^*\left(\frac{|S_n u|^{\chi+1}}{e^t e_*^\chi(u)} \int_0^{\frac{e^t}{|S_n u|}} r^\chi \mathbf{1}_{(|S_n u| r,\infty)}(\langle S_n^* R, u\rangle) dr \right)
\end{aligned}$$

$$= \frac{N^n}{e^t e_*^\chi(u)} \int_0^{e^t} r^\chi \mathbb{E}^* \left(\mathbf{1}_{(r,\infty)}(\langle S_n^* R, u\rangle) dr \right)$$

$$= \frac{N^n}{e^t e_*^\chi(u)} \int_0^{e^t} r^\chi \mathbb{E} \left(\mathbf{1}_{(r,\infty)}(\langle A_1 \cdot \dots \cdot A_n R, u\rangle) dr \right),$$

where $S_n = A_n \cdot \dots \cdot A_1$. By the continuity of $I_\mu \ni s \mapsto \kappa(s)$ (see Theorem 2.3) we can find $p < \chi$, such that $\kappa(p) = \frac{1-\varepsilon}{N}$, for some $\varepsilon > 0$, then

$$\mathbb{E} \left(\mathbf{1}_{(r,\infty)}(\langle A_1 \cdot \dots \cdot A_n R, u\rangle) \right) \le \frac{\mathbb{E}(\|A_1 \cdot \dots \cdot A_n\|^p)\, \mathbb{E}(|R|^p)}{r^p} \le \frac{C\kappa^n(p) \mathbb{E}(|R|^p)}{r^p}.$$

This implies that

$$\begin{aligned}
\Theta^n G(u,t) &= \frac{N^n}{e^t e_*^\chi(u)} \int_0^{e^t} r^\chi \mathbb{E} \left(\mathbf{1}_{(r,\infty)}(\langle A_1 \cdot \dots \cdot A_n R, u\rangle) dr \right) \\
&\le \frac{CN^n}{e^t e_*^\chi(u)} \int_0^{e^t} r^{\chi-p} \kappa^n(p) \mathbb{E}\left(|R|^p\right) dr \\
&\le \frac{CN^n}{e^t e_*^\chi(u)} \mathbb{E}\left(|R|^p\right) \left(\frac{1-\varepsilon}{N}\right)^n \int_0^{e^t} r^{\chi-p} dr \\
&\le \frac{C\mathbb{E}(|R|^p)}{e_*^\chi(u)} e^{t(\chi-p)} (1-\varepsilon)^n \xrightarrow[n\to\infty]{} 0.
\end{aligned}$$

Now it is easy to see that for any $n \in \mathbb{N}$ we have

$$G(u,t) = g(u,t) + \Theta g(u,t) + \Theta^2 g(u,t) + \dots + \Theta^{n-1} g(u,t) + \Theta^n G(u,t),$$

and (5.5) follows. This completes the proof of Lemma 5.4. □

Lemmas 5.8 and 5.16 below imply that $g(u,t)$ is direct Riemann integrable. Lemmas 5.7, 5.12 and 5.14 contain some necessary technicalities.

Lemma 5.7 *Assume that the hypotheses of Theorem 1.9 are satisfied. Then* $\mathbb{P}(\{\langle R, u\rangle = r\}) = 0$, *for every* $(u, r) \in \mathbb{S}^+ \times \mathbb{R}^+ \cup \{0\}$. *Moreover, for every* $r \ge 0$ *the functions*

$$\mathbb{S}^{d-1} \ni u \mapsto \mathbb{P}(\{\langle R, u\rangle > r\}), \quad \textit{and} \quad \mathbb{S}^{d-1} \ni u \mapsto \mathbb{P}(\{\langle AR, u\rangle > r\}),$$

are continuous.

Proof At the beginning, we assume that the law η of B is nonsingular, i.e. $\|\eta_s\| < 1$. Let ν be the law of R and μ be the law of $A \in G$. Let $*$ be the classical convolution on $\mathbb{R}^d$. Moreover, we define $\xi = \mu *_G \nu$, where $\mu *_G \nu(D) = \int_G \int_{\mathbb{R}^d} \mathbf{1}_D(ax)\nu(dx)\mu(da)$ and $D \in \mathcal{B}or(\mathbb{R}^d)$. Obviously ξ defines a probability measure on $\mathbb{R}^d$ which coincide with the distribution of AR. Notice that $\nu = \xi^{*N} * \eta$, since $R \stackrel{\mathcal{D}}{=} \sum_{i=1}^N A_i R_i + B$, and observe that by the Lebesgue decomposition we obtain

$$\begin{aligned}
\nu_a + \nu_s &= \nu = (\xi_a + \xi_s)^{*N} * (\eta_a + \eta_s) \\
&= \sum_{n=0}^{N} \binom{N}{n} \xi_a^{*n} * \xi_s^{*(N-n)} * \eta_a + \sum_{n=1}^{N} \binom{N}{n} \xi_a^{*n} * \xi_s^{*(N-n)} * \eta_s \\
&\quad + (\xi_s^{*N} * \eta_s)_a + (\xi_s^{*N} * \eta_s)_s,
\end{aligned}$$

and by its uniqueness $\nu_s = (\xi_s^{*N} * \eta_s)_s$. This gives $\|\nu_s\| \le \|\xi_s\|^N \|\eta_s\|$. Again by the Lebesgue decomposition and its uniqueness we have $\xi = \mu *_G \nu = \mu *_G \nu_a + \mu *_G \nu_s$, hence $\|\xi_s\| = \|(\mu *_G \nu)_s\| \le \|\mu *_G \nu_s\| \le \|\nu_s\|$. Now combining $\|\nu_s\| \le \|\xi_s\|^N \|\eta_s\|$ and $\|\xi_s\| \le \|\nu_s\|$ we get $\|\nu_s\| \le \|\nu_s\|^N \|\eta_s\|$, if $\|\nu_s\| > 0$, then $1 \le \|\nu_s\|^{N-1}\|\eta_s\| \le \|\eta_s\| < 1$. This contradiction shows that $\|\nu_s\| = 0$ hence ν is absolutely continuous with respect to the Lebesgue measure, which in turn implies that $\mathbb{P}(\{\langle R, u\rangle = r\}) = 0$, for every $(u, r) \in \mathbb{S}^+ \times \mathbb{R}^+ \cup \{0\}$.

