Probab. Theory Relat. Fields (2007) 139:473–519
DOI 10.1007/s00440-006-0049-7

Critical percolation exploration path and SLE_6: a proof of convergence

Federico Camia · **Charles M. Newman**

Received: 25 April 2006 / Revised: 22 November 2006 / Published online: 21 March 2007

Abstract It was argued by Schramm and Smirnov that the critical site percolation exploration path on the triangular lattice converges in distribution to the trace of chordal SLE_6. We provide here a detailed proof, which relies on Smirnov's theorem that crossing probabilities have a conformally invariant scaling limit (given by Cardy's formula). The version of convergence to SLE_6 that we prove suffices for the Smirnov–Werner derivation of certain critical percolation crossing exponents and for our analysis of the critical percolation full scaling limit as a process of continuum nonsimple loops.

Keywords Continuum scaling limit · Percolation · SLE · Critical behavior · Triangular lattice · Conformal invariance

Mathematics Subject Classification (2000) 82B27 · 60K35 · 82B43 · 60D05 · 30C35

Research of Federico Camia was partially supported by a Marie Curie Intra-European Fellowship under contract MEIF-CT-2003-500740 and by a Veni grant of the Dutch Organization for Scientific Research (NWO).
Research of Charles M.Newman was partially supported by the US NSF under grants DMS-01-04278 and DMS-06-06696.

F. Camia (✉)
Department of Mathematics, Vrije Universiteit Amsterdam,
Amsterdam, The Netherlands
e-mail: fede@few.vu.nl

C. M. Newman
Courant Institute of Mathematical Sciences, New York University,
New York, USA
e-mail: newman@courant.nyu.edu

1 Synopsis

The purpose of this paper is to present a detailed, self-contained proof of the convergence of the critical site percolation exploration path (on the triangular lattice) to the trace of chordal SLE_6 for Jordan domains. We will prove convergence in a strong sense: in the topology induced by the uniform metric on continuous curves (modulo monotonic reparametrization), and "locally uniformly" in the "shape" of the domain and the starting and ending points of the curve.

The main technical difficulty (in the approach followed here) appears in a rather surprising way—to obtain a Markov property for any scaling limit of the percolation exploration path. The surprise is that an even stronger Markov property trivially holds for the exploration path itself. To show that an analogous property holds in the scaling limit (see Theorem 4 and Remark 7.1) is largely responsible for the length of the paper. Roughly, the difficulty is that in the scaling limit the exploration path touches itself and the boundary of the domain (infinitely many times). The touching of the domain boundary in particular requires a lengthy analysis (see Lemmas 7.1–7.4) since the standard percolation bound on multiple crossings of a "semi-annulus" only applies to the case of a "flat" boundary (see [32] and Appendix A of [21]). This issue is resolved here by using the continuity of Cardy's formula with respect to changes in the domain. We remark that the results and methods developed here about touching of domain boundaries have other applications—e.g., to the existence and conformal invariance of the full scaling limit in general (non-flat) domains; these extensions of the results of [8,9] will be discussed elsewhere [10].

The proof has two parts: a characterization for SLE_6 curves (Theorem 2), which is similar to Schramm's argument identifying SLE_κ but only uses conformal invariance of hulls at special stopping times, and a series of results showing that any subsequential scaling limit of the exploration path satisfies the hypotheses of Theorem 2.

The theorem of Smirnov (Theorem 1 here) about convergence to Cardy's formula [31] is a key tool throughout. It is used in the proof of Theorem 2, which follows roughly Smirnov's sketch in [31,32] (but with one significant difference—see Remarks 5.2 and 5.3), and is also crucial, in its strengthened version, Theorem 3, in proving that the "filling" of the exploration path converges to a hull process having the Markov property necessary to apply Theorem 2. As mentioned, this step, implicitly assumed in Smirnov's sketch [31,32], turns out to be the most technically difficult one. Despite its length, we believe that a detailed proof is needed, since the result, beside its own interest, has important applications—notably the rigorous derivation of certain critical exponents [34], of Watt's crossing formula [14] and Schramm's percolation formula [28], and the derivation of the full scaling limit [8,9]. We note that Smirnov has recently sketched in [33] a proof different from that of [31,32].

2 Introduction

The percolation exploration path was introduced by Schramm in 1999 in a seminal paper [27] where it was used to give a precise formulation (beyond crossing probabilities) to the conjecture that the scaling limit of two-dimensional critical percolation is conformally invariant. Schramm's formulation of the conjecture involves his Stochastic Loewner Evolution or Schramm–Loewner Evolution (SLE_κ), and can be expressed, roughly speaking, by saying that the percolation exploration path converges in distribution to SLE_6.

A simple and elegant argument, due again to Schramm, shows that if the scaling limit of the percolation exploration path exists and is conformally invariant, then it must necessarily be an SLE_κ curve; the value $\kappa = 6$ can be determined by looking at crossing probabilities, since SLE_6 is the only SLE_κ that satisfies Cardy's formula [12].

Shortly after Schramm's paper appeared, Smirnov published a proof [31], for site percolation on the triangular lattice, of the conformal invariance of the scaling limit of crossing probabilities, opening the way to a complete proof of Schramm's conjecture. In [31] (see also [32]) Smirnov also outlined a possible strategy for using the conformal invariance of crossing probabilities to prove Schramm's conjecture. Roughly at the same time, convergence of the exploration path to SLE_6 was used by Smirnov and Werner [34] and by Lawler et al. [21] as a key step in a derivation of the values of various percolation critical exponents, most of which had been previously predicted in the physics literature (see the references in [34]). Later, it was used by the authors of this paper to obtain the full scaling limit of two-dimensional critical percolation (see [7,8]).

However, a detailed proof of the convergence of the exploration path to SLE_6 did not appear until 2005, in an appendix of [8], where we followed a modified version of Smirnov's strategy. The purpose of the present paper is to present essentially that proof in a self-contained form. (We note that Lemma A.3 in [8], whose proof had an error, has been replaced by Lemma 7.2 here.) Our proof roughly follows Smirnov's outline of [31,32], based on the convergence to Cardy's formula [31,32] and on Markov properties (see Theorem 2 below and the discussion preceding it). But there are two significant modifications, which we found necessary for a proof. The first is to use a different sequence of stopping times, which results in a different geometry for the Markov chain approximation to SLE_6 (see Remark 5.3). The second is that "close encounters" by the exploration path to the domain boundary are not handled by general results for "three-arm" events at the boundary of a half-plane, but rather by a more complex argument based partly on continuity of crossing probabilities with respect to domain boundaries (see Lemmas 7.1–7.4).

We note (see Remark 5.2 below for more discussion) that our choice of stopping times is closer in spirit than is the choice in [31,32] to the proofs of convergence of the loop erased random walk to SLE_2 [22] and of the harmonic explorer to SLE_4 [29]. It may also be applicable to other systems in which an SLE_κ limit is expected, provided that sufficient information can be obtained about conformal invariance of the scaling limit of the analogues of exploration hitting distributions in those systems.

Schramm's conjecture, as stated in Smirnov's paper [31], concerns the convergence in distribution of the percolation exploration path to the trace of chordal SLE_6 in a fairly arbitrary fixed domain. Here (see Theorem 5) we will prove a version of convergence which is slightly stronger but somewhat less general: we will show that the distribution of the percolation exploration path converges to that of the trace of chordal SLE_6 "*locally uniformly*" in the "shape" of the domain and in the positions of the starting and ending points of the path, but we will restrict attention to Jordan domains (i.e., domains whose boundary is a simple closed curve). Our main motivation in this specific formulation is to provide the key tool needed in [9] to prove that the set of *all* critical percolation interfaces converges (in distribution) in the scaling limit to a certain countable collection of continuous, nonsimple, noncrossing, fractal loops in the plane. Our formulation of convergence to SLE_6 is also sufficient for a key step in the proof of certain critical exponents [34]—namely for $j(\geq 1)$ crossings of a semi-annulus and for $j(\geq 2)$ crossings, not all of the same color, of an annulus. It does not appear to be sufficient, without at least also using some of [9], for the derivation in [21] of the "one-arm" exponent (i.e., for one crossing of an annulus) and thus not sufficient for proofs of other exponents based on the one-arm exponent (see [16] and Sect. 1 of [34]).

In the next section, we give some preliminary definitions. In Sect. 4, we define the percolation exploration path. In Sect. 5, we introduce Cardy's formula, give a characterization result for SLE_6, and state an extended version of Smirnov's result on the scaling limit of crossing probabilities. Sect. 6 contains results concerning the "envelope" of the hull of exploration paths and SLE_6 paths. Those results are needed in Sect. 7, which is devoted to the proof of the main convergence result (see Theorem 5). The paper ends with an appendix about sequences of conformal maps.

3 Preliminary definitions

We identify the real plane $\mathbb{R}^2$ and the complex plane $\mathbb{C}$ and use the open half-plane $\mathbb{H} = \{x + iy : y > 0\}$ (and its closure $\overline{\mathbb{H}}$). $\mathbb{D}$ denotes the open unit disc $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$. A domain D of the complex plane $\mathbb{C}$ is a nonempty, connected, open subset of $\mathbb{C}$; a simply connected domain D is said to be Jordan if its (topological) boundary ∂D is a Jordan curve (i.e., a simple continuous loop).

We often use Riemann's mapping theorem—that if D is any simply connected domain other than the entire plane $\mathbb{C}$ and $z_0 \in D$, then there is a unique conformal map ϕ of $\mathbb{D}$ onto D such that $\phi(0) = z_0$ and $\phi'(0) > 0$.

When taking the scaling limit $\delta \to 0$ one can focus on fixed bounded regions, $\Lambda \subset \mathbb{R}^2$, or consider the whole $\mathbb{R}^2$ at once. The second option avoids dealing with boundary conditions, but requires an appropriate choice of metric. A convenient way of dealing with the whole $\mathbb{R}^2$ is to replace the Euclidean metric with a distance function

$$\Delta(u,v) = \inf_h \int (1+|h|^2)^{-1}\,\mathrm{d}s, \tag{1}$$

where the infimum is over all smooth curves $h(s)$ joining u with v, parametrized by arclength s, and where $|\cdot|$ denotes the Euclidean norm. This metric is equivalent to the Euclidean metric in bounded regions, but it has the advantage of making $\mathbb{R}^2$ precompact. Adding a single point at infinity yields the compact space $\dot{\mathbb{R}}^2$ which is isometric, via stereographic projection, to the two-dimensional sphere.

3.1 The space of curves

In dealing with the scaling limit we use the approach of Aizenman–Burchard [2]. Denote by $\mathcal{S}_\Lambda$ the complete separable metric space of continuous curves in a closed (bounded) subset $\Lambda \subset \mathbb{R}^2$ with the metric (2) defined below. Curves are regarded as equivalence classes of continuous functions from the unit interval to $\mathbb{R}^2$, modulo monotonic reparametrizations. γ will represent a particular curve and $\gamma(t)$ a parametrization of γ; $\mathcal{F}$ will represent a set of curves (more precisely, a closed subset of $\mathcal{S}_\Lambda$). We define a metric on curves by

$$\mathrm{d}(\gamma_1,\gamma_2) \equiv \inf \sup_{t\in[0,1]} |\gamma_1(t)-\gamma_2(t)|, \tag{2}$$

where the infimum is over parametrizations of γ_1 and γ_2. The distance between two closed sets of curves is defined by the induced Hausdorff metric:

$$\mathrm{dist}(\mathcal{F},\mathcal{F}') \le \varepsilon \Leftrightarrow \forall\,\gamma \in \mathcal{F},\ \exists\,\gamma' \in \mathcal{F}' \quad \text{with } \mathrm{d}(\gamma,\gamma') \le \varepsilon, \quad \text{and vice versa.} \tag{3}$$

The space Ω_Λ of closed subsets of $\mathcal{S}_\Lambda$ (i.e., collections of curves in Λ) with the metric (3) is also a complete separable metric space. We denote by $\mathcal{B}_\Lambda$ its Borel σ-algebra. For each $\delta > 0$, the random curves we consider are polygonal paths on the edges of the hexagonal lattice $\delta\mathcal{H}$, dual to the triangular lattice $\delta\mathcal{T}$. A superscript δ indicates that the curves correspond to a lattice model with lattice spacing δ. We also consider the complete separable metric space $\mathcal{S}$ of continuous curves in $\dot{\mathbb{R}}^2$ with distance

$$\mathrm{D}(\gamma_1,\gamma_2) \equiv \inf \sup_{t\in[0,1]} \Delta(\gamma_1(t),\gamma_2(t)), \tag{4}$$

where the infimum is again over parametrizations of γ_1 and γ_2. The distance between two closed sets of curves is again defined by the induced Hausdorff metric:

$$\mathrm{Dist}(\mathcal{F},\mathcal{F}') \le \varepsilon \Leftrightarrow \forall\,\gamma \in \mathcal{F},\ \exists\,\gamma' \in \mathcal{F}' \quad \text{with } \mathrm{D}(\gamma,\gamma') \le \varepsilon, \quad \text{and vice versa.} \tag{5}$$

The space Ω of closed sets of $\mathcal{S}$ (i.e., collections of curves in $\dot{\mathbb{R}}^2$) with the metric (5) is also a complete separable metric space. We denote by $\mathcal{B}$ its Borel

σ-algebra. When we talk about convergence in distribution of random curves, we refer to the uniform metric (2), while for closed collections of curves, we refer to the metric (3) or (5).

Remark 3.1 In [7,9], the space Ω of closed sets of $\mathcal{S}$ was used for collections of exploration paths and cluster boundary loops and their scaling limits, SLE_6 paths and continuum nonsimple loops. Here, in the statements and proofs of Lemmas 7.1, 7.3 and 7.4, we apply Ω in essentially the original setting of Aizenman and Burchard [1,2], i.e., for collections of blue and yellow simple $\mathcal{T}$-paths (see Sect. 4) and their scaling limits. The slight modification needed to keep track of the colors is easily managed.

3.2 Chordal SLE_κ

SLE_κ was introduced by Schramm [27] to study two-dimensional probabilistic lattice models whose scaling limits are expected to be conformally invariant. Here we describe the chordal version of SLE_κ; for more, see [27] as well as the fine reviews by Lawler [18], Kager and Nienhuis [15], and Werner [37], and Lawler's book [19].

Let $\mathbb{H}$ denote the upper half-plane. For all $\kappa \geq 0$, chordal SLE_κ in $\mathbb{H}$ is a certain random family $(K_t, t \geq 0)$ of bounded subsets of $\overline{\mathbb{H}}$ that is generated by a continuous random curve γ [with $\gamma(0) = 0$] in the sense that, for all $t \geq 0$, $\mathbb{H}_t \equiv \mathbb{H} \setminus K_t$ is the unbounded connected component of $\mathbb{H} \setminus \gamma[0,t]$; γ is called the *trace* of chordal SLE_κ.

Let $D \subset \mathbb{C}$ ($D \neq \mathbb{C}$) be a simply connected domain whose boundary is a continuous curve. Given two distinct points $a, b \in \partial D$ (or more accurately, two distinct prime ends), there exists a conformal map f from $\mathbb{H}$ onto D such that $f(0) = a$ and $f(\infty) \equiv \lim_{|z|\to\infty} f(z) = b$. The choice of a and b only characterizes $f(\cdot)$ up to a scaling factor $\lambda > 0$, since $f(\lambda\cdot)$ would also do.

Suppose that $(K_t, t \geq 0)$ is a chordal SLE_κ in $\mathbb{H}$; chordal SLE_κ $(\tilde{K}_t, t \geq 0)$ in D from a to b as the image of $(K_t, t \geq 0)$ under f. The law of $(\tilde{K}_t, t \geq 0)$ is unchanged, up to a linear time-change, if we replace $f(\cdot)$ by $f(\lambda\cdot)$. One considers $(\tilde{K}_t, t \geq 0)$ as a process from a to b in D, ignoring the role of f.

In the case $\kappa = 6$, $(K_t, t \geq 0)$ is generated by a continuous nonsimple curve γ with Hausdorff dimension $7/4$. We will denote by $\gamma_{D,a,b}$ the image of γ under f and call it the trace of chordal SLE_6 in D from a to b; $\gamma_{D,a,b}$ is a continuous nonsimple curve inside $\overline{D}$ from a to b, and it can be given a parametrization $\gamma_{D,a,b}(t)$ such that $\gamma_{D,a,b}(0) = a$ and $\gamma_{D,a,b}(1) = b$, so that we are in the metric framework described in Sect. 3.1.

4 Lattices and paths

We will denote by $\mathcal{T}$ the two-dimensional triangular lattice, whose sites we think of as the elementary cells of a regular hexagonal lattice $\mathcal{H}$ embedded in the plane as in Fig. 1. We say that two hexagons are neighbors (or that they are adjacent) if they have a common edge. A sequence $(\xi_0, \ldots, \xi_n)$ of hexagons of $\mathcal{H}$ such that ξ_{i-1} and ξ_i are neighbors for all $i = 1, \ldots, n$ and $\xi_i \neq \xi_j$ whenever $i \neq j$ will be called a *$\mathcal{T}$-path* and denoted by π. If the first and last hexagons of the path are neighbors, the path will be called a *$\mathcal{T}$-loop*.

A set D of hexagons is *connected* if any two hexagons in D can be joined by a $\mathcal{T}$-path contained in D. We say that a finite set D of hexagons is *simply connected* if both D and $\mathcal{T} \setminus D$ are connected. For a simply connected set D of hexagons, we denote by ΔD its *external site boundary*, or *s-boundary* (i.e., the set of hexagons that do not belong to D but are adjacent to hexagons in D), and by ∂D the topological boundary of D when D is considered as a domain of $\mathbb{C}$. We will call a bounded, simply connected subset D of $\mathcal{T}$ a *Jordan set* if its s-boundary ΔD is a $\mathcal{T}$-loop.

For a Jordan set $D \subset \mathcal{T}$, a vertex $x \in \mathcal{H}$ that belongs to ∂D can be either of two types, according to whether the edge incident on x that is not in ∂D belongs to a hexagon in D or not. We call a vertex of the second type an *e-vertex* (e for "external" or "exposed").

Given a Jordan set D and two e-vertices x, y in ∂D, we denote by $\partial_{x,y} D$ the portion of ∂D traversed counterclockwise from x to y, and call it the *right boundary*; the remaining part of the boundary is denote by $\partial_{y,x} D$ and is called the *left boundary*. Analogously, the portion of $\Delta_{x,y} D$ of ΔD whose hexagons are adjacent to $\partial_{x,y} D$ is called the *right s-boundary* and the remaining part the *left s-boundary*.

A *percolation configuration* $\sigma = \{\sigma(\xi)\}_{\xi \in \mathcal{T}} \in \{-1, +1\}^{\mathcal{T}}$ on $\mathcal{T}$ is an assignment of -1 (equivalently, yellow) or $+1$ (blue) to each site of $\mathcal{T}$. For a domain D of the plane, the restriction to $D \cap \mathcal{T}$ of σ is denoted by σ_D. On the space of configurations $\Sigma = \{-1, +1\}^{\mathcal{T}}$, we consider the usual product topology and denote by $\mathbb{P}$ the uniform measure, corresponding to Bernoulli percolation with equal density of yellow (minus) and blue (plus) hexagons, which is critical percolation in the case of the triangular lattice.

