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Smoothness of densities for area-like processes of fractional Brownian motion

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Abstract We consider a process given by a n-dimensional fractional Brownian motion with Hurst parameter $\frac{1}{4} < H < \frac{1}{2}$, along with an associated Lévy area-like process, and prove the smoothness of the density for this process with respect to Lebesgue measure.

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1 Introduction

Let $B_t := (B_t^1, B_t^2, \ldots, B_t^n),\ t \in [0, T]$ be n-dimensional fractional Brownian motion of Hurst parameter $\frac{1}{4} < H < \frac{1}{2}$, and let

$$(B_m)_t := \Big((B_m^1)_t, (B_m^2)_t, \ldots, (B_m^n)_t\Big)$$

denote the mth dyadic approximation of B (as defined below in Sect. 2.2). Suppose α is an alternating bilinear map from $\mathbb{R}^n \times \mathbb{R}^n$ to $\mathbb{R}^k$. Define the sequence of area-like process approximations

$$(A_m)_t := \frac{1}{2}\left[\int_0^t \alpha\left((B_m)_s, d(B_m)_s\right)\right], \tag{1.1}$$

and $A_t := \lim_{m\to\infty}(A_m)_t$ (where this convergence is almost sure - see Theorem 2 of [8]).

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The main result of this paper is as follows:

Theorem 1.1 *Define the random process* $\{Y\}_{0\leq t\leq T}$*, taking values in* $\mathbb{R}^3$*, by*

$$\begin{aligned} Y_0 &= 0, \\ Y_t &= (B_t, A_t). \qquad (t \in (0, T]) \end{aligned}$$

Then for all $t \in [0, T]$*, the density of* Y_t *with respect to Lebesgue measure is* C^∞*.*

The investigation of this process is motivated by the potential for fractional Brownian motion to be a useful driving signal in stochastic differential equations that model a wide variety of natural and financial phenomena; in particular, the presence of long-range persistence (for $H > 1/2$) or anti-persistence (for $H < 1/2$) makes fBm a natural candidate for a driving process in many scenarios. Several examples of such applications are included in [21] and [18].

One area of interest in the study of stochastic differential equations is on finding sufficient conditions for existence and regularity of densities for solutions. More specifically, given some solution $\{Y_t\}$ to the equation

$$dY_t = \sum_{i=1}^{d} X_i(Y_t) d\xi_t, \tag{1.2}$$

where $\{X_i\}$ is some collection of vector fields and ξ_t is a Gaussian driving process on the space $\mathcal{C}([0, T], \mathbb{R}^d)$, it is natural ask whether ξ admits a density with respect to Lebesgue measure. It is due to the celebrated theorem of Hörmander (see, for example, Theorem 38.16 of [20]) that, when our driving process is standard Brownian motion, our solution admits a smooth density so long as the set of vectors $\{X_i, [X_i, X_j], [[X_i, X_j], X_k] \ldots\}$ spans $\mathbb{R}^d$.

In the case of fractional Brownian motion, one may no longer appeal to the types of martingale arguments used in proofs of the above result in the standard case. When $H > \frac{1}{2}$, the positive correlation of increments of sample paths results in better variational properties than those of Brownian motion, and so one may use Young's integration theory to attack the problem—existence of a density to a solution of (1.2) under this condition is proven in [19], and smoothness is proven in [1]. When $H < \frac{1}{2}$, one must turn to the rough path theory of T. Lyons (see [15]) in order to interpret (1.2) in a meaningful manner. The connection between fractional Brownian motion with $\frac{1}{4} < H < \frac{1}{2}$ and rough paths is investigated in [8]. Existence of a density is proven in [7] in the case when the vector fields satisfy an ellipticity condition and $\frac{1}{3} < H < \frac{1}{2}$; the same results under the more general Hörmander hypoellipticity condition above in the case of $\frac{1}{4} < H < \frac{1}{2}$ is show in [6]. As far as we are aware, Theorem 1.1 is the first positive result involving smoothness, and may give hope that similar results will hold in a more general setting.

In order to prove Theorem 1.1, we appeal to the usual technique of Malliavin calculus. It follows from Theorem 5.1 of [16] that it is enough to show that for each $t \in (0, T]$, the random variable Y_t satisfies the following two conditions:

1. $Y_t \in \mathbb{D}^\infty$;
2. If $\gamma = DY_t(DY_t)^*$ is the Malliavin covariance matrix associated to Y_t, then

$$(\det \gamma)^{-1} \in L^{\infty-}(\mathcal{W}^n, \mathbb{P}) := \bigcap_{j \geq 1} L^j(\mathcal{W}^n, \mathbb{P}),$$

where $\mathcal{W}^n$ is the path-space associated to the driving fractional Brownian motion.

Without loss of generality, it will suffice if we restrict our attention to the end-time random variable Y_T.

We will begin by calculating the derivative DY_T explicitly, and from this Condition 1 will be proven in Proposition 3.8. Condition 2 will then follow by direct analysis of the Malliavin covariance matrix; see Theorem 3.16.

2 Background

2.1 Fractional Brownian motion

A (one-dimensional) fractional Brownian motion $\{B_t^H; \ (t \in [0, T])\}$ of Hurst parameter $H \in [0, 1]$ is a continuous-time centered Gaussian process with covariance given by

$$\mathbb{E}(B_s^H B_t^H) = R(s, t) := \frac{1}{2}\left(s^{2H} + t^{2H} - |t - s|^{2H}\right)$$

Our focus will be on the case when $\frac{1}{4} < H < \frac{1}{2}$; henceforth, we shall assume that such an H has been fixed and will drop the parameter from our notation whenever possible to do so without causing confusion.

An n-dimensional fractional Brownian motion $\{B_t = (B_t^1, \ldots, B_t^n); \ t \in [0, T]\}$ is a continuous-time process comprised of n independent copies of one-dimensional fractional Brownian motion, each having the same Hurst parameter H.

It is straightforward to check that the process B satisfies a self-similarity property; that is to say, the processes B_{at} and $a^{-H} B_t$ are equal in distribution. By Kolmogorov's continuity criterion, the sample paths $t \mapsto B_t$ are almost surely Hölder continuous of order α, for any $\alpha < H$—see Theorem 1.6.1 of [3] for details of this proof.

2.2 Dyadic approximation

For each m, we will let $D_m := \{k2^{-m}T; k = 0, 1, \ldots, 2^m\}$. We define the mth dyadic approximator $\pi_m : \mathcal{C}([0, T], \mathbb{R}^d) \longrightarrow \mathcal{C}([0, T], \mathbb{R}^d)$ as the unique projection operator such that, for any given $f \in \mathcal{C}([0, T], \mathbb{R}^d)$,

$$\pi_m f(t) = f(t), \quad (t \in D_m)$$
$$\frac{d^2}{dt^2}\pi_m f(t) = 0. \quad (t \notin D_m)$$

In words, $\pi_m f$ is nothing more than the piecewise linear path agreeing with f on the set D_m. We will regularly use the shorthand notation $f_m := \pi_m f$ where convenient.

Similarly, we will define the mth dyadic approximation of fractional Brownian motion $B_m := \pi_m B$; more explicitly,

$$(B_m)_t := B_{t_-} + (t - t_-)2^m[B_{t_+} - B_{t_-}], \quad (0 \leq t \leq T)$$

where t_- is the largest member of D_m such that $t_- \leq t$ and t_+ is the smallest member of D_m such that $t \leq t_+$.

2.3 Euclidean p-variation

Let $\mathcal{P}[0, T]$ denote the set of finite partitions of $[0, T]$. Suppose we are given a path $f \in \mathcal{C}([0, T], \mathbb{R}^d)$; then for each $1 \leq p < \infty$ and $\Pi = \{0 = t_1, t_2, \ldots, t_N = 1\} \in \mathcal{P}[0, T]$, one may define the quantities

$$\begin{aligned} \Delta_i f &:= f(t_i) - f(t_{i-1}), \\ V_p(f : \Pi) &:= \left(\sum_{i=1}^{N} |\Delta_i f|^p \right)^{\frac{1}{p}}, \\ \|f\|_p &:= \sup_{\Pi \in \mathcal{P}[0,T]} V_p(f : \Pi). \end{aligned}$$

The norm $\|\cdot\|_p$ is referred to as the *p-variation norm*; we shall define the space $\mathcal{C}_p(\mathbb{R}^d) := \{f \in \mathcal{C}([0, T], \mathbb{R}^d);\ \|f\|_p < \infty\}$, which is a Banach space under $\|\cdot\|_p$. We will regularly refer to these spaces as $\mathcal{C}_p$ when the underlying Euclidean image space is clear from context. It is easy to check that for given α and p such that $\alpha < \frac{1}{p}$, any α-Hölder continuous function is in $\mathcal{C}_p$.

It will also be worthwhile to define the normalized variational spaces

$$\mathcal{C}_{0,p} := \{f \in \mathcal{C}_p : f(0) = \mathbf{0}\}.$$

Using the definition of the variational norm, it is straightforward to check that the identity embedding of $\mathcal{C}_{0,p}$ into $\mathcal{C}$ under the uniform norm $\|\cdot\|_u$ is a contraction.

Given $f \in \mathcal{C}_p$, $g \in \mathcal{C}_q$, where p and q are such that $\frac{1}{p} + \frac{1}{q} > 1$, one can develop the notion of integration of f against g in the following manner: if $\{\Pi_n := \{t_i\}\} \subset \mathcal{P}[0, T]$ is a collection of partitions such that the mesh size $|\Pi_n|$ tends to 0 as $n \to \infty$, we define

$$\int_0^T f\, dg := \lim_{n\to\infty} \sum_{i=1}^{\#(\Pi_n)} f(c_i)\Delta_i g$$

where $c_i \in (t_{i-1}, t_i)$. This limit is guaranteed to exist under the assumptions presented, and is independent of the choice we make of the family of partitions so long as their mesh size tends to zero - see Theorem 3.3.1 of [14] for further details. The element $\int_0^1 f\, dg$ is referred to as *the Young's integral of f against g*. This expression was originally formulated in [26]. We have the following estimate for this expression (see Formula 10.9 of [26]):

$$\left| \int_0^T [f - f(0)] \, dg \right| \le C \|f\|_p \|g\|_q, \tag{2.1}$$

where the constant C depends only on the values of p and q.

