Equality of averaged and quenched large deviations for random walks in random environments in dimensions four and higher

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Abstract We consider large deviations for nearest-neighbor random walk in a uniformly elliptic i.i.d. environment. It is easy to see that the quenched and the averaged rate functions are not identically equal. When the dimension is at least four and Sznitman's transience condition (T) is satisfied, we prove that these rate functions are finite and equal on a closed set whose interior contains every nonzero velocity at which the rate functions vanish.

1 Introduction

1.1 The model

Let $U := \{\pm e_i\}_{i=1}^d$ where $(e_i)_{i=1}^d$ denotes the canonical basis for the d-dimensional integer lattice $\mathbb{Z}^d$ with $d \geq 1$. Consider a discrete time Markov chain on $\mathbb{Z}^d$ with nearest-neighbor steps, i.e., with steps in U. For every $x \in \mathbb{Z}^d$ and $z \in U$, the transition probability from x to $x + z$ is denoted by $\pi(x, x + z)$, and the transition vector

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$\omega_x := (\pi(x, x + z))_{z \in U}$ is referred to as the *environment* at x. If the environment $\omega := (\omega_x)_{x \in \mathbb{Z}^d}$ is sampled from a probability space $(\Omega, \mathcal{B}, \mathbb{P})$, then this process is called *random walk in a random environment* (RWRE). Here, $\mathcal{B}$ is the Borel σ-algebra corresponding to the product topology.

For every $y \in \mathbb{Z}^d$, define the shift T_y on Ω by $(T_y \omega)_x := \omega_{x+y}$. Assume that $\mathbb{P}$ is stationary and ergodic under $(T_z)_{z \in U}$ and

$$\text{there exists a } \delta > 0 \text{ such that } \mathbb{P}\{\pi(0, z) \geq \delta\} = 1$$
$$\text{for every } z \in U. \text{ (Uniform ellipticity.)} \tag{1.1}$$

For every $x \in \mathbb{Z}^d$ and $\omega \in \Omega$, the Markov chain with environment ω induces a probability measure P_x^ω on the space of paths starting at x. Statements about P_x^ω that hold for $\mathbb{P}$-a.e. ω are referred to as *quenched*. Statements about the semi-direct product $P_x := \mathbb{P} \times P_x^\omega$ are referred to as *averaged* (or *annealed*). Expectations under $\mathbb{P}$, P_x^ω and P_x are denoted by $\mathbb{E}$, E_x^ω and E_x, respectively.

See [24] for a survey of results on RWRE.

1.2 Regeneration times

Let $(X_n)_{n \geq 0}$ denote the path of a particle taking a RWRE. Consider a unit vector $\hat{u} \in \mathcal{S}^{d-1}$. Define a sequence $(\tau_m)_{m \geq 0} = (\tau_m(\hat{u}))_{m \geq 0}$ of random times, which are referred to as *regeneration times* (relative to $\hat{u}$), by $\tau_o := 0$ and

$$\tau_m := \inf \left\{ j > \tau_{m-1} : \langle X_i, \hat{u} \rangle < \langle X_j, \hat{u} \rangle \leq \langle X_k, \hat{u} \rangle \text{ for all } i, k \text{ with } i < j < k \right\} \tag{1.2}$$

for every $m \geq 1$. If the walk is directionally transient relative to $\hat{u}$, i.e., if

$$P_o \left(\lim_{n \to \infty} \langle X_n, \hat{u} \rangle = \infty \right) = 1, \tag{1.3}$$

then $P_o(\tau_m < \infty) = 1$ for every $m \geq 1$. As noted in [19], the significance of $(\tau_m)_{m \geq 1}$ is due to the fact that

$$\left(X_{\tau_m+1} - X_{\tau_m}, X_{\tau_m+2} - X_{\tau_m}, \ldots, X_{\tau_{m+1}} - X_{\tau_m} \right)_{m \geq 1}$$

is an i.i.d. sequence under P_o when

$$\omega = (\omega_x)_{x \in \mathbb{Z}^d} \text{ is an i.i.d. collection.} \tag{1.4}$$

The walk is said to satisfy Sznitman's transience condition $(\mathbf{T}, \hat{u})$ if (1.3) holds and

$$E_o \left[\sup_{1 \leq i \leq \tau_1(\hat{u})} \exp \{\kappa_1 |X_i|\} \right] < \infty \quad \text{for some } \kappa_1 > 0. \tag{1.5}$$

When $d \geq 2$, Sznitman [18] proves that (1.1), (1.4) and $(\mathbf{T}, \hat{u})$ imply a *ballistic* law of large numbers (LLN), an averaged central limit theorem and certain large deviation estimates. Denote the LLN velocity by $\xi_o \neq 0$.

As stated below in Lemma 7, $(\mathbf{T}, \hat{u})$ is satisfied as soon as the walk is *non-nestling* relative to $\hat{u}$, i.e., when

$$\operatorname*{ess\,inf}_{\mathbb{P}} \sum_{z \in U} \pi(0, z)\langle z, \hat{u} \rangle > 0. \tag{1.6}$$

The walk is said to be non-nestling if it is non-nestling relative to some unit vector. Otherwise, it is referred to as *nestling*. In the latter case, the convex hull of the support of the law of $\sum_z \pi(0, z)z$ contains the origin.

1.3 Previous results on large deviations for RWRE

Recall that a sequence $(Q_n)_{n \geq 1}$ of probability measures on a topological space $\mathbb{X}$ satisfies the *large deviation principle* (LDP) with rate function $I : \mathbb{X} \to \mathbb{R}$ if I is non-negative, lower semicontinuous, and for any measurable set G,

$$-\inf_{x \in G^o} I(x) \leq \liminf_{n \to \infty} \frac{1}{n} \log Q_n(G) \leq \limsup_{n \to \infty} \frac{1}{n} \log Q_n(G) \leq -\inf_{x \in \bar{G}} I(x).$$

Here, G^o is the interior of G, and $\bar{G}$ its closure. See [8] for general background regarding large deviations.

Theorem 1 (Quenched LDP) *For $\mathbb{P}$-a.e. ω, $(P_o^\omega(\frac{X_n}{n} \in \cdot))_{n \geq 1}$ satisfies the LDP with a deterministic and convex rate function I_q.*

When $d = 1$, Greven and den Hollander [10] prove Theorem 1 for walks in i.i.d. environments. They provide a formula for I_q and show that its graph typically has flat pieces. Comets et al. [4] generalize the results in [10] to stationary and ergodic environments.

When $d \geq 1$, Zerner [25] proves Theorem 1 for nestling walks in i.i.d. environments. Varadhan [20] drops the nestling assumption and generalizes Theorem 1 to stationary and ergodic environments. Since both of these works rely on the subadditive ergodic theorem, they do not lead to any formulae for the rate function. Rosenbluth [16] gives an alternative proof of Theorem 1 in the case of stationary and ergodic environments. He provides a variational formula for the rate function I_q. In [23], we prove a quenched LDP for the pair empirical measure of the so-called *environment Markov chain* $(T_{X_n}\omega)_{n \geq 0}$. This implies Rosenbluth's result by an appropriate contraction.

In their aforementioned paper concerning RWRE on $\mathbb{Z}$, Comets et al. [4] prove also

Theorem 2 (Averaged LDP) $(P_o(\frac{X_n}{n} \in \cdot))_{n \geq 1}$ *satisfies the LDP with a convex rate function I_a.*

They establish this result for a class of environments including the i.i.d. case, and obtain the following variational formula for I_a:

$$I_a(\xi) = \inf_{\mathbb{Q}} \left\{ I_q^{\mathbb{Q}}(\xi) + |\xi| h \left(\mathbb{Q} \,|\mathbb{P}\right) \right\}. \tag{1.7}$$

Here, the infimum is over all stationary and ergodic probability measures on Ω, $I_q^{\mathbb{Q}}(\cdot)$ denotes the rate function for the quenched LDP when the environment measure is $\mathbb{Q}$, and $h(\cdot \,|\cdot)$ is specific relative entropy. Similar to the quenched picture, the graph of I_a is shown to typically have flat pieces.

Varadhan [20] considers walks in i.i.d. environments, and proves Theorem 2 for any $d \geq 1$. He gives a variational formula for I_a. (His formula does not resemble (1.7) in any way.) Rassoul-Agha [13] generalizes Varadhan's result to a class of mixing environments.

Let $\mathcal{N}_q := \left\{ \xi \in \mathbb{R}^d : I_q(\xi) = 0 \right\}$ and $\mathcal{N}_a := \left\{ \xi \in \mathbb{R}^d : I_a(\xi) = 0 \right\}$ denote the zero-sets of I_q and I_a. The following theorem summarizes the previous results regarding the qualitative properties of the quenched and the averaged rate functions when $d \geq 2$.

Theorem 3 *Assume $d \geq 2$, (1.1) and (1.4).*

(a) *I_q and I_a are convex, $I_q(0) = I_a(0)$ and $\mathcal{N}_q = \mathcal{N}_a$, cf. [20].*
(b) *If the walk is non-nestling, then*
    (i) *$\mathcal{N}_a$ consists of the true velocity ξ_o, cf. [20], and*
    (ii) *I_a is strictly convex and analytic on an open set $\mathcal{A}_a$ containing ξ_o, cf. [12,21].*
(c) *If the walk is nestling, then $\mathcal{N}_a$ is a line segment containing the origin, cf. [20].*
(d) *If $(\boldsymbol{T}, \hat{u})$ is satisfied for some $\hat{u} \in \mathcal{S}^{d-1}$ in the latter case, then*
    (i) *the origin is an endpoint of $\mathcal{N}_a$, cf. [18],*
    (ii) *I_a is strictly convex and analytic on an open set $\mathcal{A}_a^+$, cf. [21],*
    (iii) *there exists a $(d-1)$-dimensional smooth surface patch $\mathcal{A}_a^b$ such that $\xi_o \in \mathcal{A}_a^b \subset \partial \mathcal{A}_a^+$, cf. [21],*
    (iv) *the unit vector η_o normal to $\mathcal{A}_a^b$ (and pointing in $\mathcal{A}_a^+$) at ξ_o satisfies $\langle \eta_o, \xi_o \rangle > 0$, cf. [21], and*
    (v) *$I_a(t\xi) = t I_a(\xi)$ for every $\xi \in \mathcal{A}_a^b$ and $t \in [0, 1]$, cf. [12].*

1.4 The main result

Assume (1.1) and (1.4). It is clear that

$$\mathcal{D} := \left\{ (\xi_1, \ldots, \xi_d) \in \mathbb{R}^d : |\xi_1| + \cdots + |\xi_d| \leq 1 \right\} = \left\{ \xi \in \mathbb{R}^d : I_a(\xi) < \infty \right\}$$
$$= \left\{ \xi \in \mathbb{R}^d : I_q(\xi) \leq -\log \delta \right\}.$$

For any $\xi \in \mathbb{R}^d$, $I_a(\xi) \leq I_q(\xi)$ by Jensen's inequality and Fatou's lemma.

Proposition 4 *If the support of $\mathbb{P}$ is not a singleton, then $I_a < I_q$ at some interior points of $\mathcal{D}$.*

Proof If the support of $\mathbb{P}$ is not a singleton, then $\mathbb{P}\{\pi(0, z) = \mathbb{E}\{\pi(0, z)\}\} < 1$ for some $z \in U$, and

$$\mathbb{E}\{\log \pi(0, z)\} < \log \mathbb{E}\{\pi(0, z)\} \tag{1.8}$$

by Jensen's inequality. For every $n \geq 1$, the event $\{X_n = nz\}$ consists of a single path marching in the z-direction. In particular, this path never visits the same point more than once. Therefore,

$$\log \mathbb{E}\{\pi(0, z)\} = \lim_{n \to \infty} \frac{1}{n} \log P_o(X_n = nz) \leq -I_a(z). \tag{1.9}$$

On the other hand, for every $\epsilon > 0$,

$$-I_q(z) \leq \liminf_{n \to \infty} \frac{1}{n} \log P_o^\omega(\langle X_n, z \rangle > n(1 - \epsilon)) \leq (1 - \epsilon)\mathbb{E}\{\log \pi(0, z)\} + O(\epsilon). \tag{1.10}$$

Explanation: For every $n \geq 1$, the number of paths constituting the event $\{\langle X_n, z \rangle > n(1 - \epsilon)\}$ is $e^{n O(\epsilon)}$. The probability of each such path is bounded from above by the product of the probabilities of its jumps in the z-direction taking place at distinct points. Since there are at least $n(1 - \epsilon)$ such jumps, (1.10) follows from Jensen's inequality and the LLN for i.i.d. random variables.

Putting (1.8), (1.9) and (1.10) together, we conclude that $I_a(z) < I_q(z)$. Since the rate functions are convex and lower semicontinuous, they are in fact continuous on $\mathcal{D}$, cf. Theorem 10.2 of [15]. This implies the desired result. □

The following theorem is the main result of this paper.