A (percolation) *cluster* is a maximal, connected, monochromatic subset of $\mathcal{T}$; we will distinguish between blue (plus) and yellow (minus) clusters. The *boundary* of a cluster D is the set of edges of $\mathcal{H}$ that surround the cluster (i.e., its Peierls contour); it coincides with the topological boundary of D considered as a domain of $\mathbb{C}$. The set of all boundaries is a collection of "nested" simple loops along the edges of $\mathcal{H}$.

Given a percolation configuration σ, we associate an arrow to each edge of $\mathcal{H}$ belonging to the boundary of a cluster in such a way that the hexagon to the right of the edge with respect to the direction of the arrow is blue (plus). The set of all boundaries then becomes a collection of nested, oriented, simple loops. A *boundary path* (or *b-path*) γ is a sequence $(e_0, \ldots, e_n)$ of distinct edges of $\mathcal{H}$ belonging to the boundary of a cluster and such that e_{i-1} and e_i meet at a vertex of $\mathcal{H}$ for all $i = 1, \ldots, n$. To each b-path, we can associate a direction according to the direction of the edges in the path.

Given a b-path γ, we denote by $\Gamma_B(\gamma)$ (respectively, $\Gamma_Y(\gamma)$) the set of blue (resp., yellow) hexagons (i.e., sites of $\mathcal{T}$) adjacent to γ; we also let $\Gamma(\gamma) \equiv \Gamma_B(\gamma) \cup \Gamma_Y(\gamma)$.

4.1 The percolation exploration process and path

For a Jordan set $D \subset \mathcal{T}$ and two e-vertices x, y in ∂D, imagine coloring blue all the hexagons in $\Delta_{x,y}D$ and yellow all those in $\Delta_{y,x}D$. Then, for any percolation configuration σ_D inside D, there is a unique b-path γ from x to y which separates the blue cluster adjacent to $\Delta_{x,y}D$ from the yellow cluster adjacent to $\Delta_{y,x}D$. We call $\gamma = \gamma^1_{D,x,y}(\sigma_D)$ a *percolation exploration path* (see Fig. 1) in D from x to y with mesh size $\delta = 1$.

Notice that the exploration path γ only depends on the percolation configuration σ_D inside D and the positions of the e-vertices x and y; in particular, it does not depend on the color of the hexagons in ΔD, since it is defined by imposing fictitious boundary conditions on ΔD. To see this more clearly, we next show how to construct the percolation exploration path dynamically, via the *percolation exploration process* defined below.

Given a Jordan set $D \subset \mathcal{T}$ and two e-vertices x, y in ∂D, assign to $\partial_{x,y}D$ a counterclockwise orientation (i.e., from x to y) and to $\partial_{y,x}D$ a clockwise orientation. The edge e_x incident on x that does not belong to ∂D is oriented in the direction of x. From there one starts an exploration procedure that produces an oriented path inside D along the edges of $\mathcal{H}$, together with two *nonsimple* monochromatic paths on $\mathcal{T}$, as follows. At each step there are two possible edges (left or right edge with respect to the current direction of exploration) to choose from, both belonging to the same hexagon ξ contained in D or ΔD. If ξ belongs to D and has not been previously "explored," its color is determined by flipping a fair coin and then the edge to the left (with respect to the direction in which the exploration is moving) is chosen if ξ is blue (plus), or the edge to the right is chosen if ξ is yellow (minus). If ξ belongs to D and has been previously explored, the color already assigned to it is used to choose an edge according

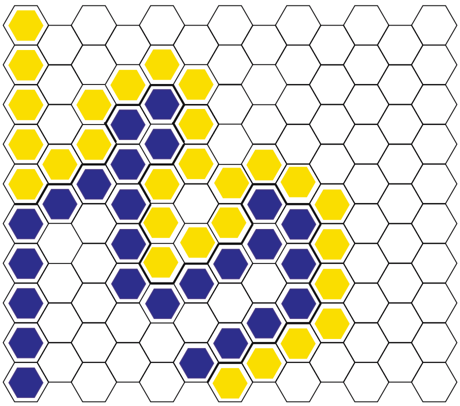


Fig. 1 Percolation exploration process in a portion of the hexagonal lattice with blue/yellow boundary conditions on the first column, corresponding to the boundary of the region where the exploration is carried out. The colored hexagons that do not belong to the first column have been "explored" during the exploration process. The heavy line between yellow (*light*) and blue (*dark*) hexagons is the exploration path produced by the exploration process

to the rule above. If ξ belongs to the right external boundary $\Delta_{x,y}D$, the left edge is chosen. If ξ belongs to the left external boundary $\Delta_{y,x}D$, the right edge is chosen. The exploration stops when it reaches y.

Next, we introduce a class of domains of the plane which will appear later in Theorems 3 and 4 and various lemmas. Let D be a bounded simply connected domain whose boundary ∂D is a continuous curve. Let $\phi : \mathbb{D} \to D$ be the (unique) conformal map from the unit disc $\mathbb{D}$ to D with $\phi(0) = z_0 \in D$ and $\phi'(0) > 0$; note that by Theorem 6 of Appendix A, ϕ has a continuous extension to $\overline{\mathbb{D}}$. Let a, c, d be three points of ∂D (or more accurately, three prime ends) in counterclockwise order—i.e., such that $a = \phi(a^*)$, $c = \phi(c^*)$ and $d = \phi(d^*)$, with a^*, c^* and d^* three distinct points of $\partial\mathbb{D}$ in counterclockwise order. We will call D *admissible* with respect to (a, c, d) if the counterclockwise arcs $J_1 \equiv \overline{da}$, $J_2 \equiv \overline{ac}$ and $J_3 \equiv \overline{cd}$ are simple curves, J_3 does not touch the interior of either J_1 or J_2, and from each point in J_3 there is a path to infinity that does not cross ∂D. (Note that a Jordan D is admissible for any counterclockwise a, c, d on ∂D.)

Notice that, according to our definition, the interiors of the arcs J_1 and J_2 can touch. If that happens, the double-points of the boundary (belonging to both arcs) are counted twice and considered as two distinct points (and are two different prime ends). The significance of the notion of admissible is that certain domains arising naturally in the proof of Theorem 4 are not Jordan but are admissible; this is because the hulls K_t generated by chordal SLE_6 paths have cut-points [4]—see Fig. 5.

With D, J_1, J_2, J_3 as just discussed, let now $\{D^\delta\}$ be a sequence of Jordan sets in $\delta\mathcal{T}$ (i.e., composed of the hexagons of the scaled hexagonal lattices $\delta\mathcal{H}$). If we can split ∂D^δ into three Jordan arcs, J_1^δ, J_2^δ, J_3^δ, such that $\mathrm{d}(J_i^\delta, J_i) \to 0$ for each $i = 1, 2, 3$ as $\delta \to 0$, we say that ∂D^δ converges to ∂D as $\delta \to 0$ and write $\partial D^\delta \to \partial D$ or, equivalently, $D^\delta \to D$.

Let a^δ and b^δ be distinct e-vertices of ∂D^δ and letγ be the exploration path in D^δ from a^δ to b^δ. If, as $\delta \to 0$, $\partial D^\delta \to \partial D$, $a^\delta \to a$ and $b^\delta \to b$, where D is a domain admissible with respect to (a, c, d) and $b \in J_3 = \overline{cd}$, we say that $(D^\delta, a^\delta, b^\delta)$ is a δ-*approximation* to (D, a, b), write $(D^\delta, a^\delta, b^\delta) \to (D, a, b)$, and denote the exploration path γ by $\gamma_{D,a,b}^\delta$. Note that $\gamma_{D,a,b}^\delta$ depends not only on (D, a, b), but also on the δ-approximation to (D, a, b). For simplicity of notation, we do not make explicit this dependence.

For a fixed $\delta > 0$, the probability measure $\mathbb{P}$ on percolation configurations induces a probability measure $\mu_{D,a,b}^\delta$ on exploration paths $\gamma_{D,a,b}^\delta$. In the continuum scaling limit, $\delta \to 0$, one is interested in the weak convergence with respect to the uniform metric (2) of $\mu_{D,a,b}^\delta$ to a probability measure $\mu_{D,a,b}$ supported on continuous curves.

Before concluding this section, we give some more definitions. Consider the exploration path $\gamma = \gamma_{D,a,b}^\delta$ and the set $\Gamma(\gamma) = \Gamma_Y(\gamma) \cup \Gamma_B(\gamma)$. The set $D \setminus \Gamma(\gamma)$ is the union of its connected components (in the lattice sense), which are simply connected. For δ small and $a, b \in \partial D$ not too close to each other, with high probability the exploration process inside D^δ will make large excursions into D^δ, so that $D^\delta \setminus \Gamma(\gamma)$ will have more than one component. Given a point $z \in \mathbb{C}$

contained in $D^\delta \setminus \Gamma(\gamma)$, we will denote by $D^\delta_{a,b}(z)$ the domain corresponding to the unique element of $D^\delta \setminus \Gamma(\gamma)$ that contains z [notice that for a deterministic $z \in D$, $D^\delta_{a,b}(z)$ is well defined with high probability for δ small, i.e., when $z \in D^\delta$ and $z \notin \Gamma(\gamma)$].

There are four types of domains which may be usefully thought of in terms of their external site boundaries: (1) those components whose site boundary contains both sites in $\Gamma_Y(\gamma)$ and $\Delta_{b^\delta,a^\delta} D^\delta$, (2) the analogous components with $\Delta_{b^\delta,a^\delta} D^\delta$ replaced by $\Delta_{a^\delta,b^\delta} D^\delta$ and $\Gamma_Y(\gamma)$ by $\Gamma_B(\gamma)$, (3) those components whose site boundary only contains sites in $\Gamma_Y(\gamma)$, and finally (4) the analogous components with $\Gamma_Y(\gamma)$ replaced by $\Gamma_B(\gamma)$. These different types will appear in the proof of Lemma 6.2.

5 Cardy's formula and a characterization of SLE_6

The existence of subsequential limits for the percolation exploration path, which follows from the work of Aizenman and Burchard [2], means that the proof of convergence to SLE_6 can be divided into two parts: first we will give a characterization of chordal SLE_6 in terms of two properties that determine it uniquely; then we will show that any subsequential scaling limit of the percolation exploration path satisfies these two properties.

The characterization part will follow from known properties of hulls and of SLE_6 (see [23] and [36]). The second part will follow from an extension of Smirnov's result about the convergence of crossing probabilities to Cardy's formula [12] (see Theorem 3 below) for sequences of Jordan domains D_k, with the domain D_k changing together with the mesh δ_k of the lattice, combined with the proof of a certain spatial Markov property for subsequential limits of percolation exploration hulls (Theorem 4). We note that although Theorem 3 represents only a slight extension to Smirnov's result on convergence of crossing probabilities, this extension and its proof play a major role in the technically important Lemmas 7.1, 7.3 and 7.4, which control the "close encounters" of exploration paths to domain boundaries. The proof of Theorem 3 is modelled after a simpler geometric argument involving only rectangles used in [11].

Let D be a bounded simply connected domain containing the origin whose boundary ∂D is a continuous curve. Let $\phi : \mathbb{D} \to D$ be the (unique) conformal map from the unit disc to D with $\phi(0) = 0$ and $\phi'(0) > 0$; note that by Theorem 6 of Appendix A, ϕ has a continuous extension to $\overline{\mathbb{D}}$. Let z_1, z_2, z_3, z_4 be four points of ∂D in counterclockwise order – i.e., such that $z_j = \phi(w_j)$, $j = 1, 2, 3, 4$, with $w_1, \dots, w_4$ in counterclockwise order. Also, let $\eta = \frac{(w_1-w_2)(w_3-w_4)}{(w_1-w_3)(w_2-w_4)}$. Cardy's formula [12] for the probability $\Phi_D(z_1, z_2; z_3, z_4)$ of a "crossing" in D from the counterclockwise arc $\overline{z_1 z_2}$ to the counterclockwise arc $\overline{z_3 z_4}$ is

$$\Phi_D(z_1, z_2; z_3, z_4) = \frac{\Gamma(2/3)}{\Gamma(4/3)\Gamma(1/3)} \eta^{1/3} {}_2F_1(1/3, 2/3; 4/3; \eta), \tag{6}$$

where ${}_2F_1$ is a hypergeometric function.

For a given mesh $\delta > 0$, the probability of a blue crossing inside D from the counterclockwise arc $\overline{z_1 z_2}$ to the counterclockwise arc $\overline{z_3 z_4}$ is the probability of the existence of a blue $\mathcal{T}$-path $(\xi_0, \dots, \xi_n)$ such that ξ_0 intersects the counterclockwise arc $\overline{z_1 z_2}$, ξ_n intersects the counterclockwise arc $\overline{z_3 z_4}$ and $\xi_1, \dots, \xi_{n-1}$ are all contained in D. Smirnov proved the following major theorem, concerning the conjectured behavior [12] of crossing probabilities in the scaling limit (see also [5]).

Theorem 1 (Smirnov [31]) *Let D be a Jordan domain whose boundary ∂D is a finite union of smooth (e.g., C^2) curves. As $\delta \to 0$, the limit of the probability of a blue crossing inside D from the counterclockwise arc $\overline{z_1^\delta z_2^\delta}$ to the counterclockwise arc $\overline{z_3^\delta z_4^\delta}$ exists, is a conformal invariant of (D, z_1, z_2, z_3, z_4) and is given by Cardy's formula (6).*

Remark 5.1 We have stated Smirnov's result in the form that will be used in this paper, but note that Smirnov does not restrict attention to Jordan domains with piecewise smooth boundary but rather allows for more general bounded domains (see [31,32]). We also remark that Theorem 3 below extends Theorem 1 to a larger class of domains, including in particular all Jordan domains.

Let us now specify the objects that we are interested in. Suppose D is a simply connected domain whose boundary ∂D is a continuous curve, and a, b are two distinct points in ∂D (or more accurately, two distinct prime ends), and let $\tilde{\mu}_{D,a,b}$ be a probability measure on continuous curves $\tilde{\gamma} = \tilde{\gamma}_{D,a,b} : [0,\infty) \to \overline{D}$ with $\tilde{\gamma}(0) = a$, $\tilde{\gamma}(\infty) \equiv \lim_{t\to\infty} \tilde{\gamma}(t) = b$, and $\tilde{\gamma}(t) \neq b$ for t finite (we remark that the use of $[0,\infty)$ instead of $[0,1]$ for the time parametrization is purely for convenience). Let $D_t \equiv D \setminus \tilde{K}_t$ denote the (unique) connected component of $D \setminus \tilde{\gamma}[0,t]$ whose closure contains b, where $\tilde{K}_t$, the *filling* of $\tilde{\gamma}[0,t]$, is a closed connected subset of $\overline{D}$. $\tilde{K}_t$ is called a *hull* if it satisfies the condition

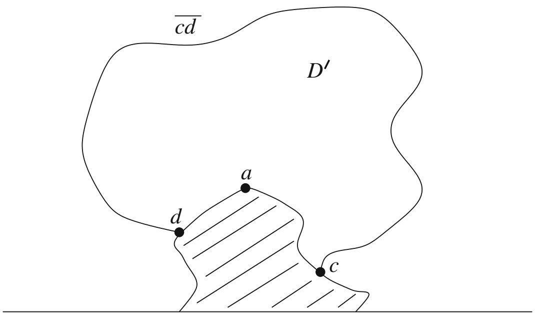
$$\overline{\tilde{K}_t \cap D} = \tilde{K}_t. \tag{7}$$

We will consider curves $\tilde{\gamma}$ such that (i) $\tilde{\gamma}$ is the limit (in distribution using the metric (2)) of random simple curves and such that, for $0 < t_1 < t_2$ with $\tilde{\gamma}(t_1) \neq \tilde{\gamma}(t_2)$, (ii) $\tilde{\gamma}(t_2) \notin \tilde{K}_{t_1} \setminus \partial \tilde{K}_{t_1}$ and (iii) $\exists t' \in (t_1, t_2)$ with $\tilde{\gamma}(t') \in D_{t_1} \equiv D \setminus \tilde{K}_{t_1}$. We note that an example of a curve satisfying these properties is the trace of chordal SLE_6.

Let $C' \subset D$ be a closed subset of $\overline{D}$ such that $a \notin C'$, $b \in C'$, and $D' = D \setminus C'$ is a bounded simply connected domain whose boundary is a continuous curve containing the counterclockwise arc $\overline{cd}$ that does not belong to ∂D (except for its endpoints c and d—see Fig. 2). Let $T' = \inf\{t : \tilde{K}_t \cap C' \neq \emptyset\}$ be the first time that $\tilde{\gamma}(t)$ hits C'. We say that the hitting distribution of $\tilde{\gamma}(t)$ at the stopping time T' is determined by Cardy's formula if, for any C' and any counterclockwise arc $\overline{xy}$ of $\overline{cd}$, the probability that $\tilde{\gamma}$ hits C' at time T' on $\overline{xy}$ is given by

$$\mathbb{P}^*(\tilde{\gamma}(T') \in \overline{xy}) = \Phi_{D'}(a, c; x, d) - \Phi_{D'}(a, c; y, d). \tag{8}$$

Fig. 2 D is the upper half-plane $\mathbb{H}$ with the shaded portion removed, $b = \infty$, C' is an unbounded subdomain, and $D' = D \setminus C'$ is indicated in the figure. The counterclockwise arc $\overline{cd}$ indicated in the figure belongs to $\partial D'$



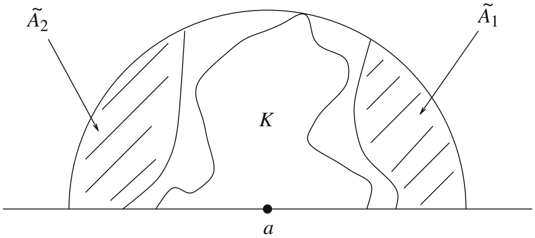
Assume that the filling $\tilde{K}_{T'}$ of $\tilde{\gamma}[0, T']$ is a hull; we denote by $\tilde{\nu}_{D',a,c,d}$ the distribution of $\tilde{K}_{T'}$. To explain what we mean by the distribution of a hull, consider the set $\tilde{\mathcal{A}}$ of closed subsets $\tilde{A}$ of $\overline{D'}$ that do not contain a and such that $\partial \tilde{A} \setminus \partial D'$ is a simple (continuous) curve contained in D' except for its endpoints, one of which is on $\partial D' \cap D$ and the other is on ∂D (see Fig. 3). Let $\mathcal{A}$ be the set of closed subsets of $\overline{D'}$ of the form $\tilde{A}_1 \cup \tilde{A}_2$, where $\tilde{A}_1, \tilde{A}_2 \in \tilde{\mathcal{A}}$ and $\tilde{A}_1 \cap \tilde{A}_2 = \emptyset$.