Similarly, given some $f \in \mathcal{C}([0,T]^2, \mathbb{R}^d)$, and partitions $\Pi_1 = \{s_i\}$, $\Pi_2 = \{t_j\} \in \mathcal{P}[0,T]$, we may define

$$\begin{aligned} \Delta_{ij} f &:= f(s_i, t_j) - f(s_i, t_{j-1}) - f(s_{i-1}, t_j) + f(s_{i-1}, t_{j-1}), \\ V_p(f : \Pi_1, \Pi_2) &:= \left(\sum_{i=1}^{\#(\Pi_1)} \sum_{j=1}^{\#(\Pi_2)} |\Delta_{ij} f|^p \right)^{\frac{1}{p}}, \\ \|f\|_p^{(2D)} &:= \sup_{\Pi_1, \Pi_2 \in \mathcal{P}[0,T]} V_p(f : \Pi_1, \Pi_2). \end{aligned}$$

As in the (one-dimensional) case above, we shall define the spaces

$$\begin{aligned} \mathcal{C}_p^{(2D)} = \mathcal{C}_p^{(2D)}(\mathbb{R}^d) &:= \{ f \in \mathcal{C}([0,T]^2, \mathbb{R}^d); \ \|f\|_p^{(2D)} < \infty \} \\ \mathcal{C}_{0,p}^{(2D)} &:= \{ f \in \mathcal{C}_p^{(2D)} : f(0, \cdot) = f(\cdot, 0) = \mathbf{0} \} \end{aligned}$$

It is helpful to note that for each $f \in \mathcal{C}_{0,p}^{(2D)}$ and fixed $s \in [0,T]$, the function $f(s,\cdot) \in \mathcal{C}_p$ and $\|f(s,\cdot)\|_p \le \|f\|_p^{(2D)}$, since

$$\begin{aligned} \sum_{i=1}^{\#(\Pi)} |\Delta_i f(s,\cdot)|^p &= \sum_{i=1}^{\#(\Pi)} |\Delta_i f(s,\cdot) - \Delta_i f(0,\cdot)|^p \\ &\le \sum_{i=1}^{\#(\Pi)} \big(|\Delta_i f(s,\cdot) - \Delta_i f(0,\cdot)|^p \\ &\quad + |\Delta_i f(T,\cdot) - \Delta_i f(s,\cdot)|^p \big) \le \left(\|f\|_p^{(2D)} \right)^p. \end{aligned}$$

Trivially, one also has that for each $f, g \in \mathcal{C}_p(\mathbb{R}^d)$, the function $f \otimes g : [0,T]^2 \to \mathbb{R}^d$ defined by

$$(f \otimes g)(s,t) := f(s) g(t)$$

is contained in $\mathcal{C}_p^{(2D)}$, and $\|f \otimes g\|_p^{(2D)} \le \|f\|_p \|g\|_p$.

Given $f \in \mathcal{C}_p^{(2D)}$ and $g \in \mathcal{C}_q^{(2D)}$, where p and q are such that $\frac{1}{p} + \frac{1}{q} > 1$, the *2D-Young's integral of f against g*, denoted by $\int_{[0,T]^2} f\, dg$, is defined by

$$\int_{[0,T]^2} f\, dg := \lim_{n\to\infty} \sum_{i=1}^{\#(\Pi_n)} \sum_{j=1}^{\#(\Psi_n)} f(c_i, d_j)\Delta_{ij} g,$$

where, as before, $\{\Pi_n := \{t_i\}\}, \{\Psi_n := \{s_j\}\} \subset \mathcal{P}[0, T]$ are collections of partitions such that the maximum mesh size $|\Pi_n| \vee |\Psi_n|$ tends to 0 as $n \to \infty$ and $c_i \in (t_{i-1}, t_i)$, $d_j \in (s_{j-1}, s_j)$. Existence of this limit under the given assumptions, independent of the choice of the family of partitions, is proven in Theorem 1.2 of [23], as is an estimate similar to that of the one-dimensional case:

$$\left| \int_{[0,T]^2} f\, dg \right| \leq C\|g\|_q^{(2D)} \Big(\|f\|_p^{(2D)} + \|f(0,\cdot)\|_p + \|f(\cdot, 0)\|_p + |f(0,0)| \Big).$$

Lemma 2.1 (1) *For each $1 \leq p < q$, $\mathcal{C}_{0,p} \subset \mathcal{C}_{0,q}$; in particular, if $f \in \mathcal{C}_{0,p}$, one has the bound*

$$\|f\|_q \leq (2\|f\|_u)^{1-\frac{p}{q}} \|f\|_p^{\frac{p}{q}},$$

where $\|\cdot\|_u$ denotes the uniform norm on $\mathcal{C}([0, T])$.

(2) *For each $1 \leq p < q$, $\mathcal{C}_{0,p}^{(2D)} \subset \mathcal{C}_{0,q}^{(2D)}$; in particular, if $f \in \mathcal{C}_{0,p}^{(2D)}$, one has the bound*

$$\|f\|_q^{(2D)} \leq (4\|f\|_u)^{1-\frac{p}{q}} \left(\|f\|_p^{(2D)}\right)^{\frac{p}{q}}.$$

Proof Pick $f \in \mathcal{C}_{0,p}$, and let $q > p$. Then it is immediately clear that for each $s, t \in [0, T]$,

$$|f(t) - f(s)| \leq 2\|f\|_u.$$

The claim then follows from noting that for each $\Pi \in \mathcal{P}[0, T]$,

$$\begin{aligned}\big(V_q(f : \Pi)\big)^q &= \sum_{i=1}^{N} |\Delta_i f|^q = \sum_{i=1}^{N} |\Delta_i f|^p\, |\Delta_i f|^{q-p} \\ &\leq \sup_{\Pi} |\Delta_i f|^{q-p} \sum_{i=1}^{N} |\Delta_i f|^p \leq (2\|f\|_u)^{q-p} \big(V_p(f : \Pi)\big)^p.\end{aligned}$$

As usual, we take the supremum over partitions to complete the proof. □

It will be helpful to record here a pair of results relating the variation of paths with their linear approximations.

Theorem 2.2 (Propositions 5.20 and 5.60 of [12])

(1) *Suppose $x \in \mathcal{C}_p(U)$, and let $x_m := \pi_m x$ be the dyadic approximation to x as defined above. Then one has that*

$$\|x_m\|_p \leq 3^{p-1}\|x\|_p.$$

(2) *Suppose $x \in \mathcal{C}_p^{(2D)}(U)$, and let $x_m := \pi_m x$ be the dyadic approximation to x as defined above. Then one has that*

$$\|x_m\|_p^{(2D)} \leq 9^{p-1}\|x\|_p^{(2D)}.$$

For further development of the theory, the interested reader may look in [10,26], and [23].

2.4 Gaussian measure spaces

Let $(\mathcal{W}, \|\cdot\|)$ denote a separable Banach space. We will say that a measure $\mathbb{P}$ on $\mathcal{W}$ is Gaussian if there exists a symmetric bilinear form $q : \mathcal{W}^* \times \mathcal{W}^* \longrightarrow \mathbb{R}$ such that for all $\varphi \in \mathcal{W}^*$,

$$\int\limits_{\mathcal{W}} \exp\left(i\varphi(\omega)\right)\, d\mathbb{P}(\omega) = \exp\left(-\frac{1}{2}q(\varphi,\varphi)\right).$$

Let $\mathcal{B}$ refer to the Borel σ-algebra on $\mathcal{W}$; we will call the triple $(\mathcal{W}, \mathcal{B}, \mathbb{P})$ a Gaussian space. Define a continuous mapping $J : L^2(\mathbb{P}) \to \mathcal{W}$ by

$$Jf := \int\limits_{\mathcal{W}} \omega f(\omega)\, d\mathbb{P}(\omega).$$

Let $\mathcal{H}$ denote the image of J restricted to the space $\overline{\mathcal{W}^*}^{L^2(\mathbb{P})}$; this space may be equipped with inner product given by

$$\langle Jf, Jg\rangle_{\mathcal{H}} = \langle f, g\rangle_{L^2(\mathbb{P})}.$$

We will refer to $\mathcal{H}$ as the *Cameron–Martin space* associated to the Gaussian space $(\mathcal{W}, \mathcal{B}, \mathbb{P})$. More information regarding the construction of these spaces may be found in [4,9], and [13].

A well-known example of a Gaussian measure space is the one associated to Brownian motion where

$$\mathcal{W} := \{\omega \in \mathcal{C}([0, T], \mathbb{R}) : \omega(0) = 0\}$$

and the Gaussian measure $\mathbb{P}$ on $\mathcal{W}$ is the law of standard Brownian motion. Details of the construction of this measure may be found in [24] and [25]. In this case, the Cameron–Martin space is

$$\mathcal{H} = \mathcal{H}_{\frac{1}{2}} := \left\{ h \in \mathcal{W} : h(s) = \int_0^s \phi(u)\, du; \phi \in L^2[0, T] \right\}$$

with inner product given by

$$\langle h, k \rangle_{\mathcal{H}_{\frac{1}{2}}} := \int_0^t h'(s) k'(s) ds.$$

A second example of a Gaussian measure space which is pertinent to the results described below, is as follows: let $\mathcal{W}$ be defined as above, and define the Gaussian measure $\mathbb{P}$ on $\mathcal{W}$ as the law of fractional Brownian motion with Hurst parameter $H < 1/2$; then by following Proposition 2.1.2 of [3], we have that $\mathcal{H}$ consists of functions of the form $h(t) = \int_0^t K_H(t, s)\hat{h}(s)\, ds$, where $\hat{h} \in L^2[0, T]$ and

$$K_H(t, s) := b_H \left[\left(\frac{t}{s} \right)^{H - \frac{1}{2}} (t - s)^{H - \frac{1}{2}} - \left(H - \frac{1}{2} \right) s^{\frac{1}{2} - H} \int_s^t (u - s) H - \frac{1}{2} u^{H - \frac{3}{2}}\, du \right],$$

where b_H is some suitable normalization constant. One may verify that relatively "nice" functions on $[0, T]$, such as continuous piecewise linear and $\mathcal{C}_0^\infty$ functions, are contained in $\mathcal{H}$. The inner product on this space is given by

$$\langle h, k \rangle_{\mathcal{H}} := \langle \hat{h}, \hat{k} \rangle_{L^2[0,T]}.$$

For each fixed $t \in [0, T]$, the function $R(t, \cdot) = \mathbb{E}[B_t B_\cdot] \in \mathcal{H}$ is a *reproducing kernel* for the space; that is to say, for any $h \in \mathcal{H}$, we have the following:

$$\langle h, R(t, \cdot) \rangle_{\mathcal{H}} = h(t).$$

Let $\mathcal{S}$ refer to the space of *cylinder functions*; that is to say, random variables of the form

$$F(\omega) = f(\phi_1(\omega), \ldots, \phi_n(\omega)),$$

where $f \in C^\infty(\mathbb{R}^d)$ with all partial derivatives having at most polynomial growth, and $\{\phi_1, \ldots, \phi_n\} \subset \mathcal{W}^*$. Let the Malliavin derivative $D : \mathcal{S} \to \mathcal{S} \otimes \mathcal{H}^*$ be the operator defined by the action by

$$
\begin{aligned}
DF(\omega)k &:= \frac{d}{dt}|_{t=0}F(\omega + tk) \\
&= \sum_{i=1}^{n} \partial_i f(\phi_1(\omega), \ldots, \phi_n(\omega))\phi_i(k) \quad (k \in \mathcal{H}) .
\end{aligned}
$$

For $1 \le q < \infty$, we will let $\mathbb{D}^{1,q}$ denote the closure of $\mathcal{S}$ with respect to the norm

$$
\|F\|_{1,q} := \left(\mathbb{E}[|F|^q] + \mathbb{E}[\|DF\|^q_{\mathcal{H}^*}]\right)
$$

One can naturally define an iterated derivate operator D^k taking values in $\mathcal{H}^{\otimes k}$; from this we can define the seminorm

$$
\|F\|_{k,q} := \left(\mathbb{E}[|F|^q] + \sum_{j=1}^{k} \mathbb{E}[\|D^j F\|^q_{\mathcal{H}^{\otimes k}}]\right)
$$

and we will denote by $\mathbb{D}^{k,q}$ the closure of $\mathcal{S}$ with respect to $\|\cdot\|_{k,q}$. Additionally, we will define

$$
\mathbb{D}^{\infty} := \bigcap_{k \in \mathbb{N}} \bigcap_{q \ge 1} \mathbb{D}^{k,q}.
$$

Finally, given some $F \in \mathbb{D}^{1,q}$, we may define the *Malliavin covariance matrix* γ as the $d \times d$ matrix given by

$$
\gamma := DF(DF)^*.
$$

3 Proof of main result

Let

$$
\mathcal{W}^n := \mathcal{W} \otimes \mathbb{R}^n \cong \{\omega \in \mathcal{C}([0, T], \mathbb{R}^n) : \omega(0) = \mathbf{0}\}.
$$

On $\mathcal{W}^n$, one may construct a unique Gaussian measure $\mathbb{P}$ such that the coordinate process $\{B_t\}_{0 \le t \le T}$ defined by

$$
B_t(\omega) = \omega(t)
$$

is an n-dimensional fractional Brownian motion with Hurst parameter H and $\mathbb{P} = \mathrm{Law}(B.)$. The Cameron–Martin space in this instance is given as $\mathcal{H}^n = \mathcal{H} \otimes \mathbb{R}^n$, where $\mathcal{H}$ is the Cameron–Martin space for one-dimensional fractional Brownian motion; for a general element $h = (h^1, \ldots, h^n) \in \mathcal{H}^n$, the norm is given by $\|h\|^2_{\mathcal{H}^n} = \sum_{i=1}^{n} \|h^i\|^2_{\mathcal{H}}$.

