

## Erratum to: On the existence of smooth densities for jump processes

Jean Picard

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**Erratum to: Probab. Theory Relat. Fields (1996) 105:481–511**  
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We correct in this note a mistake in an intermediate result of [1]. Let us first recall the structure of the main results of this article:

Theorems 2.1 and 2.2  $\Rightarrow$  Theorem 3.1  $\Rightarrow$  Theorem 4.1  $\Rightarrow$   $\left\{ \begin{array}{l} \text{Corollary 4.4} \\ \text{Corollary 4.5} \end{array} \right.$

- Theorem 2.1 is the main result about the existence of a smooth density for real variables defined on a Poisson space; Theorem 2.2 is the extension to vector-valued variables.
- Theorem 3.1 deals with the case of functionals of a Lévy process; in particular, an analogue of the Malliavin matrix of the Wiener case is introduced.
- Theorem 4.1 is the particular case of the solution of a stochastic differential equation driven by the Lévy process.

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J. Picard (✉)  
Laboratoire de Mathématiques, Clermont Université, Université Blaise Pascal, BP 10448,  
63000 Clermont-Ferrand, France  
e-mail: Jean.Picard@math.univ-bpclermont.fr

J. Picard  
Laboratoire de Mathématiques, CNRS, UMR 6620, 63177 Aubière, France

- Corollaries 4.4 and 4.5 consider respectively the case of a “uniformly elliptic” equation, and of the stochastic Lévy area.

However, there is a problem in the proof of Theorem 3.1. Technically, Theorem 3.1 is deduced from Theorem 2.1, so we have to verify that the assumptions of this latter theorem are fulfilled. In particular, we have to prove that condition (2.3) is verified for almost any  $\tau \in A(\rho)^k$ . Let us look carefully at the sequence of inequalities following (3.11); as it has been said, these inequalities should be valid for almost any  $\tau \in A(\rho)^k$ . The first inequality is trivial since the random variable  $R'$  satisfies  $R' \leq \rho$ . In the second inequality, we want to apply (3.4). A first problem is that  $R'$  is random whereas (3.4) was written for a deterministic radius. Moreover and more importantly, even if  $R'$  were deterministic, the application of (3.4) would require  $\tau \in A(R')^k$ , and this is not necessarily true since  $A(R')$  may be strictly included in  $A(\rho)$ . Thus the proof of the theorem is not correct.

Our aim in this note is to give a corrected statement for Theorem 3.1, and to consider the consequences on subsequent results. In particular, Corollaries 4.4 and 4.5 are still valid; Corollary 4.4 is used for instance in [2] which therefore is not changed by this correction.

### Correction of Theorem 3.1

Condition (b) of Theorem 3.1 has to be replaced by a modified condition (b'), and the theorem becomes the following one.

**Theorem 3.1'** *Suppose that the Lévy measure  $\mu$  satisfies the conditions of Corollary 1.2(b). Let  $T > 0$  and let  $F$  be a  $\mathbb{R}^d$  valued functional of  $(X_t; 0 \leq t \leq T)$  satisfying*

- (a) *for any  $p$  and  $k$ , the condition (3.1) holds true;*
- (b') *there exists a matrix-valued process  $\psi_t$  such that for  $|x| \leq 1, p \geq 1, k \geq 0$  and  $\tau \in A(1)^k$ ,*

$$\| (D_{tx}F - \psi_t x) \circ \varepsilon_\tau^+ \|_p \leq C_{p,k} |x|^r \tag{3.2'}$$

*for some  $r > 1$ , and*

$$\left\| \left( \det \int_0^T \psi_t \psi_t^* dt \right)^{-1} \circ \varepsilon_\tau^+ \right\|_p \leq C_{p,k}. \tag{3.3'}$$

*Then  $F$  has a  $C_b^\infty$  density.*

*Proof* Let us emphasise the changes with respect to the original Theorem 3.1. We have to check that the condition (2.3) in Theorem 2.1 is satisfied. Actually, the original proof is correct without the operator  $\varepsilon_\tau^+$ , or equivalently with  $k = 0$  and  $\tau = \emptyset$ . For the case  $k \geq 1$ , we simply have to apply the original proof to the variable  $F \circ \varepsilon_\tau^+$ . This variable

satisfies (3.1) from Lemma 3.2; it also satisfies (3.2) and (3.3) for the matrix  $\psi_t \circ \varepsilon_\tau^+$ ; thus, since  $D_u(F \circ \varepsilon_\tau^+) = (D_u F) \circ \varepsilon_\tau^+$  for almost any  $u$ , (2.3) is satisfied. Moreover, the constants involved in the estimations do not depend on  $\tau$ .  $\square$

#### Consequence on Theorem 4.1

Theorem 4.1 is correct if we replace the non-degeneracy condition (3.3) by the above (3.3').

With respect to the original proof, we have to check that the condition (3.2') is satisfied with  $\varepsilon_\tau^+$ . This means that in the computations we have to replace respectively the solution  $Y_t$  and the semi-flow  $\phi_{st}$  by  $Y_t \circ \varepsilon_\tau^+$  and  $\phi_{st} \circ \varepsilon_\tau^+$  for  $\tau = (t_1, x_1, \dots, t_k, x_k)$ ,  $|x_i| \leq 1$ ; the bounds have to be uniform in  $\tau$  for  $k$  fixed. Proving that the moments of  $Y_t \circ \varepsilon_\tau^+$  are bounded is similar to the original study of  $Y_t$ . On the other hand, we have to extend Lemma 4.3 and estimate  $\sup_{s,t} |(\phi_{st} \circ \varepsilon_\tau^+)^{(k)}(y)|$ . The original proof can be used after inserting the  $k$  jumps  $(t_i, x_i)$  in the path of the Lévy process  $X$ . The number of stopping times  $\sigma_j$  is not changed too much, and the theorem follows.

#### Consequence on Corollaries 4.4 and 4.5

The two corollaries remain correct. This means that (3.3') can be proved in the two cases.

For Corollary 4.4, it is not a restriction to assume that  $\rho_0 > 1$ , so that  $y \mapsto y + a(y, x)$  is a diffeomorphism for  $|x| \leq 1$ . Then the times  $\sigma_j$  of the original proof are not changed by  $\varepsilon_\tau^+$ ,  $\tau \in A(1)^k$ , and  $(Z_t^{\sigma_j} \circ \varepsilon_\tau^+)^{-1}$  can be estimated similarly.

For the proof of Corollary 4.5, the main point is that the variables  $X_t - X_s$ ,  $|t - s| \geq \delta$ , have uniformly bounded density; but a deterministic translation does not modify the bound on the density, so the densities of  $(X_t - X_s) \circ \varepsilon_\tau^+$  are also bounded.

## References

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