For a given C' and corresponding T', let $\mathcal{K}$ be the set whose elements are possible hulls at time T'; we claim that the events $\{K \in \mathcal{K} : K \cap A = \emptyset\}$, for $A \in \mathcal{A}$, form a π-system Π (i.e., they are closed under finite intersections; we also include the empty set in Π), and we consider the σ-algebra $\Sigma = \sigma(\Pi)$ generated by these events. To see that Π is closed under pairwise intersections, notice that, if $A_1, A_2 \in \mathcal{A}$, then $\{K \in \mathcal{K} : K \cap A_1 = \emptyset\} \cap \{K \in \mathcal{K} : K \cap A_2 = \emptyset\} = \{K \in \mathcal{K} : K \cap \{A_1 \cup A_2\} = \emptyset\}$ and $A_1 \cup A_2 \in \mathcal{A}$ (or else $\{K \in \mathcal{K} : K \cap \{A_1 \cup A_2\} = \emptyset\}$ is empty). We are interested in probability spaces of the form $(\mathcal{K}, \Sigma, \mathbb{P}^*)$.

It is easy to see that if the hitting distribution of $\tilde{\gamma}(t)$ is determined by Cardy's formula, then the probabilities of events in Π are also determined by Cardy's formula in the following way. Let $A \in \mathcal{A}$ be the union of $\tilde{A}_1, \tilde{A}_2 \in \tilde{\mathcal{A}}$, with $\partial \tilde{A}_1 \setminus \partial D'$ given by a curve from $u_1 \in \partial D' \cap D$ to $v_1 \in \partial D$ and $\partial \tilde{A}_2 \setminus \partial D'$ given by a curve from $u_2 \in \partial D' \cap D$ to $v_2 \in \partial D$; then, assuming that a, v_1, u_1, u_2, v_2 are ordered counterclockwise around $\partial D'$,

$$\mathbb{P}^*(\tilde{K}_{T'} \cap A = \emptyset) = \Phi_{D' \setminus A}(a, v_1; u_1, v_2,) - \Phi_{D' \setminus A}(a, v_1; u_2, v_2). \tag{9}$$

Fig. 3 Example of a hull K and a set $\tilde{A}_1 \cup \tilde{A}_2$ in $\mathcal{A}$. Here, $D = \mathbb{H}$ and D' is the semi-disc centered at a



Since Π is a π-system, the probabilities of the events in Π determine uniquely the distribution of the hull in the sense described above. Therefore, if we let $\gamma_{D,a,b}$ denote the trace of chordal SLE_6 inside D from a to b, K_t its hull up to time t, and $\tau = \inf\{t : K_t \cap C' \neq \emptyset\}$ the first time that $\gamma_{D,a,b}$ hits C', we have the following simple but useful lemma.

Lemma 5.1 *With the notation introduced above, if $\tilde{K}_{T'}$ is a hull and the hitting distribution of $\tilde{\gamma}_{D,a,b}$ at the stopping time T' is determined by Cardy's formula, then $\tilde{K}_{T'}$ is distributed like the hull K_τ of chordal SLE_6.*

Proof It is enough to note that the hitting distribution for chordal SLE_6 is determined by Cardy's formula [20]. □

Now let $\tilde{f}_0$ be a conformal map from the upper half-plane $\mathbb{H}$ to D such that $\tilde{f}_0^{-1}(a) = 0$ and $\tilde{f}_0^{-1}(b) = \infty$. (Since ∂D is a continuous curve, the map $\tilde{f}_0^{-1}$ has a continuous extension from D to $D \cup \partial D$—see Theorem 6 of Appendix A—and, by a slight abuse of notation, we do not distinguish between $\tilde{f}_0^{-1}$ and its extension; the same applies to $\tilde{f}_0$.) These two conditions determine $\tilde{f}_0$ only up to a scaling factor. For $\varepsilon > 0$ fixed, let $C(u, \varepsilon) = \{z : |u - z| < \varepsilon\} \cap \mathbb{H}$ denote the semi-ball of radius ε centered at u on the real line and let $\tilde{T}_1 = \tilde{T}_1(\varepsilon)$ denote the first time $\tilde{\gamma}(t)$ hits $D \setminus \tilde{G}_1$, where $\tilde{G}_1 \equiv \tilde{f}_0(C(0, \varepsilon))$. Define recursively $\tilde{T}_{j+1}$ as the first time $\tilde{\gamma}[\tilde{T}_j, \infty)$ hits $\tilde{D}_{\tilde{T}_j} \setminus \tilde{G}_{j+1}$, where $\tilde{D}_{\tilde{T}_j} \equiv D \setminus \tilde{K}_{\tilde{T}_j}$, $\tilde{G}_{j+1} \equiv \tilde{f}_{\tilde{T}_j}(C(0, \varepsilon))$, and $\tilde{f}_{\tilde{T}_j}$ is a conformal map from $\mathbb{H}$ to $\tilde{D}_{\tilde{T}_j}$ whose inverse maps $\tilde{\gamma}(\tilde{T}_j)$ to 0 and b to ∞. We also define $\tilde{\tau}_{j+1} \equiv \tilde{T}_{j+1} - \tilde{T}_j$, so that $\tilde{T}_j = \tilde{\tau}_1 + \cdots + \tilde{\tau}_j$.We choose $\tilde{f}_{\tilde{T}_j}$ so that its inverse is the composition of the restriction of $\tilde{f}_0^{-1}$ to $\tilde{D}_{\tilde{T}_j}$ with $\tilde{\varphi}_{\tilde{T}_j}$, where $\tilde{\varphi}_{\tilde{T}_j}$ is the unique conformal transformation from $\mathbb{H} \setminus \tilde{f}_0^{-1}(\tilde{K}_{\tilde{T}_j})$ to $\mathbb{H}$ that maps ∞ to ∞ and $\tilde{f}_0^{-1}(\tilde{\gamma}(\tilde{T}_j))$ to the origin of the real axis, and has derivative at ∞ equal to 1.

Notice that $\tilde{G}_{j+1}$ is a bounded simply connected domain chosen so that the conformal transformation which maps $\tilde{D}_{\tilde{T}_j}$ to $\mathbb{H}$ maps $\tilde{G}_{j+1}$ to the semi-ball $C(0, \varepsilon)$ centered at the origin on the real line. With these definitions, consider the (discrete-time) stochastic process $\tilde{X}_j \equiv (\tilde{K}_{\tilde{T}_j}, \tilde{\gamma}(\tilde{T}_j))$ for $j = 1, 2, \ldots$; we say that $\tilde{K}_t$ satisfies the *spatial Markov property* if each $\tilde{K}_{\tilde{T}_j}$ is a hull and $\tilde{X}_j$ for $j = 1, 2, \ldots$ is a Markov chain (for any choice of the map $\tilde{f}_0$). Notice that the hull of chordal SLE_6 satisfies the spatial Markov property, due to the conformal invariance and Markovian properties [27] of SLE_6.

Remark 5.2 The next theorem, our main characterization result for SLE_6, uses the choice of stopping times we have just discussed rather than that proposed by Smirnov [31,32]. A technical reason for this revision of Smirnov's strategy is discussed in Remark 5.3 below. But a conceptually more important reason is that it naturally gives rise (in the scaling limit) to a certain random walk on the real line (the sequence of points to which the tips of the hulls are conformally mapped at the stopping times) whose increments are i.i.d. random variables. As the stopping time parameter $\varepsilon \to 0$, this random walk converges to the driving Brownian motion of the SLE_6.

Theorem 2 *If the filling process $\tilde{K}_t$ of $\tilde{\gamma}_{D,a,b}$ satisfies the spatial Markov property and its hitting distribution is determined by Cardy's formula, then $\tilde{\gamma}_{D,a,b}$ is distributed like the trace $\gamma_{D,a,b}$ of chordal SLE_6 inside D started at a and aimed at b.*

Proof Since the trace $\gamma_{D,a,b}$ of chordal SLE_6 in a Jordan domain D is defined (up to a linear time change) as $f(\gamma)$, where $\gamma = \gamma_{\mathbb{H},0,\infty}$ is the trace of chordal SLE_6 in the upper half-plane started at 0 and f is any conformal map from the upper half-plane $\mathbb{H}$ to D such that $f^{-1}(a) = 0$ and $f^{-1}(b) = \infty$, it is enough to show that $\hat{\gamma} = f^{-1}(\tilde{\gamma}_{D,a,b})$ is distributed like the trace of chordal SLE_6 in the upper half-plane. Let $\hat{K}_t$ denote the filling of $\hat{\gamma}(t)$ at time t and let $\hat{g}_t(z)$ be the unique conformal transformation that maps $\mathbb{H} \setminus \hat{K}_t$ onto $\mathbb{H}$ with the following expansion at infinity:

$$\hat{g}_t(z) = z + \frac{\hat{a}(t)}{z} + o\left(\frac{1}{z}\right). \tag{10}$$

We choose to parametrize $\hat{\gamma}(t)$ so that $t = \hat{a}(t)/2$ (this is often called parametrization by capacity, $\hat{a}(t)$ being the half-plane capacity of the filling up to time t).

We want to compare $\hat{\gamma}(t)$ with the trace $\gamma(t)$ of chordal SLE_6 in the upper half-plane parameterized in the same way (i.e., with $a(t) = 2t$), so that, if K_t is the filling of γ at time t, $\mathbb{H} \setminus K_t$ is mapped onto $\mathbb{H}$ by a conformal g_t with the following expansion at infinity:

$$g_t(z) = z + \frac{2t}{z} + o\left(\frac{1}{z}\right). \tag{11}$$

Our strategy, following [31,32] but with modifications (see Remark 5.3), will be to construct suitable polygonal approximations $\hat{\gamma}_\varepsilon$ and γ_ε of $\hat{\gamma}$ and γ which converge, as $\varepsilon \to 0$, to the original curves [in the uniform metric on continuous curves (2)], and show that $\hat{\gamma}_\varepsilon$ and γ_ε have the same distribution. This implies the equidistribution of $\hat{\gamma}$ and γ.

Let us describe first the construction for $\gamma_\varepsilon(t)$; we do the same for $\hat{\gamma}_\varepsilon(t)$. The important features in the construction of the polygonal approximations are the spatial Markov property of the fillings and Cardy's formula, which are valid for both γ and $\hat{\gamma}$.

For $\varepsilon > 0$ fixed, as above let $C(u, \varepsilon) = \{z : |u-z| < \varepsilon\} \cap \mathbb{H}$ denote the semi-ball of radius ε centered at u on the real line. Let $T_1 = T_1(\varepsilon)$ denote the first time $\gamma(t)$ hits $\mathbb{H} \setminus G_1$, where $G_1 \equiv C(0, \varepsilon)$, and define recursively T_{j+1} as the first time $\gamma[T_j, \infty)$ hits $\mathbb{H}_{T_j} \setminus G_{j+1}$, where $\mathbb{H}_{T_j} = \mathbb{H} \setminus K_{T_j}$ and $G_{j+1} \equiv g^{-1}_{T_j}(C(g_{T_j}(\gamma(T_j)), \varepsilon))$. Notice that G_{j+1} is a bounded simply connected domain chosen so that the conformal transformation which maps $\mathbb{H}_{T_j}$ to $\mathbb{H}$ maps G_{j+1} to the semi-ball $C(g_{T_j}(\gamma(T_j)), \varepsilon)$ centered at the point of the real line where the "tip" $\gamma(T_j)$

of the hull K_{T_j} is mapped.The spatial Markov property and the conformal invariance of the hull of SLE_6 imply that if we write $T_j = \tau_1 + \cdots + \tau_j$, with $\tau_{j+1} \equiv T_{j+1} - T_j$, the τ_j's are i.i.d. random variables, and also that the distribution of $K_{T_{j+1}}$ is the same as that of $K_{T_j} \cup g_{T_j}^{-1}(K'_{T_1} + g_{T_j}(\gamma(T_j)))$, where K'_{T_1} is a hull equidistributed with K_{T_1}, but also is independent of K_{T_1}, and "$+g_{T_j}(\gamma(T_j))$" indicates that it is translated by $g_{T_j}(\gamma(T_j))$ along the real line. The polygonal approximation γ_ε is obtained by joining, for all j, $\gamma(T_j)$ to $\gamma(T_{j+1})$ with a straight segment, where the speed $\gamma'_\varepsilon(t)$ is constant.

Now let $\hat{T}_1 = \hat{T}_1(\varepsilon)$ denote the first time $\hat{\gamma}(t)$ hits $\mathbb{H} \setminus \hat{G}_1$, where $\hat{G}_1 \equiv C(0, \varepsilon)$, and define recursively $\hat{T}_{j+1}$ as the first time $\hat{\gamma}[\hat{T}_j, \infty)$ hits $\hat{\mathbb{H}}_{\hat{T}_j} \setminus \hat{G}_{j+1}$, where $\hat{\mathbb{H}}_{\hat{T}_j} \equiv \mathbb{H} \setminus \hat{K}_{\hat{T}_j}$ and $\hat{G}_{j+1} \equiv \hat{g}_{\hat{T}_j}^{-1}(C(\hat{g}_{\hat{T}_j}(\hat{\gamma}(\hat{T}_j)), \varepsilon))$. We also define $\hat{\tau}_{j+1} \equiv \hat{T}_{j+1} - \hat{T}_j$, so that $\hat{T}_j = \hat{\tau}_1 + \cdots + \hat{\tau}_j$. Once again, $\hat{G}_{j+1}$ is a bounded simply connected domain chosen so that the conformal transformation which maps $\hat{\mathbb{H}}_{\hat{T}_j}$ to $\mathbb{H}$ maps $\hat{G}_{j+1}$ to the semi-ball $C(\hat{g}_{\hat{T}_j}(\hat{\gamma}(\hat{T}_j)), \varepsilon)$ centered at the point on the real line where the "tip" $\hat{\gamma}(\hat{T}_j)$ of the hull $\hat{K}_{\hat{T}_j}$ is mapped. The polygonal approximation $\hat{\gamma}_\varepsilon$ is obtained by joining, for all j, $\hat{\gamma}(\hat{T}_j)$ to $\hat{\gamma}(\hat{T}_{j+1})$ with a straight segment, where the speed $\hat{\gamma}'_\varepsilon(t)$ is constant.

Consider the sequence of times $\tilde{T}_j$ defined in the natural way so that $\tilde{\gamma}(\tilde{T}_j) = f(\hat{\gamma}(\hat{T}_j))$ and the (discrete-time) stochastic processes $\hat{X}_j \equiv (\hat{K}_{\hat{T}_j}, \hat{\gamma}(\hat{T}_j))$ and $\tilde{X}_j \equiv (\tilde{K}_{\tilde{T}_j}, \tilde{\gamma}(\tilde{T}_j))$ related by $\hat{X}_j = f^{-1}(\tilde{X}_j)$. If for $x \in \mathbb{R}$ we let $\theta[x]$ denote the translation that maps x to 0 and define the family of conformal maps $(\tilde{f}_{\tilde{T}_j})^{-1} = \theta[g_{\hat{T}_j}(\hat{\gamma}(\hat{T}_j))] \circ g_{\hat{T}_j} \circ f^{-1}$ from $D \setminus \tilde{K}_{\tilde{T}_j}$ to $\mathbb{H}$, then $(\tilde{f}_{\tilde{T}_j})^{-1}$ sends $\tilde{\gamma}(\tilde{T}_j)$ to 0 and b to ∞, and $(\tilde{T}_{j+1})$ is the first time $\tilde{\gamma}[\tilde{T}_j, \infty)$ hits $\tilde{\mathbb{H}}_{\hat{T}_j} \setminus \tilde{G}_{j+1}$, where $\tilde{\mathbb{H}}_{\tilde{T}_j} = \mathbb{H} \setminus \tilde{K}_{\hat{T}_j}$ and $\tilde{G}_{j+1} = \tilde{f}_{\tilde{T}_j}(C(0, \varepsilon))$. Therefore, $\{\tilde{T}_j\}$ is a sequence of stopping times like those used in the definition of the spatial Markov property and, thanks to the relation $\hat{X}_j = f^{-1}(\tilde{X}_j)$, the fact that $\tilde{K}_t$ satisfies the spatial Markov property implies that $\hat{X}_j$ is a Markov chain. We also note that the fact that the hitting distribution of $\tilde{\gamma}(t)$ is determined by Cardy's formula implies the same for the hitting distribution of $\hat{\gamma}(t)$, thanks to the conformal invariance of Cardy's formula. We next use these properties to show that $\hat{\gamma}_\varepsilon$ is distributed like γ_ε.

To do so, we first note that g_{T_j} and $\hat{g}_{\hat{T}_j}$ are random and their distributions are functionals of those of the hulls K_{T_j} and $\hat{K}_{\hat{T}_j}$, since there is a one-to-one correspondence between hulls and conformal maps [with the normalization we have chosen in (10)and (11)]. Therefore, since $\hat{K}_{\hat{T}_1}$ is distributed like K_{T_1} (see Lemma 5.1), g_{T_1} and $\hat{g}_{\hat{T}_1}$ have the same distribution, which also implies that $\hat{T}_1$ is distributed like T_1 because, due to the parametrization by capacity of γ and $\hat{\gamma}$, $2T_1$ is exactly the coefficient of the term $1/z$ in the expansion at infinity of g_{T_1}, and $2\hat{T}_1$ is exactly the coefficient of the term $1/z$ in the expansion at infinity of $\hat{g}_{\hat{T}_1}$. Moreover, it is also clear that $\hat{\gamma}(\hat{T}_1)$ is distributed like $\gamma(T_1)$, because their distributions are both determined by Cardy's formula, and so $\hat{g}_{\hat{T}_1}(\hat{\gamma}(\hat{T}_1))$ is distributed like $g_{T_1}(\gamma(T_1))$. Notice that the law of the hull $\hat{K}_{\hat{T}_1}$ is conformally invariant because, by Lemma 5.1, it coincides with the law of the SLE_6 hull K_{T_1}.

Using now the Markovian character of $\hat{X}_j$, which implies that, conditioned on $\hat{X}_1 = (\hat{K}_{\hat{T}_1}, \hat{\gamma}(\hat{T}_1))$, $\hat{K}_{\hat{T}_2} \setminus \hat{K}_{\hat{T}_1}$ and $\hat{\gamma}(\hat{T}_2)$ are determined by Cardy's formula in $\hat{G}_2$, from the fact that $\hat{K}_{\hat{T}_1}$ is equidistributed with K_{T_1} and therefore $\hat{G}_2$ is equidistributed with G_2, we obtain that the hull $\hat{K}_{\hat{T}_2}$ is distributed like K_{T_2} and its "tip" $\hat{\gamma}(\hat{T}_2)$ is distributed like the "tip" $\gamma(T_2)$ of the hull K_{T_2}. We can then conclude that the joint distribution of $\{\hat{\gamma}(\hat{T}_1), \hat{\gamma}(\hat{T}_2)\}$ is the same as that of $\{\gamma(T_1), \gamma(T_2)\}$. It also follows immediately that $\hat{g}_{\hat{T}_2}$ is equidistributed with g_{T_2} and $\hat{\tau}_2$ is equidistributed with τ_2 or indeed with τ_1.