We shall fix

$$p \in \left(\frac{1}{H}, \frac{1}{1-2H}\right), \tag{3.1}$$

and define (following Section 5.3.3 of [12]) the spaces

$$\mathcal{W}_p := \overline{C^\infty([0,T], \mathbb{R}) \cap \mathcal{W}}^{\|\cdot\|_p};$$
$$\mathcal{W}_p^n := \mathcal{W}_p \otimes \mathbb{R}^n \cong \overline{C^\infty([0,T], \mathbb{R}^n) \cap \mathcal{W}^n}^{\|\cdot\|_p}.$$

For reasons which will become apparent as we progress, it will be beneficial for us to declare that from here on out our process $\{B_t\}$ will be restricted to the probability space $(\mathcal{W}_p^n, \mathcal{B}_{\mathcal{W}_p^n}, \mathbb{P}|_{\mathcal{W}_p^n})$; the details of this restriction are included in the Appendix. Most importantly, the Cameron–Martin space $\mathcal{H}^n$ associated to the restriction of our measure is the same as the Cameron–Martin space associated to $(\mathcal{W}^n, \mathcal{B}, \mathbb{P})$ as given previously. The following proposition shows that elements of $\mathcal{H}$ live within a smaller variational space, and will be used repeatedly in the sequel.

Proposition 3.1 (see Theorem 3 and Corollary 1 of [11]) *Let $\mathcal{H}$ be the Cameron–Martin space associated to fractional Brownian motion with Hurst parameter $\frac{1}{4} < H < \frac{1}{2}$. Then for every $r > (1+2H)^{-1}$, one has the (compact) embedding*

$$\mathcal{H} \subset \mathcal{C}_r.$$

The most important implication of Proposition 3.1 is that pathwise Young's integration of the fractional Brownian motion with respect to Cameron–Martin vectors is a well-defined operation. Indeed, for each $\frac{1}{4} < H < \frac{1}{2}$, we may set r above such that $(1+2H)^{-1} < r < 2$; one then has the Hölder-type inequality

$$\frac{1}{p} + \frac{1}{r} > 1 - 2H + \frac{1}{2} > 1.$$

This is precisely the condition sufficient to guarantee the existence of integrals of the form

$$\int_0^T \omega \, dh$$

for $\omega \in \mathcal{W}_p^n$ and $h \in \mathcal{H}^n$.

3.1 Stochastic differential equation solutions

Recall that $B = (B^1, \ldots, B^n)$ is an n-dimensional fractional Brownian motion with Hurst parameter $\frac{1}{4} < H < \frac{1}{2}$.

Let $k \in \{1, \ldots, \frac{n(n-1)}{2}\}$, and suppose that $\{\alpha_1, \ldots, \alpha_k\}$ is a collection of maps from $\mathbb{R}^n \times \mathbb{R}^n$ to $\mathbb{R}$ with the following properties:

(I) Each α_l is a skew-symmetric bilinear form;

(II) The set $\{\alpha_l\}$ is a linearly independent set; i.e., the bilinear form $\sum_l c_l \alpha_l$ is the zero map if and only if $c_l = 0$ for all l.

Define $\alpha : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^k$ by $\alpha := (\alpha_1, \ldots, \alpha_k)$; this map induces a function from $\mathcal{W}^n \times \mathcal{W}^n$ into $\mathcal{W}^k$, which we will also refer to as α; its action is given by

$$[\alpha\,(\omega, \tau)]\,(t) := \alpha(\omega(t), \tau(t)).$$

Let $\{X_i\}_{i=1}^n$ be vector fields on $\mathbb{R}^n \times \mathbb{R}^k$ defined by

$$X_i(\mathbf{v}, \mathbf{w}) = \left(e_i, \frac{1}{2}\alpha(\mathbf{v}, e_i)\right),$$

with $\{e_i\}$ denoting the standard Euclidean basis on $\mathbb{R}^n$. The objects of interest for our discussion are solutions to equations of the differential equation

$$dY = \sum_i X_i(Y) dB^i. \tag{3.2}$$

Here, we are considering the solution Y as the limiting process of solutions to (3.2) when the driving signal is replaced. by dyadic approximation processes $B_m = \pi_m B$. The stochastic differential equation in the dyadic approximation case is of the form

$$\begin{aligned} dY_m = \sum_i X_i(Y_m) dB_m^i &= \sum_i \left(e_i, \frac{1}{2}\alpha(y, e_i)\right) dB_m^i \\ &= \left(dB_m^1, \ldots, dB_m^n, \frac{1}{2}\sum_i \alpha(y, e_i) dB_m^i\right). \end{aligned}$$

One may check that the solution is given as $Y_m := (y_m, \hat{y}_m)$, where

$$\begin{aligned} (y_m)_T &= (B_m)_T, \\ (\hat{y}_m)_T &= \frac{1}{2}\int_0^T \alpha((B_m)_t, d(B_m)_t) := \frac{1}{2}\sum_i \int_0^T \alpha((B_m)_t, e_i) d(B_m^i)_t, \end{aligned}$$

with the integrals above interpreted as Riemann–Stieltjies integrals since the piecewise linearity of B_m implies that $\alpha(B_m, e_i)$ is piecewise linear as well for each i.

Theorem 2 of [8] and Theorem 4.1.1 of [15] imply that the limiting process $Y := \lim_{m\to\infty} Y_m$ exists a.s. We will suggestively write this process heuristically as

$$Y_T = \left(B_T, \frac{1}{2}\int_0^T \alpha(B_t, dB_t)\right).$$

We record here a pair of simple lemmas which allow for some control of the process Y.

Lemma 3.2 *Suppose α is a continuous bilinear form on $\mathbb{R}^n$. Then for each fixed $v \in \mathbb{R}^n$, the mapping*

$$f \mapsto \alpha(f, v)$$

is a map from $\mathcal{C}_p([0,T],\mathbb{R}^n)$ into $\mathcal{C}_p([0,T],\mathbb{R})$ for all $p \geq 1$; more explicitly, one has the bound

$$\|\alpha(f, v)\|_p \leq \|\alpha\|_{L(\mathbb{R}^n \times \mathbb{R}^n, \mathbb{R})}|v|\|f\|_p.$$

Proof Fix $f \in \mathcal{C}_p([0,T],\mathbb{R}^n)$; and let $\Pi = \{t_i\}_{i=0}^N \in \mathcal{P}[0,T]$. Then one has that

$$\begin{aligned}\sum_{i=1}^N |\alpha(f(t_{i+1}), v) - \alpha(f(t_i), v)|^p &= \sum_{i=1}^N |\alpha(f(t_{i+1}) - f(t_i), v)|^p \\ &\leq \sum_{i=1}^N \|\alpha\|^p \|v\|^p \, |f(t_{i+1}) - f(t_i)|^p .\end{aligned}$$

Taking the supremum over all such partitions of $[0,T]$ will then complete the proof. □

Lemma 3.3 *Let α be a continuous bilinear map on $\mathbb{R}^n$, and suppose $v \in \mathbb{R}^n$. If p, q be constants such that $\frac{1}{p} + \frac{1}{q} > 1$, then for any $f \in \mathcal{C}_p([0,T],\mathbb{R}^n)$, $g \in \mathcal{C}_q([0,T],\mathbb{R})$, the Young's integral*

$$\int_s^t \alpha(f(\tau), v) dg(\tau)$$

is well defined for all $0 \leq s < t \leq T$, and satisfies the bounds

$$\left| \int_s^t (\alpha(f(\tau), v) - \alpha(f(s), v)) \; dg(\tau) \right| \leq C \|f|_{[s,t]}\|_p \|g|_{[s,t]}\|_q,$$

where C is a constant depending on p, q, α, and v.

Proof This immediately follows from Lemma 3.2 and the Young's integral bound given in (2.1). □

3.2 Operator realization

For $j = 1, \ldots, k$, define a quadratic form q_j on $\mathcal{H}^n \times \mathcal{H}^n$ as follows:

$$
\begin{aligned}
q_j(h,k) &:= \frac{1}{2}\left[\int_0^T \alpha_j(h(s), dk(s)) + \int_0^T \alpha_j(k(s), dh(s))\right] \\
&= \int_0^T \alpha_j(h(s), dk(s)) - \frac{\alpha_j(h(T), k(T))}{2}.
\end{aligned} \tag{3.3}
$$

Note that the above integrals are to be interpreted in the manner of Young, and are well-defined by Lemma 3.3 along with the Cameron–Martin embedding in Proposition 3.1. Since piecewise linear continuous functions are contained in $\mathcal{H}$, we may write our approximating process Y_m as:

$$
(Y_m)_T = \left((B_m)_T, \frac{1}{2}q_1(B_m, B_m), \ldots, \frac{1}{2}q_k(B_m, B_m)\right).
$$

In order to facilitate the computation of the necessary Malliavin derivatives, we will construct the operators from $\mathcal{H} \otimes \mathbb{R}^n$ to itself which characterize the action of each q_l; that is to say, we will describe the actions of the operators Q_l, $l = 1, \ldots, k$ for which.

$$
\langle Q_l h, k\rangle_{\mathcal{H}\otimes\mathbb{R}^n} = q_l(h,k).
$$

The proposition and corollary which follow define elements of the Cameron–Martin space closely associated to these operators.

Proposition 3.4 *Fix $\omega \in \mathcal{W}_p$; for each partition $\Pi = \{t_i\}_{i=0}^N \in \mathcal{P}[0,T]$, define the vector $S_\Pi \in \mathcal{H}$ in the following manner:*

$$
S_\Pi(\cdot) := \sum_{i=1}^N \omega(c_i)\left[R(t_i, \cdot) - R(t_{i-1}, \cdot)\right],
$$

where $c_i \in (t_{i-1}, t_i)$. Then $\mathcal{H} - \lim_{k\to\infty} S_{\Pi_k}$ exists, where $\{\Pi_k\}_{k=1}^\infty \subset \mathcal{P}[0,T]$ with $|\Pi_k|$ converging to zero as $k \longrightarrow \infty$; furthermore, this limit is independent of the family of partitions. We will denote this limit by

$$
\int_0^T \omega(t) R(dt, \cdot).
$$

This limit satisfies the following properties:

(1) $\left\|\int_0^T \omega(t)R(dt,\cdot)\right\|_{\mathcal{H}}^2 = \int_{[0,T]^2} \omega\otimes\omega\, dR$; *hence, there exists a constant* $C > 0$ *such that*

$$\left\|\int_0^T \omega(t)R(dt,\cdot)\right\|_{\mathcal{H}}^2 \le C\|\omega\|_p^2\|R\|_r^{(2D)}.$$

(2) *For each* $h\in\mathcal{H}$, $\left\langle \int_0^T \omega(t)R(dt,\cdot), h\right\rangle_{\mathcal{H}} = \int_0^T \omega(t)dh(t)$.