If the law η of B is singular, i.e. $\|\eta_s\| = 1$, then for fixed $(u, r) \in \mathbb{S}^+ \times \mathbb{R}^+ \cup \{0\}$, we have $\mathbb{P}(\{\langle R, u\rangle = r\}) = 0$, since $\mathbb{P}(\{\langle B, u\rangle = r\}) = 0$.

Now we prove that $\mathbb{S}^{d-1} \ni u \mapsto \mathbb{P}(\{\langle R, u\rangle > r\})$ is continuous. Take any $(u_n)_{n\in\mathbb{N}} \subseteq \mathbb{S}^+$ such that $\lim_{n\to\infty} u_n = u \in \mathbb{S}^+$ and consider

$$\begin{aligned}
|\mathbb{P}(\{\langle R, u_n\rangle > r\}) - \mathbb{P}(\{\langle R, u\rangle > r\})| &\le \mathbb{P}(\{\langle R, u_n\rangle > r, \text{ and } \langle R, u\rangle \le r\}) \\
&\quad + \mathbb{P}(\{\langle R, u_n\rangle \le r, \text{ and } \langle R, u\rangle > r\}),
\end{aligned}$$

then

$$\begin{aligned}
\mathbb{P}(\{\langle R, u\rangle \le r < \langle R, u_n\rangle\}) &= \mathbb{P}(\{0 \le r - \langle R, u\rangle < \langle R, u_n\rangle - \langle R, u\rangle\}) \\
&\le \mathbb{P}(\{0 \le r - \langle R, u\rangle \le |R||u_n - u|\}), \quad \text{and} \\
\mathbb{P}(\{\langle R, u_n\rangle \le r < \langle R, u\rangle\}) &= \mathbb{P}(\{\langle R, u_n\rangle - \langle R, u\rangle \le r - \langle R, u\rangle < 0\}) \\
&\le \mathbb{P}(\{-|R||u_n - u| \le r - \langle R, u\rangle < 0\}).
\end{aligned}$$

If $|u_n - u| < 1/m$, then

$$\begin{aligned}
|\mathbb{P}(\{\langle R, u_n\rangle > r\}) - \mathbb{P}(\{\langle R, u\rangle > r\})| &\le \mathbb{P}(\{|\langle R, u\rangle - r| \le |R||u_n - u|\}) \\
&\le \mathbb{P}(\{|\langle R, u\rangle - r| \le |R|/m\}).
\end{aligned}$$

We also know that $\lim_{m\to\infty} \mathbb{P}(\{|\langle R, u\rangle - r| \le |R|/m\}) = \mathbb{P}(\{\langle R, u\rangle = r\}) = 0$, hence

$$\lim_{n\to\infty} |\mathbb{P}(\{\langle R, u_n\rangle > r\}) - \mathbb{P}(\{\langle R, u\rangle > r\})| = 0.$$

The same arguments work for $u \mapsto \mathbb{P}(\{\langle AR, u\rangle > r\})$, since $A \in G$ is independent of R. ◻

Lemma 5.8 *Under the assumptions of Theorem 1.9, there exists* $0 < \beta_1 < 1$ *such that for every* $\beta \in [0, \beta_1)$*, there is a finite constant* $C_\beta > 0$*, such that for every* $(u,t) \in \mathbb{S}^+ \times \mathbb{R}$ *we have*

$$\begin{aligned} g_1(u,t) &= \frac{1}{e^t e_*^\chi(u)} \int_0^{e^t} r^\chi \left| \mathbb{P}\left(\left\{\max_{1\le i\le N} \langle A_i R_i, u\rangle > r\right\}\right) - N\mathbb{P}(\{\langle AR, u\rangle > r\}) \right| dr \\ &\le C_\beta e^{-\beta|t|}, \end{aligned} \tag{5.9}$$

and

$$\begin{aligned} &\int_0^\infty \left(N\mathbb{P}(\{\langle AR, u\rangle > r\}) - \mathbb{P}\left(\left\{\max_{1\le i\le N} \langle A_i R_i, u\rangle > r\right\}\right)\right) r^{\chi+\beta-1} dr \\ &\quad = \frac{1}{\chi+\beta} \mathbb{E}\left(\sum_{i=1}^N \langle A_i R_i, u\rangle^{\chi+\beta} - \left(\max_{1\le i\le N} \langle A_i R_i, u\rangle\right)^{\chi+\beta}\right). \end{aligned} \tag{5.10}$$

Moreover, $\mathbb{S}^+ \times \mathbb{R} \ni (u,t) \mapsto g_1(u,t)$ *is continuous.*

In the proof we extend the approach developed in [13].

Proof Let $\beta_1 \in (0, \min\{1, \chi/2\})$ and take any $0 \le \beta < \beta_1$. Then for every $t > 0$

$$\begin{aligned} I_1(t) &= e^{-\beta t} e^{-(1-\beta)t} \int_0^{e^t} r^\chi \left| \mathbb{P}\left(\left\{\max_{1\le i\le N} \langle A_i R_i, u\rangle > r\right\}\right) - N\mathbb{P}(\{\langle AR, u\rangle > r\}) \right| dr \\ &\le e^{-\beta t} \int_0^{e^t} r^{\chi+\beta-1} \left| \mathbb{P}\left(\left\{\max_{1\le i\le N} \langle A_i R_i, u\rangle > r\right\}\right) - N\mathbb{P}(\{\langle AR, u\rangle > r\}) \right| dr. \end{aligned}$$

Now observe that $N\mathbb{P}(\{\langle AR, u\rangle > r\}) \ge \mathbb{P}\left(\left\{\max_{1\le i\le N} \langle A_i R_i, u\rangle > r\right\}\right)$, then

$$\begin{aligned} &\int_0^1 \left(N\mathbb{P}(\{\langle AR, u\rangle > r\}) - \mathbb{P}\left(\left\{\max_{1\le i\le N} \langle A_i R_i, u\rangle > r\right\}\right)\right) r^{\chi+\beta-1} dr \\ &\quad \le N \int_0^1 r^{\chi+\beta-1} dr < \infty. \end{aligned}$$