Theorem 5 *Assume $d \geq 4$, (1.1), (1.4) and (T, $\hat{u}$) for some $\hat{u} \in \mathcal{S}^{d-1}$.*

(a) *If the walk is non-nestling, then $I_q = I_a$ on an open set $\mathcal{A}_{eq}$ containing ξ_o.*
(b) *If the walk is nestling, then*
    (i) *$I_q = I_a$ on an open set $\mathcal{A}_{eq}^+$,*
    (ii) *there exists a $(d - 1)$-dimensional smooth surface patch $\mathcal{A}_{eq}^b$ such that $\xi_o \in \mathcal{A}_{eq}^b \subset \partial \mathcal{A}_{eq}^+$,*
    (iii) *the unit vector η_o normal to $\mathcal{A}_{eq}^b$ (and pointing in $\mathcal{A}_{eq}^+$) at ξ_o satisfies $\langle \eta_o, \xi_o \rangle > 0$, and*
    (iv) *$I_q(t\xi) = t I_q(\xi) = t I_a(\xi) = I_a(t\xi)$ for every $\xi \in \mathcal{A}_{eq}^b$ and $t \in [0, 1]$.*

Some remarks

1. Since I_q and I_a are both continuous on $\mathcal{D}$, it is clear that $\mathcal{E} := \{\xi \in \mathcal{D} : I_q(\xi) = I_a(\xi)\}$ is closed. Proposition 4 and Theorem 5 imply that $\mathcal{D} \setminus \mathcal{E}$ and $\mathcal{E}$ both have nonempty interiors.
2. Assuming $d = 1$, (1.1) and (1.4), Comets et al. [4] use (1.7) to show that $I_q(\xi) = I_a(\xi)$ if and only if $\xi = 0$ or $I_a(\xi) = 0$. In particular, Theorem 5 cannot be generalized to $d \geq 1$. Whether it can be generalized to $d \geq 2$ is an open problem.

3. The analog of Theorem 5 for so-called *space-time* RWRE is proved in [22].
4. Related results have been obtained for random walks in random potentials, cf.
   [9,26], for directed polymers in random environments, cf. [5], and for random
   walks on Galton-Watson trees, cf. [1,6,7].

2 Proof of the main result

2.1 Outline

For every $\theta \in \mathbb{R}^d$, consider the logarithmic moment generating functions

$$\Lambda_q(\theta) := \lim_{n \to \infty} \frac{1}{n} \log E_o^\omega \left[\exp\{\langle \theta, X_n \rangle\} \right] \quad \text{and}$$

$$\Lambda_a(\theta) := \lim_{n \to \infty} \frac{1}{n} \log E_o \left[\exp\{\langle \theta, X_n \rangle\} \right].$$

By Varadhan's Lemma, cf. [8], $\Lambda_q(\theta) = \sup_{\xi \in \mathbb{R}^d} \left\{ \langle \theta, \xi \rangle - I_q(\xi) \right\} = I_q^*(\theta)$, the convex conjugate of I_q at θ. Similarly, $\Lambda_a(\theta) = I_a^*(\theta)$.

Assume $d \geq 4$ and $(\mathbf{T}, \hat{u})$ for some $\hat{u} \in \mathcal{S}^{d-1}$. For every $n \geq 0$, $\theta \in \mathbb{R}^d$ and $\omega \in \Omega$, define

$$H_n = H_n(\hat{u}) := \inf \left\{ i \geq 0 : \langle X_i, \hat{u} \rangle \geq n \right\},$$
$$\beta = \beta(\hat{u}) := \inf \left\{ i \geq 0 : \langle X_i, \hat{u} \rangle < \langle X_o, \hat{u} \rangle \right\} \quad \text{and} \quad (2.1)$$
$$g_n(\theta, \omega) := E_o^\omega \left[\exp \left\{ \langle \theta, X_{H_n} \rangle - \Lambda_a(\theta) H_n \right\}, H_n = \tau_k \text{ for some } k \geq 1, \beta = \infty \right].$$

When $|\theta|$ is sufficiently small (and $\Lambda_a(\theta) > 0$ in the nestling case), we show that $(g_n(\theta, \cdot))_{n \geq 1}$ is bounded in $L^2(\mathbb{P})$ and $\mathbb{E}\{g_n(\theta, \cdot)\}$ converges to a nonzero limit as $n \to \infty$. These two facts imply that $\Lambda_q(\theta) = \Lambda_a(\theta)$.

Section 3 is devoted to the L^2 estimate regarding $(g_n(\theta, \cdot))_{n \geq 1}$ which constitutes the core of this paper. Assuming that, the equality of the logarithmic moment generating functions is established in Sect. 2.3. Finally, convex duality is used in Sect. 2.4 to prove Theorem 5 by showing that the local equality of Λ_q and Λ_a implies the equality of I_q and I_a on certain subsets of $\mathcal{D}$.

We find it more convenient to work with regeneration times relative to a $z \in U$ rather than any $\hat{u} \in \mathcal{S}^{d-1}$. In Sect. 2.2, we give some results which imply that there is no loss of generality in doing so.

2.2 Some preliminaries regarding regenerations

Assume $d \geq 2$, (1.1) and (1.4).

Lemma 6 (Sznitman [18]) *Assume* $(\mathbf{T}, \hat{u})$ *for some* $\hat{u} \in \mathcal{S}^{d-1}$.

(a) $P_o(\beta(\hat{u}) = \infty) > 0$, *and* $\tau_1(\hat{u})$ *has finite* P_o-*moments of arbitrary order.*
(b) *The LLN holds with a limiting velocity* ξ_o *such that* $\langle \xi_o, \hat{u} \rangle > 0$.
(c) $(\mathbf{T}, \hat{v})$ *is satisfied for every* $\hat{v} \in \mathcal{S}^{d-1}$ *such that* $\langle \xi_o, \hat{v} \rangle > 0$.

Lemma 7 (Sznitman [17]) *If the walk is non-nestling relative to some $\hat{u} \in \mathcal{S}^{d-1}$, then*

$$E_o\left[\exp\left\{\kappa_2\tau_1(\hat{u})\right\}\right] < \infty$$

for some $\kappa_2 > 0$. In particular, $(\mathbf{T}, \hat{u})$ is satisfied.

Lemma 8 *If the walk is non-nestling and some $\hat{v} \in \mathcal{S}^{d-1}$ satisfies $\langle \xi_o, \hat{v} \rangle > 0$, then*

$$E_o\left[\exp\left\{c\tau_1(\hat{v})\right\}\right] < \infty$$

for some $c > 0$.

Proof Since the walk is non-nestling, (1.6) holds for some $\hat{u} \in \mathcal{S}^{d-1}$ with rational coordinates. Let $a \geq 1$ be an integer such that $a\hat{u}$ has integer coordinates. Note that $\langle x, \hat{u} \rangle > 0$ if and only if $\langle x, \hat{u} \rangle \geq \frac{1}{a}$ for $x \in \mathbb{Z}^d$. Therefore, $\left|X_{\tau_{ak+1}(\hat{u})}\right| \geq \langle X_{\tau_{ak+1}(\hat{u})}, \hat{u} \rangle > k$ for every $k \geq 1$.

For every $c, c' > 0$ and $\hat{v} \in \mathcal{S}^{d-1}$ such that $\langle \xi_o, \hat{v} \rangle > 0$,

$$
\begin{aligned}
E_o\left[\exp\left\{c\tau_1(\hat{v})\right\}\right] &= \sum_{k=1}^{\infty} E_o\left[\exp\left\{c\tau_1(\hat{v})\right\}, \sup_{1\leq i\leq\tau_1(\hat{v})}|X_i| \in (k-1, k]\right] \\
&\leq \sum_{k=1}^{\infty} E_o\left[\exp\left\{c\tau_{ak+1}(\hat{u})\right\}, \sup_{1\leq i\leq\tau_1(\hat{v})}|X_i| \in (k-1, k]\right] \\
&\leq \sum_{k=1}^{\infty} E_o\left[\exp\left\{c\tau_{ak+1}(\hat{u})\right\}\left(\sup_{1\leq i\leq\tau_1(\hat{v})}\exp\left\{c'|X_i|\right\}\right)\right] \\
&\quad\times \exp\{-c'(k-1)\} \\
&\leq E_o\left[\sup_{1\leq i\leq\tau_1(\hat{v})}\exp\left\{2c'|X_i|\right\}\right]^{1/2}\sum_{k=1}^{\infty}E_o\left[\exp\left\{2c\tau_{ak+1}(\hat{u})\right\}\right]^{1/2} \\
&\quad\times \exp\{-c'(k-1)\}. \quad\quad (2.2)
\end{aligned}
$$

Note that $(\mathbf{T}, \hat{u})$ is satisfied by Lemma 7. Since $\langle \xi_o, \hat{v} \rangle > 0$, it follows from Lemma 6 that $(\mathbf{T}, \hat{v})$ is satisfied as well. Therefore, (1.5) implies that the first term in (2.2) is finite when $c' > 0$ is small enough.

It is immediate from the renewal structure that

$$
\begin{aligned}
&E_o\left[\exp\left\{2c\tau_{ak+1}(\hat{u})\right\}\right]^{1/2} \\
&= E_o\left[\exp\left\{2c\tau_1(\hat{u})\right\}\right]^{1/2}E_o\left[\exp\left\{2c\tau_1(\hat{u})\right\}\middle|\beta(\hat{u})=\infty\right]^{ak/2}.
\end{aligned}
$$

By Lemma 7, $E_o\left[\exp\left\{\kappa_2\tau_1(\hat{u})\right\}\middle|\beta(\hat{u})=\infty\right] < \infty$ for some $\kappa_2 > 0$. When $c > 0$ is small enough,

$$E_o\left[\exp\left\{2c\tau_1(\hat{u})\right\}\middle|\beta(\hat{u})=\infty\right]^{a/2} \leq E_o\left[\exp\left\{\kappa_2\tau_1(\hat{u})\right\}\middle|\beta(\hat{u})=\infty\right]^{ac/\kappa_2} < e^{c'}$$

and the summation in (2.2) is finite. This implies the desired result. $\quad\square$

Corollary 9 *Assume (**T**, û) for some û ∈ $\mathcal{S}^{d-1}$. Since $\xi_o \neq 0$, $\langle \xi_o, z \rangle > 0$ for some $z \in U$.*

(a) $P_o(\beta(z) = \infty) > 0$, *and* $\tau_1(z)$ *has finite* P_o-*moments of arbitrary order.*

(b) *If the walk is non-nestling, then there exists a* $\kappa_3 > 0$ *such that*

$$E_o \left[\exp \{ 2\kappa_3 \tau_1(z) \} \right] < \infty.$$

(c) *If the walk is nestling, then there exists a* $\kappa_3 > 0$ *such that*

$$E_o \left[\sup_{1 \leq i \leq \tau_1(z)} \exp \{ \kappa_3 |X_i| \} \right] < \infty.$$

2.3 Equality of the logarithmic moment generating functions

Assume $d \geq 4$, (1.1), (1.4) and (**T**, û) for some û ∈ $\mathcal{S}^{d-1}$. Since $\xi_o \neq 0$, $\langle \xi_o, z \rangle > 0$ for some $z \in U$. Assume WLOG that $\langle \xi_o, e_1 \rangle > 0$. Refer to (1.2) and (2.1) for the definitions of

$$(\tau_m)_{m \geq 1} = (\tau_m(e_1))_{m \geq 1}, \quad (H_n)_{n \geq 0} = (H_n(e_1))_{n \geq 0} \quad \text{and} \quad \beta = \beta(e_1).$$

Fix κ_3 as in Corollary 9. For every $\kappa \in (0, \kappa_3]$, define

$$\mathcal{C}_a(\kappa) := \begin{cases} \{ \theta \in \mathbb{R}^d : |\theta| < \kappa \} & \text{if the walk is non-nestling,} \\ \{ \theta \in \mathbb{R}^d : |\theta| < \kappa \, , \, \Lambda_a(\theta) > 0 \} & \text{if the walk is nestling.} \end{cases} \quad (2.3)$$

By Jensen's inequality,

$$\langle \theta, \xi_o \rangle = \lim_{n \to \infty} \frac{1}{n} E_o \left[\langle \theta, X_n \rangle \right] \leq \lim_{n \to \infty} \frac{1}{n} \log E_o \left[\exp \{ \langle \theta, X_n \rangle \} \right]$$

$$= \Lambda_a(\theta) \leq \lim_{n \to \infty} \frac{1}{n} \log E_o \left[e^{|\theta| n} \right] = |\theta|. \quad (2.4)$$

In the nestling case, $\{ \theta \in \mathbb{R}^d : |\theta| < \kappa, \langle \theta, \xi_o \rangle > 0 \} \subset \mathcal{C}_a(\kappa)$ by (2.4). Hence, $\mathcal{C}_a(\kappa)$ is a non-empty open set both for nestling and non-nestling walks.

Lemma 10 $E_o \left[\exp \{ \langle \theta, X_{\tau_1} \rangle - \Lambda_a(\theta) \tau_1 \} \big| \beta = \infty \right] = 1$ *for every* $\theta \in \mathcal{C}_a(\kappa_3)$.