(3) $\left(\int_0^T \omega(t)R(dt,\cdot)\right)(s) = \int_0^T \omega(t)R(dt,s)$.

Proof First note that $\frac{1}{p}+\frac{1}{r} > (1-2H)+2H = 1$, which implies that

(1) the Young's integral of ω against $R(\cdot,s)$ for any $s\in[0,T]$ is well-defined, and
(2) the 2D-Young's integral of $\omega\otimes\omega$ against R is well-defined.

It follows from Theorem 2.1 of [23] that for each k,

$$\begin{aligned}\|S_{\Pi_k}\|_{\mathcal{H}}^2 &= \sum_{i=1}^{\#(\Pi_k)}\sum_{j=1}^{\#(\Pi_k)} \omega(c_i)\omega(c_j)\left\langle \Delta_i R(t_i,\cdot), \Delta_j R(t_j,\cdot)\right\rangle_{\mathcal{H}} \\ &\le C\|\omega\|_p^2\|R\|_r^{(2D)},\end{aligned} \tag{3.4}$$

where C is a constant depending only on p and r. Given any two partitions $\Pi_n = \{s_i\}$, $\Pi_m = \{t_k\}$ in the family, for $c_i\in[s_{i-1},s_i]$, $d_k\in[t_{k-1},t_k]$,

$$\begin{aligned}\|S_{\Pi_n}-S_{\Pi_m}\|_{\mathcal{H}}^2 &= \|S_{\Pi_n}\|_{\mathcal{H}}^2 + \|S_{\Pi_m}\|_{\mathcal{H}}^2 - 2\langle S_{\Pi_n}, S_{\Pi_m}\rangle_{\mathcal{H}} \\ &= \sum_{i=1}^{\#(\Pi_n)}\sum_{j=1}^{\#(\Pi_n)} \omega(c_i)\omega(c_j)\left[\Delta_{ij}R(s_i,s_j)\right] \\ &\quad + \sum_{k=1}^{\#(\Pi_m)}\sum_{l=1}^{\#(\Pi_m)} \omega(d_k)\omega(d_l)\left[\Delta_{kl}R(t_k,t_l)\right] \\ &\quad - 2\sum_{i=1}^{\#(\Pi_n)}\sum_{k=1}^{\#(\Pi_m)} \omega(c_i)\omega(d_k)\left[\Delta_{ik}R(s_i,t_k)\right] \\ &\xrightarrow{n,m\to\infty} \int_{[0,T]^2} (\omega\otimes\omega)(s,t)\, dR(s,t)\end{aligned}$$

$$
+ \int_{[0,T]^2} (\omega \otimes \omega)(s,t) \, dR(s,t) \\
-2 \int_{[0,T]^2} (\omega \otimes \omega)(s,t) \, dR(s,t) = 0.
$$

Hence, the completeness of $\mathcal{H}$ implies the existence of $\int_0^T \omega(t) R(dt, \cdot)$. Since the 2D-Young's integral is independent of choice of partitions, one may also see from the calculation above that the limit of S_{Π_k} is also independent of choice of partition, as claimed. Letting k tend to infinity in (3.4) and applying bounds as in Theorem 1.2 of [23] proves (1). For an arbitrary $h \in \mathcal{H}$, we note that

$$
\begin{aligned}
\left\langle \int_0^T \omega(t) \; R(dt, \cdot), h \right\rangle_{\mathcal{H}} &= \lim_{|\Pi| \to 0} \langle S_\Pi, h \rangle_{\mathcal{H}} \\
&= \lim_{|\Pi| \to 0} \sum_{i=1}^{\#(\Pi)} \omega(c_i) \big\langle R(t_{i+1}, \cdot) - R(t_i, \cdot), h \big\rangle_{\mathcal{H}} \\
&= \lim_{|\Pi| \to 0} \sum_{i=1}^{\#(\Pi)} \omega(c_i) \left[h(t_{i+1}) - h(t_i) \right] \\
&= \int_0^T \omega(t) \; dh(t),
\end{aligned}
$$

and so (2) holds. In particular, by setting $h = R(s, \cdot)$, (3) is a consequence of (2). □

Corollary 3.5 *Suppose α is a continuous skew-symmetric bilinear form on $\mathbb{R}^n$. Then for any $h \in \mathcal{H}^n$ and for each partition $\Pi = \{t_j\}_{j=0}^N \in \mathcal{P}[0, T]$, define the vector $S_\Pi \in \mathcal{H}$ in the following manner:*

$$
S_\Pi(\cdot) := \sum_{j=1}^{N} \alpha(h(c_j), \left[R(t_j, \cdot) - R(t_{j-1}, \cdot) \right] e_i)
$$

where $c_j \in (t_{j-1}, t_j)$. Then $\mathcal{H} - \lim_{k \to \infty} S_{\Pi_k}$ exists, where $\{\Pi_k\}_{k=1}^\infty \subset \mathcal{P}[0, T]$ with $|\Pi_k|$ converging to zero as $k \longrightarrow \infty$; furthermore, this limit is independent of the family of partitions. We will denote this limit by

$$
\int_0^T \alpha(h(t), R(dt, \cdot) e_i).
$$

This limit satisfies the following properties:

(1) $\left\| \int_0^T \alpha(h(t), R(dt, \cdot)e_i) \right\|_{\mathcal{H}}^2 = \int_{[0,T]^2} \alpha(h(s), e_i)\alpha(h(t), e_i)\, dR(s,t)$; *hence, there exists a constant* $C > 0$ *such that*

$$\left\| \int_0^T \alpha(h(t), R(dt, \cdot)e_i) \right\|_{\mathcal{H}}^2 \leq C \|\alpha(h, e_i)\|_p^2 \|R\|_r^{(2D)}.$$

(2) *For each* $k \in \mathcal{H}$,

$$\left\langle \int_0^T \alpha(h(t), R(dt, \cdot)e_i), k \right\rangle_{\mathcal{H}} = \int_0^T \alpha(h(t), e_i)\, dk(t).$$

(3) $\left(\int_0^T \alpha(h(t), R(dt, \cdot)e_i) \right)(s) = \int_0^T \alpha(h(t), R(dt, s)e_i).$

Proof This is an application of Proposition 3.4 along with Lemma 3.2. □

Define the linear mapping $a : \mathcal{W}_p \to \mathcal{H}$ by

$$a\omega := \frac{1}{2}\omega(T)R(T, \cdot) - \int_0^T \omega(t)R(dt, \cdot).$$

The above integral is to be interpreted in the manner of Young, and the mapping above is well-defined as a result of Proposition 3.4. Additionally, one may verify using Proposition 3.4 that

$$\|a\omega\|_{\mathcal{H}}^2 = \frac{1}{4}T^{2H}|\omega(T)|^2 - \omega(T)\int_0^T \omega(t)\ R(dt, T) + \int_{[0,T]^2} \omega(s)\omega(t)\, dR(s,t).$$

By applying one- and two-dimensional Young's integral bounds to the right-hand side of this expression, one may conclude that a is a bounded operator on $\mathcal{W}_p$.

Given a skew-symmetric bilinear form ξ on $\mathbb{R}^n$, we define J_ξ as the linear map on $\mathbb{R}^n$ with action given by

$$J_\xi x = \sum_i \xi(e_i, x)e_i. \tag{3.5}$$

We will regularly refer to $J_l := J_{\alpha_l}$ for the operators $\{\alpha_l\}$ defined in Sect. 3.1.

Using this notation, one has the identity

$$q_l(B_m, B_m) = \int_0^T (B_m)_t \, dJ_l(B_m)_t.$$

One may take the tensor product of these two operators to form an operator on $\mathcal{H} \otimes \mathbb{R}^n$:

$$\begin{aligned}(a \otimes J_\xi)h &= \frac{1}{2}R(T,\cdot) \otimes J_\xi h(T) - \int_0^T R(dt,\cdot) \otimes J_\xi h(t) \\ &= \frac{1}{2}\left[\int_0^T R(t\cdot) \otimes dJ_\xi h(t) - \int_0^T R(dt,\cdot) \otimes J_\xi h(t)\right].\end{aligned} \tag{3.6}$$

Of course, we are particularly interested in such operators arising from our skew-symmetric α_l; we will write $Q_l := a \otimes J_l$ to denote these mappings in the sequel.

Proposition 3.6 *Let $Q_l : \mathcal{H} \otimes \mathbb{R}^n \to \mathcal{H} \otimes \mathbb{R}^n$, $l = 1, \ldots, k$ be the linear operators described above. For each symmetric form q_l as defined in Eq. (3.3) with $l = 1, \ldots, k$, one has the identity*

$$q_l(h, \tilde{h}) = \left\langle Q_l h, \tilde{h}\right\rangle_{\mathcal{H}\otimes\mathbb{R}^n}.$$

Proof This result follows from computing the right-hand inner product, using part 2 of Corollary 3.5. □

3.3 Malliavin derivative

Let $R_T^j : \mathcal{H} \otimes \mathbb{R}^n \to \mathbb{R}$ be the evaluation operator on the jth coordinate; i.e.,

$$R_T^j h := h^j(T) = \langle R(T,\cdot) \otimes e_j, h\rangle_{\mathcal{H}\otimes\mathbb{R}^n}.$$

Proposition 3.7 *For each skew-symmetric bilinear form ξ on $\mathbb{R}^n$, the mapping $a \otimes J_\xi$ may be extended to a bounded linear functional from $\mathcal{W}_p \otimes \mathbb{R}^n$ to $\mathcal{H} \otimes \mathbb{R}^n$; we will also denote this extension by $a \otimes J_\xi$. Its action is given by*

$$(a \otimes J_\xi)\omega = \frac{1}{2}\left[\int_0^T R(t,\cdot) \otimes dJ_\xi \omega(t) - \int_0^T R(dt,\cdot) \otimes J_\xi \omega(t)\right].$$

Proof The result follows immediately from the variational properties of the path-space and Cameron–Martin space along with the boundedness of a and J_ξ; we will prove the result by hand here in order to record an explicit upper bound on the operator norm of $a \otimes J_\xi$. To that end, given an $\omega \in \mathcal{W}_p \otimes \mathbb{R}^n$, we have

$$\begin{aligned}
\|(a \otimes J_\xi)\omega\|^2_{\mathcal{H}\otimes\mathbb{R}^n} &= \sum_{i=1}^n \left(\left\| \frac{\xi(\omega(T), e_i) R(t, \cdot)}{2} \right\|^2 + \left\| \int_0^T \xi(e_i, \omega(t)) R(dt, \cdot) \right\|^2 \right. \\
&\quad \left. - \left\langle \xi(\omega(T), e_i) R(T, \cdot), \int_0^T \xi(e_i, \omega(t)) R(dt, \cdot) \right\rangle \right) \\
&= \sum_{i=1}^n \left((\xi(\omega(T), e_i))^2 \frac{T^{2H}}{4} \right. \\
&\quad + \int_{[0,T]^2} \xi(e_i, \omega(s)) \xi(e_i, \omega(t)) dR(s, t) \\
&\quad \left. + \xi(\omega(T), e_i) \int_0^T \xi(e_i, \omega(t)) R(dt, T) \right) \\
&\le \sum_{i=1}^n \left(\|\xi\|^2 \left(|\omega(T)|^2 \frac{T^{2H}}{4} + \|\omega\|_p^2 \|R\|_r^{(2D)} \right. \right. \\
&\quad \left. \left. + |\omega(T)| \|\omega\|_p \|R\|_r \right) \right) \\
&\le n \|\xi\|^2 \left(\frac{T^{2H}}{4} + \|R\|_r^{(2D)} + \|R\|_r \right) \|\omega\|_p^2.
\end{aligned}$$