Let us define $\overline{F}(y) = \mathbb{P}(\{\langle AR, u\rangle > y\})$, and $\gamma = \chi + \beta - \beta_1$, and notice $N\mathbb{P}(\{\langle AR, u\rangle > r\}) - \mathbb{P}\left(\left\{\max_{1\le i\le N} \langle A_i R_i, u\rangle > r\right\}\right) = (1-\overline{F}(r))^N - 1 + N\overline{F}(r) \le e^{-N\overline{F}(r)} - 1 + N\overline{F}(r)$, and for some $c > 0$

$$\overline{F}(r) = \mathbb{P}(\{\langle AR, u\rangle > r\}) \leq r^{-\gamma}\mathbb{E}\left(\langle AR, u\rangle^{\gamma}\right) \leq cr^{-\gamma}.$$

Clearly, $1 < \frac{\chi+\beta}{\gamma}$, and $\beta_1 < \chi/2$ implies $\gamma = \chi + \beta - \beta_1 \geq \chi/2 + \beta/2$, hence $\frac{\chi+\beta}{\gamma} < 2$. Then

$$\begin{aligned}
&\int_1^\infty \left(N\mathbb{P}(\{\langle AR, u\rangle > r\}) - \mathbb{P}\left(\left\{\max_{1\leq i\leq N}\langle A_i R_i, u\rangle > r\right\}\right)\right) r^{\chi+\beta-1}dr \\
&\leq \int_1^\infty \left(e^{-N\overline{F}(r)} - 1 + N\overline{F}(r)\right) r^{\chi+\beta-1}dr \leq \int_1^\infty \left(e^{-cNr^{-\gamma}} - 1 + cNr^{-\gamma}\right) r^{\chi+\beta-1}dr \\
&= \int_1^\infty \left(e^{-cNr^{-\gamma}} - 1 + cNr^{-\gamma}\right)\left(\left(cNr^{-\gamma}\right)\frac{1}{cN}\right)^{-\frac{\chi+\beta}{\gamma}} \frac{dr}{r} \\
&= \frac{(cN)^{\frac{\chi+\beta}{\gamma}}}{\gamma}\int_0^{cN} (e^{-r} - 1 + r) r^{-\frac{\chi+\beta}{\gamma}-1}dr \leq \frac{(cN)^{\frac{\chi+\beta}{\gamma}}}{\gamma}\int_0^\infty (e^{-r} - 1 + r) r^{-\frac{\chi+\beta}{\gamma}-1}dr \\
&\leq \frac{(cN)^{\frac{\chi+\beta}{\gamma}}}{\gamma}\left(\frac{1}{2}\int_0^1 r^{1-\frac{\chi+\beta}{\gamma}}dr + \int_1^\infty r^{-\frac{\chi+\beta}{\gamma}}dr\right) \\
&= \frac{(cN)^{\frac{\chi+\beta}{\gamma}}}{\gamma}\left(\frac{1}{2\left(2-\frac{\chi+\beta}{\gamma}\right)} + \frac{1}{\frac{\chi+\beta}{\gamma} - 1}\right) < \infty.
\end{aligned}$$

We have shown that $I_1(t) \leq C_\beta e^{-\beta t}$, for every $\beta \in [0, \beta_1)$ and $t \geq 0$ with the constant $C_\beta > 0$ which does not depend on $u \in \mathbb{S}^+$. If is not difficult to see that the statement is clear for $t \leq 0$. A straightforward applications of Fubini theorem yields

$$\begin{aligned}
&\int_0^\infty \left(N\mathbb{P}(\{\langle AR, u\rangle > r\}) - \mathbb{P}\left(\left\{\max_{1\leq i\leq N}\langle A_i R_i, u\rangle > r\right\}\right)\right) r^{\chi+\beta-1}dr \\
&= \int_0^\infty \left(\mathbb{E}\left(\sum_{i=1}^N \mathbf{1}_{\{\langle A_i R_i, u\rangle > r\}}\right) - \mathbb{E}\left(\mathbf{1}_{\{\max_{1\leq i\leq N}\langle A_i R_i, u\rangle > r\}}\right)\right) r^{\chi+\beta-1}dr \\
&= \mathbb{E}\left(\int_0^\infty \left(\sum_{i=1}^N \mathbf{1}_{\{\langle A_i R_i, u\rangle > r\}} - \mathbf{1}_{\{\max_{1\leq i\leq N}\langle A_i R_i, u\rangle > r\}}\right) r^{\chi+\beta-1}dr\right)
\end{aligned}$$

$$= \mathbb{E}\left(\sum_{i=1}^{N} \int_0^{\langle A_i R_i, u\rangle} r^{\chi+\beta-1} dr - \int_0^{\max_{1\le i\le N}\langle A_i R_i, u\rangle} r^{\chi+\beta-1} dr \right)$$

$$= \frac{1}{\chi+\beta}\mathbb{E}\left(\sum_{i=1}^{N} \langle A_i R_i, u\rangle^{\chi+\beta} - \left(\max_{1\le i\le N} \langle A_i R_i, u\rangle \right)^{\chi+\beta} \right).$$

In order to show the continuity of $\mathbb{S}^+ \times \mathbb{R} \ni (u, t) \mapsto g_1(u, t)$ it is enough to prove the continuity of

$$u \mapsto \frac{1}{e^t} \int_0^{e^t} r^{\chi} \left(N\mathbb{P}(\{\langle AR, u\rangle > r\}) - \mathbb{P}\left(\left\{ \max_{1\le i\le N} \langle A_i R_i, u\rangle > r \right\} \right) \right) dr. \quad (5.11)$$

