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ERRATUM

Erratum to: Percolation on random Johnson–Mehl tessellations and related models

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The proof presented in [2] of the result that the critical probability for percolation on a random Johnson–Mehl tessellation is $1/2$ contains an obvious error; we are very grateful to Rob van den Berg for bringing this to our attention. Fortunately, this error is easy to correct. There is a class of influence results, especially those of Talagrand [9] and Friedgut and Kalai [5] (in both cases extending results of Kahn, Kalai and Linial [6] and Bourgain, Kahn, Kalai, Katznelson and Linial [4]), whose application to prove sharp thresholds for 'symmetric' events works particularly well. As so often in such settings, to prove that a threshold is sharp, one needs only enough symmetry to ensure that many variables are equivalent, rather than total symmetry; the latter is not present in [2].

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There are many applications of these and other influence results which do not require any symmetry, starting with Russo's proof of Kesten's Theorem [8]; however, such arguments do not seem to apply (at least without extra work) in the context here, where the discrete product probability space is made up of a very large number of variables each of which has only a very small probability of being non-zero, and we require the width of the threshold to be small in terms of the *relative* changes in these probabilities. This situation arises naturally when discretizing continuous processes such as Poisson processes.

Let $\mathbb{P}^n_{p_-,p_+}$ denote the probability measure on $\{-1,0,1\}^n$ in which each coordinate is independent, and is equal to $+1$ with probability p_+, and to -1 with probability p_-. An event $E \subset \{-1,0,1\}^n$ is *increasing* if whenever $x \in E$ and the inequality $x \le x'$ holds coordinatewise, then $x' \in E$. We say that E has *symmetry of order* m if there is a group action on $[n] = \{1,2,\dots,n\}$ in which each orbit has size at least m, such that the induced action on $\{-1,0,1\}^n$ preserves E. To correct the proof in [2] we need the following lemma.

Lemma 1 *There is an absolute constant $c_3 > 0$ such that if $0 < q_- < p_- < 1/e$, $0 < p_+ < q_+ < 1/e$, $E \subset \{-1,0,1\}^n$ is increasing and has symmetry of order m, and $\mathbb{P}^n_{p_-,p_+}(E) > \eta$, then $\mathbb{P}^n_{q_-,q_+}(E) > 1-\eta$ whenever*

$$\min\{q_+ - p_+, p_- - q_-\} \ge c_3 \log(1/\eta) p_{\max} \log(1/p_{\max})/\log m, \tag{1}$$

where $p_{\max} = \max\{q_+, p_-\}$.

Using this result in place of Theorem 2.2 of [1] (which is simply the special case when $m = n$, i.e., when E is symmetric), the proof in [2] may be corrected with essentially no changes. Indeed, the event E_3 considered at the bottom of page 329, that some $3s/4$ by $s/12$ rectangle in $\mathbb{T}(s)$ has a robustly black horizontal crossing, is symmetric under translations of the space $\mathbb{T}(s) \times [0,s]$ in which the Poisson point processes live through vectors of the form $(x, y, 0)$. Hence the corresponding discrete event E_3^{crude} considered on the next page has symmetry of order $m = (s/\delta)^2$. To deduce Theorem 6 of [2] from Lemma 1 one needs the inequality in the middle of page 330 of [2], but with $\log N$ replaced by $\log m$. Since $N = (s/\delta)^3$, this corresponds simply to a change in the constant, and all remaining calculations are unaffected.

To prove Lemma 1 one needs a suitable influence result. Such a result was proved for a product of 2-element spaces by Talagrand [9]; later, Friedgut and Kalai [5] used a different method to obtain slightly weaker results. One can adapt Talagrand's proof to the 3-element setting, obtaining a slightly weaker form of Lemma 1 (see the remark at the end of the full version [3] of this note), but it seems easier to follow the method of [5]. Unfortunately, even in the two element case, Friedgut and Kalai did not prove the precise result we need, although their method gives it.

Given a function f on a product probability space Ω^n, let $I_f(k)$ denote the *influence* of the kth coordinate with respect to f, i.e., the probability of the set of configurations ω with the property that there is some ω' differing from ω only in the kth coordinate for which $f(\omega') \ne f(\omega)$. For $A \subset \Omega^n$, let $I_A(k) = I_f(k)$ where f is the characteristic function of A.

Following the notation of Friedgut and Kalai [5], let $V_n(p)$ denote the *weighted cube*, that is the nth power of the 2-element probability space in which $\mathbb{P}(0) = 1 - p$ and $\mathbb{P}(1) = p$. Bourgain, Kahn, Kalai, Katznelson and Linial [4] showed that if f is any $0/1$-valued function on the nth power of a 'standard' probability space (such as a discrete space, or the interval $[0, 1]$), then some influence $I_f(k)$ is at least a constant times $t \log n/n$, where $t = \min\{\mathbb{P}(f^{-1}(0)), \mathbb{P}(f^{-1}(1))\}$. Friedgut and Kalai [5] adapted their proof to prove two extensions (Theorems 3.1 and Theorem 3.4 in [5]). Although they did not do so, there is no problem combining these two extensions, to obtain the following result; for details see [3].