□

We may thus define random processes $Q_l B$ for each $l = 1, \dots, k$ by the formula

$$Q_l B = \frac{1}{2} \left[\int_0^T R(t, \cdot) \otimes d J_l B_t - \int_0^T R(dt, \cdot) \otimes J_l B_t) \right]. \qquad (a.s.)$$

Proposition 3.8 *The process Y_T has derivative DY_T taking values in $\mathcal{L}(\mathcal{H} \otimes \mathbb{R}^n, \mathbb{R}^{n+k})$, with action given by*

$$DY_T h = \left(R_T^1 h, \dots, R_T^n h, \langle Q_1 B, h \rangle_{\mathcal{H}\otimes\mathbb{R}^n}, \dots, \langle Q_k B, h \rangle_{\mathcal{H}\otimes\mathbb{R}^n} \right) \quad (a.s.)$$

Proof We begin by computing the derivative of $q_l(B_m, B_m)$ for $l = 1, \dots, k$ in the direction of some $h \in \mathcal{H} \otimes \mathbb{R}^n$. Define $T_j := \frac{j}{2^m} T$.

$$Dq_l(B_m, B_m)h = D \left[\sum_{j=1}^{2^m} \frac{B_{T_j} + B_{T_{j-1}}}{2} \left(J_l B_{T_j} - J_l B_{T_{j-1}} \right) \right] h$$

$$
\begin{aligned}
&= \frac{1}{2}\sum_{j=1}^{2^m} \left(h(T_j)+h(T_{j-1})\right)\left(J_l B_{T_j} - J_l B_{T_{j-1}}\right) \\
&\quad + \left(B_{T_j} + B_{T_{j-1}}\right)\left(J_l h(T_j) - J_l h(T_{j-1})\right) \\
&= \frac{1}{2}\sum_{j=1}^{2^m} \left(h(T_j)+h(T_{j-1})\right)\left(J_l B_{T_j} - J_l B_{T_{j-1}}\right) \\
&\quad - \left(J_l B_{T_j} + J_l B_{T_{j-1}}\right)\left(h(T_j) - h(T_{j-1})\right) \\
&= \frac{1}{2}\left[\int_0^T h_m(t)d(J_l B_m)_t - \int_0^T (J_l B_m)_t dh_m(t)\right] \\
&= \langle Q_l B_m, h\rangle_{\mathcal{H}\otimes\mathbb{R}^n}\,.
\end{aligned}
$$

It is easy to check that $D^2 q_l$ is entirely deterministic, and that higher Malliavin derivatives are equivalently zero. Thus, in order to prove the claim, it suffices to show that for each $l = 1, \dots, k$,

$$
\begin{aligned}
&\mathbb{E}\ \|\langle Q_l B, \cdot\rangle - \langle Q_l B_m, h\rangle\|^2_{(\mathcal{H}\otimes\mathbb{R}^n)^*} \\
&\quad = \mathbb{E}\left\|\int_0^T R(dt,\cdot)\otimes(J_l B)_t - \int_0^T R_m(dt,\cdot)\otimes(J_l B_m)_t\right\|^2_{\mathcal{H}\otimes\mathbb{R}^n}
\end{aligned}
$$

tends to zero as $m \to \infty$. Yet one can dominate each term by

$$
C_l \mathbb{E}\left\|\int_0^T R(dt,\cdot)\otimes B_t - \int_0^T R_m(dt,\cdot)\otimes(B_m)_t\right\|^2_{\mathcal{H}\otimes\mathbb{R}^n}
$$

for some suitable constant C_l depending only on α_l. Our proof then rests upon demonstrating that the above quantity vanishes in the limit. Let us begin by noting that

$$
\begin{aligned}
&\left\|\int_0^T B^i(s)\quad R(ds,\cdot) - \int_0^T B^i_m(s)\ R_m(ds,\cdot)\right\|^2_{\mathcal{H}} \\
&\qquad \le \Bigg(\left\|\int_0^T B^i(s)\ (R-R_m)(ds,\cdot)\right\|^2_{\mathcal{H}} \\
&\qquad\quad + \left\|\int_0^T (B^i - B^i_m)(s) R_m(ds,\cdot)\right\|^2_{\mathcal{H}}\Bigg).
\end{aligned}
$$

In order to prove convergence of the above expression, it will be helpful to introduce the following notation:

$$\begin{aligned}\Psi_m(u,v) &:= \langle (R-R_m)(u,\cdot),(R-R_m)(v,\cdot)\rangle_{\mathcal{H}}\\ &= R(u,v)-R_m(u,v)-R_m(v,u)+\langle R_m(u,\cdot),R_m(v,\cdot)\rangle_{\mathcal{H}}\\ &= R(u,v)-R_m(u,v)-R_m(v,u)+\pi_m\left[R_m(u,\cdot)\right](v).\end{aligned}$$

Note that Ψ_m converges uniformly to zero as m tends to infinity. As a linear combination of R and R_m, Ψ_m has finite two-dimensional r-variation for $r=\frac{1}{2H}$. Let r' be a number such that $r'>r$ and $\frac{1}{r'}+\frac{1}{r}>1$; then it follows that Ψ_m has finite two-dimensional r'-variation; furthermore, Lemma 2.1 implies that

$$\lim_{m\to\infty}\|\Psi_m\|_{r'}^{(2D)}=0.$$

Using the continuity of the inner product, we find that

$$\begin{aligned}&\left\|\int_0^T B^i(s)\,(R-R_m)(ds,\cdot)\right\|_{\mathcal{H}}^2\\ &=\lim_{|\Pi|\to 0}\sum_{u_j,v_k\in\Pi} B^i(c_j)B^i(d_k)\langle\Delta_i(R-R_m)(u_j,\cdot),\Delta_j(R-R_m)(v_k,\cdot)\rangle_{\mathcal{H}}\\ &=\lim_{|\Pi|\to 0}\sum_{u_j,v_k\in\Pi}\Big(B^i(c_j)B^i(d_k)\Big)\,\Delta_{ij}\Psi_m(u_j,v_k)=\int_{[0,T]^2}B^i\otimes B^i\;d\Psi_m\\ &\qquad\le C\|\Psi_m\|_{r'}^{(2D)}\|B^i\|_p^2.\end{aligned}$$

Hence,

$$\mathbb{E}\left\|\int_0^T B^i(s)\,(R-R_m)(ds,\cdot)\right\|_{\mathcal{H}}^2\le C\|\Psi_m\|_{r'}^{(2D)}\mathbb{E}\|B^i\|_p^2;$$

Fernique's Theorem guarantees that the right-hand expression is finite; from our above remarks, we know that its value tends to zero as $m\to\infty$. We may then conclude that

$$\mathbb{E}\left\|\int_0^T B^i(s)\,(R-R_m(ds,\cdot))\right\|_{\mathcal{H}}^2\overset{m\to\infty}{\longrightarrow}0.$$

We can approach the convergence of the second term in a similar manner. Choose p' such that $p'>p$ and $\frac{1}{p'}+\frac{1}{r}>1$; then the sample paths of $B^i-B^i_m$ has finite r-variation and Lemma 2.1 tells us that $\|B^i-B^i_m\|_{p'}$ tends to zero as m tends to infinity.

Therefore, it follows that

$$\mathbb{E}\left\| \int_0^T (B^i - B_m^i)(s) R_m(ds, \cdot) \right\|_{\mathcal{H}}^2 = \mathbb{E} \int_{[0,T]^2} (B^i - B_m^i) \otimes (B^i - B_m^i)\, dR_m$$
$$\leq \mathbb{E}\left[\|B^i - B_m^i\|_{p'}^2 \right] \|R\|_r^{(2D)} \to 0$$

□

3.4 Integrability of the Malliavin covariance determinant

Recall that the Malliavin covariance matrix is defined as the operator $DY_T\,(DY_T)^*$.

We will indicate by Ψ the Gram matrix for the operators $\{Q_l\}_{l=1}^k$; more precisely, Ψ will be the function from $\mathcal{W}_p \otimes \mathbb{R}^n$ to $M_k(\mathbb{R})$ defined by

$$[\Psi(\omega)]_{ij} = \left\langle Q_i\omega, Q_j\omega \right\rangle_{\mathcal{H}\otimes\mathbb{R}^n}.$$

We will let Θ denote the linear mapping from $\mathcal{W}_p \otimes \mathbb{R}^n$ into $M_{n,k}(\mathbb{R})$ with entries given by

$$[\Theta\omega]_{ij} := \left[Q_j\omega \right](T) \cdot e_i.$$

Each of these matrices may be extended to matrix-valued random variables $\Psi(B)$ and ΘB in the usual manner.

Proposition 3.9 *Suppose $\gamma : \mathcal{W}_p \otimes \mathbb{R}^n \to M^{n+k}(\mathbb{R})$ is given by*

$$\gamma(\omega) := \begin{bmatrix} T^{2H} I_n & \Theta\omega \\ (\Theta\omega)^{tr} & \Psi(\omega) \end{bmatrix}.$$

then $DY_T\,(DY_T)^* = \gamma(B)$ almost surely.

Proof Given $\mathbf{x} \in \mathbb{R}^{n+k}$ and $h \in \mathcal{H} \otimes \mathbb{R}^n$,

$$\left\langle (DY_T)^*\mathbf{x}, h \right\rangle = \mathbf{x} \cdot \left(R_T^1 h, \ldots, R_T^n h, \langle Q_1 B, h\rangle_{\mathcal{H}\otimes\mathbb{R}^n}, \ldots, \langle Q_k B, h\rangle_{\mathcal{H}\otimes\mathbb{R}^n} \right)$$
$$= \left\langle \sum_{i=1}^n x^i R(T, \cdot) \otimes e_i + \sum_{j=1}^k x^{n+j} Q_j B, h \right\rangle_{\mathcal{H}\otimes\mathbb{R}^n}.$$

Direct calculations reveal the following identities:

$$R_T^l(R(T, \cdot) \otimes e_i) = \delta_{il} T^{2H};$$
$$\langle Q_l B, R(T, \cdot) \otimes e_i\rangle_{\mathcal{H}\otimes\mathbb{R}^n} = R_T^l\,(Q_l B) = [Q_l B]_T \cdot e_i.$$

By linearity, it follows that

$$\begin{aligned} DY_T(DY_T)^*\mathbf{x} &= \sum_{i=1}^{n} x^i DY_T\left[R(T,\cdot)\otimes e_i\right] + \sum_{j=1}^{k} x^{n+j} DY_T\left[Q_j B\right] \\ &= \sum_{i=1}^{n} x^i \left(T^{2H} e_i, [Q_1 B]_T \cdot e_i, \dots, [Q_k B]_T \cdot e_i\right) \\ &\quad + \sum_{j=1}^{k} x^{n+j} \Big(\left[Q_j B\right]_T \cdot e_1, \dots, \left[Q_j B\right]_T \cdot e_n, \\ &\quad \langle Q_1 B, Q_j B\rangle, \dots, \langle Q_k B, Q_j B\rangle\Big). \end{aligned}$$

One may readily verify that this is almost surely equivalent to $[\gamma(B)]\mathbf{x}$. □

Recall that proving smoothness of the density of Y with respect to Lebesgue measure requires showing that

$$(\det \gamma(B))^{-1} \in L^{\infty-}. \tag{3.7}$$

We will use the following elementary lemma from matrix algebra in order to compute the determinant of the covariance matrix.