In this purpose observe that $\mathbb{P}\left(\left\{\max_{1\le i\le N} \langle A_i R_i, u\rangle > r\right\}\right) = 1-(1-\mathbb{P}(\{\langle AR, u\rangle > r\}))^N = 1-(1-\overline{F}(r))^N$, where $\overline{F}(y) = \mathbb{P}(\{\langle AR, u\rangle > y\})$, hence Lemma 5.7 guarantees that

$$\begin{aligned} u \mapsto\ & N\mathbb{P}(\{\langle AR, u\rangle > r\}) - \mathbb{P}\left(\left\{ \max_{1\le i\le N} \langle A_i R_i, u\rangle > r \right\} \right) \\ =\ & (1-\overline{F}(r))^N - 1 + N\overline{F}(r), \end{aligned}$$

is continuous. Observe that

$$\begin{aligned} & N\mathbb{P}(\{\langle AR, u\rangle > r\}) - \mathbb{P}\left(\left\{ \max_{1\le i\le N} \langle A_i R_i, u\rangle > r \right\} \right) \\ & \le \begin{cases} N, & \text{if } r \le 1, \\ e^{-N\overline{F}(r)} - 1 + N\overline{F}(r), & \text{if } r > 1, \end{cases} \end{aligned}$$

then arguing in a similar way as above with $\beta = 0$, and using Lebesgue dominated convergence theorem we obtain the continuity of (5.11) and the lemma follows. □

Now we are going to prove inequality (5.13) and (5.15), that will provide necessary estimates for Lemma 5.16. The first one was proved in [13] and was sufficient in the one dimensional case discussed there. The second one is more subtle and allows us to deal with our situation.

Here and subsequently $\lceil \alpha \rceil$ denotes the smallest integer $\ge \alpha$.

Lemma 5.12 *Let $\alpha > 1$ and $p = \lceil \alpha \rceil \ge 2$. For any sequence of nonnegative i.i.d. random variables $Y, Y_1, Y_2, \ldots$ such that $\mathbb{E}(Y^{p-1}) < \infty$, and any $k \in \mathbb{N}$ we have*

$$\mathbb{E}\left(\left(\sum_{i=1}^{k} Y_i \right)^{\alpha} - \sum_{i=i}^{k} Y_i^{\alpha} \right) \le k^{\alpha} \mathbb{E}\left(Y^{p-1} \right)^{\frac{\alpha}{p-1}}. \quad (5.13)$$

Proof As mentioned before the proof is contained in [13]. □

Lemma 5.14 *Let $p \in \mathbb{N}$ and $\beta \in (0,1)$. Then for any $\delta \in \left(0, \frac{p(1-\beta)}{p+1}\right)$, for any sequence of nonnegative i.i.d. random variables $Y, Y_1, Y_2, \dots$ such that $\mathbb{E}(Y^{p-\delta}) < \infty$, and any $k \in \mathbb{N}$ we have*

$$\mathbb{E}\left(\left(\sum_{i=1}^{k} Y_i\right)^{p+\beta} - \sum_{i=1}^{k} Y_i^{p+\beta}\right) \le k^{p+1}\mathbb{E}\left(Y^{p-\delta}\right)^{\frac{p+\beta}{p-\delta}}. \tag{5.15}$$

Proof Define $A_p(k) = \{(j_1, \dots, j_k) \in \mathbb{Z}^k : j_1 + \dots + j_k = p, \text{ and } 0 \le j_i < p\}$ and observe that

$$\begin{aligned}\left(\sum_{i=1}^{k} Y_i\right)^{p-\delta} &= \left(\left(\sum_{i=1}^{k} Y_i\right)^{p}\right)^{\frac{p-\delta}{p}} \\ &= \left(\sum_{i=1}^{k} Y_i^p + \sum_{(j_1,\dots,j_k)\in A_p(k)} \binom{p}{j_1,\dots,j_k} Y_1^{j_1}\cdot\dots\cdot Y_k^{j_k}\right)^{\frac{p-\delta}{p}} \\ &\le \sum_{i=1}^{k} Y_i^{p-\delta} + \sum_{(j_1,\dots,j_k)\in A_p(k)} \binom{p}{j_1,\dots,j_k} \left(Y_1^{j_1}\cdot\dots\cdot Y_k^{j_k}\right)^{\frac{p-\delta}{p}}.\end{aligned}$$

Now observe that $\beta + \delta < \beta + \frac{p(1-\beta)}{p+1} < 1$. By the above inequality

$$\begin{aligned}\left(\sum_{i=1}^{k} Y_i\right)^{p+\beta} &= \left(\sum_{i=1}^{k} Y_i\right)^{p-\delta}\left(\sum_{i=1}^{k} Y_i\right)^{\beta+\delta} \\ &= \left(\left(\sum_{i=1}^{k} Y_i\right)^{p-\delta} - \sum_{i=1}^{k} Y_i^{p-\delta}\right)\left(\sum_{i=1}^{k} Y_i\right)^{\beta+\delta} \\ &\quad + \left(\sum_{i=1}^{k} Y_i^{p-\delta}\right)\left(\sum_{i=1}^{k} Y_i\right)^{\beta+\delta} \\ &\le \left(\sum_{(j_1,\dots,j_k)\in A_p(k)} \binom{p}{j_1,\dots,j_k}\left(Y_1^{j_1}\cdot\dots\cdot Y_k^{j_k}\right)^{\frac{p-\delta}{p}}\right)\left(\sum_{i=1}^{k} Y_i^{\beta+\delta}\right) \\ &\quad + \left(\sum_{i=1}^{k} Y_i^{p-\delta}\right)\left(\sum_{i=1}^{k} Y_i^{\beta+\delta}\right).\end{aligned}$$

It follows that

$$\left(\sum_{i=1}^{k} Y_i\right)^{p+\beta} - \sum_{i=1}^{k} Y_i^{p+\beta} \le \sum_{i=1}^{k} \sum_{(j_1,\dots,j_k)\in A_p(k)} \binom{p}{j_1,\dots,j_k} \left(Y_1^{j_1}\cdot\dots\cdot Y_k^{j_k}\right)^{\frac{p-\delta}{p}} Y_i^{\beta+\delta}$$
$$+ \sum_{i\neq j} Y_i^{p-\delta} Y_j^{\beta+\delta}.$$