Proof This is Lemma 12 of [21].   □

For every $\theta \in \mathcal{C}_a(\kappa_3)$ and $y \in \mathbb{Z}^d$, let

$$q^\theta(y) := E_o \left[\exp \{ \langle \theta, X_{\tau_1} \rangle - \Lambda_a(\theta) \tau_1 \}, X_{\tau_1} = y \big| \beta = \infty \right]. \quad (2.5)$$

Since $\sum_{y \in \mathbb{Z}^d} q^\theta(y) = 1$ by Lemma 10, $\left(q^\theta(y) \right)_{y \in \mathbb{Z}^d}$ defines a random walk $(Y_k)_{k \geq 0}$ on $\mathbb{Z}^d$. For every $x \in \mathbb{Z}^d$, this walk induces a probability measure $\hat{P}^\theta_x$ on paths starting at x. As usual, $\hat{E}^\theta_x$ denotes the corresponding expectation. It follows from Corollary 9 and Hölder's inequality that

$$\hat{E}^\theta_o \left[|Y_1|^m \right] < \infty \quad \text{for every } m \geq 1. \tag{2.6}$$

For every $n \geq 1$, $\theta \in C_a(\kappa_3)$ and $\omega \in \Omega$, recall from Sect. 2.1 that

$$g_n(\theta, \omega) := E^\omega_o \left[\exp\left\{ \langle \theta, X_{H_n} \rangle - \Lambda_a(\theta) H_n \right\}, H_n = \tau_k \text{ for some } k \geq 1, \beta = \infty \right].$$

Lemma 11 *For every $\theta \in C_a(\kappa_3)$,*

$$\lim_{n \to \infty} \mathbb{E}\{g_n(\theta, \cdot)\} = P_o(\beta = \infty)/\hat{E}^\theta_o \left[\langle Y_1, e_1 \rangle \right] > 0.$$

Proof For every $n \geq 1$ and $\theta \in C_a(\kappa_3)$,

$$\mathbb{E}\{g_n(\theta, \cdot)\} = E_o \left[\exp\left\{ \langle \theta, X_{H_n} \rangle - \Lambda_a(\theta) H_n \right\}, H_n = \tau_k \text{ for some } k \geq 1, \beta = \infty \right]$$

$$= P_o(\beta = \infty) \sum_{k=1}^{\infty} E_o \left[\exp\left\{ \langle \theta, X_{H_n} \rangle - \Lambda_a(\theta) H_n \right\}, H_n = \tau_k \,\big|\, \beta = \infty \right]$$

$$= P_o(\beta = \infty) \sum_{k=1}^{\infty} E_o \left[\exp\left\{ \langle \theta, X_{\tau_k} \rangle - \Lambda_a(\theta) \tau_k \right\}, \langle X_{\tau_k}, e_1 \rangle = n \,\big|\, \beta = \infty \right]$$

$$= P_o(\beta = \infty) \sum_{k=1}^{\infty} \hat{P}^\theta_o \left(\langle Y_k, e_1 \rangle = n \right).$$

Note that $\hat{P}^\theta_o \left(\langle Y_1, e_1 \rangle = 1 \right) > 0$ by (1.1) and part (a) of Lemma 6. Hence, the desired result follows from the renewal theorem for aperiodic sequences, cf. Theorem 10.8 of [3]. □

Lemma 12 *There exists a $\kappa_{eq} \in (0, \kappa_3)$ such that*

$$\sup_{n \geq 1} \mathbb{E}\left\{ g_n(\theta, \cdot)^2 \right\} < \infty$$

for every $\theta \in C_a(\kappa_{eq})$.

Remark 13 Lemma 12 is proved in Sect. 3.

Lemma 14 *For every $\theta \in C_a(\kappa_{eq})$,*

$$\mathbb{P}\left\{ \omega : \lim_{n \to \infty} g_n(\theta, \omega) = 0 \right\} < 1. \tag{2.7}$$

Proof Take any $\theta \in \mathcal{C}_a(\kappa_{eq})$. Note that $(g_n(\theta, \cdot))_{n \geq 1}$ is uniformly integrable by Lemma 12. If $g_n(\theta, \cdot)$ were to converge $\mathbb{P}$-a.s. to 0 as $n \to \infty$, then $\lim_{n\to\infty} \mathbb{E}\{g_n(\theta, \cdot)\} = 0$ would hold. However, this would contradict Lemma 11. □

Lemma 15 *For every $\theta \in \mathbb{R}^d$, $\epsilon > 0$ and $\mathbb{P}$-a.e. ω,*

$$\lim_{n\to\infty} E_o^\omega \left[\exp\left\{\langle \theta, X_{H_n}\rangle - (\Lambda_q(\theta) + \epsilon)H_n\right\}\right] = 0.$$

Proof For every $n \geq 1$, $\theta \in \mathbb{R}^d$, $\epsilon > 0$ and $\mathbb{P}$-a.e. ω,

$$E_o^\omega \left[\exp\left\{\langle \theta, X_{H_n}\rangle - (\Lambda_q(\theta) + \epsilon)H_n\right\}\right]$$

$$= \sum_{i=n}^\infty E_o^\omega \left[\exp\left\{\langle \theta, X_{H_n}\rangle - (\Lambda_q(\theta) + \epsilon)H_n\right\}, H_n = i\right]$$

$$\leq \sum_{i=n}^\infty E_o^\omega \left[\exp\left\{\langle \theta, X_i\rangle - (\Lambda_q(\theta) + \epsilon)i\right\}\right] = \sum_{i=n}^\infty e^{o(i)-\epsilon i} \leq \sum_{i=n}^\infty e^{-\epsilon i/2}$$

when n is sufficiently large. Therefore,

$$\limsup_{n\to\infty} E_o^\omega \left[\exp\left\{\langle \theta, X_{H_n}\rangle - (\Lambda_q(\theta) + \epsilon)H_n\right\}\right] \leq \limsup_{n\to\infty} e^{-\epsilon n/2}\left(1 - e^{-\epsilon/2}\right)^{-1} = 0.$$

□

Lemma 16 $\Lambda_q(\theta) = \Lambda_a(\theta)$ *for every $\theta \in \mathcal{C}_a(\kappa_{eq})$.*

Proof For every $\theta \in \mathbb{R}^d$, it follows from Jensen's inequality and the bounded convergence theorem that

$$\Lambda_q(\theta) = \mathbb{E}\left\{\lim_{n\to\infty} \frac{1}{n} \log E_o^\omega \left[\exp\{\langle \theta, X_n\rangle\}\right]\right\} = \lim_{n\to\infty} \frac{1}{n}\mathbb{E}\left\{\log E_o^\omega \left[\exp\{\langle \theta, X_n\rangle\}\right]\right\}$$

$$\leq \lim_{n\to\infty} \frac{1}{n} \log E_o \left[\exp\{\langle \theta, X_n\rangle\}\right] = \Lambda_a(\theta).$$

Let us now establish the reverse inequality. For every $\theta \in \mathcal{C}_a(\kappa_{eq})$ and $\epsilon > 0$,

$$\mathbb{P}\left\{\omega : \lim_{n\to\infty} E_o^\omega \left[\exp\left\{\langle \theta, X_{H_n}\rangle - \Lambda_a(\theta)H_n\right\}\right] = 0\right\} < 1 \quad \text{and}$$

$$\mathbb{P}\left\{\omega : \lim_{n\to\infty} E_o^\omega \left[\exp\left\{\langle \theta, X_{H_n}\rangle - (\Lambda_q(\theta) + \epsilon)H_n\right\}\right] = 0\right\} = 1$$

by Lemmas 14 and 15, respectively. Therefore, $\Lambda_q(\theta) + \epsilon > \Lambda_a(\theta)$. Since $\epsilon > 0$ is arbitrary, we conclude that $\Lambda_q(\theta) \geq \Lambda_a(\theta)$ for every $\theta \in \mathcal{C}_a(\kappa_{eq})$. □

2.4 Equality of the rate functions

Since $\Lambda_q = \Lambda_a$ on $\mathcal{C}_a(\kappa_{eq})$, it will follow from convex duality that $I_q(\xi) = I_a(\xi)$ for every $\xi \in \mathcal{D}$ that defines a supporting hyperplane of Λ_a at some $\theta \in \mathcal{C}_a(\kappa_{eq})$. In order to show that the set of such ξ satisfies the properties stated in Theorem 5, we need two preliminary lemmas.

Lemma 17 *Assume that the walk is nestling. Define*

$$\mathcal{C}_a^b(\kappa_{eq}) := \left\{ \theta \in \partial \mathcal{C}_a(\kappa_{eq}) : |\theta| < \kappa_{eq} \right\}. \tag{2.8}$$

(a) *If* $|\theta| < \kappa_{eq}$, *then* $\theta \notin \mathcal{C}_a(\kappa_{eq})$ *if and only if* $E_o\left[\exp\{\langle \theta, X_{\tau_1} \rangle\} \big| \beta = \infty \right] \leq 1$.
(b) *If* $|\theta| < \kappa_{eq}$, *then* $\theta \in \mathcal{C}_a^b(\kappa_{eq})$ *if and only if* $E_o\left[\exp\{\langle \theta, X_{\tau_1} \rangle\} \big| \beta = \infty \right] = 1$.

Proof This is Lemma 13 of [21]. □

Lemma 18 Λ_a *is analytic on* $\mathcal{C}_a(\kappa_{eq})$. *Its gradient* $\nabla \Lambda_a$ *extends smoothly to* $\overline{\mathcal{C}_a(\kappa_{eq})}$, *the closure of* $\mathcal{C}_a(\kappa_{eq})$. *Moreover, the extension of the Hessian* $\mathcal{H}_a$ *of* Λ_a *is positive definite on* $\overline{\mathcal{C}_a(\kappa_{eq})}$.

Proof This follows immediately from the proof of Lemma 6 of [21]. □

Proof of Theorem 5 (a) *The non-nestling case:* Recall that Λ_a is analytic on $\mathcal{C}_a(\kappa_{eq})$. Define $\mathcal{A}_{eq} := \left\{ \nabla \Lambda_a(\theta) : \theta \in \mathcal{C}_a(\kappa_{eq}) \right\}$. $\nabla \Lambda_a : \mathcal{C}_a(\kappa_{eq}) \to \mathcal{A}_{eq}$ is invertible since the Hessian $\mathcal{H}_a$ of Λ_a is positive definite on $\mathcal{C}_a(\kappa_{eq})$. The inverse, denoted by $\Gamma_a : \mathcal{A}_{eq} \to \mathcal{C}_a(\kappa_{eq})$, is analytic by the inverse function theorem (cf. Theorem 6.1.2 of [11]), and $\mathcal{A}_{eq}$ is open.
For every $\xi \in \mathcal{A}_{eq}$,

$$I_a(\xi) = \sup_{\theta \in \mathbb{R}^d} \{\langle \theta, \xi \rangle - \Lambda_a(\theta)\} = \langle \Gamma_a(\xi), \xi \rangle - \Lambda_a(\Gamma_a(\xi)). \tag{2.9}$$

Thus, I_a is analytic on $\mathcal{A}_{eq}$. Differentiating (2.9) twice with respect to ξ shows that the Hessian of I_a at ξ is equal to $\mathcal{H}_a(\Gamma_a(\xi))^{-1}$, a positive definite matrix. Therefore, I_a is strictly convex on $\mathcal{A}_{eq}$.
It is shown in [19] that $\xi_o = \left(E_o\left[X_{\tau_1} \big| \beta = \infty \right] \right) / (E_o\left[\tau_1 \big| \beta = \infty \right])$. Since $0 \in \mathcal{C}_a(\kappa_{eq})$, it follows that $\xi_o = \nabla \Lambda_a(0) \in \mathcal{A}_{eq}$.
$\Lambda_q = \Lambda_a$ on $\mathcal{C}_a(\kappa_{eq})$ by Lemma 16. For every $\xi \in \mathcal{A}_{eq}$,

$$I_q(\xi) = \sup_{\theta \in \mathbb{R}^d} \left\{ \langle \theta, \xi \rangle - \Lambda_q(\theta) \right\} = \langle \Gamma_a(\xi), \xi \rangle - \Lambda_a(\Gamma_a(\xi)) = I_a(\xi).$$

(b) *The nestling case:* Recall that $\nabla \Lambda_a$ extends smoothly to $\overline{\mathcal{C}_a(\kappa_{eq})}$. Refer to the extension by $\overline{\nabla \Lambda_a}$. Define $\mathcal{A}_{eq}^+ := \left\{ \nabla \Lambda_a(\theta) : \theta \in \mathcal{C}_a(\kappa_{eq}) \right\}$ and $\mathcal{A}_{eq}^b := \left\{ \overline{\nabla \Lambda_a}(\theta) : \theta \in \mathcal{C}_a^b(\kappa_{eq}) \right\}$ with $\mathcal{C}_a^b(\kappa_{eq})$ as in (2.8). Note that $0 \in \mathcal{C}_a^b(\kappa_{eq}) \subset \partial \mathcal{C}_a(\kappa_{eq})$ by Lemma 17, and $\xi_o = \overline{\nabla \Lambda_a}(0) \in \mathcal{A}_{eq}^b \subset \partial \mathcal{A}_{eq}^+$.