Lemma 3.10 *Given any $a \neq 0$, $C \in M_{m,n}(\mathbb{R})$, and $D \in M_n(\mathbb{R})$, one has that*

$$\det \begin{bmatrix} aI_m & C \\ C^{tr} & D \end{bmatrix} = a^m \left(\det(D - a^{-1}C^{tr}C)\right).$$

Proof This claim follows immediately when one writes

$$\begin{bmatrix} aI_m & C \\ C^{tr} & D \end{bmatrix} = \begin{bmatrix} aI_m & 0 \\ C^{tr} & I_n \end{bmatrix} \begin{bmatrix} I_m & a^{-1}C \\ 0 & D - a^{-1}C^{tr}C \end{bmatrix}.$$

□

As an application of Lemma 3.10, we have that

$$\det \gamma(B) = T^{2Hn} \det \left(\Psi(B) - T^{-2H}[\Theta B]^{tr}\Theta B\right).$$

For each $\mathbf{y} \in S^{k-1}$, define the operator $\Phi_{\mathbf{y}}$ on $\mathcal{W}_p \otimes \mathbb{R}^n$ by

$$\Phi_{\mathbf{y}}(\omega) := \left(\Psi(\omega) - T^{-2H}[\Theta\omega]^{tr}\Theta\omega\right)\mathbf{y}\cdot\mathbf{y} \tag{3.8}$$

This operator is continuous both in ω and $\mathbf{y}$. If for each $\mathbf{y} \in S^{k-1}$ we let $\mathbf{y} \cdot \alpha := \sum_{i=1}^{k} y_i \alpha_i$, then one has the identities

$$(\Psi(B))\mathbf{y} \cdot \mathbf{y} = \left\| (a \otimes J_{\mathbf{y}\cdot\alpha}) B \right\|^2_{\mathcal{H}\otimes\mathbb{R}^n}$$
$$\left([\Theta B]^{tr} \Theta B \right) \mathbf{y} \cdot \mathbf{y} = \left| ([a \otimes J_{\mathbf{y}\cdot\alpha}) B](T) \right|^2,$$

where $\left(a \otimes J_{\mathbf{y}\cdot\alpha}\right)$ is the almost sure extension of the operator as defined in Eqs. (3.5) and (3.6). Thus, we may write

$$\Phi_{\mathbf{y}}(\omega) = \left\| (a \otimes J_{\mathbf{y}\cdot\alpha}) B \right\|^2_{\mathcal{H}\otimes\mathbb{R}^n} - T^{-2H} \left| ([a \otimes J_{\mathbf{y}\cdot\alpha}) B](T) \right|^2;$$

an application of the Cauchy–Schwarz inequality gives that each $\Phi_{\mathbf{y}}$ is non-negative.

Once again, we may almost surely identify this operator with a random variable $\Phi_{\mathbf{y}}(B)$. We note that

$$(\det \gamma(B))^{-1} \leq T^{-2Hn} \left(\min_{\mathbf{y}\in S^{k-1}} \Phi_{\mathbf{y}}(B) \right)^{-k};$$

hence our desired integrability condition will be implied by showing that

$$\left(\min_{\mathbf{y}\in S^{k-1}} \Phi_{\mathbf{y}}(B) \right)^{-1} \in L^{\infty-}.$$

Lemma 3.11 *Suppose that X is a non-negative random variable such that, for each $j \geq 1$, there exists a constant $C_j > 0$ for which*

$$\mathbb{E}\left[e^{-sX} \right] \leq C_j s^{-j} \quad \forall s \geq 1.$$

Then $X^{-1} \in L^{\infty-}$.

Proof Fix $j \geq 1$. We note that for any $k \geq 0$,

$$\int_0^\infty s^{j-1} e^{-ks}\, ds = k^{-j} \Gamma(j),$$

where Γ denotes the standard Gamma function. By replacing k with the random variable X, we find that

$$\mathbb{E}[X^{-j}] = \frac{1}{\Gamma(j)} \mathbb{E}\left[\int_0^\infty s^{j-1} e^{-sX}\, ds \right] = \frac{1}{\Gamma(j)} \int_0^\infty s^{j-1} \mathbb{E}\left[e^{-sX} \right] ds.$$

It is sufficient for completion of the proof to note that, under the assumption given, the right-hand expression is finite:

$$\int_0^1 s^{j-1} E\left[e^{-sX}\right] ds \leq \int_0^1 s^{j-1} ds = \frac{1}{j};$$
$$\int_1^\infty s^{j-1} E\left[e^{-sX}\right] ds \leq \int_1^\infty s^{j-1}(C_{j+1} s^{-(j+1)})\, ds = C_{j+1}.$$

$\square$

Theorem 3.12 (see Melcher [17, pp. 26–27]) *Let $(\mathcal{W}, \mathcal{B}, \mathbb{P})$ be a Gaussian measure space with associated Cameron–Martin space $\mathcal{H}$, and suppose $\Phi : \mathcal{W} \times \mathcal{W} \to \mathbb{R}$ is a bounded non-negative quadratic form. Then the operator $\hat{\Phi} : \mathcal{H} \to \mathcal{H}$ given by*

$$\Phi(h, k) = \langle \hat{\Phi} h, k \rangle_{\mathcal{H}}$$

is trace-class. In addition, if $\hat{\Phi}$ is not a finite rank operator, then

$$\Phi^{-1} \in L^{\infty-}(\mathcal{W}, \mathbb{P}).$$

Proof By Theorem 5.3.32 of [22], we have that there exists a set of independent, identically distributed standard normal random variables $\{\xi_n\}_{n=1}^\infty$ such that the series $B^N := \sum_{n=1}^N \xi_n h_n$ converges in $\mathcal{W}$ to B $\mathbb{P}$-a.s. and in all L^j, $j \geq 1$ as $N \to \infty$, and

$$\mathrm{Law}\left(\sum_{n=1}^\infty \xi_n h_n\right) = \mathbb{P}.$$

In particular, the fact that $\mathbb{E}\|B^N - B\|_{\mathcal{W}}^2 \to 0$ implies that $\Phi(B^N, B^N) \to \Phi(B, B)$ in L^1. Thus, Fernique's Theorem allows us to conclude that

$$\begin{aligned}
\sum_{n=1}^\infty \left\langle \hat{\Phi} h_n, h_n \right\rangle_{\mathcal{H}} &= \lim_{N\to\infty} \sum_{n=1}^N \Phi(h_n, h_n) \\
&= \lim_{N\to\infty} \mathbb{E}\left[\Phi(B^N, B^N)\right] \\
&= \mathbb{E}[\Phi(B)] \leq C\mathbb{E}[\|B\|_{\mathcal{W}}^2] < \infty.
\end{aligned}$$

Thus, $\hat{\Phi}$ is trace-class.

Suppose that $\hat{\Phi}$ is not finite rank. Since $\hat{\Phi}$ is compact, there exists an orthonormal basis $\{h_n\}_{n=1}^\infty \subset \mathcal{H}$ for which $\hat{\Phi} h_n = \lambda_n h_n$; our assumption guarantees that $\#\{n : \lambda_n > 0\} = \infty$. Using this, it is easy to check that

$$\Phi(B^N, B^N) = \left\langle \hat{\Phi} B^N, B^N \right\rangle_{\mathcal{H}} = \sum_{n=1}^N \lambda_n \xi_n^2$$

and so

$$\Phi(B,B) = L^1 - \lim_{N\to\infty} \sum_{n=1}^{N} \lambda_n \xi_n^2.$$

We will let $K_N := \#\{1 \le n \le N : \lambda_n > 0\}$; it is clear that $\{K_N\}$ is an non-decreasing sequence with $K_N \overset{N\to\infty}{\longrightarrow} \infty$. Therefore, for each fixed N and positive s,

$$\begin{aligned}\mathbb{E}\left[\exp\left(-s\Phi\left(B,B\right)\right)\right] &= \mathbb{E}\left[\exp\left(-s \lim_{N\to\infty}\sum_{n=1}^{N}\lambda_n\xi_n^2\right)\right] \\ &\le \mathbb{E}\left[\exp\left(-s\sum_{n=1}^{N}\lambda_n\xi_n^2\right)\right] \\ &= \prod_{n=1}^{N}\left(\frac{1}{2\lambda_n s+1}\right)^{\frac{1}{2}} \le C_N s^{-\frac{K_N}{2}}.\end{aligned}$$

Applying Lemma 3.11 finishes the proof. □

Theorem 3.12 implies that the operator $\hat{\Phi}_{\mathbf{y}} : \mathcal{H}^n \to \mathcal{H}^n$ defined by

$$\left\langle \hat{\Phi}_{\mathbf{y}} h, h \right\rangle_{\mathcal{H}^n} = \Phi_{\mathbf{y}} h$$

is trace-class.

Proposition 3.13 *For each fixed* $\mathbf{y} \in S^{k-1}$, *the map* $\Phi_{\mathbf{y}}$ *is contained in* $L^{\infty-}(\mathbb{P})$.

Proof Again recalling Theorem 3.12, it suffices to show that $\hat{\Phi}_{\mathbf{y}}$ is not a finite rank operator. To that aim, we begin by noting that $\hat{\Phi}_{\mathbf{y}} h = 0$ implies that $\Phi_{\mathbf{y}}(h) = 0$. By Cauchy–Schwarz, this is true if and only if

$$\left(a \otimes J_{\mathbf{y}\cdot\alpha}\right) h = \mathbf{c} R(T,\cdot)$$

for some $\mathbf{c} \in \mathbb{R}^n$. By definition, this is equivalent to the statement that

$$\int_0^T R(dt,\cdot) \otimes \left(J_{\mathbf{y}\cdot\alpha} h(T) - J_{\mathbf{y}\cdot\alpha} h(t) - \mathbf{c}\right) = 0.$$

Taking the inner product of each side against an arbitrary test function $\varphi \in \mathcal{C}_c^\infty \otimes \mathbb{R}^n \subset \mathcal{H} \otimes \mathbb{R}^n$ and applying integration by parts, we obtain the identity

$$0 = \int_0^T \varphi(t) \otimes dJ_{\mathbf{y}\cdot\alpha} h(t) \qquad (\forall \varphi \in \mathcal{C}_c^\infty \otimes \mathbb{R}^n)$$

which implies $J_{\mathbf{y}\cdot\alpha}h \equiv 0$; that is, $h(t) \in \mathrm{Null}\left(J_{\mathbf{y}\cdot\alpha}\right)$ for all $t \in [0, T]$. By the assumption on our skew-symmetric operators, $\mathbf{y} \cdot \alpha$ is not the zero map. Thus, we may pick some $\mathbf{v} \neq 0$ for which $J_{\mathbf{y}\cdot\alpha}\mathbf{v}$ is non-zero; it follows that for each fixed non-zero $h \in \mathcal{H}$, $h(t) \otimes J_{\mathbf{y}\cdot\alpha}\mathbf{v} \neq 0$ for all $t \in (0, T]$ such that $h(t) \neq 0$. Thus the set

$$\{h \otimes \mathbf{v} : 0 \neq h \in \mathcal{H}\}$$

is contained in the complement of the kernel of $\hat{\Phi}$; it is clear that the cardinality of this set is infinite. □