But $j_i \le p-1$. Hence $\frac{j_i(p-\delta)}{p}+\beta+\delta \le \frac{1}{p}(p-1)(p-\delta)+\beta+\delta \le p+\beta-1+\frac{\delta}{p} < p-\delta$, since

$$0 < \delta < \frac{p(1-\beta)}{p+1} \Longrightarrow \delta\left(1+\frac{1}{p}\right) < 1-\beta \Longrightarrow \beta-1+\frac{\delta}{p} < -\delta$$
$$\Longrightarrow p+\beta-1+\frac{\delta}{p} < p-\delta.$$

Now we have

$$\mathbb{E}\left(Y_1^{\frac{j_1(p-\delta)}{p}}\cdot\dots\cdot Y_i^{\frac{j_i(p-\delta)}{p}+\beta+\delta}\cdot\dots\cdot Y_k^{\frac{j_k(p-\delta)}{p}}\right)$$
$$\le \|Y\|_{p-\delta}^{\frac{j_1(p-\delta)}{p}}\cdot\dots\cdot \|Y\|_{p-\delta}^{\frac{j_i(p-\delta)}{p}+\beta+\delta}\cdot\dots\cdot \|Y\|_{p-\delta}^{\frac{j_k(p-\delta)}{p}}$$
$$= \|Y\|_{p-\delta}^{p+\beta},$$

because $j_1+\dots+j_k = p$. Observe that

$$\delta < \frac{p(1-\beta)}{p+1} \Longrightarrow \delta < \frac{p-\beta}{2} \Longrightarrow \beta+\delta < p-\delta,$$

hence $\mathbb{E}\left(Y_i^{p-\delta}Y_j^{\beta+\delta}\right) = \|Y\|_{p-\delta}^{p-\delta}\|Y\|_{\beta+\delta}^{\beta+\delta} \le \|Y\|_{p-\delta}^{p+\beta}$, and so

$$\mathbb{E}\left(\left(\sum_{i=1}^{k} Y_i\right)^{p+\beta} - \sum_{i=1}^{k} Y_i^{p+\beta}\right) \le k(k^p-k)\mathbb{E}\left(Y^{p-\delta}\right)^{\frac{p+\beta}{p-\delta}} + k^2\mathbb{E}\left(Y^{p-\delta}\right)^{\frac{p+\beta}{p-\delta}}$$
$$= k^{p+1}\mathbb{E}\left(Y^{p-\delta}\right)^{\frac{p+\beta}{p-\delta}}.$$

□

Lemma 5.16 *Under the assumptions of Theorem 1.9, there exists* $0 < \beta_2 < 1$ *such that for every* $\beta \in [0, \beta_2)$*, there is a finite constant* $C_\beta > 0$*, such that for every* $(u,t) \in \mathbb{S}^+ \times \mathbb{R}$ *we have*

$$g_2(u,t) = \frac{1}{e^t e_*^{\chi}(u)} \int_0^{e^t} r^{\chi} \left| \mathbb{P}(\{\langle R,u\rangle > r\}) - \mathbb{P}\left(\left\{\max_{1\le i\le N} \langle A_i R_i, u\rangle > r\right\}\right)\right| dr$$
$$\le C_\beta e^{-\beta|t|}, \tag{5.17}$$

and

$$\int_0^{\infty} r^{\chi+\beta-1} \left(\mathbb{P}(\{\langle R,u\rangle > r\}) - \mathbb{P}\left(\left\{\max_{1\le i\le N} \langle A_i R_i, u\rangle > r\right\}\right)\right) dr \tag{5.18}$$
$$= \frac{1}{\chi+\beta} \mathbb{E}\left(\langle R,u\rangle^{\chi+\beta} - \left(\max_{1\le i\le N} \langle A_i R_i, u\rangle\right)^{\chi+\beta}\right).$$

Moreover, $\mathbb{S}^+ \times \mathbb{R} \ni (u,t) \mapsto g_2(u,t)$ *is continuous.*

Proof Let $0 < \beta_2 < \min\{\varepsilon, \beta_1\}$ ($\varepsilon > 0$ as in Theorem 1.9 and $\beta_1 > 0$ as in Lemma 5.8) and take $\beta \in [0, \beta_2)$. Then for every $t > 0$

$$I_2(t) = e^{-\beta t} e^{-(1-\beta)t} \int_0^{e^t} r^{\chi} \left| \mathbb{P}(\{\langle R,u\rangle > r\}) - \mathbb{P}\left(\left\{\max_{1\le i\le N} \langle A_i R_i, u\rangle > r\right\}\right)\right| dr$$
$$\le e^{-\beta t} \int_0^{\infty} r^{\chi+\beta-1} \left| \mathbb{P}(\{\langle R,u\rangle > r\}) - \mathbb{P}\left(\left\{\max_{1\le i\le N} \langle A_i R_i, u\rangle > r\right\}\right)\right| dr.$$

Observe that $\langle R,u\rangle \ge \max_{1\le i\le N} \langle A_i R_i, u\rangle$. Then applying Fubini theorem as in Lemma 5.8 we obtain

$$\int_0^{\infty} r^{\chi+\beta-1} \left(\mathbb{P}(\{\langle R,u\rangle > r\}) - \mathbb{P}\left(\left\{\max_{1\le i\le N} \langle A_i R_i, u\rangle > r\right\}\right)\right) dr$$
$$= \frac{1}{\chi+\beta} \mathbb{E}\left(\langle R,u\rangle^{\chi+\beta} - \left(\max_{1\le i\le N} \langle A_i R_i, u\rangle\right)^{\chi+\beta}\right).$$

If $0 < \chi < 1$, take any $\beta \in [0, \beta_2)$ such that $0 < \chi + \beta \le 1$ and notice

$$\mathbb{E}\left(\langle R,u\rangle^{\chi+\beta} - \left(\max_{1\le i\le N} \langle A_i R_i, u\rangle\right)^{\chi+\beta}\right)$$
$$\le \mathbb{E}\left(\langle B,u\rangle^{\chi+\beta}\right) + \mathbb{E}\left(\sum_{i=1}^{N} \langle A_i R_i, u\rangle^{\chi+\beta} - \left(\max_{1\le i\le N} \langle A_i R_i, u\rangle\right)^{\chi+\beta}\right) < \infty,$$

since $\mathbb{E}(|B|^{\chi+\varepsilon}) < \infty$ for some $\varepsilon > 0$, and the second term is finite by Lemma 5.8.