Similar to the non-nestling case, I_a is strictly convex and analytic on $\mathcal{A}_{eq}^+$ which is an open set, and $I_q(\xi) = I_a(\xi)$ for every $\xi \in \mathcal{A}_{eq}^+$. Moreover, $\mathcal{A}_{eq}^b$ is a $(d-1)$-dimensional smooth surface patch and item (iii) is satisfied. (The latter facts follow from part (d) of Theorem 3 since $\mathcal{A}_{eq}^b \subset \mathcal{A}_a^b$, cf. [21].)

It remains to show that $I_q(t\xi) = t I_q(\xi) = t I_a(\xi) = I_a(t\xi)$ for every $\xi \in \mathcal{A}_{eq}^b$ and $t \in [0, 1]$. The rest of this proof focuses on this statement.

For every $\xi \in \mathcal{A}_{eq}^b$, there exists a $\theta \in \mathcal{C}_a^b(\kappa_{eq})$ such that $\xi = \overline{\nabla \Lambda_a}(\theta)$ and

$$\langle \xi, e_1 \rangle = \langle \overline{\nabla \Lambda_a}(\theta), e_1 \rangle = \frac{E_o\left[\langle X_{\tau_1}, e_1 \rangle \exp\{\langle \theta, X_{\tau_1} \rangle\} \middle| \beta = \infty\right]}{E_o\left[\tau_1 \exp\{\langle \theta, X_{\tau_1} \rangle\} \middle| \beta = \infty\right]} > 0. \quad (2.10)$$

Suppose $\xi = \overline{\nabla \Lambda_a}(\theta')$ for some $\theta' \in \mathcal{C}_a^b(\kappa_{eq})$ such that $\theta \neq \theta'$. Then, for every $t \in (0, 1)$, ξ defines a supporting hyperplane of Λ_a at $\theta_t := t\theta + (1-t)\theta'$. Recall Lemma 17. $E_o[\exp\{\langle \theta_t, X_{\tau_1} \rangle\} | \beta = \infty] < 1$ by Jensen's inequality, and θ_t is an interior point of $\mathcal{C}_a(\kappa_{eq})^c$. Therefore, $\nabla \Lambda_a(\theta_t) = 0$ since Λ_a is identically equal to zero on $\{\theta : |\theta| < \kappa_{eq}\} \setminus \mathcal{C}_a(\kappa_{eq})$. However, this contradicts (2.10). We conclude that there exists a *unique* $\theta \in \mathcal{C}_a^b(\kappa_{eq})$ such that $\xi = \overline{\nabla \Lambda_a}(\theta)$. Denote the inverse of $\overline{\nabla \Lambda_a}$ by $\overline{\Gamma_a}$.

For every $\xi \in \mathcal{A}_{eq}^b$ and $t \in [0, 1]$, $\exists \theta_n \in \mathcal{C}_a(\kappa_{eq})$ such that $\theta_n \to \overline{\Gamma_a}(\xi)$ and $\xi_n := \nabla \Lambda_a(\theta_n) \to \xi$ as $n \to \infty$. Note that $\Lambda_a(\overline{\Gamma_a}(\xi)) = 0$ since $\overline{\Gamma_a}(\xi) \in \mathcal{C}_a^b(\kappa_{eq})$. By the continuity of I_a and Λ_a,

$$I_a(\xi) = \lim_{n \to \infty} I_a(\xi_n) = \lim_{n \to \infty} \langle \theta_n, \xi_n \rangle - \Lambda_a(\theta_n) = \langle \overline{\Gamma_a}(\xi), \xi \rangle - \Lambda_a(\overline{\Gamma_a}(\xi)) = \langle \overline{\Gamma_a}(\xi), \xi \rangle \quad \text{and}$$

$$I_a(t\xi) = \sup_{\theta \in \mathbb{R}^d} \{\langle \theta, t\xi \rangle - \Lambda_a(\theta)\} \geq \langle \overline{\Gamma_a}(\xi), t\xi \rangle - \Lambda_a(\overline{\Gamma_a}(\xi)) = t\langle \overline{\Gamma_a}(\xi), \xi \rangle = t I_a(\xi).$$

Conversely, $I_a(t\xi) \leq t I_a(\xi) + (1-t) I_a(0) = t I_a(\xi)$ by Jensen's inequality (and the fact that $I_a(0) = 0$, cf. Theorem 3). Hence, $I_a(t\xi) = t I_a(\xi)$.

The continuity of the rate functions implies that $I_q = I_a$ on $\mathcal{A}_{eq}^b$. Recall that $I_q(0) = 0$, cf. Theorem 3. Since the averaged rate function is always less than or equal to the quenched rate function, we conclude that

$$I_q(t\xi) \leq t I_q(\xi) + (1-t) I_q(0) = t I_q(\xi) = t I_a(\xi) = I_a(t\xi) \leq I_q(t\xi).$$

$\square$

Remark 19 The argument above, due to its structure, not only proves Theorem 5, but also reproduces some of the proofs of the statements in Theorem 3 that are given in [21]. Moreover, it provides a new and concise proof of item (v) of part (d) of Theorem 3 which is originally obtained in [12].

3 The L^2 estimate

In our proof of Theorem 5 given in Sect. 2, we assumed Lemma 12. In this section, we will verify this assumption. The following fact will play a central role in our argument:

if the dimension is at least four, then, with positive averaged probability, the paths of two independent ballistic walks in the same environment do not intersect.

3.1 Some preliminaries regarding two walks

Assume $d \geq 4$, (1.1), (1.4) and $(\mathbf{T}, \hat{u})$ for some $\hat{u} \in \mathcal{S}^{d-1}$. Also, like in Sects. 2.3 and 2.4, assume WLOG that $\langle \xi_o, e_1 \rangle > 0$.

For every x and $\tilde{x} \in \mathbb{Z}^d$, consider two independent walks $X = X(x) := (X_i)_{i \geq 0}$ and $\tilde{X} = \tilde{X}(\tilde{x}) := (\tilde{X}_j)_{j \geq 0}$ starting at x and $\tilde{x}$ in the same environment. Denote their joint quenched law and joint averaged law by $P^\omega_{x,\tilde{x}} := P^\omega_x \times P^\omega_{\tilde{x}}$ and $P_{x,\tilde{x}}(\cdot) := \mathbb{E}\{P^\omega_{x,\tilde{x}}(\cdot)\}$. As usual, $E^\omega_{x,\tilde{x}}$ and $E_{x,\tilde{x}}$ refer to expectations under $P^\omega_{x,\tilde{x}}$ and $P_{x,\tilde{x}}$, respectively.

Clearly, $P_{x,\tilde{x}} \neq P_x \times P_{\tilde{x}}$. On the other hand, the two walks don't know that they are in the same environment unless their paths intersect. In particular, for any event A involving X and $\tilde{X}$,

$$P_{x,\tilde{x}}(A \cap \{\nu_1 = \infty\}) = P_x \times P_{\tilde{x}}(A \cap \{\nu_1 = \infty\}) \qquad (3.1)$$

where

$$\nu_1 := \inf\{m \in \mathbb{Z} : X_i = \tilde{X}_j \text{ for some } i \geq 0, j \geq 0, \text{ and } \langle X_i, e_1 \rangle = m\}. \quad (3.2)$$

Similar to the random times $(\tau_m)_{m \geq 1} = (\tau_m(e_1))_{m \geq 1}$, $(H_n)_{n \geq 0} = (H_n(e_1))_{n \geq 0}$ and $\beta = \beta(e_1)$ defined in (1.2) and (2.1) for X, consider $(\tilde{\tau}_m)_{m \geq 1} = (\tilde{\tau}_m(e_1))_{m \geq 1}$, $(\tilde{H}_n)_{n \geq 0} = (\tilde{H}_n(e_1))_{n \geq 0}$ and $\tilde{\beta} = \tilde{\beta}(e_1)$ for $\tilde{X}$. In our proof of Lemma 12, we will make use of the *joint regeneration levels* of X and $\tilde{X}$, which are elements of

$$\mathcal{L} := \{n \geq 0 : \langle X_i, e_1 \rangle \geq n \text{ and } \langle \tilde{X}_j, e_1 \rangle \geq n \text{ for every } i \geq H_n \text{ and } j \geq \tilde{H}_n\}.$$

This random set has been previously introduced and studied by Rassoul-Agha and Seppäläinen [14]. Note that if the starting points x and $\tilde{x}$ are both in $\mathbb{V}_d := \{z \in \mathbb{Z}^d : \langle z, e_1 \rangle = 0\}$, then

$$0 \in \mathcal{L} \iff \beta = \tilde{\beta} = \infty \iff l_1 := \inf \mathcal{L} = 0.$$

Let $\mathbb{V}'_d := \mathbb{V}_d \setminus \{0\}$. As mentioned in the opening paragraph of this section, the following lemma is central to our proof of Lemma 12.

Lemma 20 (Berger and Zeitouni [2], Proposition 3.4)

$$\inf_{z \in \mathbb{V}'_d} P_{o,z}(l_1 = 0) \geq \inf_{z \in \mathbb{V}'_d} P_{o,z}(\nu_1 = \infty, l_1 = 0) > 0.$$

The proof of Lemma 20 is based on certain Green's function estimates which fail to hold unless $d \geq 4$.

3.2 A renewal argument

For every $n \geq 1$, $\theta \in \mathcal{C}_a(\kappa_3)$, $z \in \mathbb{V}_d$ and $\omega \in \Omega$,

$$g_n(\theta, \omega) = E_o^\omega \left[\exp\left\{ \langle \theta, X_{H_n} \rangle - \Lambda_a(\theta) H_n \right\}, H_n = \tau_k \quad \text{for some } k \geq 1, \beta = \infty \right] \quad \text{and}$$

$$g_n(\theta, T_z\omega) = E_o^{T_z\omega} \left[\exp\left\{ \langle \theta, X_{H_n} \rangle - \Lambda_a(\theta) H_n \right\}, H_n = \tau_k \quad \text{for some } k \geq 1, \beta = \infty \right]$$

$$= \mathrm{e}^{-\langle \theta, z \rangle} E_z^\omega \left[\exp\left\{ \langle \theta, X_{H_n} \rangle - \Lambda_a(\theta) H_n \right\}, H_n = \tau_k \quad \text{for some } k \geq 1, \beta = \infty \right].$$

Thus,

$$G_{n,z}(\theta) := \mathbb{E}\left\{ g_n(\theta, \cdot) g_n(\theta, T_z\cdot) \right\} = \mathrm{e}^{-\langle \theta, z \rangle} E_{o,z} \left[f(\theta, n, X, \tilde{X}), n \in \mathcal{L}, l_1 = 0 \right] \quad \text{where}$$

$$f(\theta, n, X, \tilde{X}) := \exp\left\{ \langle \theta, X_{H_n} \rangle - \Lambda_a(\theta) H_n \right\} \exp\{ \langle \theta, \tilde{X}_{\tilde{H}_n} \rangle - \Lambda_a(\theta) \tilde{H}_n \}.$$

Our aim is to show that $\left(G_{n,o}(\theta) \right)_{n \geq 1}$ is bounded. We start the argument by considering a related family of functions $\left(F_{n,z}(\theta) \right)_{n \geq 1, z \in \mathbb{V}_d'}$ where

$$F_{n,z}(\theta) := \mathrm{e}^{-\langle \theta, z \rangle} E_{o,z} \left[f(\theta, n, X, \tilde{X}), n \in \mathcal{L}, X_{H_n} \neq \tilde{X}_{\tilde{H}_n} \middle| l_1 = 0 \right].$$

Recall (3.2). It follows from the definitions and the regeneration structure that

$$F_{n,z}(\theta) = \sum_{k=1}^n \sum_{z' \in \mathbb{V}_d'} \mathrm{e}^{-\langle \theta, z \rangle} E_{o,z} \left[f(\theta, n, X, \tilde{X}), k = \inf\{ l \in \mathcal{L} : l > \nu_1, X_{H_l} \neq \tilde{X}_{\tilde{H}_l} \}, \right.$$

$$\left. \tilde{X}_{\tilde{H}_k} - X_{H_n} = z', \quad n \in \mathcal{L}, X_{H_n} \neq \tilde{X}_{\tilde{H}_n} \middle| l_1 = 0 \right]$$

$$+ \mathrm{e}^{-\langle \theta, z \rangle} E_{o,z} \left[f(\theta, n, X, \tilde{X}), n \leq \nu_1, n \in \mathcal{L}, X_{H_n} \neq \tilde{X}_{\tilde{H}_n} \middle| l_1 = 0 \right]$$

$$= \sum_{k=1}^n \sum_{z' \in \mathbb{V}_d'} \mathrm{e}^{-\langle \theta, z \rangle} E_{o,z} \left[f(\theta, k, X, \tilde{X}), k = \inf\{ l \in \mathcal{L} : l > \nu_1, X_{H_l} \neq \tilde{X}_{\tilde{H}_l} \}, \right.$$

$$\left. \tilde{X}_{\tilde{H}_k} - X_{H_k} = z' \middle| l_1 = 0 \right]$$

$$\times \mathrm{e}^{-\langle \theta, z' \rangle} E_{o,z'} \left[f(\theta, n - k, X, \tilde{X}), n - k \in \mathcal{L}, X_{H_{n-k}} \neq \tilde{X}_{\tilde{H}_{n-k}} \middle| l_1 = 0 \right]$$

$$+ \mathrm{e}^{-\langle \theta, z \rangle} E_{o,z} \left[f(\theta, n, X, \tilde{X}), n \leq \nu_1, n \in \mathcal{L}, X_{H_n} \neq \tilde{X}_{\tilde{H}_n} \middle| l_1 = 0 \right].$$