Proposition 3.14 *For each $1 \leq p < \infty$, the expectations $\mathbb{E}[\Phi_{\mathbf{y}}^{-p}]$ are uniformly bounded over S^{k-1}; i.e.,*

$$\sup_{\mathbf{y}\in S^{k-1}} \mathbb{E}[\Phi_{\mathbf{y}}^{-p}] < \infty.$$

Proof We begin by noting that the operator $\hat{\Phi}_{\mathbf{y}}$ may be written as

$$\begin{aligned}\hat{\Phi}_{\mathbf{y}} &= (a \otimes J_{\mathbf{y}\cdot\alpha})^*(a \otimes J_{\mathbf{y}\cdot\alpha}) - T^{-2H}(a \otimes J_{\mathbf{y}\cdot\alpha})^* R_T^* R_T (a \otimes J_{\mathbf{y}\cdot\alpha}) \\ &= (a^*a - T^{-2H} a^* R_T^* R_T a) \otimes J_{\mathbf{y}\cdot\alpha}^* J_{\mathbf{y}\cdot\alpha},\end{aligned}$$

where R_T denotes the evaluation operator at time T. We note that the quadratic form A on $\mathcal{W}_p$ given by

$$A(\omega, \tau) = \langle a\omega, a\tau\rangle_{\mathcal{H}} - (a\omega(T) \cdot a\tau(T))$$

is non-negative and bounded; hence Theorem 3.12 implies that $(a^*a - T^{-2H}a^*R_T^*R_Ta)$ is trace-class and a fortiori compact. For each $\mathbf{y}$, the (non-negative) eigenvalues $\{\lambda_n^{\mathbf{y}}\}$ of the operator $\hat{\Phi}_{\mathbf{y}}$ are given by the products of eigenvalues of $(a^*a - T^{-2H}a^*R_T^*R_Ta)$ and $J_{\mathbf{y}\cdot\alpha}^* J_{\mathbf{y}\cdot\alpha}$. Recall from the proof of Theorem 3.12 that one has the equation

$$\Phi_{\mathbf{y}}(B) = L^1 - \lim_{N\to\infty} \sum_{n=1}^{N} \lambda_n^{\mathbf{y}} \xi_n^2$$

where $\{\xi_n\}$ are a set of independent standard normal random variables.

For each $\mathbf{y} \in S^{k-1}$, let $\rho_{\mathbf{y}}$ denote the spectral radius of $J_{\mathbf{y}\cdot\alpha}^* J_{\mathbf{y}\cdot\alpha}$. Additionally, let $\{\sigma_n\}$ denote the eigenvalues of the operator $(a^*a - T^{-2H}a^*R_T^*R_Ta)$. Define the set $\mathcal{E}_{\mathbf{y}}$ as the collection of non-zero eigenvalues of $\hat{\Phi}_{\mathbf{y}}$ of the form $\rho_{\mathbf{y}}\sigma_n$. Since a has a trivial kernel, and $J_{\mathbf{y}\cdot\alpha}$ is not the zero map, we have that $\#(\mathcal{E}_{\mathbf{y}}) = \infty$. Without loss of generality, we may order our eigenvalues such that members of $\mathcal{E}_{\mathbf{y}}$ are listed "first"; i.e., $\lambda_n^{\mathbf{y}} \in \mathcal{E}_{\mathbf{y}}$ for any $n \in \mathbb{N}$.

For each $1 \leq p < \infty$, let N be the first integer for which $N > 2p$. Then

$$\begin{aligned}\mathbb{E}[\big(\Phi_{\mathbf{y}}(B)\big)^{-p}] &\leq \mathbb{E}\left[\left(\sum_{n=1}^{N} \lambda_n^{\mathbf{y}} \xi_n^2\right)^{-p}\right] \\ &\leq \left(\min_{n=1,\ldots,N} \lambda_n^{\mathbf{y}}\right)^{-p} \mathbb{E}\left[\left(\sum_{n=1}^{N} \xi_n^2\right)^{-p}\right] \\ &= \left(\min_{n=1,\ldots,N} \lambda_n^{\mathbf{y}}\right)^{-p} \int_0^\infty r^{-2p} r^{N-1} e^{\frac{-r^2}{2}} dr\end{aligned}$$

which is certainly a finite expression. In particular, a uniform bound on $\mathbb{E}[\big(\Phi_{\mathbf{y}}(B)\big)^{-p}]$ will be proven if we can find a constant M for which

$$\left(\min_{n=1,\ldots,N} \lambda_n^{\mathbf{y}}\right)^{-p} \leq M$$

for all $\mathbf{y} \in S^{k-1}$. We note that

$$\min_{n=1,\ldots,N} \lambda_n^{\mathbf{y}} = \rho_{\mathbf{y}} \min_{n=1,\ldots,N} \sigma_n \geq C\rho_{\mathbf{y}}$$

where the constant C is dependent only on the value of N. Thus it is only left for us to prove that

$$\max_{\mathbf{y}\in S^{k-1}} \rho_{\mathbf{y}}^{-p} \leq M;$$

yet this is equivalent to the statement that

$$\min_{\mathbf{y}\in S^{k-1}} \rho_{\mathbf{y}} > 0,$$

which is true by the compactness of the unit sphere and the non-degeneracy condition imposed upon α. □

With a uniform bound on the moments of $\Phi_{\mathbf{y}}^{-1}$, we may now approach proving our desired integrability condition by way of the compactness of the unit sphere and the following lemma.

Lemma 3.15 (Lemma 6.6 of [2]) *Suppose X is a non-negative random variable such that*

$$\mathbb{P}(X < \varepsilon) = O(\varepsilon^{\infty-}) \qquad (\varepsilon \longrightarrow 0).$$

Then $X^{-1} \in L^{\infty-}(\mathbb{P})$.

Proof Fix some $p \geq 1$. Pick some $q > p$; then by assumption, there exists some constants $K = K_q$, $M = M_q$ such that

$$\mathbb{P}(X < \varepsilon) \leq K\varepsilon^q,$$

provided $\varepsilon < \frac{1}{M}$. Using this, we see that

$$\begin{aligned}
\mathbb{E}[X^{-p}] &= \int_0^\infty \tau^{p-1}\mathbb{P}(X^{-1} > \tau)\, d\tau \\
&= \int_0^\infty \tau^{p-1}\mathbb{P}(X < \tau^{-1})\, d\tau \\
&= \int_0^M \tau^{p-1}\mathbb{P}(X < \tau^{-1})\, d\tau + \int_M^\infty \tau^{p-1}\mathbb{P}(X < \tau^{-1})\, d\tau \\
&\leq \int_0^M \tau^{p-1}\, d\tau + K\int_M^\infty \tau^{p-1}\tau^{-q}\, d\tau \\
&\leq \frac{M^p}{p} + \frac{M^{p-q}}{q-p} < \infty.
\end{aligned}$$

□

Theorem 3.16 *Let $\Phi_{\mathbf{y}}$ be defined as in Eq. (3.8). Then*

$$\left(\min_{\mathbf{y}\in S^{k-1}} \Phi_{\mathbf{y}}(B)\right)^{-1} \in L^{\infty-}(\mathcal{W}_p^n, \mathbb{P}).$$

Proof By Lemma 3.15, it suffices to check that for all q,

$$\mathbb{P}\left\{\left(\min_{\mathbf{y}\in S^{k-1}} \Phi_{\mathbf{y}}(B)\right) < \varepsilon\right\} \leq C_q\varepsilon^q$$

for some suitable constant C_q dependent only on q. Fix $\varepsilon > 0$. We pick a natural number $N(\varepsilon)$ and vectors $\{\mathbf{y}_i\}_{i=1}^{N(\varepsilon)}$ such that

$$\bigcup_{i=1}^{N(\varepsilon)} B(\mathbf{y}_i; \varepsilon^2)$$

form an open cover of S^{k-1}. Note that the value of $N(\varepsilon)$ is bounded above by $2^k\varepsilon^{-2k}$; one may see this by slicing the cube $[-1, 1]^k$ into disjoint cubes of size length ε^2.

Define the following sets:

$$\mathcal{A}_i := \left\{ \inf_{\mathbf{z}\in B(\mathbf{y}_i;\varepsilon^2)} \Phi_{\mathbf{z}} B < \varepsilon : \|B\|_p^2 \leq \frac{1}{\varepsilon} \right\};$$
$$\mathcal{B}_i := \left\{ \inf_{\mathbf{z}\in B(\mathbf{y}_i;\varepsilon^2)} \Phi_{\mathbf{z}} B < \varepsilon : \|B\|_p^2 > \frac{1}{\varepsilon} \right\}.$$

Then one has that

$$\mathbb{P}\left\{ \left(\min_{\mathbf{y}\in S^{k-1}} \Phi_{\mathbf{y}}(B) \right) < \varepsilon \right\} \leq \sum_{i=1}^{N} \Big(\mathbb{P}(\mathcal{A}_i) + \mathbb{P}(\mathcal{B}_i) \Big).$$

Suppose $\mathbf{z} \in B(\mathbf{y}_i; \varepsilon^2)$. Then on $\mathcal{A}_i$, one has the inequality

$$|\Phi_{\mathbf{y}_i}(B)| \leq |\Phi_{\mathbf{y}_i}(B) - \Phi_{\mathbf{z}}(B)| + |\Phi_{\mathbf{z}}(B)| < C\|B\|_p^2 \varepsilon^2 + \varepsilon = (1 + C)\varepsilon$$

for a suitable constant C. Therefore $\mathcal{A}_i \subset \left\{ \Phi_{\mathbf{y}_i}(B) < (1 + C)\varepsilon \right\}$. Letting $M_q := \sup_{\mathbf{y}\in S^{k-1}} \mathbb{E}[\big(\Phi_{\mathbf{y}}(B)\big)^{-q}]$ (a finite quantity by Proposition 3.14) and using Markov's inequality, we obtain the bound

$$\begin{aligned}\mathbb{P}(\mathcal{A}_i) &\leq \mathbb{P}\left\{ \Phi_{\mathbf{y}_i}(B) < 2\varepsilon \right\} = \mathbb{P}\left\{ \big(\Phi_{\mathbf{y}_i}(B)\big)^{-q} > (2\varepsilon)^{-q} \right\} \\ &\leq ((1 + C)\varepsilon)^q \mathbb{E}[\big(\Phi_{\mathbf{y}_i}(B)\big)^{-q}] \leq (2^q M_q)\varepsilon^q.\end{aligned}$$

Another application of Markov's inequality gives us that

$$\begin{aligned}\mathbb{P}(\mathcal{B}_i) &\leq \mathbb{P}\left\{ \|B\|_p^2 > \frac{1}{\varepsilon} \right\} = \mathbb{P}\left\{ \|B\|_p^{2q} > \frac{1}{\varepsilon^q} \right\} \\ &\leq \varepsilon^q \mathbb{E}[\|B\|_p^{2q}],\end{aligned}$$

which is finite as a consequence of Fernique's Theorem.