By repeating this recursively, using the Markovian character of the hulls and tips, we obtain that, for all j, $\{\hat{\gamma}(\hat{T}_1), \dots, \hat{\gamma}(\hat{T}_j)\}$ is equidistributed with $\{\gamma(T_1), \dots, \gamma(T_j)\}$. This immediately implies that $\hat{\gamma}_\varepsilon$ is equidistributed with γ_ε.

To conclude the proof, we just have to show that, as $\varepsilon \to 0$, $\hat{\gamma}_\varepsilon$ converges to $\hat{\gamma}$ and γ_ε to γ in the uniform metric (2) on continuous curves. This, however, follows easily from the properties of the continuous curves we are considering [see the discussion after (7)], if we can show that $\hat{T}_{j+1} - \hat{T}_j = \hat{\tau}_{j+1}$ and $T_{j+1} - T_j = \tau_{j+1}$ go to 0 as $\varepsilon \to 0$. To see this, we recall that $\hat{\tau}_{j+1}$ and τ_{j+1} are distributed like τ_1 and use Lemma 2.1 of [22], which implies the (deterministic) bound $\tau_1(\varepsilon) \le \varepsilon^2/2$, which follows from the well-known bound $a(t) \le \varepsilon^2$ for the half-plane capacity $a(t) = 2t$ of (11). □

Remark 5.3 The procedure for constructing the polygonal approximations of $\hat{\gamma}$ and γ and the recursive strategy for proving that they have the same distribution include significant modifications to the sketched argument for convergence of the percolation exploration process to chordal SLE_6 proposed by Smirnov in [31] and [32]. One modification is that we use "conformal semi-balls" instead of balls (see [31,32]) to define the sequences of stopping times $\{\hat{T}_j\}$ and $\{T_j\}$. Since the paths we are dealing with touch themselves (or almost do), if one were to use ordinary balls, some of them would intersect multiple disjoint pieces of the past hull, making it impossible to use Cardy's formula in the "triangular setting" proposed by (Carleson and) Smirnov and used here. The use of conformally mapped semi-balls ensures, thanks to the choice of the conformal maps, that the domains used to define the stopping times intersect a single piece of the past hull. This is a natural choice (exploiting the conformal invariance) to obtain a good polygonal approximation of the paths while still being able to use Cardy's formula to determine hitting distributions.

We will next prove a version of Smirnov's result (Theorem 1 above) extended to cover the convergence of crossing probabilities to Cardy's formula for the case of sequences of admissible domains (see the definition of admissible in Sect. 4.1). The statement of Theorem 3 below is certainly not optimal, but it is sufficient for our purposes. We remark that a weaker statement restricted, for instance, only to Jordan domains would not be sufficient—see Fig. 5 and the discussion referring to it in the proof of Theorem 4 below.

Theorem 3 *Consider a sequence* $\{(D_k, a_k, c_k, b_k, d_k)\}$ *of domains* D_k*, containing the origin, admissible with respect to the points* a_k, c_k, d_k *on* ∂D_k*, and with* b_k *belonging to the interior of the counterclockwise arc* $\overline{c_k d_k}$ *of* ∂D_k*. Assume that, as* $k \to \infty$*,* $b_k \to b$ *and there is convergence in the metric* (2) *of the counterclockwise arcs* $\overline{d_k a_k}$*,* $\overline{a_k c_k}$*,* $\overline{c_k d_k}$ *to the corresponding counterclockwise arcs* $\overline{da}$*,* $\overline{ac}$*,* $\overline{cd}$ *of* ∂D*, where* D *is a domain containing the origin, admissible with respect to* (a, c, d)*, and* b *belongs to the interior of* $\overline{cd}$*. Then, for any sequence* $\delta_k \downarrow 0$*, the probability* $\Phi_k^{\delta_k} (\equiv \Phi_{D_k}^{\delta_k})$ *of a blue crossing inside* D_k *from* $\overline{a_k c_k}$ *to* $\overline{b_k d_k}$ *converges, as* $k \to \infty$*, to Cardy's formula* Φ_D [see (6)] *for a blue crossing inside* D *from* $\overline{ac}$ *to* $\overline{cd}$*.*

Proof When D is Jordan, one can use essentially the same arguments as in Lemma 7.3 below (see also Theorem 1 of [11]) to construct for each $\varepsilon > 0$, ε-approximations, $(\tilde{D}, \tilde{a}, \tilde{b}, \tilde{c}, \tilde{d})$ and $(\hat{D}, \hat{a}, \hat{b}, \hat{c}, \hat{d})$, to (D, a, b, c, d) so that $\Phi_{\tilde{D}(\varepsilon)}, \Phi_{\hat{D}(\varepsilon)} \overset{\varepsilon \to 0}{\longrightarrow} \Phi_D$ while

$$\Phi_{\tilde{D}(\varepsilon)} = \liminf_{k \to \infty} \Phi_{\tilde{D}(\varepsilon)}^{\delta_k} \le \liminf_{k \to \infty} \Phi_{D_k}^{\delta_k} \le \limsup_{k \to \infty} \Phi_{D_k}^{\delta_k} \le \limsup_{k \to \infty} \Phi_{\hat{D}(\varepsilon)}^{\delta_k} = \Phi_{\hat{D}(\varepsilon)}. \tag{12}$$

When D is not Jordan but admissible, one can do a similar construction on a Riemann surface with a cut starting from a to separate the touching arcs $\overline{da}$ and $\overline{ac}$ of ∂D. This is similar to an argument in [34] replacing an annulus in the plane by its universal cover. □

Remark 5.4 A construction for the non-Jordan case of approximating $\tilde{D}$ and $\hat{D}$ without the use of a cut surface may be found in the appendix of [8]. It has been suggested to us by a referee and by Beffara that existing proofs of convergence to Cardy's formula for fixed domains (see, in particular [5]) should also work in the context of Theorem 3.

6 Boundary of the hull and the scaling limit

We give here some important results which are needed in the proofs of the main theorems. We start with two lemmas from [8,9], which are consequences of [2] and of standard bounds on the probability of events corresponding to having a certain number of disjoint monochromatic crossings of an annulus (see Lemma 5 of [17], Appendix A of [21], and also [3]). Afterwards we give two related lemmas that are more suited to this paper and whose proofs are revisions of those of the first two lemmas.

Lemma 6.1 *Let* $\gamma_{\mathbb{D},-i,i}^{\delta}$ *be the percolation exploration path on the edges of* $\delta\mathcal{H}$ *inside (a* δ*-approximation of)* $\mathbb{D}$ *between (e-vertices close to)* $-i$ *and* i*. For any* *fixed point $z \in \mathbb{D}$, chosen independently of $\gamma^{\delta}_{\mathbb{D},-i,i}$, as $\delta \to 0$, $\gamma^{\delta}_{\mathbb{D},-i,i}$ and the boundary $\partial\mathbb{D}^{\delta}_{-i,i}(z)$ of the domain $\mathbb{D}^{\delta}_{-i,i}(z)$ that contains z jointly have limits in distribution along subsequences of δ with respect to the uniform metric (2) on continuous curves. Moreover, any subsequence limit of $\partial\mathbb{D}^{\delta}_{-i,i}(z)$ is almost surely a simple loop* [3].

Proof The first part of the lemma is a direct consequence of [2]; it is enough to notice that the (random) polygonal curves $\gamma^{\delta}_{\mathbb{D},-i,i}$ and $\partial\mathbb{D}^{\delta}_{-i,i}(z)$ satisfy the conditions in [2] and thus have a scaling limit in terms of continuous curves, at least along subsequences of δ.

To prove the second part, we use standard percolation bounds (see Lemma 5 of [17] and Appendix A of [21]) to show that, in the limit $\delta \to 0$, the loop $\partial\mathbb{D}^{\delta}_{-i,i}(z)$ does not collapse on itself but remains a simple loop [3].

Let us assume that this is not the case and that the limit $\tilde{\gamma}$ of $\partial\mathbb{D}^{\delta_k}_{-i,i}(z)$ along some subsequence $\{\delta_k\}_{k\in\mathbb{N}}$ touches itself, i.e., $\tilde{\gamma}(t_0) = \tilde{\gamma}(t_1)$ for $t_0 \neq t_1$ with positive probability. If so, we can take $\varepsilon > \varepsilon' > 0$ so small that the annulus $B(\tilde{\gamma}(t_1),\varepsilon) \setminus B(\tilde{\gamma}(t_1),\varepsilon')$ is crossed at least four times by $\tilde{\gamma}$ [here $B(u,r)$ is the ball of radius r centered at u].

Because of the choice of topology, the convergence in distribution of $\partial\mathbb{D}^{\delta_k}_{-i,i}(z)$ to $\tilde{\gamma}$ implies that we can find coupled versions of $\partial\mathbb{D}^{\delta_k}_{-i,i}(z)$ and $\tilde{\gamma}$ on some $(\Omega',\mathcal{B}',\mathbb{P}')$ such that $\mathrm{d}(\partial\mathbb{D}^{\delta}_{-i,i}(z),\tilde{\gamma}) \to 0$, for all $\omega' \in \Omega'$ as $k \to \infty$ (see, e.g., Corollary 1 of [6]).

Using this coupling, we can choose k large enough (depending on ω') so that $\partial\mathbb{D}^{\delta_k}_{-i,i}(z)$ stays in an $\varepsilon'/2$-neighborhood $\mathcal{N}(\tilde{\gamma},\varepsilon'/2) \equiv \bigcup_{u\in\tilde{\gamma}} B(u,\varepsilon'/2)$ of $\tilde{\gamma}$. This however would correspond to an event $\mathcal{A}_{\tilde{\gamma}(t_1)}(\varepsilon,\varepsilon')$ that (at least) four paths of one color (corresponding to the four crossings by $\partial\mathbb{D}^{\delta_k}_{-i,i}(z)$) and two of the other color cross the annulus $B(\tilde{\gamma}(t_1),\varepsilon-\varepsilon'/2)\setminus B(\tilde{\gamma}(t_1),3\,\varepsilon'/2)$. As $\delta_k \to 0$, we can let $\varepsilon' \to 0$ (keeping ε fixed), in which case, we claim that the probability of seeing the event just described somewhere inside $\mathbb{D}$ goes to zero,leading to a contradiction. This is because a standard bound [17] on the probability of six disjoint crossings (not all of the same color) of an annulus gives that the probability of $\mathcal{A}_w(\varepsilon,\varepsilon')$ scales as $(\frac{\varepsilon'}{\varepsilon})^{2+\alpha}$ with $\alpha > 0$. As $\delta \to 0$, we can let $\varepsilon' \to 0$ (keeping ε fixed); then the probability of $\mathcal{A}_w(\varepsilon,\varepsilon')$ goes to zero sufficiently rapidly with ε' to conclude that the probability to see such an event anywhere in $\mathbb{D}$ goes to zero. □

The second lemma states that, for every subsequence limit, the discrete boundaries converge to the boundaries of the domains generated by the limiting continuous curve. In order to insure this, we need to show that whenever the discrete exploration path comes at distance of order δ from the boundary of the exploration domain or from its past filling, producing a "fjord" (see [3]) and causing touching in the limit $\delta \to 0$, with high probability the discrete path already closes the fjord by touching the boundary of the exploration domain or by "touching" itself (i.e., getting to distance δ of itself, just one hexagon away), so that no discrepancy arises, as $\delta \to 0$, between the limit of the discrete filling and the filling of the limiting continuous curve. This issue will come up in the proof of Theorem 4 below, and is one of the main technical issues of this paper.

Lemma 6.2 *Using the notation of Lemma 6.1, let* $\gamma_{\mathbb{D},-i,i}$ *be the limit in distribution of* $\gamma^{\delta}_{\mathbb{D},-i,i}$ *as* $\delta \to 0$ *along some convergent subsequence* $\{\delta_k\}$ *and* $\partial\mathbb{D}_{-i,i}(z)$ *be the boundary of the domain* $\mathbb{D}_{-i,i}(z)$ *of* $\mathbb{D} \setminus \gamma_{D,-i,i}[0,1]$ *that contains* z. *Then, as* $k \to \infty$, $(\gamma^{\delta_k}_{\mathbb{D},-i,i}, \partial\mathbb{D}^{\delta_k}_{-i,i}(z))$ *converges in distribution to* $(\gamma_{\mathbb{D},-i,i}, \partial\mathbb{D}_{-i,i}(z))$.

Proof Let $\{\delta_k\}_{k\in\mathbb{N}}$ be a convergent subsequence for $\gamma^{\delta}_{\mathbb{D},-i,i}$ and $\gamma \equiv \gamma_{\mathbb{D},-i,i}$ the limit in distribution of $\gamma^{\delta_k}_{\mathbb{D},-i,i}$ as $k \to \infty$. For simplicity of notation, we now drop the k and write δ instead of δ_k. Because of the choice of topology, the convergence in distribution of $\gamma^{\delta} \equiv \gamma^{\delta}_{\mathbb{D},-i,i}$ to γ implies that we can find coupled versions of γ^{δ} and γ on some probability space $(\Omega', \mathcal{B}', \mathbb{P}')$ such that $\mathrm{d}(\gamma^{\delta}(\omega'), \gamma(\omega')) \to 0$, for all ω' as $k \to \infty$ (see, for example, Corollary 1 of [6]). Using this coupling, our first task will be to prove the following claim:

(C) For two (deterministic) points $u, v \in \mathbb{D}$, the probability that $\mathbb{D}_{-i,i}(u) = \mathbb{D}_{-i,i}(v)$ but $\mathbb{D}^{\delta}_{-i,i}(u) \neq \mathbb{D}^{\delta}_{-i,i}(v)$ or vice versa goes to zero as $\delta \to 0$.

Let us consider first the case of u, v such that $\mathbb{D}_{-i,i}(u) = \mathbb{D}_{-i,i}(v)$ but $\mathbb{D}^{\delta}_{-i,i}(u) \neq \mathbb{D}^{\delta}_{-i,i}(v)$. Since $\mathbb{D}_{-i,i}(u)$ is an open subset of $\mathbb{C}$, there exists a continuous curve $\gamma_{u,v}$ joining u and v and a constant $\varepsilon > 0$ such that the ε-neighborhood $\mathcal{N}(\gamma_{u,v}, \varepsilon)$ of the curve is contained in $\mathbb{D}_{-i,i}(u)$, which implies that γ does not intersect $\mathcal{N}(\gamma_{u,v}, \varepsilon)$. Now, if γ^{δ} does not intersect $\mathcal{N}(\gamma_{u,v}, \varepsilon/2)$, for δ small enough, then there is a $\mathcal{T}$-path π of unexplored hexagons connecting the hexagon that contains u with the hexagon that contains v, and we conclude that $\mathbb{D}^{\delta}_{-i,i}(u) = \mathbb{D}^{\delta}_{-i,i}(v)$.

This shows that the event that $\mathbb{D}_{-i,i}(u) = \mathbb{D}_{-i,i}(v)$ but $\mathbb{D}^{\delta}_{-i,i}(u) \neq \mathbb{D}^{\delta}_{-i,i}(v)$ implies the existence of a curve $\gamma_{u,v}$ whose ε-neighborhood $\mathcal{N}(\gamma_{u,v}, \varepsilon)$ is not intersected by γ but whose $\varepsilon/2$-neighborhood $\mathcal{N}(\gamma_{u,v}, \varepsilon/2)$ is intersected by γ^{δ}. This implies that $\forall u, v \in \mathbb{D}, \exists \varepsilon > 0$ such that $\mathbb{P}'(\mathbb{D}_{-i,i}(u) = \mathbb{D}_{-i,i}(v)$ but $\mathbb{D}^{\delta}_{-i,i}(u) \neq \mathbb{D}^{\delta}_{-i,i}(v)) \leq \mathbb{P}'(\mathrm{d}(\gamma^{\delta}, \gamma) \geq \varepsilon/2)$. But the right hand side goes to zero for every $\varepsilon > 0$ as $\delta \to 0$, which concludes the proof of one direction of the claim.

To prove the other direction, we consider two points $u, v \in \mathbb{D}$ such that $\mathbb{D}_{-i,i}(u) \neq \mathbb{D}_{-i,i}(v)$ but $\mathbb{D}^{\delta}_{-i,i}(u) = \mathbb{D}^{\delta}_{-i,i}(v)$. Assume that u is trapped before v by γ and suppose for the moment that $\mathbb{D}_{-i,i}(u)$ is a domain of type 3 or 4 (as defined at the end of Sect. 4); the case of a domain of type 1 or 2 is analogous and will be treated later. Let t_1 be the first time u is trapped by γ with $\gamma(t_0) = \gamma(t_1)$ the double point of γ where the domain $\mathbb{D}_{-i,i}(u)$ containing u is "sealed off." At time t_1, a new domain containing u is created and v is disconnected from u.

Choose $\varepsilon > 0$ small enough so that neither u nor v is contained in the ball $B(\gamma(t_1), \varepsilon)$ of radius ε centered at $\gamma(t_1)$, nor in the ε-neighborhood $\mathcal{N}(\gamma[t_0, t_1], \varepsilon)$ of the portion of γ which surrounds u. Then it follows from the coupling that, for δ small enough, there are appropriate parameterizations of γ and γ^{δ} such that the portion $\gamma^{\delta}[t_0, t_1]$ of $\gamma^{\delta}(t)$ is inside $\mathcal{N}(\gamma[t_0, t_1], \varepsilon)$, and $\gamma^{\delta}(t_0)$ and $\gamma^{\delta}(t_1)$ are contained in $B(\gamma(t_1), \varepsilon)$.

For u and v to be contained in the same domain in the discrete construction, there must be a $\mathcal{T}$-path π of unexplored hexagons connecting the hexagon that contains u to the hexagon that contains v. From what we said in the previous paragraph, any such $\mathcal{T}$-path connecting u and v would have to go though a "bottleneck" in $B(\gamma(t_1), \varepsilon)$.

Assume now, for concreteness but without loss of generality, that $\mathbb{D}_{-i,i}(u)$ is a domain of type 3, which means that γ winds around u counterclockwise, and consider the hexagons to the "left" of $\gamma^{\delta}[t_0, t_1]$. Those hexagons form a "quasi-loop" around u since they wind around it (counterclockwise) and the first and last hexagons are both contained in $B(\gamma(t_1), \varepsilon)$. The hexagons to the left of $\gamma^{\delta}[t_0, t_1]$ belong to the set $\Gamma_Y(\gamma^{\delta})$, which can be seen as a (nonsimple) path by connecting the centers of the hexagons in $\Gamma_Y(\gamma^{\delta})$ by straight segments. Such a path shadows γ^{δ}, with the difference that it can have double (or even triple) points, since the same hexagon can be visited more than once. Consider $\Gamma_Y(\gamma^{\delta})$ as a path $\hat{\gamma}^{\delta}$ with a given parametrization $\hat{\gamma}^{\delta}(t)$, chosen so that $\hat{\gamma}^{\delta}(t)$ is inside $B(\gamma(t_1), \varepsilon)$ when $\gamma^{\delta}(t)$ is, and it winds around u together with $\gamma^{\delta}(t)$.