Therefore,

$$F_{n,z}(\theta) \leq \sum_{k=1}^n \mathrm{e}^{-\langle \theta, z \rangle} E_{o,z} \left[f(\theta, k, X, \tilde{X}), k = \inf\{ l \in \mathcal{L} : l > \nu_1, X_{H_l} \neq \tilde{X}_{\tilde{H}_l} \} \middle| l_1 = 0 \right]$$

$$\times \sup_{z' \in \mathbb{V}_d'} F_{n-k,z'}(\theta)$$

$$+ \mathrm{e}^{-\langle \theta, z \rangle} E_{o,z} \left[f(\theta, n, X, \tilde{X}), n \leq \nu_1, n \in \mathcal{L}, X_{H_n} \neq \tilde{X}_{\tilde{H}_n} \middle| l_1 = 0 \right].$$

In other words,

$$F_{n,z}(\theta) \le \sum_{k=1}^{n} B_{k,z}(\theta) \sup_{z' \in \mathbb{V}'_d} F_{n-k,z'}(\theta) + C_{n,z}(\theta)$$

where

$$B_{k,z}(\theta) := e^{-\langle \theta, z \rangle} E_{o,z}\left[f(\theta, k, X, \tilde{X}), k = \inf\{l \in \mathcal{L} : l > v_1, X_{H_l} \ne \tilde{X}_{\tilde{H}_l}\} \,\Big|\, l_1 = 0 \right] \text{ and}$$

$$C_{n,z}(\theta) := e^{-\langle \theta, z \rangle} E_{o,z}\left[f(\theta, n, X, \tilde{X}), n \le v_1, n \in \mathcal{L}, X_{H_n} \ne \tilde{X}_{\tilde{H}_n} \,\Big|\, l_1 = 0 \right].$$

Lemma 21 *There exists a $\kappa_{eq} \in (0, \kappa_3)$ such that*

(a) $C(\theta) := \displaystyle\sup_{n \ge 1} \sup_{z \in \mathbb{V}'_d} C_{n,z}(\theta) < \infty$ *and* (b) $B(\theta) := \displaystyle\sup_{z \in \mathbb{V}'_d} \sum_{k=1}^{\infty} B_{k,z}(\theta) < 1$

for every $\theta \in \mathcal{C}_a(\kappa_{eq})$.

Remark 22 Lemma 21 is proved in Sect. 3.3.

Lemma 23 *For every $\theta \in \mathcal{C}_a(\kappa_{eq})$,*

$$\sup_{n \ge 1} \sup_{z \in \mathbb{V}'_d} F_{n,z}(\theta) < \infty.$$

Proof For every $n \ge 1$, $N \ge n$ and $z \in \mathbb{V}'_d$,

$$\begin{aligned}
F_{n,z}(\theta) &\le \sum_{k=1}^{n} B_{k,z}(\theta) \sup_{z' \in \mathbb{V}'_d} F_{n-k,z'}(\theta) + C_{n,z}(\theta) \\
&\le \left(\sum_{k=1}^{n} B_{k,z}(\theta) \right) \sup_{m \le N} \sup_{z' \in \mathbb{V}'_d} F_{m,z'}(\theta) + C_{n,z}(\theta) \\
&\le B(\theta) \sup_{m \le N} \sup_{z' \in \mathbb{V}'_d} F_{m,z'}(\theta) + C(\theta).
\end{aligned}$$

Therefore,

$$\sup_{n \le N} \sup_{z \in \mathbb{V}'_d} F_{n,z}(\theta) \le B(\theta) \sup_{n \le N} \sup_{z \in \mathbb{V}'_d} F_{n,z}(\theta) + C(\theta).$$

Finally, by Lemma 21,

$$\sup_{n \ge 1} \sup_{z \in \mathbb{V}'_d} F_{n,z}(\theta) \le C(\theta) \, (1 - B(\theta))^{-1} < \infty.$$

$\square$

Proof of Lemma 12 For every $n \geq 1$, $\theta \in \mathcal{C}_a(\kappa_{eq})$ and $z \in \mathbb{V}'_d$,

$$
\begin{aligned}
F_{n+1,z}(\theta) &= \mathrm{e}^{-\langle \theta, z \rangle} E_{o,z} \left[f(\theta, n+1, X, \tilde{X}), n+1 \in \mathcal{L}, X_{H_{n+1}} \neq \tilde{X}_{\tilde{H}_{n+1}} \,\Big|\, l_1 = 0 \right] \\
&\geq \mathrm{e}^{-\langle \theta, z \rangle} E_{o,z} \Big[f(\theta, n+1, X, \tilde{X}), n \in \mathcal{L}, \\
&\qquad\qquad X_{H_n} = \tilde{X}_{\tilde{H}_n}, n+1 \in \mathcal{L}, X_{H_{n+1}} \neq \tilde{X}_{\tilde{H}_{n+1}} \,\Big|\, l_1 = 0 \Big] \\
&= \mathrm{e}^{-\langle \theta, z \rangle} E_{o,z} \left[f(\theta, n, X, \tilde{X}), n \in \mathcal{L}, X_{H_n} = \tilde{X}_{\tilde{H}_n} \,\Big|\, l_1 = 0 \right] \\
&\quad \times E_{o,o} \left[f(\theta, 1, X, \tilde{X}), 1 \in \mathcal{L}, X_{H_1} \neq \tilde{X}_{\tilde{H}_1} \,\Big|\, l_1 = 0 \right].
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\frac{G_{n,z}(\theta)}{P_{o,z}(l_1 = 0)} &- F_{n,z}(\theta) \\
&= \mathrm{e}^{-\langle \theta, z \rangle} E_{o,z} \left[f(\theta, n, X, \tilde{X}), n \in \mathcal{L}, X_{H_n} = \tilde{X}_{\tilde{H}_n} \,\Big|\, l_1 = 0 \right] \\
&\leq E_{o,o} \left[f(\theta, 1, X, \tilde{X}), 1 \in \mathcal{L}, X_{H_1} \neq \tilde{X}_{\tilde{H}_1} \,\Big|\, l_1 = 0 \right]^{-1} F_{n+1,z}(\theta). \quad (3.3)
\end{aligned}
$$

By the uniform ellipticity assumption (1.1), Lemma 20, and part (a) of Lemma 6, the first term in (3.3) is bounded from above. This, in combination with Lemma 23, implies that

$$
\sup_{n \geq 1} \sup_{z \in \mathbb{V}'_d} G_{n,z}(\theta) < \infty.
$$

For every $\hat{z} \in U \cap \mathbb{V}_d$,

$$
\begin{aligned}
g_n(\theta, T_{\hat{z}} \omega) &= \mathrm{e}^{-\langle \theta, \hat{z} \rangle} E_{\hat{z}}^{\omega} \left[\exp\{ \langle \theta, X_{H_n} \rangle - \Lambda_a(\theta) H_n \}, H_n = \tau_k \text{ for some } k \geq 1, \beta = \infty \right] \\
&\geq \mathrm{e}^{-\langle \theta, \hat{z} \rangle} E_{\hat{z}}^{\omega} \left[\exp\{ \langle \theta, X_{H_n} \rangle - \Lambda_a(\theta) H_n \}, H_n = \tau_k \text{ for some } k \geq 1, X_1 = 0, \beta = \infty \right] \\
&\geq \delta \mathrm{e}^{-\langle \theta, \hat{z} \rangle - \Lambda_a(\theta)} E_o^{\omega} \left[\exp\{ \langle \theta, X_{H_n} \rangle - \Lambda_a(\theta) H_n \}, H_n = \tau_k \text{ for some } k \geq 1, \beta = \infty \right] \\
&= \delta \mathrm{e}^{-\langle \theta, \hat{z} \rangle - \Lambda_a(\theta)} g_n(\theta, \omega) \geq \delta \mathrm{e}^{-2\kappa_{eq}} g_n(\theta, \omega).
\end{aligned}
$$

Hence,

$$
\begin{aligned}
\sup_{n \geq 1} \mathbb{E}\left\{ g_n(\theta, \cdot)^2 \right\} &\leq \delta^{-1} \mathrm{e}^{2\kappa_{eq}} \sup_{n \geq 1} \sup_{z \in \mathbb{V}'_d} \mathbb{E}\{ g_n(\theta, \cdot) g_n(\theta, T_z \cdot) \} \\
&= \delta^{-1} \mathrm{e}^{2\kappa_{eq}} \sup_{n \geq 1} \sup_{z \in \mathbb{V}'_d} G_{n,z}(\theta) < \infty.
\end{aligned}
$$

$\square$

3.3 Proof of Lemma 21

Let us start by proving the easy part.

Proof of part (a) of Lemma 21 For every $n \geq 1$, $\theta \in C_a(\kappa_3)$ and $z \in \mathbb{V}'_d$,

$$
\begin{aligned}
C_{n,z}(\theta) &\leq \frac{e^{-\langle \theta, z \rangle}}{P_{o,z}(l_1 = 0)} E_{o,z} \left[f(\theta, n, X, \tilde{X}), n \leq v_1, n \in \mathcal{L}, \beta = \infty, \tilde{\beta} = \infty \right] \\
&\leq \frac{e^{-\langle \theta, z \rangle}}{P_{o,z}(l_1 = 0)} E_{o,z} \left[f(\theta, n, X, \tilde{X}), \{X_i : 0 \leq i < H_n\} \cap \{\tilde{X}_j : 0 \leq j < \tilde{H}_n\} = \emptyset, \right. \\
&\qquad\qquad \left. \beta \geq H_n, \tilde{\beta} \geq \tilde{H}_n \right] \\
&= \frac{e^{-\langle \theta, z \rangle}}{P_{o,z}(l_1 = 0)} E_o \times E_z \left[f(\theta, n, X, \tilde{X}), \{X_i : 0 \leq i < H_n\} \cap \{\tilde{X}_j : 0 \leq j < \tilde{H}_n\} = \emptyset, \right. \\
&\qquad\qquad \left. \beta \geq H_n, \tilde{\beta} \geq \tilde{H}_n \right] \qquad\qquad\qquad\qquad (3.4) \\
&\leq \frac{e^{-\langle \theta, z \rangle}}{P_{o,z}(l_1 = 0)} E_o \times E_z \left[f(\theta, n, X, \tilde{X}), \beta \geq H_n, \tilde{\beta} \geq \tilde{H}_n \right] \\
&= \frac{1}{P_{o,z}(l_1 = 0)} E_o \left[\exp\{\langle \theta, X_{H_n} \rangle - \Lambda_a(\theta) H_n\}, \beta \geq H_n \right]^2. \qquad (3.5)
\end{aligned}
$$

Here, (3.4) is similar to (3.1). Both facts follow from a standard coupling argument (cf. [2], Proposition 3.7.) Note that

$$
\begin{aligned}
&E_o \left[\exp\{\langle \theta, X_{H_n} \rangle - \Lambda_a(\theta) H_n\}, \beta \geq H_n \right] \\
&= \frac{P_o(\beta = \infty)}{P_o(\beta = \infty)} E_o \left[\exp\{\langle \theta, X_{H_n} \rangle - \Lambda_a(\theta) H_n\}, \beta \geq H_n \right] \\
&= \frac{1}{P_o(\beta = \infty)} E_o \left[\exp\{\langle \theta, X_{H_n} \rangle - \Lambda_a(\theta) H_n\}, \beta \geq H_n \right] \\
&\quad \times P_o(\langle X_i, e_1 \rangle \geq n \text{ for every } i \geq H_n) \\
&= \frac{1}{P_o(\beta = \infty)} E_o \left[\exp\{\langle \theta, X_{H_n} \rangle - \Lambda_a(\theta) H_n\}, H_n = \tau_k \text{ for some } k \geq 1, \beta = \infty \right] \\
&= \mathbb{E}\{g_n(\theta, \cdot)\} / P_o(\beta = \infty).
\end{aligned}
$$

Therefore, (3.5), Lemma 11 and Lemma 20 imply that

$$
\sup_{n \geq 1} \sup_{z \in \mathbb{V}'_d} C_{n,z}(\theta) \leq P_o(\beta = \infty)^{-2} \sup_{z \in \mathbb{V}'_d} P_{o,z}(l_1 = 0)^{-1} \left(\sup_{n \geq 1} \mathbb{E}\{g_n(\theta, \cdot)\} \right)^2 < \infty.
$$

$\square$

The proof of part (b) of Lemma 21 is more technical. At $\theta = 0$,

$$
\begin{aligned}
B(0) &= \sup_{z \in \mathbb{V}'_d} \sum_{k=1}^{\infty} B_{k,z}(0) \\
&= \sup_{z \in \mathbb{V}'_d} \sum_{k=1}^{\infty} P_{o,z} \left(k = \inf\{l \in \mathcal{L} : l > \nu_1, X_{H_l} \neq \tilde{X}_{\tilde{H}_l}\} \Big| l_1 = 0 \right) \\
&= \sup_{z \in \mathbb{V}'_d} P_{o,z} (\nu_1 < \infty | l_1 = 0) = 1 - \inf_{z \in \mathbb{V}'_d} P_{o,z} (\nu_1 = \infty | l_1 = 0) < 1 \quad (3.6)
\end{aligned}
$$

by Lemma 20. For every $\theta \in \mathcal{C}_a(\kappa_3)$ and $z \in \mathbb{V}'_d$,

$$
\sum_{k=1}^{\infty} B_{k,z}(\theta) = \sum_{k=1}^{\infty} B_{k,z}(0) + \sum_{k=1}^{\infty} \left(B_{k,z}(\theta) - B_{k,z}(0) \right) \quad \text{and}
$$

$$
\begin{aligned}
B(\theta) &= \sup_{z \in \mathbb{V}'_d} \sum_{k=1}^{\infty} B_{k,z}(\theta) \leq \sup_{z \in \mathbb{V}'_d} \sum_{k=1}^{\infty} B_{k,z}(0) + \sum_{k=1}^{\infty} \sup_{z \in \mathbb{V}'_d} \left(B_{k,z}(\theta) - B_{k,z}(0) \right) \\
&= B(0) + \sum_{k=1}^{\infty} \sup_{z \in \mathbb{V}'_d} \left(B_{k,z}(\theta) - B_{k,z}(0) \right). \quad (3.7)
\end{aligned}
$$

The next three lemmas control the sum in (3.7).