If $\chi \geq 1$ we write

$$\mathbb{E}\left(\langle R, u\rangle^{\chi+\beta} - \left(\max_{1\leq i\leq N}\langle A_i R_i, u\rangle\right)^{\chi+\beta}\right)$$
$$= \mathbb{E}\left(\langle R, u\rangle^{\chi+\beta} - \sum_{i=1}^{N}\langle A_i R_i, u\rangle^{\chi+\beta}\right)$$
$$+ \mathbb{E}\left(\sum_{i=1}^{N}\langle A_i R_i, u\rangle^{\chi+\beta} - \left(\max_{1\leq i\leq N}\langle A_i R_i, u\rangle\right)^{\chi+\beta}\right).$$

We have to estimate only the first term, since the second one is finite by Lemma 5.8. In this purpose we use Lemmas 5.12 and 5.14. Notice that

$$\mathbb{E}\left(\langle R, u\rangle^{\chi+\beta} - \sum_{i=1}^{N}\langle A_i R_i, u\rangle^{\chi+\beta}\right)$$
$$= \mathbb{E}\left(\left\langle \sum_{i=1}^{N} A_i R_i + B, u\right\rangle^{\chi+\beta} - \left\langle \sum_{i=1}^{N} A_i R_i, u\right\rangle^{\chi+\beta}\right)$$
$$+ \mathbb{E}\left(\left\langle \sum_{i=1}^{N} A_i R_i, u\right\rangle^{\chi+\beta} - \sum_{i=1}^{N}\langle A_i R_i, u\rangle^{\chi+\beta}\right)$$
$$\leq (\chi+\beta)\mathbb{E}\left(|B|\left(\sum_{i=1}^{N}|A_i R_i| + |B|\right)^{\chi+\beta-1}\right)$$
$$+ \mathbb{E}\left(\left\langle \sum_{i=1}^{N} A_i R_i, u\right\rangle^{\chi+\beta} - \sum_{i=1}^{N}\langle A_i R_i, u\rangle^{\chi+\beta}\right).$$

$\mathbb{E}\left(|B|\left(\sum_{i=1}^{N}|A_i R_i| + |B|\right)^{\chi+\beta-1}\right)$ is finite, since $\mathbb{E}(\|A\|^{\chi+\beta-1}) < \infty$, $\mathbb{E}(|B|^{\chi+\varepsilon}) < \infty$ and Theorem 1.7 yields $\mathbb{E}(|R|^{\chi+\beta-1}) < \infty$.

If $\chi \notin \mathbb{N}$ we assume additionally that $\lceil \chi + \beta_2 \rceil = \lceil \chi \rceil$, (which holds for sufficiently small $\beta_2 > 0$). Applying inequality (5.13) with $p = \lceil \chi \rceil = \lceil \chi + \beta \rceil$ and $\beta \in [0, \beta_2)$ we obtain

$$\mathbb{E}\left(\left\langle \sum_{i=1}^{N} A_i R_i, u\right\rangle^{\chi+\beta} - \sum_{i=1}^{N}\langle A_i R_i, u\rangle^{\chi+\beta}\right) \leq N^{\chi+\beta}\left(\mathbb{E}\left(\langle AR, u\rangle^{p-1}\right)\right)^{\frac{\chi+\beta}{p-1}} < \infty,$$

since $p - 1 < \chi$.

If $\chi \in \mathbb{N}$ and $\beta \in [0, \beta_2)$ take any $\delta \in \left(0, \frac{p(1-\beta)}{p+1}\right)$ as in Lemma 5.14 with $p = \chi$, then by inequality (5.15) we get

$$\mathbb{E}\left(\left\langle \sum_{i=1}^{N} A_i R_i, u \right\rangle^{\chi+\beta} - \sum_{i=1}^{N} \langle A_i R_i, u\rangle^{\chi+\beta}\right) \le N^{\chi+1}\left(\mathbb{E}\left(\langle AR, u\rangle^{\chi-\delta}\right)\right)^{\frac{\chi+\beta}{\chi-\delta}} < \infty.$$

Finally, we have proved $I_2(t) \le C_\beta e^{-\beta|t|}$, for every $\beta \in [0, \beta_2)$ and $t > 0$ with $C_\beta < \infty$ independent of $u \in \mathbb{S}^+$. If $t \le 0$ there is nothing to do and the statement follows.

It remains to prove that $\mathbb{S}^+ \times \mathbb{R} \ni (u, t) \mapsto g_2(u, t)$ is continuous. In this purpose it suffices to show continuity of

$$u \mapsto \frac{1}{e^t}\int_0^{e^t} r^\chi \left(\mathbb{P}(\{\langle R, u\rangle > r\}) - \mathbb{P}\left(\left\{\max_{1\le i\le N} \langle A_i R_i, u\rangle > r\right\}\right)\right) dr. \tag{5.19}$$

Observe that

$$\begin{aligned}
&\frac{1}{e^t}\int_0^{e^t} r^\chi \Bigg|\mathbb{P}(\{\langle R, u_n\rangle > r\}) - \mathbb{P}\left(\left\{\max_{1\le i\le N} \langle A_i R_i, u_n\rangle > r\right\}\right) \\
&\quad - \left(\mathbb{P}(\{\langle R, u_0\rangle > r\}) - \mathbb{P}\left(\left\{\max_{1\le i\le N} \langle A_i R_i, u_0\rangle > r\right\}\right)\right)\Bigg| dr \\
&\le \int_0^{\infty} r^{\chi-1} \Bigg|\mathbb{P}(\{\langle R, u_n\rangle > r\}) - \mathbb{P}\left(\left\{\max_{1\le i\le N} \langle A_i R_i, u_n\rangle > r\right\}\right) \\
&\quad - \left(\mathbb{P}(\{\langle R, u_0\rangle > r\}) - \mathbb{P}\left(\left\{\max_{1\le i\le N} \langle A_i R_i, u_0\rangle > r\right\}\right)\right)\Bigg| dr.
\end{aligned}$$