We note that each inequality is independent of i, and so, for suitable constant K_q, we obtain the bound

$$\begin{aligned}\mathbb{P}\left\{ \left(\min_{\mathbf{y}\in S^{k-1}} \Phi_{\mathbf{y}}(B) \right) < \varepsilon \right\} &\leq N(\varepsilon)((1 + C)^q M_q + \mathbb{E}[\|B\|_p^{2q}])\varepsilon^q \\ &\leq K_q \varepsilon^{q-2k}.\end{aligned}$$

As such a bound holds for all $q \geq 1$, we may conclude that

$$\mathbb{P}\left\{ \left(\min_{\mathbf{y}\in S^{k-1}} \Phi_{\mathbf{y}}(B) \right) < \varepsilon \right\} = O(\varepsilon^{\infty-}),$$

as desired. □

4 Appendix

At first blush, it may seem natural to have our process B have the classical Wiener space $\mathcal{W}^n := \mathcal{C}([0, T], \mathbb{R}^n)$ as its sample space. However, doing so is not ideal, since many of the operators we will be considering are only defined on smaller spaces, such as the p-variation spaces.

We begin with a general result regarding σ-algebras.

Lemma 4.1 *Let X be any real separable Banach space and $\mathcal{L}$ be any non-empty subset of X^*. Then $\|\cdot\|_X$ is $\sigma(\mathcal{L})$-measurable if and only if $\mathcal{B}_X = \sigma(\mathcal{L})$.*

Proof It it easy to see that, in any case, $\sigma(\mathcal{L}) \subset \mathcal{B}_X$. Also, since $\|\cdot\|_X$ is continuous it is always Borel measurable; therefore, if $\mathcal{B}_X = \sigma(\mathcal{L})$ then $\|\cdot\|_X$ is clearly $\sigma(\mathcal{L})$-measurable.

Suppose that $\|\cdot\|_X$ is $\sigma(\mathcal{L})$-measurable; then for each $x_0 \in \sigma(\mathcal{L})$, $\|\cdot - x_0\|_X$ is also $\sigma(\mathcal{L})$ – measurable, and $x \to x - x_0$ is $\sigma(\mathcal{L})/\sigma(\mathcal{L})$-measurable. From this observation, it follows that $\sigma(\mathcal{L})$ contains all balls in X. Since X is separable, every open subset of X may be written as a countable union of open balls. It follows, then, that $\sigma(\mathcal{L})$ contains all open subsets of X and therefore that $\mathcal{B}_X \subset \sigma(\mathcal{L})$. □

Theorem 4.2 *Suppose $(X, \mathcal{B} = \mathcal{B}_X, \mu)$ is a Gaussian probability space, and $\tilde{X}$ is a linear subspace of X. Also let $\|\cdot\|_{\tilde{X}}$ is a norm on $\tilde{X}$ such that*

(1) *The space $(\tilde{X}, \|\cdot\|_{\tilde{X}})$ is a separable Banach space,*
(2) *The embedding of $\tilde{X}$ into X is continuous,*
(3) *$\tilde{X} \in \mathcal{B}$ and $\mu(\tilde{X}) = 1$,*
(4) *$\tilde{\mathcal{B}} := \mathcal{B}_{\tilde{X}} = \{A \cap \tilde{X} : A \in \mathcal{B}\}$.*

Then $\tilde{\mu} := \mu|_{\tilde{X}}$ is a Gaussian measure and $(\tilde{X}, \tilde{\mathcal{B}}, \tilde{\mu})$ is a Gaussian probability space. Furthermore, (X, μ) and $(\widetilde{X}, \tilde{\mu})$ share the same Cameron–Martin space $\mathcal{H}$.

Proof Let $R_{\pi/4} : X \times X \to X \times X$ is the rotation map defined by

$$R_{\pi/4}(x, y) = \left(\frac{\sqrt{2}}{2}(x - y), \frac{\sqrt{2}}{2}(x + y)\right);$$

then by the rotational invariance of Gaussian measures (see, for example, Theorem 3.1.1 of [5]), proving the statement that $\tilde{\mu}$ is Gaussian is equivalent to proving that

$$\int\limits_{\tilde{X}\times\tilde{X}} f(x, y)\, d\tilde{\mu}(x) d\tilde{\mu}(y) = \int\limits_{\tilde{X}\times\tilde{X}} f \circ R_{\pi/4}(x, y)\, d\tilde{\mu}(x) d\tilde{\mu}(y)$$

for any bounded $\tilde{\mathcal{B}} \times \tilde{\mathcal{B}}$-measurable function f. Let f be such a function; since $\widetilde{X}$ is of full μ measure, we may extend f to an $\mathcal{B} \times \mathcal{B}$-measurable function (which we shall also refer to as f) such that $\int_{\tilde{X}\times\tilde{X}} f d\mu d\mu = \int_{X\times X} f d\mu d\mu$ (this extension may be done by setting a function equal to f on $\widetilde{X} \times \widetilde{X}$ and equal to zero on the complement, for example). Then it follows that

$$\begin{aligned}\int_{\tilde{X}\times\tilde{X}} f(x,y)\, d\tilde{\mu}(x)d\tilde{\mu}(y) &= \int_{\tilde{X}\times\tilde{X}} f(x,y)\, d\mu(x)d\mu(y) \\ &= \int_{X\times X} f(x,y)\, d\mu(x)d\mu(y) \\ &= \int_{X\times X} f\circ R_{\pi/4}(x,y)\, d\mu(x)d\mu(y) \\ &= \int_{\widetilde{X}\times\widetilde{X}} f\circ R_{\pi/4}(x,y)\, d\mu(x)d\mu(y) \\ &= \int_{\widetilde{X}\times\widetilde{X}} f\circ R_{\pi/4}(x,y)\, d\tilde{\mu}(x)d\tilde{\mu}(y).\end{aligned}$$

This proves the first assertion.

To see the equivalence of Cameron–Martin spaces, we recall that $J : L^2(X,\mu) \to X$, defined by

$$Jf := \int_X x f(x)\, d\mu(x),$$

maps onto $\mathcal{H}$. Again, by virtue of μ being fully supported on $\widetilde{X}$, we may extend any element of $L^2(\widetilde{X},\tilde{\mu})$ to an element of $L^2(X,\mu)$; thus it is easy to see that $J(L^2(\widetilde{X},\tilde{\mu})) = J(L^2(X,\mu)) = \mathcal{H}$, as desired. $\square$

Remark 4.3 An alternate proof of the equivalence of Cameron–Martin spaces may be found in Proposition 2.8 of [9].

Let us now focus on restricting the law of fractional Brownian motion with Hurst parameter $\frac{1}{4} < H < \frac{1}{2}$ to a variational space. The standard Gaussian space on which fBm is realized is $(\mathcal{W},\mathcal{B},\mathbb{P})$, where $\mathcal{W} = \{\omega \in \mathcal{C}([0,T],\mathbb{R}) : \omega(0) = 0\}$ and $\mathbb{P} = \text{Law}(B^H)$. Pick $0 < \epsilon << 1$ and fix $p := 1/H + \epsilon$. Let $\phi_t, 0 \le t \le T$ denote the evaluation map on $\mathcal{W}$; i.e., $\phi_t(x) = x(t)$ for any $x \in \mathcal{W}$. Since

$$\|\cdot\|_{\mathcal{W}} = \sup_{0\le t\le T} \phi_t,$$

it follows that $\|\cdot\|_{\mathcal{W}}$ is a $\sigma(\{\phi_t : 0 \le t \le T\})$-measurable function, and by Lemma 4.1, it then follows that $\sigma(\{\phi_t : 0 \le t \le T\}) = \mathcal{B}_{\mathcal{W}}$. Recall that we have defined the

p-variation norm on $\mathcal{W}$ by

$$\|x\|_p = \sup_{\Pi \in \mathcal{P}[0,T]} \left(\sum_{i=1}^{(\#\Pi)} |\Delta_i x|^p \right)^{\frac{1}{p}}.$$

Recall that we have defined the space

$$\mathcal{W}_p = \overline{\{x \in \mathcal{C}_\infty([0,T], \mathbb{R}) : x(0) = 0\}}^{\|\cdot\|_p}.$$

By Corollary 5.35 and Proposition 5.38 of [10], this space is a separable Banach space under the p-variation norm and contains all q-variation paths starting at zero for any $1 \le q < p$. Note that for $x \in \mathcal{W}_p$, Hölder's inequality implies that for any $t \in [0, T]$,

$$\begin{aligned} |x(t)| &= |x(t) - x(0)| \\ &\le |x(t) - x(0)| + |x(T) - x(0)| \\ &\le 2^{\frac{p-1}{p}} \left(|x(t) - x(0)|^p + |x(T) - x(0)|^p \right)^{\frac{1}{p}} \\ &\le 2^{\frac{p-1}{p}} \|x\|_p, \end{aligned}$$

from which it follows that $\|x\|_{\mathcal{W}} \le \|x\|_p$, and so the embedding of $\mathcal{W}_0^p$ into $\mathcal{W}$ is continuous. Observe that we may rewrite the p-variation norm as

$$\|\cdot\|_p = \sup_{\Pi \in \mathcal{P}[0,T]} \left(\sum_{i=1}^{(\#\Pi)} |\phi_{t_i} - \phi_{t_{i-1}}|^p \right)^{\frac{1}{p}}.$$

Thus, $\|\cdot\|_p$ is $\sigma(\{\phi_t|_{\mathcal{W}_p} : 0 \le t \le T\})$-measurable, which implies that $\sigma(\mathcal{L}) = \mathcal{B}_{\mathcal{W}_p}$. Furthermore, by Theorem 5.33 of [10], we know that the space $\mathcal{W}_p$ is equivalent to

$$\left\{ x \in \mathcal{W}_p : \lim_{\delta \to 0} \sup_{\Pi \in \mathcal{P}[0,T] : |\Pi| < \delta} \sum_{i=1}^{\#(\Pi)} |x(t_i) - x(t_{i-1})|^p = 0 \right\}$$

If we now define

$$\alpha_p(x) := \lim_{n \to \infty} \sup_{\Pi \in \mathcal{P}[0,T] \cap \mathbb{N} : |\Pi| < \frac{1}{n}} \sum_{i=1}^{\#(\Pi)} |x(t_i) - x(t_{i-1})|^p,$$

then it follows that α_p is a $\sigma(\{\phi_t|_{\mathcal{W}_p} : 0 \le t \le T\})$-measurable function, and that

$$\mathcal{W}_p = \mathcal{W}_p \cap \{\alpha_p = 0\} \in \mathcal{B}_{\mathcal{W}}.$$

Additionally, we may now use Lemma 4.1 to conclude that

$$\begin{aligned}\mathcal{B}_{\mathcal{W}_p} &= \sigma(\{\phi_t|_{\mathcal{W}_p} : 0 \le t \le T\}) \\ &= \{A \cap \mathcal{W}_p : A \in \sigma(\{\phi_t|_{\mathcal{W}} : 0 \le t \le T\})\} \\ &= \{A \cap \mathcal{W}_p : A \in \mathcal{B}_{\mathcal{W}}\}.\end{aligned}$$

Finally, we note that since the paths $t \mapsto B_t^H$ are a.s. Hölder continuous of order $\beta := H\left(1 + \frac{\epsilon H}{2}\right)^{-1} < H$, each such path has finite q-variation for $q = \frac{1}{\beta} = \frac{1}{H} + \frac{\epsilon}{2}$. So by Corollary 5.35 of [10], $\mathbb{P}(\mathcal{W}_p) \ge \mathbb{P}(\mathcal{W}_q) = 1$. Thus, we may appeal to Theorem 4.2 to conclude that $(\mathcal{W}_p, \mathcal{B}_{\mathcal{W}_p}, \mathbb{P}|_{\mathcal{W}_p})$ is also a Gaussian probability space, and that the associated Cameron–Martin space $\mathcal{H}$ coincides with the usual Cameron–Martin space corresponding to $\mathbb{P}$ on $\mathcal{W}$.

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