Now suppose that there were two times, $\hat{t}_0$ and $\hat{t}_1$, such that $\hat{\gamma}^{\delta}(\hat{t}_1) = \hat{\gamma}^{\delta}(\hat{t}_0) \in B(\gamma(t_1), \varepsilon)$ and $\hat{\gamma}^{\delta}[\hat{t}_0, \hat{t}_1]$ winds around u. This would imply that the "quasi-loop" of explored yellow hexagons around u is actually completed, and that $\mathbb{D}^{\delta}_{-i,i}(v) \neq \mathbb{D}^{\delta}_{-i,i}(u)$. Thus, for u and v to belong to the same discrete domain, this cannot happen.

For any $0 < \varepsilon' < \varepsilon$, if we take δ small enough, $\hat{\gamma}^{\delta}$ will be contained inside $\mathcal{N}(\gamma, \varepsilon')$, due to the coupling. Following the considerations above, the fact that u and v belong to the same domain in the discrete construction but to different domains in the continuum construction implies, for δ small enough, that there are four disjoint yellow $\mathcal{T}$-paths crossing the annulus $B(\gamma(t_1), \varepsilon) \setminus B(\gamma(t_1), \varepsilon')$ (the paths have to be disjoint because, as we said, $\hat{\gamma}^{\delta}$ cannot, when coming back to $B(\gamma(t_1), \varepsilon)$ after winding around u, touch itself inside $B(\gamma(t_1), \varepsilon)$). Since $B(\gamma(t_1), \varepsilon) \setminus B(\gamma(t_1), \varepsilon')$ is also crossed by at least two blue $\mathcal{T}$-paths from $\Gamma_B(\gamma^{\delta})$, there is a total of at least six $\mathcal{T}$-paths, not all of the same color, crossing the annulus $B(\gamma(t_1), \varepsilon) \setminus B(\gamma(t_1), \varepsilon')$. As $\delta \to 0$, we can let $\varepsilon' \to 0$ (keeping ε fixed) and conclude, like in the proof of Lemma 6.1, that the probability to see such an event anywhere in $\mathbb{D}$ goes to zero.

In the case in which u belongs to a domain of type 1 or 2, let $\mathcal{E}$ be the excursion that traps u and $\gamma(t_0) \in \partial\mathbb{D}$ be the point on the boundary of $\mathbb{D}$ where $\mathcal{E}$ starts and $\gamma(t_1) \in \partial\mathbb{D}$ the point where it ends. Choose $\varepsilon > 0$ small enough so that neither u nor v is contained in the balls $B(\gamma(t_0), \varepsilon)$ and $B(\gamma(t_1), \varepsilon)$ of radius ε centered at $\gamma(t_0)$ and $\gamma(t_1)$, nor in the ε-neighborhood $\mathcal{N}(\mathcal{E}, \varepsilon)$ of the excursion $\mathcal{E}$. Because of the coupling, for δ small enough (depending on ε), γ^{δ} shadows γ along $\mathcal{E}$, staying within $\mathcal{N}(\mathcal{E}, \varepsilon)$. If this is the case, any $\mathcal{T}$-path of unexplored hexagons connecting the hexagon that contains u with the hexagon that contains v would have to go through one of two "bottlenecks," one contained in $B(\gamma(t_0), \varepsilon)$ and the other in $B(\gamma(t_1), \varepsilon)$.

Assume for concreteness (but without loss of generality) that u is in a domain of type 1, which means that γ winds around u counterclockwise. If we parameterize γ and γ^{δ} so that $\gamma^{\delta}(t_0) \in B(\gamma(t_0), \varepsilon)$ and $\gamma^{\delta}(t_1) \in B(\gamma(t_1), \varepsilon)$, $\gamma^{\delta}[t_0, t_1]$

forms a "quasi-excursion" around u since it winds around it (counterclockwise) and it starts inside $B_\varepsilon(\gamma(t_0))$ and ends inside $B_\varepsilon(\gamma(t_1))$. Notice that if γ^δ touched $\partial\mathbb{D}^\delta$, inside both $B_\varepsilon(\gamma(t_0))$ and $B_\varepsilon(\gamma(t_1))$, this would imply that the "quasi-excursion" is a real excursion and that $\mathbb{D}^\delta_{-i,i}(v) \neq \mathbb{D}^\delta_{-i,i}(u)$.

For any $0 < \varepsilon' < \varepsilon$, if we take δ small enough, γ^δ will be contained inside $\mathcal{N}(\gamma, \varepsilon')$, due to the coupling. Therefore, the fact that $\mathbb{D}^\delta_{-i,i}(v) = \mathbb{D}^\delta_{-i,i}(u)$ implies, with probability going to one as $\delta \to 0$, that for $\varepsilon > 0$ fixed and any $0 < \varepsilon' < \varepsilon$, γ^δ enters the ball $B(\gamma(t_i), \varepsilon')$ and does not touch $\partial\mathbb{D}^\delta$ inside the larger ball $B(\gamma(t_i), \varepsilon)$, for $i = 0$ or 1. This is equivalent to having at least two yellow and one blue $\mathcal{T}$-paths (contained in $\mathbb{D}^\delta$) crossing the annulus $B(\gamma(t_i), \varepsilon) \setminus B(\gamma(t_i), \varepsilon')$. Let us call $\mathcal{B}_w(\varepsilon, \varepsilon')$ the event described above, where $\gamma(t_i) = w$; a standard bound [21] (this bound can also be derived from the one obtained in [17]) on the probability of disjoint crossings (not all of the same color) of a semi-annulus in the upper half-plane gives that the probability of $\mathcal{B}_w(\varepsilon, \varepsilon')$ scales as $(\frac{\varepsilon'}{\varepsilon})^{1+\beta}$ with $\beta > 0$. (We can apply the bound to our case because the unit disc is a convex subset of the half-plane $\{x + iy : y > -1\}$ and therefore the intersection of an annulus centered at say $-i$ with the unit disc is a subset of the intersection of the same annulus with the half-plane $\{x + iy : y > -1\}$.) As $\delta \to 0$, we can let $\varepsilon' \to 0$ (keeping ε fixed), concluding that the probability that such an event occurs anywhere on the boundary of the disc goes to zero.

We have shown that, for two fixed points $u, v \in \mathbb{D}$, having $\mathbb{D}_{-i,i}(u) \neq \mathbb{D}_{-i,i}(v)$ but $\mathbb{D}^\delta_{-i,i}(u) = \mathbb{D}^\delta_{-i,i}(v)$ or vice versa implies the occurrence of an event whose probability goes to zero as $\delta \to 0$, and the proof of the claim is concluded.

The Hausdorff distance between two closed nonempty subsets of $\overline{\mathbb{D}}$ is

$$\mathrm{d_H}(A, B) \equiv \inf\{\ell \geq 0 : B \subset \cup_{a \in A} B(a, \ell),\ A \subset \cup_{b \in B} B(b, \ell)\}. \tag{13}$$

With this metric, the collection of closed subsets of $\overline{\mathbb{D}}$ is a compact space. We will next prove that $\partial\mathbb{D}^\delta_{-i,i}(z)$ converges in distribution to $\partial\mathbb{D}_{-i,i}(z)$ as $\delta \to 0$, in the topology induced by (13). (Notice that the coupling between γ^δ and γ provides a coupling between $\partial\mathbb{D}^\delta_{-i,i}(z)$ and $\partial\mathbb{D}_{-i,i}(z)$, seen as boundaries of domains produced by the two paths.)

We will now use Lemma 6.1 and take a further subsequence k_n of the δ's that for simplicity of notation we denote by $\{\delta_n\}_{n\in\mathbb{N}}$ such that, as $n \to \infty$, $\{\gamma^{\delta_n}, \partial\mathbb{D}^{\delta_n}_{-i,i}(z)\}$ converge jointly in distribution to $\{\gamma, \tilde{\gamma}\}$, where $\tilde{\gamma}$ is a simple loop. For any $\varepsilon > 0$, since $\tilde{\gamma}$ is a compact set, we can find a covering of $\tilde{\gamma}$ by a finite number of balls of radius $\varepsilon/2$ centered at points on $\tilde{\gamma}$. Each ball contains both points in the interior $\mathrm{int}(\tilde{\gamma})$ of $\tilde{\gamma}$ and in the exterior $\mathrm{ext}(\tilde{\gamma})$ of $\tilde{\gamma}$, and we can choose (independently of n) one point from $\mathrm{int}(\tilde{\gamma})$ and one from $\mathrm{ext}(\tilde{\gamma})$ inside each ball.

Once again, the convergence in distribution of $\partial\mathbb{D}^{\delta_n}_{-i,i}(z)$ to $\tilde{\gamma}$ implies the existence of a coupling such that, for n large enough, the selected points that are in $\mathrm{int}(\tilde{\gamma})$ are contained in $\mathbb{D}^{\delta_n}_{-i,i}(z)$, and those that are in $\mathrm{ext}(\tilde{\gamma})$ are contained in the complement of $\overline{\mathbb{D}^{\delta_n}_{-i,i}(z)}$. But by claim (C), each one of the selected points that is contained in $\mathbb{D}^{\delta_n}_{-i,i}(z)$ is also contained in $\mathbb{D}_{-i,i}(z)$ with probability going to 1 as $n \to \infty$; analogously, each one of the selected points contained in the complement of $\overline{\mathbb{D}^{\delta_n}_{-i,i}(z)}$ is also contained in the complement of $\overline{\mathbb{D}_{-i,i}(z)}$ with probability going to 1 as $n \to \infty$. This implies that $\partial\mathbb{D}_{-i,i}(z)$ crosses each one of the balls in the covering of $\tilde{\gamma}$, and therefore $\tilde{\gamma} \subset \cup_{u \in \partial\mathbb{D}_{-i,i}(z)} B(u,\varepsilon)$. From this and the coupling between $\partial\mathbb{D}^{\delta_n}_{-i,i}(z)$ and $\tilde{\gamma}$, it follows immediately that, for n large enough, $\partial\mathbb{D}^{\delta_n}_{-i,i}(z) \subset \cup_{u \in \partial\mathbb{D}_{-i,i}(z)} B(u,\varepsilon)$ with probability close to one.

A similar argument (analogous to the previous one but simpler, since it does not require the use of $\tilde{\gamma}$), with the roles of $\mathbb{D}^{\delta_n}_{-i,i}(z)$ and $\mathbb{D}_{-i,i}(z)$ inverted, shows that $\partial\mathbb{D}_{-i,i}(z) \subset \cup_{u \in \partial\mathbb{D}^{\delta_n}_{-i,i}(z)} B(u,\varepsilon)$ with probability going to 1 as $n \to \infty$. Therefore, for all $\varepsilon > 0$, $\mathbb{P}(\mathrm{d_H}(\partial\mathbb{D}^{\delta_n}_{-i,i}(z), \partial\mathbb{D}_{-i,i}(z)) > \varepsilon) \to 0$ as $n \to \infty$, which implies convergence in distribution of $\partial\mathbb{D}^{\delta_n}_{-i,i}(z)$ to $\partial\mathbb{D}_{-i,i}(z)$, as $\delta_n \to 0$, in the topology of (13). But Lemma 6.1 implies that $\partial\mathbb{D}^{\delta_n}_{-i,i}(z)$ converges in distribution (using (2)) to a simple loop; therefore $\partial\mathbb{D}_{-i,i}(z)$ must also be simple and we have convergence in the topology of (2).

It is also clear that the argument above is independent of the subsequence $\{\delta_n\}$, so the limit of $\partial\mathbb{D}^{\delta}_{-i,i}(z)$ is unique and coincides with $\partial\mathbb{D}_{-i,i}(z)$. Hence, we have convergence in distribution of $\partial\mathbb{D}^{\delta}_{-i,i}(z)$ to $\partial\mathbb{D}_{-i,i}(z)$, as $\delta \to 0$, in the topology of (2), and indeed joint convergence of $(\gamma^{\delta}, \partial\mathbb{D}^{\delta}_{-i,i}(z))$ to $(\gamma, \partial\mathbb{D}_{-i,i}(z))$. □

We next give two new lemmas which mostly follow from the previous ones (or their proofs) but are more suitable for the purposes of this paper.Let D be a Jordan domain, with a and b two distinct points on ∂D, and consider Jordan sets D^{δ}, for $\delta > 0$, from $\delta\mathcal{H}$ such that $(D^{\delta}, a^{\delta}, b^{\delta}) \to (D, a, b)$, as $\delta \to 0$, where $a^{\delta}, b^{\delta} \in \partial D^{\delta}$ are two distinct e-vertices on ∂D^{δ}. This means that $(D^{\delta}, a^{\delta}, b^{\delta})$ is a δ-approximation of (D, a, b). Denote by $\gamma^{\delta}_{D,a,b}$ the percolation exploration path inside D^{δ} from a^{δ} to b^{δ}.

Let f be a conformal map from the upper half-plane $\mathbb{H}$ to the Jordan domain D and assume that $f^{-1}(a) = 0$ and $f^{-1}(b) = \infty$. (Since D is a Jordan domain, the map f^{-1} has a continuous extension from D to $D \cup \partial D$—see Theorem 7 of Appendix A—and, by a slight abuse of notation, we do not distinguish between f^{-1} and its extension; the same applies to f.) Denote by $C(0,\varepsilon) = \{z : |z| < \varepsilon\} \cap \mathbb{H}$ the semi-ball of radius ε centered at the origin of the real line. Let $G \equiv f(C(0,\varepsilon))$, $c' \equiv f(\varepsilon)$, $d' \equiv f(-\varepsilon)$. Also denote by $\partial^* G$ the following subset of the boundary of G: $\partial^* G \equiv f(\{z : |z| = \varepsilon\} \cap \mathbb{H})$.

Analogously, let f^{δ} be a conformal map from the upper half-plane $\mathbb{H}$ to the Jordan set D^{δ}, assume that $(f^{\delta})^{-1}(a^{\delta}) = 0$ and $(f^{\delta})^{-1}(b^{\delta}) = \infty$, and define $G^{\delta} \equiv f^{\delta}(C(0,\varepsilon))$ and $\partial^* G^{\delta} \equiv f^{\delta}(\{z : |z| = \varepsilon\} \cap \mathbb{H})$. Note that since $\partial D^{\delta} \to \partial D$, by an application of Corollary A.2 of Appendix A, we can and do choose f^{δ} so that it converges to f uniformly in $\overline{\mathbb{H}}$. (We remark that the full strength of Corollary A.2 is not needed here since we are dealing with Jordan domains). With this choice, $\partial^* G^{\delta} \to \partial^* G$ in the metric (2).

Let T^{δ} be the first time that $\gamma^{\delta}_{D,a,b}$ intersects $\partial^* G^{\delta}$, and let $K^{\delta}_{T^{\delta}}$ be the (*discrete*) *filling* of $\gamma^{\delta}_{D,a,b}[0, T^{\delta}]$, i.e., the union of the hexagons explored up to time T^δ and those unexplored hexagons from which it is not possible to reach b without crossing an explored hexagon or ∂D (in other words, this is the set of hexagons that at time T^δ have been explored or are disconnected from b by the exploration path). Notice that even though time variables appear explicitly in the next two lemmas, the time parametrizations of the curves are irrelevant and do not need to be specified.

Lemma 6.3 *With the above notation, as $\delta \to 0$, $\gamma^\delta_{D,a,b}$ and the boundary of $K^\delta_{T^\delta}$ jointly have limits in distribution along subsequences of δ with respect to the uniform metric (2) on continuous curves. Moreover any subsequence limit of $K^\delta_{T^\delta}$ is almost surely a hull that touches $\partial^* G$ at a single point.*

Proof As in Lemma 6.1, the first part of this lemma is a direct consequence of [2]. The fact that the scaling limit of $K^{\delta_k}_{T^{\delta_k}}$ along any convergent subsequence $\delta_k \downarrow 0$ touches $\partial^* G$ at a single point, is a consequence of Lemma 7.1 of Sect. 7 below. (For any fixed k, the statement that $K^{\delta_k}_{T^{\delta_k}}$ touches $\partial^* G^{\delta_k}$ at a single point is a consequence of the definition of the stopping time T^{δ_k}, but a priori, this could fail to be true in the limit $k \to \infty$.) Therefore, if we remove that single point, the scaling limit of the boundary of $K^{\delta_k}_{T^{\delta_k}}$ splits into a left and a right part (corresponding to the scaling limit of the leftmost yellow and the rightmost blue $\mathcal{T}$-paths of hexagons explored by $\gamma^{\delta_k}_{D,a,b}$, respectively) each of which does not touch $\partial^* G$ (like in Fig. 4).

Moreover, Lemmas 7.1 and 7.2 below (with $\tilde{D} = D$, $\hat{D} = G$ and in the limit where the target region $J' = \partial^* G$ is fixed while $J \to \partial D \setminus \{a\}$) imply that if $\gamma^{\delta_k}_{D,a,b}$ has a "close encounter" with ∂D^{δ_k}, then the fjord produced by $\gamma^{\delta_k}_{D,a,b}$ is closed nearby with probability going to 1. Analogously, the standard bound on the probability of six crossings of an annulus [17], used repeatedly before, implies that wherever $\gamma^{\delta_k}_{D,a,b}$ has a "close encounter" with itself, there is "touching" (see the proof of Lemma 6.1). These two observations assure that the complement of the scaling limit of $K^{\delta_k}_{T^{\delta_k}}$ is almost surely connected, which means that the scaling limit of $K^{\delta_k}_{T^{\delta_k}}$ has (almost surely) the properties of a filling. From the same bound on the probability of six crossings of an annulus, we can also conclude

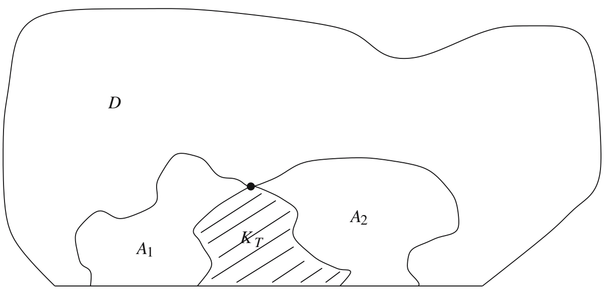


Fig. 4 Schematic figure representing $G \setminus K_T = A_1 \cup A_2$

that the scaling limits of the left and right boundaries of $K^{\delta_k}_{T^{\delta_k}}$ are almost surely simple, as in the proof of Lemma 6.1.