It is enough to show that the last integral converges to 0 as $\lim_{n\to\infty} u_n = u_0$. In this purpose we will use an extended version of Lebesgue dominated convergence theorem (see for instance in [1]). Namely,

Theorem 5.20 *Given a measure space* $(X, \mathcal{M}, \mu)$ (*where* μ *may take values in* $[0, \infty]$). *Let* $(f_n)_{n\in\mathbb{N}}$ *and* $(h_n)_{n\in\mathbb{N}}$, f *and* h *be* $\mathcal{M}$ *measurable, real valued functions on* X. *Suppose*

- $\lim_{n\to\infty} f_n = f$ *and* $\lim_{n\to\infty} h_n = h$ *a.e. on* X,
- $(h_n)_{n\in\mathbb{N}}$ *and* h *are all* μ *integrable on* X *and* $\lim_{n\to\infty} \int_X h_n d\mu = \int_X h d\mu$,
- $|f_n| \le h_n$ *a.e. on* X *for every* $n \in \mathbb{N}$.

Then f *is* μ *integrable on* X *and* $\lim_{n\to\infty} \int_X f_n d\mu = \int_X f d\mu$.

We will apply Theorem 5.20 with

$$f_n(r) = r^{\chi-1}\Bigg|\mathbb{P}(\{\langle R, u_n\rangle > r\}) - \mathbb{P}\left(\left\{\max_{1\le i\le N}\langle A_i R_i, u_n\rangle > r\right\}\right) - \left(\mathbb{P}(\{\langle R, u_0\rangle > r\}) - \mathbb{P}\left(\left\{\max_{1\le i\le N}\langle A_i R_i, u_0\rangle > r\right\}\right)\right)\Bigg|,$$

$$h_n(r) = r^{\chi-1}\Bigg(\mathbb{P}(\{\langle R, u_n\rangle > r\}) - \mathbb{P}\left(\left\{\max_{1\le i\le N}\langle A_i R_i, u_n\rangle > r\right\}\right) + \left(\mathbb{P}(\{\langle R, u_0\rangle > r\}) - \mathbb{P}\left(\left\{\max_{1\le i\le N}\langle A_i R_i, u_0\rangle > r\right\}\right)\right)\Bigg),$$

and

$$h(r) = 2r^{\chi-1}\left(\mathbb{P}(\{\langle R, u_0\rangle > r\}) - \mathbb{P}\left(\left\{\max_{1\le i\le N}\langle A_i R_i, u_0\rangle > r\right\}\right)\right).$$

Clearly, $|f_n| \le h_n$ for every $n \in \mathbb{N}$, and $(h_n)_{n\in\mathbb{N}}$ and h are all integrable. Lemma 5.7 guarantees that $\lim_{n\to\infty} f_n(r) = 0$ and $\lim_{n\to\infty} h_n(r) = h(r)$. In order to show that $\lim_{n\to\infty}\int_0^\infty h_n(r)dr = \int_0^\infty h(r)dr$, notice that by (5.18) with $\beta = 0$ we have to show that

$$\lim_{n\to\infty}\mathbb{E}\left(\langle R, u_n\rangle^\chi - \left(\max_{1\le i\le N}\langle A_i R_i, u_n\rangle\right)^\chi\right) = \mathbb{E}\left(\langle R, u_0\rangle^\chi - \left(\max_{1\le i\le N}\langle A_i R_i, u_0\rangle\right)^\chi\right). \tag{5.21}$$

But in view of the first part of this lemma and the estimates given there (5.21) is a simple consequence of a classical Lebesgue dominated convergence theorem. This finishes the proof of Lemma 5.16. □

Proof of Theorem 1.9 From Lemma 5.2 we know that

$$G(u, t) = \sum_{n=0}^{\infty}\Theta^n g(u, t),$$

where

$$g(u, t) = \frac{1}{e^t e_*^\chi(u)}\int_0^{e^t} r^\chi\left(\mathbb{P}(\{\langle R, u\rangle > r\}) - N\mathbb{P}(\{\langle AR, u\rangle > r\})\right)dr.$$

As a consequence of Lemmas 5.8 and 5.16 the function $\mathbb{S}^+ \times \mathbb{R} \ni (u, t) \mapsto g(u, t)$ is jointly continuous. Moreover, it is possible to find $\beta > 0$ and a positive constant $C_\beta < \infty$ such that

$$|g(u,t)| \le C_\beta e^{-\beta|t|}, \quad \text{for every } (u,t) \in \mathbb{S}^+ \times \mathbb{R},$$

since $|g(u,t)| \le g_1(u,t) + g_2(u,t)$, for $g_1(u,t)$ and $g_2(u,t)$ defined in Lemmas 5.8 and 5.16, respectively. This shows that $g(u,t)$ satisfies condition (4.24). By the Kesten's renewal theorem 4.19 we obtain

$$\lim_{t\to\infty} G(u,t) = \lim_{t\to\infty} \mathbb{E}_x^{\chi,*}\left(\sum_{n=0}^{\infty} g(X_n, t - V_n)\right)$$
$$= \frac{1}{\alpha(\chi)} \int_{\mathbb{S}^+} \left(\int_{\mathbb{R}} g(y,x)dx\right) \pi_*^\chi(dy) = C_\chi.$$

In other words we have proved that for every $u \in \mathbb{S}^+$

$$\lim_{t\to\infty} G(u,t) = \lim_{t\to\infty} \frac{1}{e^t e_*^\chi(u)} \int_0^{e^t} r^\chi \mathbb{P}(\{\langle R, u\rangle > r\})dr = C_\chi \ge 0.$$

Hence in view of Lemma 9.3 of [5], for every $u \in \mathbb{S}^+$

$$\lim_{t\to\infty} t^\chi \mathbb{P}(\{\langle R, u\rangle > t\}) = C_\chi e_*^\chi(u).$$