Lemma 24 *For every $k \geq 1$ and $\epsilon > 0$, there exists a $\kappa_4 = \kappa_4(k, \epsilon) \in (0, \kappa_3)$ such that*

$$
\sup_{z \in \mathbb{V}'_d} \sup_{\theta \in \mathcal{C}_a(\kappa_4)} \left(B_{k,z}(\theta) - B_{k,z}(0) \right) < \epsilon. \quad (3.8)
$$

Proof For every $k \geq 1$, $\theta \in \mathcal{C}_a(\kappa_3)$ and $z \in \mathbb{V}'_d$,

$$
\begin{aligned}
B_{k,z}(\theta) - B_{k,z}(0) &= E_{o,z} \left[e^{-\langle \theta, z \rangle} f(\theta, k, X, \tilde{X}) - 1, \right. \\
&\qquad\qquad \left. k = \inf\{l \in \mathcal{L} : l > \nu_1, X_{H_l} \neq \tilde{X}_{\tilde{H}_l}\} \Big| l_1 = 0 \right] \\
&\leq P_{o,z}(l_1 = 0)^{-1} E_{o,z} \left[\left(e^{-\langle \theta, z \rangle} f(\theta, k, X, \tilde{X}) - 1 \right)^2 \right]^{1/2} \\
&\quad \times P_{o,z} \left(k = \inf\{l \in \mathcal{L} : l > \nu_1, X_{H_l} \neq \tilde{X}_{\tilde{H}_l}\} \right)^{1/2} \\
&\leq P_{o,z}(l_1 = 0)^{-1} \mathbb{E} \left\{ E_o^{\omega} \times E_o^{T_z \omega} \left[\left(f(\theta, k, X, \tilde{X}) - 1 \right)^2 \right] \right\}^{1/2} \\
&\quad \times \left(P_{o,z} \left(\sup_{1 \leq i \leq H_k} |X_i| \geq \frac{|z|}{2} \right) + P_{o,z} \left(\sup_{1 \leq j \leq \tilde{H}_k} \left| \tilde{X}_j - z \right| \geq \frac{|z|}{2} \right) \right)^{1/2}
\end{aligned}
$$

$$\leq P_{o,z}(l_1 = 0)^{-1} \left(\mathbb{E} \left\{ E_o^\omega \times E_o^{T_z \omega} \left[f(\theta, k, X, \tilde{X})^2 \right] \right\} + 1 \right)^{1/2}$$

$$\times \sqrt{2} P_o \left(\sup_{1 \leq i \leq H_k} |X_i| \geq \frac{|z|}{2} \right)^{1/2}$$

$$\leq P_{o,z}(l_1 = 0)^{-1} \left(E_o \left[\exp \left\{ 4 \langle \theta, X_{H_k} \rangle - 4 \Lambda_a(\theta) H_k \right\} \right] + 1 \right)^{1/2}$$

$$\times \sqrt{2} P_o \left(\sup_{1 \leq i \leq \tau_k} |X_i| \geq \frac{|z|}{2} \right)^{1/2}. \tag{3.9}$$

For every $\epsilon > 0$, it follows from (3.9), Corollary 9 and Lemma 20 that there exists an $N \geq 1$ such that

$$\sup_{\substack{z \in \mathbb{V}'_d \\ |z| > N}} \sup_{\theta \in \mathcal{C}_a(\kappa_3/4)} \left(B_{k,z}(\theta) - B_{k,z}(0) \right) < \epsilon. \tag{3.10}$$

Note that $\theta \mapsto f(\theta, k, X, \tilde{X})$ is continuous. Hence, for every $k \geq 1$ and $z \in \mathbb{V}'_d$, the map $\theta \mapsto B_{k,z}(\theta)$ is continuous at 0 by Schwarz's inequality, Corollary 9 and the dominated convergence theorem. Consequently, there exists a $\kappa_4 \in (0, \kappa_3/4)$ such that

$$\sup_{\substack{z \in \mathbb{V}'_d \\ |z| \leq N}} \sup_{\theta \in \mathcal{C}_a(\kappa_4)} \left| B_{k,z}(\theta) - B_{k,z}(0) \right| < \epsilon. \tag{3.11}$$

Clearly, (3.10) and (3.11) imply (3.8). □

Lemma 25 *There exists a $\kappa_5 \in (0, \kappa_3)$ such that*

$$\sum_{k=1}^\infty \sup_{z \in \mathbb{V}'_d} \sup_{\theta \in \mathcal{C}_a(\kappa_5)} B_{k,z}(\theta) < \infty.$$

Proof For every $k \geq 1$, $\kappa \in (0, \kappa_3)$, $\theta \in \mathcal{C}_a(\kappa)$ and $z \in \mathbb{V}'_d$,

$$B_{k,z}(\theta) = e^{-\langle \theta, z \rangle} E_{o,z} \left[f(\theta, k, X, \tilde{X}), k = \inf \{ l \in \mathcal{L} : l > v_1, X_{H_l} \neq \tilde{X}_{\tilde{H}_l} \} \Big| l_1 = 0 \right]$$

$$= \sum_{j=0}^{k-1} \sum_{z' \in \mathbb{V}_d} e^{-\langle \theta, z \rangle} E_{o,z} \left[f(\theta, k, X, \tilde{X}), j = \sup \{ l \in \mathcal{L} : l \leq v_1 \}, \tilde{X}_{\tilde{H}_j} - X_{H_j} = z', \right.$$

$$\left. k = \inf \{ l \in \mathcal{L} : l > v_1, X_{H_l} \neq \tilde{X}_{\tilde{H}_l} \} \Big| l_1 = 0 \right]$$

$$\leq \sum_{j=0}^{k-1} \sum_{z' \in \mathbb{V}_d} e^{-\langle \theta, z \rangle} E_{o,z} \left[f(\theta, j, X, \tilde{X}), j \in \mathcal{L}, j \leq v_1, \tilde{X}_{\tilde{H}_j} - X_{H_j} = z' \Big| l_1 = 0 \right]$$

$$\times e^{-\langle \theta, z' \rangle} E_{o,z'} \Big[f(\theta, k - j, X, \tilde{X}),$$

$$k - j = \inf\{l \in \mathcal{L} : l > 0, X_{H_l} \neq \tilde{X}_{\tilde{H}_l}\}, k - j > \nu_1 \Big| l_1 = 0 \Big]$$

$$= \sum_{j=0}^{k-1} \sum_{z' \in \mathbb{V}_d} e^{-\langle \theta, z \rangle} E_{o,z} \Big[f(\theta, j, X, \tilde{X}), j \in \mathcal{L}, j \leq \nu_1, \tilde{X}_{\tilde{H}_j} - X_{H_j} = z' \Big| l_1 = 0 \Big] h_{k-j,z'}(\theta)$$

$$\leq \sum_{j=0}^{k-1} \sup_{z' \in \mathbb{V}_d} e^{-\langle \theta, z \rangle} E_{o,z} \Big[f(\theta, j, X, \tilde{X}), j \in \mathcal{L}, j \leq \nu_1, \tilde{X}_{\tilde{H}_j} - X_{H_j} = z' \Big| l_1 = 0 \Big]$$

$$\times \sum_{z' \in \mathbb{V}_d} h_{k-j,z'}(\theta) \tag{3.12}$$

where, for every $i \geq 1$,

$$h_{i,z'}(\theta) := e^{-\langle \theta, z' \rangle} E_{o,z'} \Big[f(\theta, i, X, \tilde{X}), i = \inf\{l \in \mathcal{L} : l > 0, X_{H_l} \neq \tilde{X}_{\tilde{H}_l}\},$$

$$i > \nu_1 \Big| l_1 = 0 \Big]. \tag{3.13}$$

For every $z' \in \mathbb{V}_d$,

$$e^{-\langle \theta, z \rangle} E_{o,z} \Big[f(\theta, j, X, \tilde{X}), j \in \mathcal{L}, j \leq \nu_1, \tilde{X}_{\tilde{H}_j} - X_{H_j} = z', l_1 = 0 \Big]$$

$$\leq e^{-\langle \theta, z \rangle} E_{o,z} \Big[f(\theta, j, X, \tilde{X}), \{X_n : 0 \leq n < H_j\} \cap \{\tilde{X}_m : 0 \leq m < \tilde{H}_j\} = \emptyset,$$

$$\tilde{X}_{\tilde{H}_j} - X_{H_j} = z', \beta \geq H_j, \tilde{\beta} \geq \tilde{H}_j \Big]$$

$$= e^{-\langle \theta, z \rangle} E_o \times E_z \Big[f(\theta, j, X, \tilde{X}), \{X_n : 0 \leq n < H_j\} \cap \{\tilde{X}_m : 0 \leq m < \tilde{H}_j\} = \emptyset,$$

$$\tilde{X}_{\tilde{H}_j} - X_{H_j} = z', \beta \geq H_j, \tilde{\beta} \geq \tilde{H}_j \Big]$$

$$\leq e^{-\langle \theta, z \rangle} E_o \times E_z \Big[f(\theta, j, X, \tilde{X}), \tilde{X}_{\tilde{H}_j} - X_{H_j} = z', \beta \geq H_j, \tilde{\beta} \geq \tilde{H}_j \Big]$$

$$= E_o \times E_o \Big[f(\theta, j, X, \tilde{X}), \tilde{X}_{\tilde{H}_j} - X_{H_j} = z' - z, \beta \geq H_j, \tilde{\beta} \geq \tilde{H}_j \Big]$$

$$= \frac{P_o \times P_o(j \in \mathcal{L})}{P_o \times P_o(l_1 = 0)} E_o \times E_o \Big[f(\theta, j, X, \tilde{X}), \tilde{X}_{\tilde{H}_j} - X_{H_j} = z' - z, \beta \geq H_j, \tilde{\beta} \geq \tilde{H}_j \Big]$$

$$= E_o \times E_o \Big[f(\theta, j, X, \tilde{X}), j \in \mathcal{L}, \tilde{X}_{\tilde{H}_j} - X_{H_j} = z' - z \Big| l_1 = 0 \Big]$$

$$= \hat{P}_o^\theta \times \hat{P}_o^\theta \Big(\exists n, m \text{ such that } \langle Y_n, e_1 \rangle = j \text{ and } \tilde{Y}_m - Y_n = z' - z \Big) \tag{3.14}$$

where $(Y_n)_{n \geq 0}$ and $(\tilde{Y}_m)_{m \geq 0}$ denote two independent random walks on $\mathbb{Z}^d$, both with transition kernel $q^\theta(y)_{y \in \mathbb{Z}^d}$ given in (2.5).

Let $\mu = \mu(\theta) := \hat{E}_o^\theta[\langle Y_1, e_1 \rangle]$. For every $j \geq 1$, (3.14) is equal to

$$\sum_{\langle x, e_1 \rangle = j} \hat{P}_o^\theta \left(\exists n \text{ such that } Y_n = x \right) \hat{P}_o^\theta \left(\exists m \text{ such that } \tilde{Y}_m = x + z' - z \right)$$

$$\leq \sup_{\langle x, e_1 \rangle = j} \hat{P}_o^\theta \left(\exists n \text{ such that } Y_n = x \right) \sum_{\langle x, e_1 \rangle = j} \hat{P}_o^\theta \left(\exists m \text{ such that } \tilde{Y}_m = x + z' - z \right)$$

$$= \sup_{\langle x, e_1 \rangle = j} \sum_{n \geq 1} \hat{P}_o^\theta \left(Y_n = x \right) \hat{P}_o^\theta \left(\exists m \text{ such that } \langle \tilde{Y}_m, e_1 \rangle = j \right)$$

$$\leq \sup_{\langle x, e_1 \rangle = j} \sum_{n \geq 1} \hat{P}_o^\theta \left(Y_n = x \right)$$

$$= \sup_{\langle x, e_1 \rangle = j} \left(\sum_{|n - j/\mu| \leq \sqrt{j/\mu}} \hat{P}_o^\theta \left(Y_n = x \right) + \sum_{|n - j/\mu| > \sqrt{j/\mu}} \hat{P}_o^\theta \left(Y_n = x \right) \right)$$

$$\leq S(\theta) j^{-(d-1)/2}. \tag{3.15}$$

Here, (3.15) follows from (2.6) and the local CLT. $S(\theta)$ depends on the mean and covariance of $\left(q^\theta(y) \right)_{y \in \mathbb{Z}^d}$. In particular, $\sup_{\theta \in \mathcal{C}_a(\kappa_3)} S(\theta) < \infty$.