It is also possible to conclude that the intersection of the scaling limit of the left and right boundaries of $K^{\delta_k}_{T^{\delta_k}}$ with the boundary of D almost surely does not contain arcs of positive length. In fact, if that were the case, one could find a subdomain D' with three points z_1, z_2, z_3 in counterclockwise order on $\partial D'$ such that the probability that an exploration path started at z_1 and stopped when it first hits the arc $\overline{z_2 z_3}$ of $\partial D'$ has a positive probability, in the scaling limit, of hitting at z_2 or z_3, contradicting Cardy's formula (which, by Theorem 3, holds for all subsequential scaling limits). Thus the scaling limit of $K^{\delta_k}_{T^{\delta_k}}$ almost surely satisfies the condition in (7) and is therefore a hull. □

Lemma 6.4 *Using the notation of Lemma 6.3, let $\gamma_{D,a,b}$ be the limit in distribution of $\gamma^{\delta}_{D,a,b}$ as $\delta \to 0$ along some convergent subsequence $\{\delta_k\}$. Denote by T the first time that $\gamma_{D,a,b}$ exits G and by K_T the filling of $\gamma_{D,a,b}[0,T]$. Then, as $k \to \infty$, $(\gamma^{\delta_k}_{D,a,b}, K^{\delta_k}_{T^{\delta_k}})$ converges in distribution to $(\gamma_{D,a,b}, K_T)$. Moreover, $\gamma_{D,a,b}$ satisfies the properties* (i)–(iii) *stated after* (7) *above, and its hull K_T is equidistributed with that of chordal SLE_6 at the corresponding stopping time.*

Proof Let $A^{\delta_k}_1$ and $A^{\delta_k}_2$ be the two domains of $G^{\delta_k} \setminus K^{\delta_k}_{T^{\delta_k}}$, and A_1 and A_2 the two domains of $G \setminus K_T$ (see Fig. 4). Since hulls are characterized by their "envelope" (see Lemma 5.1 and the discussion preceding it), the joint convergence in distribution of $\{\partial A^{\delta_k}_1, \partial A^{\delta_k}_2\}$ to $\{\partial A_1, \partial A_2\}$ would be enough to conclude that $K^{\delta_k}_{T^{\delta_k}}$ converges to K_T as $k \to \infty$, and in fact that $(\gamma^{\delta_k}_{D,a,b}, K^{\delta_k}_{T^{\delta_k}})$ converges in distribution to $(\gamma_{D,a,b}, K_T)$.

In order to obtain the convergence of $\{\partial A^{\delta_k}_1, \partial A^{\delta_k}_2\}$, we can use the convergence in distribution of $\gamma^{\delta_k}_{D,a,b}$ to $\gamma_{D,a,b}$ and apply almost the same arguments as used in the proof of Lemma 6.2. In fact, the domains $A^{\delta_k}_1, A^{\delta_k}_2$ and A_1, A_2 are of the same type as those treated in Lemma 6.2. We just need to extend the definitions of the domains $D^{\delta}_{a,b}(z)$ and $D_{a,b}(z)$, as at the end of Section 4 and in Lemma 6.2 [where (D,a,b) was taken to be $(\mathbb{D}, -i, i)$], to cover the case in which the domain D is replaced by a subset G of D and the target point b on the boundary of D by an arc $\partial^* G$ of the boundary of G. In our case, the subdomain G and the arc $\partial^* G$ are defined just before Lemma 6.3.

The definitions are as before but with a deterministic target point replaced by the random hitting point at the stopping time, i.e., we define $G^{\delta}_{a,\partial^* G}(z) \equiv G^{\delta}_{a,\gamma^{\delta}_{D,a,b}(T^{\delta})}(z)$ and $G_{a,\partial^* G}(z) \equiv G_{a,\gamma_{D,a,b}(T)}(z)$. A^{δ}_i (resp., A_i) for $i = 1, 2$ is a domain of type $G^{\delta}_{a,\partial^* G}(z_i)$ (resp., $G_{a,\partial^* G}(z_i)$) for some $z_i \in G$. With these definitions, we need to prove the following claim.

(C′) For two (deterministic) points $u, v \in G$, the probability that $D_{a,\partial^* G}(u) = D_{a,\partial^* G}(v)$ but $D^{\delta}_{a,\partial^* G}(u) \neq D^{\delta}_{a,\partial^* G}(v)$ or vice versa goes to zero as $\delta \to 0$.

The proof is the same as that of claim (C) in Lemma 6.2, except that here we cannot use the bound on the probability of three crossings of an annulus centered at a boundary point because we may not have a convex domain. To replace that bound we use once again Lemmas 7.1 and 7.2 below (as in the proof of Lemma 6.3).

We have proved the convergence in distribution of $(\gamma^{\delta_k}_{D,a,b}, K^{\delta_k}_{T^{\delta_k}})$ to $(\gamma_{D,a,b}, K_T)$. As a consequence of that and Smirnov's result on the convergence of crossing probabilities (see Theorem 3), the hitting distribution of $\gamma_{D,a,b}$ at the stopping time T is determined by Cardy's formula, which allows us to apply Lemma 5.1 to conclude that K_T is equidistributed with the hull of chordal SLE_6 at the corresponding stopping time.

It remains to prove (i)–(iii) stated after (7). Property (i) is immediate in our case. Properties (ii) and (iii) are consequences of Lemmas 7.1, 7.2 and six-arm estimates. □

7 Convergence of the exploration path

Next we show that the filling of any subsequential scaling limit of the percolation exploration process satisfies the spatial Markov property. Let us start with some notation. First, suppose that D is Jordan, with a and b distinct points on ∂D, and consider a sequence D^δ of Jordan sets from $\delta\mathcal{H}$ such that $(D^\delta, a^\delta, b^\delta) \to (D, a, b)$, as $\delta \to 0$, where $a^\delta, b^\delta \in \partial D^\delta$ are two distinct e-vertices on ∂D^δ. This means that $(D^\delta, a^\delta, b^\delta)$ is a δ-approximation of (D, a, b). As before, denote by $\gamma^\delta_{D,a,b}$ the percolation exploration path inside D^δ from a^δ to b^δ.

We can apply the results of [2] to conclude that there exist subsequences $\delta_k \downarrow 0$ such that the law of $\gamma^{\delta_k}_k \equiv \gamma^{\delta_k}_{D,a,b}$ (i.e., the percolation exploration path inside D^{δ_k} from a^{δ_k} to b^{δ_k}) converges to some limiting law for a process $\tilde{\gamma}$ supported on (Hölder) continuous curves inside D from a to b. The curves are defined up to (monotonic) reparametrizations; in the next theorem and its proof, even where the time variable appears explicitly, we do not specify a parametrization since it is irrelevant. The filling $\tilde{K}_t$ of $\tilde{\gamma}[0, t]$, appearing in the next theorem, is defined just above (7).

Theorem 4 *For any subsequential limit $\tilde{\gamma}$ of the percolation exploration path $\gamma^\delta_{D,a,b}$ defined above, the filling $\tilde{K}_t$ of $\tilde{\gamma}[0, t]$, as a process, satisfies the spatial Markov property.*

Proof Let $\delta_k \downarrow 0$ be a subsequence such that the law of $\gamma^{\delta_k}_k$ converges to some limiting law supported on continuous curves $\tilde{\gamma}$ in D from a to b. We will prove the spatial Markov property by showing that $(\tilde{K}_{\tilde{T}_j}, \tilde{\gamma}(\tilde{T}_j))$ as defined in the proof of Theorem 2 are *jointly* distributed like the corresponding SLE_6 hull variables, which do have the spatial Markov property. Since $\gamma^{\delta_k}_k$ converges in distribution to $\tilde{\gamma}$, we can find coupled versions of $\gamma^{\delta_k}_k$ and $\tilde{\gamma}$ on some probability space $(\Omega', \mathcal{B}', \mathbb{P}')$ such that $\gamma^{\delta_k}_k$ converges to $\tilde{\gamma}$ for all $\omega' \in \Omega'$; in the rest of the proof we work with these new versions which, with a slight abuse of notation, we denote with the same names as the original ones.

Let $\tilde{f}_0$ be a conformal transformation that maps $\mathbb{H}$ to D such that $\tilde{f}_0^{-1}(a) = 0$ and $\tilde{f}_0^{-1}(b) = \infty$ and let $\tilde{T}_1 = \tilde{T}_1(\varepsilon)$ denote the first time $\tilde{\gamma}(t)$ hits $D \setminus \tilde{G}_1$, with $\tilde{G}_1 \equiv \tilde{f}_0(C(0,\varepsilon))$ and (as in the proof of Lemma 6.2) $C(0,\varepsilon) = \{z : |z| < \varepsilon\} \cap \mathbb{H}$. Define recursively $\tilde{T}_{j+1}$ as the first time $\tilde{\gamma}(t)$ hits $D_{\tilde{T}_j} \setminus \tilde{G}_{j+1}$, with $\tilde{G}_{j+1} \equiv \tilde{f}_{\tilde{T}_j}(C(0,\varepsilon))$ and $D_{\tilde{T}_j} \equiv D \setminus \tilde{K}_{\tilde{T}_j}$, where $\tilde{f}_{\tilde{T}_j}$ is a conformal map from $\mathbb{H}$ to $D_{\tilde{T}_j}$ whose inverse maps $\tilde{\gamma}(\tilde{T}_j)$ to 0 and b to ∞, chosen as in the definition (before Theorem 2 above) of the spatial Markov property. We also define $\tilde{\tau}_j \equiv \tilde{T}_{j+1} - \tilde{T}_j$, so that $\tilde{T}_j = \tilde{\tau}_1 + \cdots + \tilde{\tau}_j$, and the (discrete-time) stochastic process $\tilde{X}_j \equiv (\tilde{K}_{\tilde{T}_j}, \tilde{\gamma}(\tilde{T}_j))$.

Analogous quantities can be defined for the trace of chordal SLE_6. For clarity, they will be indicated here by the superscript SLE_6; e.g., $f_{T_j}^{SLE_6}$, $K_{T_j}^{SLE_6}$, $G_j^{SLE_6}$ and $X_j^{SLE_6}$. We choose $f_0^{SLE_6} = \tilde{f}_0$, so that $G_1^{SLE_6} = \tilde{G}_1$.

For each k, let K_t^k denote the filling at time t of $\gamma_k^{\delta_k}$ (see the definition of discrete filling just before Lemma 6.3). It follows from the Markovian character of the percolation exploration process that, for all k, the filling K_t^k satisfies a suitably adapted (to the discrete setting) spatial Markov property. (In fact, the percolation exploration path satisfies a stronger property – roughly speaking, that for *all* times t the future of the path given the filling of the past is distributed as a percolation exploration path in the original domain from which the filling up to time t has been removed.)

Let now f_0^k be a conformal transformation that maps $\mathbb{H}$ to $D_k \equiv D^{\delta_k}$ such that $(f_0^k)^{-1}(a_k) = 0$ and $(f_0^k)^{-1}(b_k) = \infty$ and let $T_1^k = T_1^k(\varepsilon)$ denote the first exit time of $\gamma_k^{\delta_k}(t)$ from $G_1^k \equiv f_0^k(C(0,\varepsilon))$ defined as the first time that $\gamma_k^{\delta_k}$ intersects the image under f_0^k of the semi-circle $\{z : |z| = \varepsilon\} \cap \mathbb{H}$. Define recursively T_{j+1}^k as the first exit time of $\gamma_k^{\delta_k}[T_j^k, \infty)$ from $G_{j+1}^k \equiv f_{T_j^k}^k(C(0,\varepsilon))$, where $f_{T_j^k}^k$ is a conformal map from $\mathbb{H}$ to $D_k \setminus K_{T_j^k}^k$ whose inverse maps $\gamma_k^{\delta_k}(T_j^k)$ to 0 and b_k to ∞. The maps $f_{T_j^k}^k$, for $j \geq 1$, are defined only up to a scaling factor. We also define $\tau_{j+1}^k \equiv T_{j+1}^k - T_j^k$, so that $T_j^k = \tau_1^k + \cdots + \tau_j^k$, and the (discrete-time) stochastic process $X_j^k \equiv (K_{T_j^k}^k, \gamma_k^{\delta_k}(T_j^k))$ for $j = 1, 2, \ldots$. The Markovian character of the percolation exploration process implies that, for every k, X_j^k is a Markov chain (in j).

We want to show recursively that, for any j, as $k \to \infty$, $\{X_1^k, \ldots, X_j^k\}$ converge jointly in distribution to $\{\tilde{X}_1, \ldots, \tilde{X}_j\}$. By recursively applying Theorem 3 and Lemma 5.1, we will then be able to conclude that $\{\tilde{X}_1, \ldots, \tilde{X}_j\}$ are jointly equidistributed with the corresponding SLE_6 hull variables (at the corresponding stopping times) $\{X_1^{SLE_6}, \ldots, X_j^{SLE_6}\}$. Since the latter do satisfy the spatial Markov property, so will the former, as desired.

The zeroth step consists in noticing that the convergence of (D_k, a_k, b_k) to (D, a, b) as $k \to \infty$ allows us to use Corollary A.2 to select a sequence of conformal maps f_0^k that converge to $f_0^{SLE_6} = \tilde{f}_0$ uniformly in $\overline{\mathbb{H}}$ as $k \to \infty$, which implies that the boundary ∂G_1^k of $G_1^k = f_0^k(C(0,\varepsilon))$ converges to the boundary $\partial \tilde{G}_1$ of $\tilde{G}_1 = \tilde{f}_0(C(0,\varepsilon))$ in the uniform metric on continuous curves.

Starting from there, the first step of our recursion argument is organized as follows, where all limits and equalities are in distribution:

(i) $K^k_{T_1^k} \to \tilde{K}_{\tilde{T}_1} = K^{SLE_6}_{T_1}$ by Lemma 6.4.

(ii) by i), $D_k \setminus K^k_{T_1^k} \to D \setminus \tilde{K}_{\tilde{T}_1} = D \setminus K^{SLE_6}_{T_1}$.

(iii) by (ii), $f^{SLE_6}_{T_1} = \tilde{f}_{\tilde{T}_1}$, and by Corollary A.2 of Appendix A we can select a sequence $f^k_{T_1^k} \to \tilde{f}_{\tilde{T}_1} = f^{SLE_6}_{T_1}$.

(iv) by (iii), $G_2^k \to \tilde{G}_2 = G_2^{SLE_6}$.

We remark that Lemmas 6.3 and 6.4 imply that the filling $\tilde{K}_{\tilde{T}_1}$ is a hull, and its "envelope" is therefore composed of two simple curves. It follows that $D \setminus \tilde{K}_{\tilde{T}_1}$ and $\tilde{G}_2$ are admissible, since the part of the boundary of either $D \setminus \tilde{K}_{\tilde{T}_1}$ or $\tilde{G}_2$ that belongs to the boundary of $\tilde{K}_{\tilde{T}_1}$ can be split up, by removing the single point $\tilde{\gamma}(\tilde{T}_1)$, into two simple curves, while the remaining part of the boundary of either $D \setminus \tilde{K}_{\tilde{T}_1}$ or $\tilde{G}_2$ is a Jordan arc whose interior does not touch the hull $\tilde{K}_{\tilde{T}_1}$. This allows us to use Theorem 3 (and therefore Lemma 5.1), Corollary A.2 and Lemmas 7.1 and 7.2. (Note that $D \setminus \tilde{K}_{\tilde{T}_1}$ and $\tilde{G}_2$ need not be Jordan because $\tilde{K}_{\tilde{T}_1}$ has cut-points with positive probability—see Fig. 5.)

At this point, we are in the same situation as at the zeroth step, but with G_1^k, $\tilde{G}_1$ and $G_1^{SLE_6}$, replaced by G_2^k, $\tilde{G}_2$ and $G_2^{SLE_6}$, respectively, and we can proceed by induction, as follows. (As explained above, the theorem will then follow from the fact that the SLE_6 hull variables do possess the spatial Markov property.)

The next step consists in proving that $((K^k_{T_1^k}, \gamma_k^{\delta_k}(T_1^k)), (K^k_{T_2^k}, \gamma_k^{\delta_k}(T_2^k)))$ converges in distribution to $((\tilde{K}_{\tilde{T}_1}, \tilde{\gamma}(\tilde{T}_1)), (\tilde{K}_{\tilde{T}_2}, \tilde{\gamma}(\tilde{T}_2)))$. Since we have already proved the convergence of $(K^k_{T_1^k}, \gamma_k^{\delta_k}(T_1^k))$ to $(\tilde{K}_{\tilde{T}_1}, \tilde{\gamma}(\tilde{T}_1))$, we claim that all we really need to prove is the convergence of $(K^k_{T_2^k} \setminus K^k_{T_1^k}, \gamma_k^{\delta_k}(T_2^k))$ to $(\tilde{K}_{\tilde{T}_2} \setminus \tilde{K}_{\tilde{T}_1}, \tilde{\gamma}(\tilde{T}_2))$. To do this, notice that $K^k_{T_2^k} \setminus K^k_{T_1^k}$ is distributed like the filling of a percolation exploration path inside $D_k \setminus K^k_{T_1^k}$. Besides, the convergence in distribution of $(K^k_{T_1^k}, \gamma_k^{\delta_k}(T_1^k))$ to $(\tilde{K}_{\tilde{T}_1}, \tilde{\gamma}(\tilde{T}_1))$ implies that we can find versions of $(\gamma_k^{\delta_k}, K^k_{T_1^k})$ and $(\tilde{\gamma}, \tilde{K}_{\tilde{T}_1})$ on some probability space $(\Omega', \mathcal{B}', \mathbb{P}')$ such that $\gamma_k^{\delta_k}(\omega')$ converges to $\tilde{\gamma}(\omega')$ and $(K^k_{T_1^k}, \gamma_k^{\delta_k}(T_1^k))$ converges to $(\tilde{K}_{\tilde{T}_1}, \tilde{\gamma}(\tilde{T}_1))$ for all $\omega' \in \Omega'$. These two observations imply that, if we work with the coupled versions of $(\gamma_k^{\delta_k}, K^k_{T_1^k})$ and $(\tilde{\gamma}, \tilde{K}_{\tilde{T}_1})$, we are in the same situation as before, but with D_k

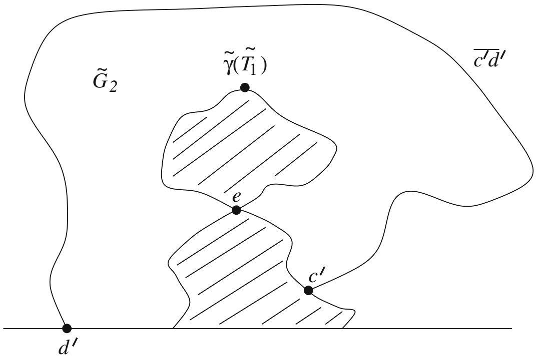


Fig. 5 Schematic figure representing a hull (*shaded*) with a cut-point e, resulting in a non-Jordan, but admissible, $\tilde{G}_2$

(resp., D) replaced by $D_k \setminus K^k_{T^k_1}$ (resp., $D \setminus \tilde{K}_{\tilde{T}_1}$) and a_k (resp., a) by $\gamma^{\delta_k}_k(T^k_1)$ (resp., $\tilde{\gamma}(\tilde{T}_1)$).