It remains to prove that $C_\chi > 0$ for every $\chi \ge 1$. (It is worth emphasizing, as we mentioned in the discussion given after Theorem 1.9, that the case when $\chi < 1$ is unavailable at the moment. However, some positive results in this direction can be found in [4].) In this purpose notice that

$$C_\chi = \frac{1}{\alpha(\chi)} \int_{\mathbb{S}^+} \left(\int_{\mathbb{R}} g(u,t)dt\right) \pi_*^\chi(du)$$
$$= \frac{1}{\alpha(\chi)} \int_{\mathbb{S}^+}\int_{\mathbb{R}} \left(\frac{1}{e^t e_*^\chi(u)} \int_0^{e^t} r^\chi (\mathbb{P}(\{\langle R,u\rangle > r\}) - N\mathbb{P}(\{\langle AR,u\rangle > r\}))dr\right) dt\pi_*^\chi(du)$$
$$= \frac{1}{\alpha(\chi)} \int_{\mathbb{S}^+}\int_{\mathbb{R}} \left(\frac{1}{e^t e_*^\chi(u)} \int_{-\infty}^{t} e^{s(\chi+1)} (\mathbb{P}(\{\langle R,u\rangle > e^s\}) - N\mathbb{P}(\{\langle AR,u\rangle > e^s\}))ds\right) dt\pi_*^\chi(du)$$
$$= \frac{1}{\alpha(\chi)} \int_{\mathbb{S}^+}\int_{\mathbb{R}}\int_s^\infty \left(\frac{e^{s(\chi+1)}}{e^t e_*^\chi(u)} (\mathbb{P}(\{\langle R,u\rangle > e^s\}) - N\mathbb{P}(\{\langle AR,u\rangle > e^s\}))dt\right) ds\pi_*^\chi(du)$$
$$= \frac{1}{\alpha(\chi)} \int_{\mathbb{S}^+}\int_{\mathbb{R}} \frac{e^{s\chi}}{e_*^\chi(u)} (\mathbb{P}(\{\langle R,u\rangle > e^s\}) - N\mathbb{P}(\{\langle AR,u\rangle > e^s\}))ds\pi_*^\chi(du)$$
$$= \frac{1}{\alpha(\chi)} \int_{\mathbb{S}^+} \frac{1}{e_*^\chi(u)} \int_0^\infty r^{\chi-1} (\mathbb{P}(\{\langle R,u\rangle > r\}) - N\mathbb{P}(\{\langle AR,u\rangle > r\}))dr\pi_*^\chi(du)$$

$$
\begin{aligned}
&= \frac{1}{\alpha(\chi)} \int_{\mathbb{S}^+} \frac{1}{e_*^\chi(u)} \int_0^\infty \mathbb{E}\left(\mathbf{1}_{\left\{\left\langle \sum_{i=1}^N A_i R_i + B, u\right\rangle > r\right\}} - \sum_{i=1}^N \mathbf{1}_{\{\langle A_i R_i, u\rangle > r\}} \right) r^{\chi-1} dr \pi_*^\chi(du) \\
&= \frac{1}{\alpha(\chi)\chi} \int_{\mathbb{S}^+} \frac{1}{e_*^\chi(u)} \mathbb{E}\left(\left\langle \sum_{i=1}^N A_i R_i + B, u \right\rangle^\chi - \sum_{i=1}^N \langle A_i R_i, u\rangle^\chi \right) \pi_*^\chi(du) \\
&\geq \frac{1}{\alpha(\chi)\chi} \int_{\mathbb{S}^+} \frac{1}{e_*^\chi(u)} \mathbb{E}\left(\langle B, u\rangle^\chi\right) \pi_*^\chi(du),
\end{aligned}
$$

since we have used (the fact that we are working with positive matrices and $\chi \geq 1$ are indispensable)

$$
\left(\sum_{i=1}^N \langle A_i R_i, u\rangle^\chi + \langle B, u\rangle^\chi \right)^{1/\chi} \leq \sum_{i=1}^N \langle A_i R_i, u\rangle + \langle B, u\rangle.
$$

We need only to show that

$$
\int_{\mathbb{S}^+} \frac{1}{e_*^\chi(u)} \mathbb{E}\left(\langle B, u\rangle^\chi\right) \pi_*^\chi(du) > 0. \tag{5.22}
$$

We will show that there exists $c_\chi > 0$ such that

$$
\int_{\mathbb{S}^+} \langle x, u\rangle^\chi \, \pi_*^\chi(du) \geq c_\chi \|x\|^\chi, \tag{5.23}
$$

for every $x \in \mathbb{R}_+^d$. Observe that $\mathbb{S}^+ \ni x \mapsto \int_{\mathbb{S}^+} \langle x, u\rangle^\chi \pi_*^\chi(du)$ is continuous and nonzero for every $x \in \mathbb{S}^+$, since $\mathrm{supp}\pi_*^\chi$ is not contained in any proper subspace of $\mathbb{S}^+$ (see Sect. 2). This allows us to conclude that $x \mapsto \int_{\mathbb{S}^+} \langle x, u\rangle^\chi \pi_*^\chi(du)$ attains its minimum $c_\chi > 0$ on $\mathbb{S}^+$, and in fact this proves (5.23).

In order to prove (5.22) notice that by (5.23) we obtain

$$
\begin{aligned}
&\int_{\mathbb{S}^+} \frac{1}{e_*^\chi(u)} \mathbb{E}\left(\langle B, u\rangle^\chi\right) \pi_*^\chi(du) \\
&\quad \geq \frac{1}{\sup_{u\in\mathbb{S}^+} e_*^\chi(u)} \int_{\mathbb{S}^+} \mathbb{E}\left(\langle B, u\rangle^\chi\right) \pi_*^\chi(du) \\
&\quad \geq \frac{1}{\sup_{u\in\mathbb{S}^+} e_*^\chi(u)} \mathbb{E}\left(\int_{\mathbb{S}^+} \langle B, u\rangle^\chi \pi_*^\chi(du) \right) \\
&\quad \geq \frac{c_\chi}{\sup_{u\in\mathbb{S}^+} e_*^\chi(u)} \mathbb{E}\left(\|B\|^\chi\right) > 0,
\end{aligned}
$$

since $\mathbb{P}(\{B > 0\}) > 0$. This completes the proof of Theorem 1.9. □

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