Putting (3.12), (3.14) and (3.15) together, we see that

$$\sup_{z \in \mathbb{V}_d'} \sup_{\theta \in \mathcal{C}_a(\kappa)} B_{k,z}(\theta) \leq \frac{\sup_{\theta \in \mathcal{C}_a(\kappa)} S(\theta)}{\inf_{z \in \mathbb{V}_d'} P_{o,z}(l_1 = 0)} \sum_{j=0}^{k-1} \max(1, j)^{-(d-1)/2}$$

$$\times \sum_{z' \in \mathbb{V}_d} \sup_{\theta \in \mathcal{C}_a(\kappa)} h_{k-j,z'}(\theta) \quad \text{and}$$

$$\sum_{k=1}^{\infty} \sup_{z \in \mathbb{V}_d'} \sup_{\theta \in \mathcal{C}_a(\kappa)} B_{k,z}(\theta) \leq \frac{\sup_{\theta \in \mathcal{C}_a(\kappa)} S(\theta)}{\inf_{z \in \mathbb{V}_d'} P_{o,z}(l_1 = 0)} \left(1 + \sum_{j=1}^{\infty} j^{-(d-1)/2} \right)$$

$$\times \sum_{i=1}^{\infty} \sum_{z' \in \mathbb{V}_d} \sup_{\theta \in \mathcal{C}_a(\kappa)} h_{i,z'}(\theta).$$

The desired result follows from Lemma 20 and Lemma 26 (stated below.) □

Lemma 26 *Recall* (3.13). *There exists a* $\kappa_5 \in (0, \kappa_3)$ *such that*

$$\sum_{i=1}^{\infty} \sum_{z \in \mathbb{V}_d} \sup_{\theta \in \mathcal{C}_a(\kappa_5)} h_{i,z}(\theta) < \infty.$$

Remark 27 Lemma 26 is proved in Sect. 3.4.

Finally, we are ready to give the

Proof of part (b) of Lemma 21 Let $\epsilon := 1 - B(0)$. Note that $\epsilon > 0$ by (3.6). Lemma 25 implies that

$$\sum_{k=N+1}^{\infty} \sup_{z \in \mathbb{V}'_d} \sup_{\theta \in \mathcal{C}_a(\kappa_5)} \left(B_{k,z}(\theta) - B_{k,z}(0) \right) \leq \sum_{k=N+1}^{\infty} \sup_{z \in \mathbb{V}'_d} \sup_{\theta \in \mathcal{C}_a(\kappa_5)} B_{k,z}(\theta) < \epsilon/2$$

for some $\kappa_5 \in (0, \kappa_3)$ and $N \geq 1$. Also, for every $k \geq 1$, there exists a $\kappa_4 = \kappa_4(k, \epsilon/2N) \in (0, \kappa_3)$ such that

$$\sup_{z \in \mathbb{V}'_d} \sup_{\theta \in \mathcal{C}_a(\kappa_4)} \left(B_{k,z}(\theta) - B_{k,z}(0) \right) < \epsilon/2N$$

by Lemma 24.

Let $\kappa_{eq} := \min \left(\kappa_5, \kappa_4(1, \epsilon/2N), \kappa_4(2, \epsilon/2N), \ldots, \kappa_4(N, \epsilon/2N) \right)$. Recall (3.7). For every $\theta \in \mathcal{C}_a(\kappa_{eq})$,

$$B(\theta) \leq B(0) + \sum_{k=1}^{\infty} \sup_{z \in \mathbb{V}'_d} \left(B_{k,z}(\theta) - B_{k,z}(0) \right)$$

$$= 1 - \epsilon + \sum_{k=1}^{N} \sup_{z \in \mathbb{V}'_d} \left(B_{k,z}(\theta) - B_{k,z}(0) \right) + \sum_{k=N+1}^{\infty} \sup_{z \in \mathbb{V}'_d} \left(B_{k,z}(\theta) - B_{k,z}(0) \right)$$

$$< 1 - \epsilon + N(\epsilon/2N) + \epsilon/2 = 1.$$

$\square$

3.4 Tail estimates for joint regenerations

Recall that our proof of Theorem 5 given in Sect. 2 relies on Lemma 12 which, in turn, is proved in Sect. 3.2 assuming Lemma 21. In Sect. 3.3, the latter assumption is verified using yet another result, namely Lemma 26. Therefore, in order to complete the proof of Theorem 5, we need to prove Lemma 26.

For every $i \geq 1$, $\theta \in \mathcal{C}_a(\kappa_3)$ and $z \in \mathbb{V}_d$, it follows from Hölder's inequality that

$$h_{i,z}(\theta) = \frac{e^{-\langle \theta, z \rangle}}{P_{o,z}(l_1 = 0)} E_{o,z} \left[f(\theta, i, X, \tilde{X}), i = \inf\{l \in \mathcal{L} : l > 0, X_{H_l} \neq \tilde{X}_{\tilde{H}_l}\}, \right.$$

$$\left. i > v_1, \beta = \infty, \tilde{\beta} = \infty \right]$$

$$\leq E_{o,z} \left[\exp\left\{ 4\langle \theta, X_{H_i} \rangle - 4\Lambda_a(\theta) H_i \right\}, \beta = \infty \right]^{1/4}$$

$$\times e^{-\langle \theta, z \rangle} E_{o,z} \left[\exp\{ 4\langle \theta, \tilde{X}_{\tilde{H}_i} \rangle - 4\Lambda_a(\theta) \tilde{H}_i \}, \tilde{\beta} = \infty \right]^{1/4}$$

$$\times P_{o,z}(l_1 = 0)^{-1}$$

$$\times P_{o,z} \left(i = \inf\{l \in \mathcal{L} : l > 0, X_{H_l} \neq \tilde{X}_{\tilde{H}_l}\}, l_1 = 0 \right)^{1/4} P_{o,z}(i > v_1)^{1/4}$$

$$= E_o \left[\exp \left\{ 4 \langle \theta, X_{H_i} \rangle - 4 \Lambda_a(\theta) H_i \right\}, \beta = \infty \right]^{1/2}$$

$$\times P_{o,z}(l_1 = 0)^{-1} P_{o,z} \left(i = \inf\{l \in \mathcal{L} : l > 0, X_{H_l} \neq \tilde{X}_{\tilde{H}_l}\}, l_1 = 0 \right)^{1/4}$$

$$\times P_{o,z} \left(i > \nu_1 \right)^{1/4}. \tag{3.16}$$

The next four lemmas control the terms in (3.16).

Lemma 28 *There exists an $a_1 < \infty$ such that*

$$E_o \left[\exp \left\{ 4 \langle \theta, X_{H_i} \rangle - 4 \Lambda_a(\theta) H_i \right\} \big| \beta = \infty \right] \leq i e^{a_1 |\theta| i}$$

for every $i \geq 1$ and $\theta \in \mathcal{C}_a(\kappa_3/4)$.

Proof Recall κ_3 from Corollary 9.

(a) *The non-nestling case:* For every $i \geq 1$ and $\theta \in \mathcal{C}_a(\kappa_3/4)$,

$$E_o \left[\exp \left\{ 4 \langle \theta, X_{H_i} \rangle - 4 \Lambda_a(\theta) H_i \right\} \big| \beta = \infty \right]$$
$$\leq E_o \left[\exp \left\{ (4|\theta| + 4|\Lambda_a(\theta)|) H_i \right\} \big| \beta = \infty \right]$$
$$\leq E_o \left[\exp \left\{ 8|\theta|\tau_i \right\} \big| \beta = \infty \right] = E_o \left[\exp \left\{ 8|\theta|\tau_1 \right\} \big| \beta = \infty \right]^i$$
$$\leq E_o \left[\exp \left\{ 2\kappa_3 \tau_1 \right\} \big| \beta = \infty \right]^{4|\theta| i / \kappa_3}$$

by Jensen's inequality. Since $a_1 := \log E_o \left[\exp \left\{ 2\kappa_3 \tau_1 \right\} \big| \beta = \infty \right]^{4/\kappa_3} < \infty$ by Corollary 9, we are done.

(b) *The nestling case:* For every $i \geq 1$ and $\theta \in \mathcal{C}_a(\kappa_3/4)$,

$$E_o \left[\exp \left\{ 4 \langle \theta, X_{H_i} \rangle - 4 \Lambda_a(\theta) H_i \right\} \big| \beta = \infty \right] \leq E_o \left[\exp \left\{ 4|\theta| \left| X_{H_i} \right| \right\} \big| \beta = \infty \right]$$

$$= \sum_{j=0}^{i-1} \sum_{k=j}^{i-1} E_o \left[\exp \left\{ 4|\theta| \left| X_{H_i} \right| \right\}, \tau_j < H_i \leq \tau_{j+1}, \langle X_{\tau_j}, e_1 \rangle = k \big| \beta = \infty \right]$$

$$\leq \sum_{j=0}^{i-1} \sum_{k=j}^{i-1} E_o \left[\exp \left\{ 4|\theta| \left| X_{\tau_j} \right| \right\}, \tau_j = H_k \big| \beta = \infty \right]$$

$$\times E_o \left[\exp \left\{ 4|\theta| \left| X_{H_{i-k}} \right| \right\}, H_{i-k} \leq \tau_1 \big| \beta = \infty \right]$$

$$\leq \sum_{j=0}^{i-1} E_o \left[\exp \left\{ 4|\theta| \left| X_{\tau_j} \right| \right\} \big| \beta = \infty \right] E_o \left[\sup_{1 \leq n \leq \tau_1} \exp \left\{ 4|\theta| |X_n| \right\} \bigg| \beta = \infty \right]$$

$$\leq \sum_{j=0}^{i-1} E_o \left[\sup_{1 \leq n \leq \tau_1} \exp \left\{ 4|\theta| |X_n| \right\} \bigg| \beta = \infty \right]^{j+1}$$

$$\leq i E_o \left[\sup_{1 \leq n \leq \tau_1} \exp \{4|\theta| \, |X_n|\} \,\Big|\, \beta = \infty \right]^i$$

$$\leq i E_o \left[\sup_{1 \leq n \leq \tau_1} \exp \{\kappa_3 \, |X_n|\} \,\Big|\, \beta = \infty \right]^{4|\theta| i / \kappa_3}.$$

Since $a_1 := \log E_o \left[\sup_{1 \leq n \leq \tau_1} \exp \{\kappa_3 \, |X_n|\} \,\big|\, \beta = \infty \right]^{4/\kappa_3} < \infty$ by Corollary 9, we are done.   □

Lemma 29 *For every $p \geq 1$, there exists an $A_1 = A_1(p) < \infty$ such that*

$$P_{o,z} \, (i > \nu_1) \leq A_1 i^p \max(1, |z|)^{-p}$$

for every $i \geq 1$ and $z \in \mathbb{V}_d$.