Then, the conclusion that $(K^k_{T^k_2} \setminus K^k_{T^k_1}, \gamma^{\delta_k}_k(T^k_2))$ converges in distribution to $(\tilde{K}_{\tilde{T}_2} \setminus \tilde{K}_{\tilde{T}_1}, \tilde{\gamma}(\tilde{T}_2))$ follows, as before, by arguments like those used for Lemma 6.4— i.e., by using a standard bound on the probability of six disjoint monochromatic crossings [17] and Lemmas 7.1 and 7.2, as we now explain. G^k_2 (resp., $\tilde{G}_2$) is a domain admissible with respect to $(\gamma^{\delta_k}_k(T^k_1), c'_k, d'_k)$ (resp., $(\tilde{\gamma}(\tilde{T}_1), c', d')$), where c'_k and d'_k (resp., c' and d') are the unique points where the image of $\overline{\partial\mathbb{D} \cap \mathbb{H}}$ under $f^k_{T^1_k}$ (resp., $\tilde{f}_{\tilde{T}_1}$) meets either the envelope of $K^k_{T^k_1}$ (resp., $\tilde{K}_{\tilde{T}_1}$) or ∂D_k (resp., ∂D)—see Fig. 5. The envelope of the filling $K^k_{T^k_1}$ (resp., $\tilde{K}_{\tilde{T}_1}$) is part of the boundary of the domain explored by $\gamma^{\delta_k}_k[T^k_1, T^k_2]$ (resp., $\tilde{\gamma}[\tilde{T}_1, \tilde{T}_2]$). Close encounters of $\gamma^{\delta_k}_k[T^k_1, T^k_2]$ (resp., $\tilde{\gamma}[\tilde{T}_1, \tilde{T}_2]$) with this part of the boundary can be dealt with using again a standard bound [17] on the probability of six disjoint monochromatic crossings, as explained in the proofs of Lemmas 6.1 and 6.2 (see also [3]). If part of the boundary of G^k_2 (resp., $\tilde{G}_2$) coincides with part of ∂D_k (resp., ∂D), we can use Lemmas 7.1 and 7.2 as in Lemmas 6.3 and 6.4, to obtain the same conclusions. Notice that Lemmas 7.1 and 7.2 are adapted to the situation we encounter here, with the boundary of the exploration domain divided in three parts (corresponding here to the envelope of the past filling, part of the boundary of the original domain, and the semi-circle conformally mapped from the upper half-plane—see Figs. 5 and 6).

We can now iterate the above arguments j times, for any $j > 1$. It is in fact easy to see by induction that the domains $D \setminus \tilde{K}_{\tilde{T}_j}$ and $\tilde{G}_j$ that appear in the successive steps are admissible for all j. Therefore we can keep using Theorem 3 (and Lemma 5.1), Corollary A.2 and Lemmas 7.1 and 7.2. If we keep track at each step of the previous ones, this provides the *joint* convergence of all the curves and fillings involved at each step, and concludes the proof of Theorem 4. □

Remark 7.1 The key technical problem in proving Theorem 4 is showing that for the exploration path $\gamma^{\delta_k}_k$, one can interchange the limit $\delta_k \to 0$ with the process of filling. This requires showing two things about the exploration path: (1) the return of a (macroscopic) segment of the path close to an earlier segment (and away from ∂D_k) without nearby (microscopic) touching does not occur (probably), and (2) the close approach of a (macroscopic) segment to ∂D_k without nearby (microscopic) touching either of ∂D_k itself or else of another segment that touches ∂D_k does not occur (probably). In the related proof of Lemma 6.2 where D was just the unit disk, these were controlled by known estimates on probabilities of six-arm events in the full plane for (1) and of three-arm events in the half-plane for (2). When D is not necessarily convex, as in Theorem 4, the three-arm event argument for (2) appears to break down. Our replacement is the use of Lemmas 7.1-7.4. Basically, these control (2) by a novel argument about "mushroom events" on ∂D (see Lemma 7.4), which is based on continuity of Cardy's formula with respect to changes in ∂D.

The situation described in the next lemma is depicted in Fig. 6 and corresponds to those in the proof of Theorem 4 and in Lemmas 6.3 and 6.4. In the next lemmas and their proofs, when we write that a percolation exploration path in $\delta\mathcal{H}$ touches itself we mean that it gets to distance δ of itself, just one hexagon away.

Lemma 7.1 *Let $\{(\hat{D}_k, a_k, c_k, c'_k, d'_k, d_k)\}$ be a sequence of Jordan domains with five points (not necessarily all distinct) on their boundaries in counterclockwise order. Assume that $\hat{D}_k \subset \tilde{D}_k$, where $\tilde{D}_k$ is a Jordan set from $\delta_k\mathcal{H}$, that $a_k, c_k, c'_k, d'_k, d_k \in \partial\tilde{D}_k$, that the counterclockwise arcs $\overline{d'_k c'_k}$ of $\partial\hat{D}_k$ and $\partial\tilde{D}_k$ coincide, and that a_k is an e-vertex of $\partial\tilde{D}_k$. Consider a second e-vertex $b_k \in \partial\tilde{D}_k, b_k \notin \partial\hat{D}_k$, and denote by $\gamma_k^{\delta_k}$ the percolation exploration path in $\tilde{D}_k$ started at a_k, aimed at b_k, and stopped when it first hits the counterclockwise arc $J'_k = \overline{c'_k d'_k} \subset J_k = \overline{c_k d_k}$ of $\partial\hat{D}_k$. Assume that, as $k \to \infty$, $\delta_k \downarrow 0$ and $(\tilde{D}_k, a_k, c_k, d_k) \to$*

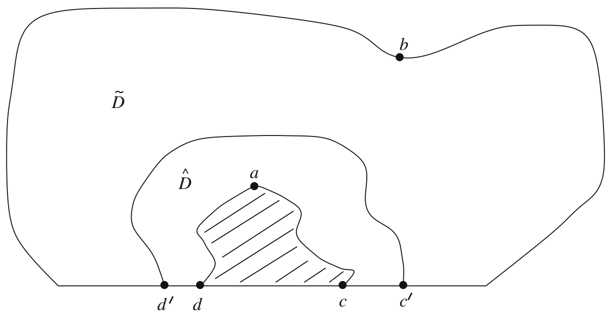


Fig. 6 Schematic figure representing the situation described in Lemma 7.1. Note that the shaded region is not part of the domains $\tilde{D}$ and $\hat{D}$ and that $\tilde{D}$ contains $\hat{D}$. In the application to Theorem 4, the shaded region represents the hull of the past (which may have cut-points, as in Fig. 5) and the counterclockwise arc $\overline{c'd'}$ of $\partial\hat{D}$ represents the conformal image of a semicircle. Notice that c and c' (resp., d and d') coincide if the right (resp., left) endpoint of the conformal image of the semicircle lies on the boundary of the shaded region (see, e.g., Fig. 5). In the application to Lemmas 6.3-6.4, $D = \tilde{D}$, $G = \hat{D}$ and $\partial^* G$ corresponds to the counterclockwise arc $\overline{c'd'}$ of $\partial\hat{D}$

$(\tilde{D}, a, c, d)$, $(\hat{D}_k, a_k, c_k, d_k) \to (\hat{D}, a, c, d)$, where $\tilde{D}$ and $\hat{D}$ are domains admissible with respect to (a, c, d). Assume also that J'_k converges in the metric (2) to the counterclockwise arc $J' \equiv \overline{c'd'}$ of $\partial\hat{D}$, a subset of the counterclockwise arc $J \equiv \overline{cd}$ of $\partial\hat{D}$, and that $b_k \to b \in \partial\tilde{D}$, $b \notin \partial\hat{D}$.

Let $\mathcal{E}_k(J_k; \varepsilon, \varepsilon') = \{\bigcup_{v \in J_k \setminus J'_k} \mathcal{B}_k(v; \varepsilon, \varepsilon')\} \cup \{\bigcup_{v \in J'_k} \mathcal{A}_k(v; \varepsilon, \varepsilon')\}$, where $\mathcal{A}_k(v; \varepsilon, \varepsilon')$ is the event that $\gamma_k^{\delta_k}$ contains a segment that stays within $B(v, \varepsilon)$ and has a double crossing of the annulus $B(v, \varepsilon) \setminus B(v, \varepsilon')$ without that segment touching $\partial\hat{D}_k$, and $\mathcal{B}_k(v; \varepsilon, \varepsilon')$ is the event that $\gamma_k^{\delta_k}$ enters $B(v, \varepsilon')$, but is stopped outside $B(v, \varepsilon)$ and does not touch $\partial\hat{D}_k \cap B(v, \varepsilon)$. Then, for any $\varepsilon > 0$,

$$\lim_{\varepsilon' \to 0} \limsup_{k \to \infty} \mathbb{P}(\mathcal{E}_k(J_k; \varepsilon, \varepsilon')) = 0. \tag{14}$$

Our next result is a lemma in which the part of Lemma 7.1 concerning $J \setminus J'$ can be strengthened to conclude that if $\tilde{\gamma}$ touches $J \setminus J'$ somewhere, then for k large enough, $\gamma_k^{\delta_k}$ touches either $J_k \setminus J'_k$ or its own past hull "nearby." Lemmas 7.1 and 7.2 are used in the proof of Theorem 4 to show that in the limit there is no discrepancy between the hull generated by $\tilde{\gamma}$ and the limit as $k \to \infty$ of the hull generated by $\gamma_k^{\delta_k}$.

Lemma 7.2 *With the notation of Lemma 7.1, let $\mathcal{C}_k(v; \varepsilon, \varepsilon')$ be the event that $\gamma_k^{\delta_k}$ contains a segment that stays within $B(v, \varepsilon)$ and has a double crossing of the annulus $B(v, \varepsilon) \setminus B(v, \varepsilon')$ without that segment touching either $\partial\hat{D}_k$ or any other segment of $\gamma_k^{\delta_k}$ that stays within $B(v, \varepsilon)$ and touches $\partial\hat{D}_k$. Then, for any $\varepsilon > 0$,*

$$\lim_{\varepsilon' \to 0} \limsup_{k \to \infty} \mathbb{P}\left(\bigcup_{v \in J_k \setminus J'_k} \mathcal{C}_k(v; \varepsilon, \varepsilon')\right) = 0. \tag{15}$$

The proofs of Lemmas 7.1 and 7.2 are partly based on relating the failure of (14) or (15) to the occurrence with strictly positive probability of certain continuum limit "mushroom" events (see Lemma 7.4) that we will show must have zero probability because otherwise there would be a contradiction to Lemma 7.3, which itself is a consequence of the continuity of Cardy's formula with respect to the domain boundary (see Lemma A.2 of Appendix A). In both of the next two lemmas, we denote by μ any subsequence limit as $\delta = \delta_k \to 0$ of the probability measures for the collection of all colored (blue and yellow) $\mathcal{T}$-paths on *all* of $\mathbb{R}^2$, in the Aizenman-Burchard sense (see Remark 3.1). We recall that in our notation, D represents an open domain and $\overline{z_1 z_2}$, $\overline{z_3 z_4}$ represent closed segments of its boundary. In Lemma 7.3 below, we restrict attention to a Jordan domain D since that case suffices for the use of Lemma 7.3 in the proofs of Lemmas 7.1 and 7.2.

Lemma 7.3 *For (D, z_1, z_2, z_3, z_4), with D a Jordan domain, consider the following crossing events, $\mathcal{C}_i^* = \mathcal{C}_i^*(D, z_1, z_2, z_3, z_4)$, where $*$ denotes either blue or yellow, a $*$ path denotes a segment of a $*$ curve, and $i = 1, 2, 3$:*

$\mathcal{C}_1^* = \{\exists$ *a* $*$ *path in the closure* $\overline{D}$ *from* $\overline{z_1 z_2}$ *to* $\overline{z_3 z_4}\}$,
$\mathcal{C}_2^* = \{\exists$ *a* $*$ *path in D from the interior of* $\overline{z_1 z_2}$ *to the interior of* $\overline{z_3 z_4}\}$,
$\mathcal{C}_3^* = \{\exists$ *a* $*$ *path starting and ending* outside $\overline{D}$ *whose restriction to D is as in* $\mathcal{C}_2^*\}$.

Then $\mu(\mathcal{C}_1^*) = \mu(\mathcal{C}_2^*) = \mu(\mathcal{C}_3^*) = \Phi_D(z_1, z_2; z_3, z_4)$.

Proof Recall that convergence to μ implies a coupling of the lattice and continuum processes on some $(\Omega', \mathcal{B}', \mathbb{P}')$ such that the distance between the set of $\mathcal{T}$-paths and the set of continuum paths tends to zero as $\delta_k \to 0$ for all $\omega' \in \Omega'$ (see, e.g., Corollary 1 of [6]).

We will construct for each small $\varepsilon > 0$, two domains with boundary points, denoted by $(\tilde{D}, \tilde{z}_1, \tilde{z}_2, \tilde{z}_3, \tilde{z}_4)$ and $(\hat{D}, \hat{z}_1, \hat{z}_2, \hat{z}_3, \hat{z}_4)$, approximating (D, z_1, z_2, z_3, z_4) in such a way that $\tilde{\Phi}_\varepsilon \equiv \Phi_{\tilde{D}}(\tilde{z}_1, \tilde{z}_2; \tilde{z}_3, \tilde{z}_4) \overset{\varepsilon \to 0}{\longrightarrow} \Phi \equiv \Phi_D(z_1, z_2; z_3, z_4)$ and the same for $\hat{\Phi}_\varepsilon$, and with the property that $\tilde{\Phi}_\varepsilon \le \mu(\mathcal{C}_i^*) \le \hat{\Phi}_\varepsilon$ for $i = 1, 2, 3$. This will yield the desired result. The construction of the approximating domains uses fairly straightforward conformal mapping arguments. We provide details for $\tilde{D}$; the construction of $\hat{D}$ is analogous.

To construct $\tilde{D}$ we will need continuous simple loops, $\underline{E}(D, \varepsilon)$ and $\overline{E}(D, \varepsilon)$, that are inner and outer approximations to ∂D in the sense that $\underline{E}$ is surrounded by ∂D which is surrounded by $\overline{E}$ with

$$\mathrm{d}(\partial D, \underline{E}) \le \varepsilon, \quad \mathrm{d}(\partial D, \overline{E}) \le \varepsilon. \tag{16}$$

We will also need four simple curves $\{\partial_1, \partial_2, \partial_3, \partial_4\}$ in the interior of the (topological) annulus between $\underline{E}$ and $\overline{E}$ and connecting their endpoints $\{(\underline{z}_1, \overline{z}_1), (\underline{z}_2, \overline{z}_2), (\underline{z}_3, \overline{z}_3), (\underline{z}_4, \overline{z}_4)\}$ on $\underline{E}$ and $\overline{E}$ with each touching ∂D at exactly one point which is either in the interior of the counterclockwise segment $\overline{z_1 z_2}$ (for ∂_1 and ∂_2) or else the counterclockwise segment $\overline{z_3 z_4}$ (for ∂_3 and ∂_4). Furthermore each of these connecting curves is close to its corresponding point z_1, z_2, z_3, or z_4; i.e., $\mathrm{d}(\partial_1, z_1) \le \varepsilon$, etc. In the special case where D is a rectangle, the construction of $\overline{E}, \underline{E}, \tilde{D}$ (and $\hat{D}$) is easily done—see Fig. 7.

Returning to a general Jordan domain D, we will take $\tilde{z}_1 = \overline{z}_1$, $\tilde{z}_2 = \overline{z}_2$, $\tilde{z}_3 = \overline{z}_3$, and $\tilde{z}_4 = \overline{z}_4$ with $\partial\tilde{D}$ the concatenation of: ∂_1 from $\underline{z}_1$ to $\overline{z}_1$, the portion of $\overline{E}$ from $\overline{z}_1$ to $\overline{z}_2$ counterclockwise, ∂_2 from $\overline{z}_2$ to $\underline{z}_2$, the portion of $\underline{E}$ from

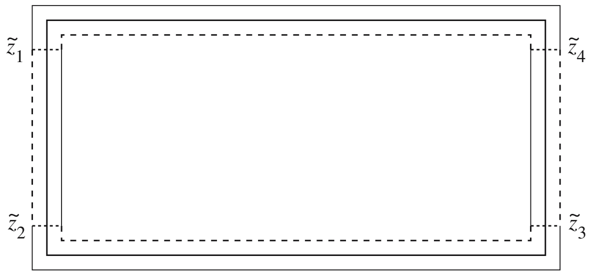


Fig. 7 Here, the middle rectangle is the domain D while the boundaries of the two rectangles outside and inside D are $\overline{E}$ and $\underline{E}$, respectively. $\tilde{D}$ is the domain with a dashed boundary

$\underline{z}_2$ to $\underline{z}_3$ counterclockwise, ∂_3 from $\underline{z}_3$ to $\overline{z}_3$, the portion of $\overline{E}$ from $\overline{z}_3$ to $\overline{z}_4$ counterclockwise, ∂_4 from $\overline{z}_4$ to $\underline{z}_4$, and the portion of $\underline{E}$ from $\underline{z}_4$ to $\underline{z}_1$ counterclockwise. It is important that (for fixed ε and $\tilde{D}$) there is a strictly positive minimal distance between $\underline{E} \cup \overline{E}$ and ∂D, and between $\partial \tilde{D}$ and the union of the two counterclockwise segments $\overline{z_2 z_3}$ and $\overline{z_4 z_1}$ of ∂D (see Fig. 7). These features will guarantee that for fixed ε, once k is large enough, a δ_k-lattice crossing within $\tilde{D}$ that corresponds to the crossing event whose (limiting) probability is $\Phi_{\tilde{D}}(\tilde{z}_1, \tilde{z}_2; \tilde{z}_3, \tilde{z}_4)$ must have a subsegment that satisfies the conditions defining $\mathcal{C}_i^*$ (in D).