Lemma 2 *Let $0 < p \leq 1/2$ and let $f : V_n(p) \to \{0, 1\}$ with $\mathbb{P}(f^{-1}(1)) = t$. If $I_f(k) \leq \delta$ for every k then $I_f = \sum_{k=1}^{n} I_f(k)$ satisfies the inequality*

$$I_f \geq c \frac{t(1-t)}{p \log(1/p)} \log\left(\frac{ct(1-t)}{\delta^{1/2} I_f}\right),$$

where $c > 0$ is an absolute constant. In particular, if for some $a \leq 1/2$ we have $I_f(k) \leq ap^2 \log(1/p)^2$ for every k, then

$$I_f \geq c' \frac{t(1-t)}{p \log(1/p)} \log(1/a),$$

where $c' > 0$ is an absolute constant.

Lemma 2 is also implied by Corollary 1.2 of Talagrand [9] (see [3]). However, a key feature of the approach in [5] is that, as in [4], the first step is to replace each factor $V_1(p)$ in the product space $V_n(p)$ by the probability space $Y = \{0, 1\}^m$ with uniform measure; one can assume that p is a dyadic rational, choose m so that $2^m p$ is an integer, and take the first $(1 - p)2^m$ points of Y (in the binary order) to correspond to $0 \in V_1(p)$ and the last $p2^m$ to $1 \in V_1(p)$. Then, as noted in [5], for any function $f : V_1(p) \to \{0, 1\}$, the sum $w(f)$ of the influences of the corresponding function on Y satisfies

$$w(f) \leq c_1 p \log(1/p) \tag{2}$$

for some absolute constant c_1. As in [1], this implies that the extension of Lemma 2 to the probability space $W^n_{p_-,p_+}$, i.e., $\{-1, 0, 1\}^n$ with the product measure $\mathbb{P}^n_{p_-,p_+}$ (with $p_-, p_+ \leq 1/e$), is immediate. Indeed, replacing each factor W_{p_-,p_+} by $Y = \{0, 1\}^m$, one now has

$$w(f) \leq cp_+ \log(1/p_+) + cp_- \log(1/p_-) \leq 2cp_{\max} \log(1/p_{\max})$$

in place of (2). From this point on the original probability space is irrelevant, and one obtains the following result (see [3] for details).

Corollary 3 *For every* $0 < p_-, p_+ \le 1/e$ *and every function* $f : W^n_{p_-,p_+} \to \{0,1\}$ *with* $\mathbb{P}(f^{-1}(1)) = t$, *if* $a \le 1/2$ *and* $I_f(k) \le a p_{\max}^2 \log(1/p_{\max})^2$ *for every* k, *then*

$$I_f \ge c \frac{t(1-t)}{p_{\max}\log(1/p_{\max})} \log(1/a),$$

where $p_{\max} = \max\{p_-, p_+\}$ *and* $c > 0$ *is an absolute constant.*

Corollary 3 easily implies Lemma 1.

Proof of Lemma 1 Since the left-hand side of (1) is at most $p_{\max}$, taking c_3 large we may assume that $\log m \ge 100\log(1/p_{\max})$, say, i.e., that $m \ge p_{\max}^{-100}$.

For $0 \le h \le h_{\max} = \min\{q_+ - p_+, p_- - q_-\}$ let $r_+ = p_+ + h$ and $r_- = p_- - h$, and let $g(h) = \mathbb{P}^n_{r_-,r_+}(E)$. Then, by a Margulis–Russo-type formula, the derivative of $g(h)$ is at least $I_f = \sum_k I_f(k)$, where f is the characteristic function of E considered on the product space $\mathbb{P}^n_{r_-,r_+}$. A simple calculation shows that Lemma 1 follows if for every h we have

$$I_f \ge 2c_3^{-1} \frac{t(1-t)}{p_{\max}\log(1/p_{\max})} \log m, \tag{3}$$

where $t = g(h) = \mathbb{P}^n_{r_-,r_+}(E)$.

To see (3), suppose first that some influence $I_f(k)$ is at least $m^{-1/2}$, say. Then, from the symmetry assumption, at least m influences are at least this large, and $I_f \ge m^{1/2} \ge m^{1/3} p_{\max}^{-2}$. Taking c_3 large enough, this is much larger than the bound in (3). (The factor $t(1-t)$ works in our favour.) On the other hand, if $I_f(k) \le m^{-1/2}$ for all k, then $a = \max I_f(k) p_{\max}^{-2} \log(1/p_{\max})^{-2} \le m^{-1/3}$, say, and Corollary 3 gives (3).

An alternative approach to proving results such as Lemma 1 is to adapt Talagrand's argument in [9] (see [3]); it may also be possible to use the approach of Rossignol [7]. His Corollary 3.1 does not seem to be strong enough as it is, however.

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