Proof For every $i \geq 1$, $z \in \mathbb{V}'_d$ and $p \geq 1$,

$$
\begin{aligned}
P_{o,z} \, (i > \nu_1) &\leq P_{o,z} \left(\{X_n : 0 \leq n \leq \tau_i\} \cap \left\{ \tilde{X}_m : 0 \leq m \leq \tilde{\tau}_i \right\} \neq \emptyset \right) \\
&\leq P_{o,z} \left(\tau_i \geq \frac{|z|}{2} \right) + P_{o,z} \left(\tilde{\tau}_i \geq \frac{|z|}{2} \right) = 2 P_o \left(\tau_i \geq \frac{|z|}{2} \right) \\
&\leq 2 \left(\frac{|z|}{2} \right)^{-p} E_o \left[\tau_i^p \right] = 2^{p+1} |z|^{-p} E_o \left[\{\tau_1 + \cdots + (\tau_i - \tau_{i-1})\}^p \right] \\
&\leq 2^{p+1} |z|^{-p} i^{p-1} E_o \left[\tau_1^p + \cdots + (\tau_i - \tau_{i-1})^p \right] \\
&= 2^{p+1} |z|^{-p} i^{p-1} \left(E_o \left[\tau_1^p \right] + (i-1) E_o \left[\tau_1^p \,|\, \beta = \infty \right] \right) \\
&\leq 2^{p+1} P_o(\beta = \infty)^{-1} E_o \left[\tau_1^p \right] i^p |z|^{-p}
\end{aligned}
$$

by Hölder's inequality. Since $A_1(p) := 2^{p+1} P_o(\beta = \infty)^{-1} E_o \left[\tau_1^p \right] < \infty$ by Corollary 9, we are done.   □

Lemma 30 $\sup_{z \in \mathbb{V}_d} E_{o,z} \left[e^{a_3 l^+} \,\Big|\, l_1 = 0 \right] < \infty$ *for some $a_3 > 0$, where $l^+ := \inf\{l \in \mathcal{L} : l > 0\}$.*

Proof For every nearest-neighbor path $(x_i)_{i \geq 0}$ on $\mathbb{Z}^d$, define

$$
\begin{aligned}
\beta'((x_i)_{i \geq 0}) &:= \inf\{i \geq 1 : \langle x_i, e_1 \rangle < \langle x_o, e_1 \rangle\} \quad \text{and} \\
M((x_i)_{i \geq 0}) &:= \sup\{\langle x_i, e_1 \rangle : 0 \leq i < \beta'((x_i)_{i \geq 0})\}.
\end{aligned}
$$

In particular, $\beta'(X) = \beta$ and $\beta'(\tilde{X}) = \tilde{\beta}$ for $X = (X_n)_{n \geq 0}$ and $\tilde{X} = (\tilde{X}_m)_{m \geq 0}$. Define

$$\lambda = \lambda(X, \tilde{X}) := \left(M(X) \wedge M(\tilde{X}) \right) + 1, \quad \lambda_1 := 1 \quad \text{and}$$

$$\lambda_{j+1} := \lambda \left((X_n)_{n \geq H_{\lambda_j}}, (\tilde{X}_m)_{m \geq \tilde{H}_{\lambda_j}} \right)$$

for every $j \geq 1$. It is easy to see that $l^+ := \sup\{\lambda_j : \lambda_j < \infty\}$ when $X_o \in \mathbb{V}_d$ and $\tilde{X}_o \in \mathbb{V}_d$.

For every $z \in \mathbb{V}_d$,

$$E_{o,z}\left[e^{\kappa_3 \lambda}, \lambda < \infty\right] \leq E_{o,z}\left[e^{\kappa_3(M(X)+1)}, \beta < \infty\right] + E_{o,z}\left[e^{\kappa_3(M(\tilde{X})+1)}, \tilde{\beta} < \infty\right]$$

$$= 2E_o\left[e^{\kappa_3(M(X)+1)}, \beta < \infty\right] \leq 2E_o\left[\sup_{1 \leq n \leq \tau_1} e^{\kappa_3|X_n|}, \beta < \infty\right].$$

By Hölder's inequality,

$$E_{o,z}\left[e^{a\lambda}, \lambda < \infty\right] \leq E_{o,z}\left[e^{\kappa_3 \lambda}, \lambda < \infty\right]^{a/\kappa_3} P_{o,z}(\lambda < \infty)^{1-a/\kappa_3}$$

$$\leq \left(2E_o\left[\sup_{1 \leq n \leq \tau_1} e^{\kappa_3|X_n|}\right]\right)^{a/\kappa_3} \left(1 - P_{o,z}(l_1 = 0)\right)^{1-a/\kappa_3}$$

for every $a \in (0, \kappa_3)$. Therefore, it follows from Corollary 9 and Lemma 20 that

$$\sup_{z \in \mathbb{V}_d} E_{o,z}\left[e^{a_3 \lambda}, \lambda < \infty\right] \leq \left(2E_o\left[\sup_{1 \leq n \leq \tau_1} e^{\kappa_3|X_n|}\right]\right)^{a_3/\kappa_3}$$

$$\times \left(1 - \inf_{z \in \mathbb{V}_d} P_{o,z}(l_1 = 0)\right)^{1-a_3/\kappa_3} < 1 \quad (3.17)$$

for some $a_3 \in (0, \kappa_3)$.

For every $j \geq 2$,

$$E_{o,z}\left[e^{a_3 \lambda_j}, \lambda_j < \infty\right] = \sum_{z' \in \mathbb{V}_d} E_{o,z}\left[e^{a_3 \lambda_j}, \tilde{X}_{\tilde{H}_{\lambda_{j-1}}} - X_{H_{\lambda_{j-1}}} = z', \lambda_j < \infty\right]$$

$$= \sum_{z' \in \mathbb{V}_d} E_{o,z}\left[e^{a_3 \lambda_{j-1}}, \lambda_{j-1} < \infty, \tilde{X}_{\tilde{H}_{\lambda_{j-1}}} - X_{H_{\lambda_{j-1}}} = z'\right]$$

$$\times E_{o,z'}\left[e^{a_3 \lambda}, \lambda < \infty\right]$$

$$\leq E_{o,z}\left[e^{a_3 \lambda_{j-1}}, \lambda_{j-1} < \infty\right] \sup_{z' \in \mathbb{V}_d} E_{o,z'}\left[e^{a_3 \lambda}, \lambda < \infty\right].$$

By induction,

$$\sup_{z \in \mathbb{V}_d} E_{o,z}\left[e^{a_3 \lambda_j}, \lambda_j < \infty\right] \leq e^{a_3}\left(\sup_{z \in \mathbb{V}_d} E_{o,z}\left[e^{a_3 \lambda}, \lambda < \infty\right]\right)^{j-1}.$$

Therefore,

$$
\sup_{z \in \mathbb{V}_d} E_{o,z}\left[e^{a_3 l^+}\right] \le \sum_{j=1}^{\infty} \sup_{z \in \mathbb{V}_d} E_{o,z}\left[e^{a_3 l^+}, l^+ = \lambda_j\right] \le \sum_{j=1}^{\infty} \sup_{z \in \mathbb{V}_d} E_{o,z}\left[e^{a_3 \lambda_j}, \lambda_j < \infty\right]
$$

$$
\le \sum_{j=1}^{\infty} e^{a_3}\left(\sup_{z \in \mathbb{V}_d} E_{o,z}\left[e^{a_3 \lambda}, \lambda < \infty\right]\right)^{j-1}
$$

$$
= e^{a_3}\left(1 - \sup_{z \in \mathbb{V}_d} E_{o,z}\left[e^{a_3 \lambda}, \lambda < \infty\right]\right)^{-1} < \infty
$$

by (3.17). This implies the desired result since $\inf_{z \in \mathbb{V}_d} P_{o,z}(l_1 = 0) > 0$ by Lemma 20. $\square$

Lemma 31 *There exist $a_2 > 0$ and $A_2 < \infty$ such that $P_{o,z}\left(l^* = i \,|\, l_1 = 0\right) \le A_2 e^{-a_2 i}$ for every $i \ge 1$ and $z \in \mathbb{V}_d$, where $l^* := \inf\{l \in \mathcal{L} : l > 0, X_{H_l} \ne \tilde{X}_{\tilde{H}_l}\}$.*

Proof Fix a_3 as in Lemma 30. Define $v_1^+ := \inf\{m > 0 : X_i = \tilde{X}_j$ for some $i \ge 0, j \ge 0$, and $\langle X_i, e_1 \rangle = m\}$. For every $z \in \mathbb{V}_d$ and $a \in (0, a_3)$, by Hölder's inequality,

$$
E_{o,z}\left[e^{a l^+}, v_1^+ < \infty \,\middle|\, l_1 = 0\right] \le E_{o,z}\left[e^{a_3 l^+} \,\middle|\, l_1 = 0\right]^{a/a_3} P_{o,z}\left(v_1^+ < \infty \,\middle|\, l_1 = 0\right)^{1 - a/a_3} \text{ and}
$$

$$
\sup_{z \in \mathbb{V}_d} E_{o,z}\left[e^{a l^+}, v_1^+ < \infty \,\middle|\, l_1 = 0\right] \le \left(\sup_{z \in \mathbb{V}_d} E_{o,z}\left[e^{a_3 l^+} \,\middle|\, l_1 = 0\right]\right)^{a/a_3}
$$

$$
\times \left(\sup_{z \in \mathbb{V}_d} P_{o,z}\left(v_1^+ < \infty \,\middle|\, l_1 = 0\right)\right)^{1 - a/a_3}.
$$

It is an easy consequence of (1.1) and Lemma 20 that $\sup_{z \in \mathbb{V}_d} P_{o,z}\left(v_1^+ < \infty \,\middle|\, l_1 = 0\right) < 1$. Hence,

$$
\sup_{z \in \mathbb{V}_d} E_{o,z}\left[e^{a_2 l^+}, v_1^+ < \infty \,\middle|\, l_1 = 0\right] < 1
$$

for some $a_2 \in (0, a_3)$.

It follows from the regeneration structure that

$$E_{o,z}\left[e^{a_2 l^*}\middle|l_1 = 0\right] = E_{o,z}\left[e^{a_2 l^*}, l^+ < l^*\middle|l_1 = 0\right] + E_{o,z}\left[e^{a_2 l^*}, l^+ = l^*\middle|l_1 = 0\right]$$

$$= E_{o,z}\left[e^{a_2 l^+}, X_{H_{l^+}} = \tilde{X}_{\tilde{H}_{l^+}}\middle|l_1 = 0\right] E_{o,o}\left[e^{a_2 l^*}\middle|l_1 = 0\right]$$

$$+ E_{o,z}\left[e^{a_2 l^+}, l^+ = l^*\middle|l_1 = 0\right]$$

$$\leq E_{o,z}\left[e^{a_2 l^+}, v_1^+ < \infty\middle|l_1 = 0\right] E_{o,o}\left[e^{a_2 l^*}\middle|l_1 = 0\right]$$

$$+ E_{o,z}\left[e^{a_2 l^+}\middle|l_1 = 0\right].$$

Therefore,

$$\sup_{z\in\mathbb{V}_d} E_{o,z}\left[e^{a_2 l^*}\middle|l_1 = 0\right] \leq \sup_{z\in\mathbb{V}_d} E_{o,z}\left[e^{a_2 l^+}, v_1^+ < \infty\middle|l_1 = 0\right]$$

$$\times \sup_{z\in\mathbb{V}_d} E_{o,z}\left[e^{a_2 l^*}\middle|l_1 = 0\right] + \sup_{z\in\mathbb{V}_d} E_{o,z}\left[e^{a_2 l^+}\middle|l_1 = 0\right]$$

and

$$A_2 := \sup_{z\in\mathbb{V}_d} E_{o,z}\left[e^{a_2 l^*}\middle|l_1 = 0\right] \leq \left(1 - \sup_{z\in\mathbb{V}_d} E_{o,z}\left[e^{a_2 l^+}, v_1^+ < \infty\middle|l_1 = 0\right]\right)^{-1}$$

$$\times \sup_{z\in\mathbb{V}_d} E_{o,z}\left[e^{a_2 l^+}\middle|l_1 = 0\right] < \infty.$$

Finally, for every $i \geq 1$ and $z \in \mathbb{V}_d$,

$$P_{o,z}\left(l^* = i\middle|l_1 = 0\right) \leq E_{o,z}\left[e^{a_2 l^*}\middle|l_1 = 0\right] e^{-a_2 i} \leq A_2 e^{-a_2 i}$$

by Chebyshev's inequality.   □

Proof of Lemma 26 Recall (3.16). For every $\kappa_5 \in (0, \kappa_3/4)$ and $p \geq 1$, Lemmas 28, 29 and 31 imply that

$$\sum_{i=1}^{\infty} \sum_{z\in\mathbb{V}_d} \sup_{\theta\in\mathcal{C}_a(\kappa_5)} h_{i,z}(\theta)$$

$$\leq \sum_{i=1}^{\infty} \sum_{z\in\mathbb{V}_d} \sup_{\theta\in\mathcal{C}_a(\kappa_5)} E_o\left[\exp\left\{4\langle\theta, X_{H_i}\rangle - 4\Lambda_a(\theta)H_i\right\}, \beta = \infty\right]^{1/2}$$

$$\times P_{o,z}(l_1 = 0)^{-1}$$

$$\times P_{o,z}\left(i = \inf\{l \in \mathcal{L} : l > 0, X_{H_l} \neq \tilde{X}_{\tilde{H}_l}\}, l_1 = 0\right)^{1/4}$$

$$\times P_{o,z}(i > v_1)^{1/4}$$

$$\leq \sum_{i=1}^{\infty} \sum_{z \in \mathbb{V}_d} \sup_{\theta \in \mathcal{C}_a(\kappa_5)} \left(i\mathrm{e}^{a_1|\theta|i}\right)^{1/2} P_{o,z}(l_1 = 0)^{-1} \left(A_2 \mathrm{e}^{-a_2 i}\right)^{1/4}$$

$$\times \left(A_1 i^p \max(1, |z|)^{-p}\right)^{1/4}$$

$$\leq \frac{(A_1 A_2)^{1/4}}{\inf_{z \in \mathbb{V}_d} P_{o,z}(l_1 = 0)} \left(\sum_{i=1}^{\infty} i^{1/2 + p/4} \exp\{(2a_1\kappa_5 - a_2)i/4\}\right)$$

$$\times \left(1 + \sum_{z \in \mathbb{V}'_d} |z|^{-p/4}\right) \tag{3.18}$$

for some $a_1 < \infty$, $a_2 > 0$, $A_1 < \infty$ and $A_2 < \infty$. Clearly, (3.18) is finite when $p > 4d$ and $\kappa_5 \in (0, a_2/2a_1)$. □

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