We construct the parts of $\partial \tilde{D}$ that are inside and outside ∂D separately and then paste them together (with some care to make sure that they "match up"). Let ϕ be the conformal map from $\mathbb{D}$ onto D with $\phi(0) = 0$ and $\phi'(0) > 0$, and consider the image $\phi(\partial \mathbb{D}_{1-\varepsilon'})$ of the circle $\partial \mathbb{D}_{1-\varepsilon'} = \{z : |z| = 1 - \varepsilon'\}$ under ϕ and the inverse images $z_1^*, z_2^*, z_3^*, z_4^*$ under ϕ^{-1} of z_1, z_2, z_3, z_4. Let $\partial_1^*(\varepsilon', \varepsilon_1)$ be the straight line between $\mathrm{e}^{-i\varepsilon_1} z_1$ on the unit circle $\partial \mathbb{D}$, and $(1-\varepsilon')\mathrm{e}^{-i\varepsilon_1} z_1$ on the circle $\partial \mathbb{D}_{1-\varepsilon'}$, and define $\partial_2^*, \partial_3^*$, and ∂_4^* similarly, but using *clockwise* rotations by $e^{+i\varepsilon_2}$ and $e^{+i\varepsilon_4}$ for z_2 and z_4 (see Fig. 8). $\phi(\partial \mathbb{D}_{1-\varepsilon'})$ is a candidate for $\underline{E}(D, \varepsilon)$ and $\phi(\partial_\sharp^*(\varepsilon', \varepsilon_\sharp))$ is a candidate for half of $\partial_\sharp$ (where $\sharp = 1$ or 2 or 3 or 4), so we must choose ε' and the $\varepsilon_\sharp$'s small enough so that $\mathrm{d}(\partial D, \phi(\mathbb{D}_{1-\varepsilon'})) \le \varepsilon$, $\mathrm{d}(\phi(\partial_1^*(\varepsilon', \varepsilon_1)), z_1) \le \varepsilon$, etc. We do an analogous construction using a conformal mapping from the exterior of $\mathbb{D}$ onto the exterior of D, to obtain candidates for $\overline{E}(D, \varepsilon)$ and for the exterior half of the $\partial_\sharp$'s. Finally we use the freedom to choose the exterior values for ε' and the $\varepsilon_\sharp$'s differently from the interior ones to make sure that the interior and exterior halves of the $\varepsilon_\sharp$'s match up.

It should be clear that for a given approximation $\partial \tilde{D}$ of ∂D constructed as described above there is a strictly positive $\tilde{\varepsilon}$ such that the distance between ∂D and the portions of $\partial \tilde{D}$ that belong to $\underline{E}$ and $\overline{E}$ is not smaller than $\tilde{\varepsilon}$, and the distance between the union $\overline{z_2 z_3} \cup \overline{z_4 z_1}$ of two counterclockwise segments of ∂D and $\partial_1 \cup \partial_2 \cup \partial_3 \cup \partial_4$ is also not smaller than $\tilde{\varepsilon}$. This implies (see the definition of blue crossing just before Theorem 1) that for k large enough, any (lattice) blue path crossing inside $\tilde{D}$ from the counterclockwise segment $\overline{\overline{z}_1 \overline{z}_2}$ of $\partial \tilde{D}$ to the counterclockwise segment $\overline{\overline{z}_3 \overline{z}_4}$ of $\partial \tilde{D}$ (and thanks to the coupling, any

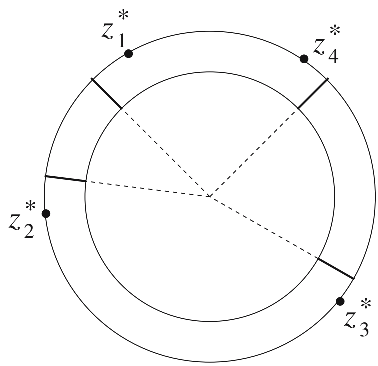


Fig. 8 The figure shows $\partial_1^*, \partial_2^*, \partial_3^*, \partial_4^*$ represented as heavy segments between the unit circle and the circle of radius $1 - \varepsilon'$ near $z_1^*, z_2^*, z_3^*, z_4^*$

limiting continuum counterpart) must (with high probability) have a subpath that satisfies the conditions of $\mathcal{C}_i^*$. Thus, for $i = 1, 2, 3$,

$$\mu(\mathcal{C}_i^*) \geq \lim_{k\to\infty} \tilde{\Phi}^{\delta_k}_{\tilde{D}(\varepsilon)} = \tilde{\Phi}_\varepsilon \tag{17}$$

as desired (the equality uses Theorem 1 for the Jordan domain $\tilde{D}(\varepsilon)$).

We now note that as $\varepsilon \to 0$, $(\tilde{D}, \tilde{z}_1, \tilde{z}_2, \tilde{z}_3, \tilde{z}_4) \to (D, z_1, z_2, z_3, z_4)$. This allows us to use the continuity of Cardy's formula (Lemma A.2 in Appendix A) to obtain

$$\lim_{\varepsilon\to 0} \tilde{\Phi}_\varepsilon = \Phi. \tag{18}$$

From this and (17) it follows that for $i = 1, 2, 3$,

$$\mu(\mathcal{C}_i^*) \geq \Phi. \tag{19}$$

The remaining part of the proof involves defining a domain $\hat{D}$ analogous to $\tilde{D}$ (i.e., such that $\lim_{\varepsilon\to 0}\lim_{k\to\infty} \hat{\Phi}^{\delta_k}_{\hat{D}(\varepsilon)} = \Phi$) but with the property that any blue δ_k-lattice path crossing inside D from the counterclockwise segment $\overline{z_1 z_2}$ to the counterclockwise segment $\overline{z_3 z_4}$ of ∂D must have a subpath in $\hat{D}$ that crosses between the counterclockwise segment $\overline{\hat{z}_1\hat{z}_2}$ and the counterclockwise segment $\overline{\hat{z}_3\hat{z}_4}$ of $\partial\hat{D}$. (The details of the construction of $\hat{D}$ are analogous to those of $\tilde{D}$; we leave them to the reader.) Then, for $i = 1, 2, 3$,

$$\mu(\mathcal{C}_i^*) \leq \Phi, \tag{20}$$

which, combined with (19), implies $\mu(\mathcal{C}_i^*) = \Phi$ and concludes the proof. □

Lemma 7.4 *For* $(\hat{D}, a, c, d)$ *as in Lemma 7.1,* $v \in J \equiv \overline{cd}$, *and* $\varepsilon > 0$, *we define* $U^{yellow}(\hat{D}, \varepsilon, v)$, *the yellow "mushroom" event* (*at* v), *to be the event that there is a yellow path in the closure of* $\hat{D}$ *from* v *to* $\partial B(v, \varepsilon)$ *and a blue path in the closure of* $\hat{D}$, *between some pair of distinct points* v_1, v_2 *in* $\partial\hat{D} \cap \{B(v, \varepsilon/3) \setminus B(v, \varepsilon/8)\}$, *that passes through* v *and such that this blue path is* between $\partial\hat{D}$ *and the yellow path* (*see Fig. 9*). *We similarly define* $U^{blue}(\hat{D}, \varepsilon, v)$ *with the colors interchanged and* $U^*(\hat{D}, \varepsilon, J) = \cup_{v\in J} U^*(\hat{D}, \varepsilon, v)$ *where* $*$ *denotes blue or yellow. Then for any deterministic domain* $\hat{D}$ *and any* $0 < \varepsilon < \min\{|a-c|, |a-d|\}$, $\mu(U^*(\hat{D}, \varepsilon, J)) = 0$.

Proof If $\mu(U^*(\hat{D}, \varepsilon, J)) > 0$ for some $\varepsilon > 0$, then there is some segment $\overline{z_1 z_2} \subset J$ of $\partial\hat{D}$ of diameter not larger than $\varepsilon/10$ such that

$$\mu\left(\cup_{v\in\overline{z_1 z_2}} U^*(\hat{D}, \varepsilon, v)\right) > 0. \tag{21}$$

Choose any point $v_0 \in \overline{z_1 z_2}$ and consider the new domain D' whose boundary consists of the correctly chosen (as we explain below) segment of the circle

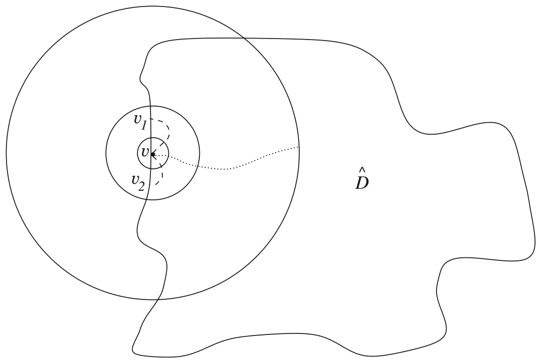


Fig. 9 A yellow "mushroom" event. The *dashed path is blue* and the *dotted path is yellow*. The three *circles* centered at v in the figure have radii $\varepsilon/8$, $\varepsilon/3$, and ε, respectively

$\partial B(v_0, \varepsilon/2)$ between the two points z_3, z_4 where $\partial\hat{D}$ first hits $\partial B(v_0, \varepsilon/2)$ on either side of v_0, together with the segment from z_4 to z_3 of $\partial\hat{D}$ (see Fig. 10).

The correct circle segment between z_3 and z_4 is the (counter)clockwise one if v_0 comes after z_4 and before z_3 along $\partial\hat{D}$ when $\partial\hat{D}$ is oriented (counter)clockwise. It is also not hard to see that since $\varepsilon < \min\{|a-c|, |a-d|\}$, D' is a Jordan domain, so that Lemma 7.3 can be applied. In the new domain D', $\overline{z_1 z_2}$ is the same curve segment as it was in the old domain $\hat{D}$, but $\overline{z_3 z_4}$ is now a segment of the circle $\partial B(v_0, \varepsilon/2)$. It should be clear that

$$\bigcup_{v \in \overline{z_1 z_2}} U^*(\hat{D}, \varepsilon, v) \subset \mathcal{C}_1^*(D', z_1, z_2, z_3, z_4) \setminus \mathcal{C}_3^*(D', z_1, z_2, z_3, z_4) \qquad (22)$$

which yields a contradiction of Lemma 7.3 if $\mu(U^*(\hat{D}, \varepsilon, J)) > 0$. □

Proof of Lemma 7.1. Let us first consider the simpler case of $\mathcal{A}_k(v; \varepsilon, \varepsilon')$ in which $v \in J'_k$. For $\varepsilon > 0$, let $J'_k(\varepsilon) \equiv J'_k \setminus \{B(c'_k, \varepsilon) \cup B(d'_k, \varepsilon)\}$ and $\hat{D}_k(\varepsilon) \equiv \hat{D}_k \setminus \{B(c'_k, \varepsilon) \cup B(d'_k, \varepsilon)\}$. With this notation, we have

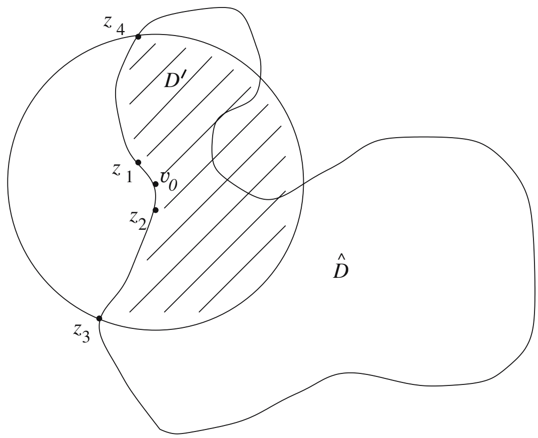


Fig. 10 Construction of the domain D' (shaded) used in the proof of Lemma 7.4

$$\mathbb{P}\left(\bigcup_{v\in J'_k} \mathcal{A}_k(v;\varepsilon,\varepsilon')\right) \le \mathbb{P}\left(\bigcup_{v\in J'_k(\varepsilon)} \mathcal{A}_k(v;\varepsilon,\varepsilon')\right) + F(\varepsilon;\hat{D}_k,c'_k,d'_k), \tag{23}$$

where $F(\varepsilon;\hat{D}_k,c'_k,d'_k)$ is the probability that $\gamma_k^{\delta_k}$ enters $B(c'_k,\varepsilon)$ or $B(d'_k,\varepsilon)$ before touching J'_k. $F(\varepsilon;\hat{D}_k,c'_k,d'_k)$ can be expressed as the sum of two crossing probabilities in $\hat{D}_k(\varepsilon)$: (1) the probability of a blue crossing from the (portion of the) counterclockwise arc $\overline{ac}$ of $\partial\hat{D}_k(\varepsilon)$ to the "first exposed" arc of $B(d',\varepsilon)$ contained in $\hat{D}_k(\varepsilon)$ and (2) the analogous probability of a yellow crossing from the (portion of the) counterclockwise arc $\overline{da}$ of $\partial\hat{D}_k(\varepsilon)$ to the "first exposed" arc of $B(c',\varepsilon)$ contained in $\hat{D}_k(\varepsilon)$. Since crossing probabilities converge to Cardy's formula, we easily conclude that

$$\lim_{\varepsilon\to 0}\limsup_{k\to\infty} F((\varepsilon;\hat{D}_k,c'_k,d'_k)) = 0. \tag{24}$$

Noting that the probability in the left hand side of (23) is nonincreasing in ε, we see that in order to obtain

$$\lim_{\varepsilon'\to 0}\limsup_{k\to\infty}\mathbb{P}\left(\bigcup_{v\in J'_k} \mathcal{A}_k(v;\varepsilon,\varepsilon')\right) = 0, \tag{25}$$

it is enough to show that the limit as $\varepsilon\to 0$ of the left hand side of (25) is zero. Therefore, thanks to (23) and (24), it suffices to prove that for every $\varepsilon > 0$

$$\lim_{\varepsilon'\to 0}\limsup_{k\to\infty}\mathbb{P}\left(\bigcup_{v\in J'_k(\varepsilon)} \mathcal{A}_k(v;\varepsilon,\varepsilon')\right) = 0. \tag{26}$$

To do so, we follow the exploration process until time T, when it first touches $\partial B(v,\varepsilon')$ for some $v\in J'_k(\varepsilon)$, and consider the annulus $B(v,\varepsilon)\setminus B(v,\varepsilon')$. Let π_Y be the leftmost yellow $\mathcal{T}$-path and π_B the rightmost blue $\mathcal{T}$-path in $\Gamma(\gamma_k^{\delta_k})$ at time T that cross $B(v,\varepsilon)\setminus B(v,\varepsilon')$. π_Y and π_B split the annulus $B(v,\varepsilon)\setminus B(v,\varepsilon')$ into three sectors that, for simplicity, we will call the central sector, containing the crossing segment of the exploration path, the yellow (left) sector, with π_Y as part of its boundary, and the blue (right) sector, the remaining one, with π_B as part of its boundary.

We then look for a yellow "lateral" crossing within the yellow sector from π_Y to $\partial\hat{D}_k$ and a blue lateral crossing within the blue sector from π_B to $\partial\hat{D}_k$. Notice that the yellow sector may contain "excursions" of the exploration path coming off $\partial B(v,\varepsilon)$, producing nested yellow and blue excursions off $\partial B(v,\varepsilon)$, and the same for the blue sector. But for topological reasons, those excursions are such that for every group of nested excursions, the outermost one is always yellow in the yellow sector and blue in the blue sector. Therefore, by standard percolation theory arguments, the conditional probability (conditioned on $\Gamma(\gamma_k^{\delta_k})$ at time T) to find a yellow lateral crossing of the yellow sector from π_Y to $\partial\hat{D}_k$ is bounded below by the probability to find a yellow circuit in an annulus with inner radius ε' and outer radius ε. An analogous statement holds for the conditional probability (conditioned on $\Gamma(\gamma_k^{\delta_k})$ at time T and on the entire percolation configuration in the yellow sector) to find a blue lateral crossing of the blue sector from π_B to $\partial\hat{D}_k$. Thus for any fixed $\varepsilon > 0$, by the Russo–Seymour–Welsh lemma [26,30], the conditional probability to find both a yellow lateral crossing within the yellow sector from π_Y to $\partial\hat{D}_k$ and a blue lateral crossing within the blue sector from π_B to $\partial\hat{D}_k$ goes to one as $\varepsilon' \to 0$.

But if such yellow and blue crossings are present, the exploration path is forced to touch J'_k before exiting $B(v,\varepsilon)$, and if that happens, the exploration process is stopped, so that it will never exit $B(v,\varepsilon)$ and the union over $v \in J'_k(\varepsilon)$ of $\mathcal{A}_k(v;\varepsilon,\varepsilon')$ cannot occur. This concludes the proof of this case.

Let us now consider the remaining case in which $v \notin J'_k$. The basic idea of the proof is then that by straightforward weak convergence and related coupling arguments, the failure of (14) would imply that *some* subsequence limit μ would satisfy $\mu(U^{\text{yellow}}(\hat{D},\varepsilon,J) \cup U^{\text{blue}}(\hat{D},\varepsilon,J)) > 0$, which would contradict Lemma 7.4. This is essentially because the close approach of an exploration path on the δ_k-lattice to $J_k \setminus J'_k$ without quickly touching nearby yields one two-sided colored $\mathcal{T}$-path (the "perimeter" of the portion of the hull of the exploration path seen from a boundary point of close approach) and a one-sided $\mathcal{T}$-path of the other color belonging to the percolation cluster not seen from the boundary point (i.e., shielded by the two-sided path). Both the two-sided path and the one-sided one are subsets of $\Gamma(\gamma_k^{\delta_k})$.

We first note that since the probability in (14) is nonincreasing in ε, we may assume that $\varepsilon < \min\{|a-c|,|a-d|\}$, as requested by Lemma 7.4. Assume by contradiction that (14) is false, so that close encounters without touching happen with bounded away from zero probability. Consider for concreteness an exploration path $\gamma_k^{\delta_k}$ that has a close approach to a point v in the counterclockwise arc $\overline{d'_k d_k}$. The exploration path may have multiple close approaches to v with differing colors of the perimeter as seen from v, but for topological reasons, the last time the exploration path comes close to v, it must do so in such a way as to produce a yellow $\mathcal{T}$-path π_Y (seen from v) that crosses $B(v,\varepsilon) \setminus B(v,\varepsilon')$ twice, and a blue path π_B that crosses it once (see Fig. 11). This is so because the exploration process that produced $\gamma_k^{\delta_k}$ ended somewhere on J'_k (and outside $B(v,\varepsilon)$), which is to the right of (i.e., clockwise to) v.

The portion of π_Y inside $B(v,\varepsilon) \setminus B(v,\varepsilon')$ contains at least two yellow $\mathcal{T}$-paths crossing the annulus. We denote by π_L the leftmost (looking at v from inside $\hat{D}_k$) such path and denote by π_R the rightmost one. The paths π_L and π_R split $B(v,\varepsilon) \setminus B(v,\varepsilon')$ into three sectors, that we will call the central sector, containing π_{B}, the left sector, with π_{L} as part of its boundary, and the right sector, with π_{R} as part of its boundary. Again for topological reasons, all other monochromatic

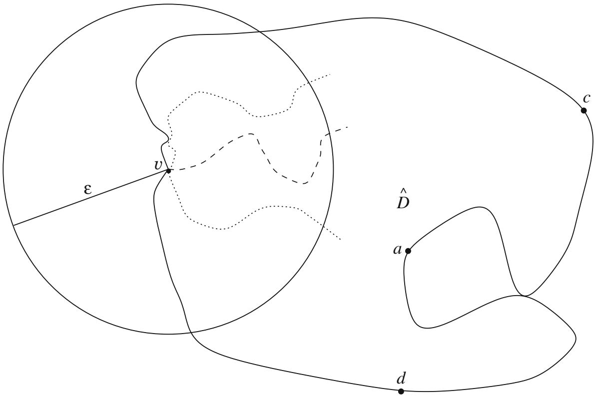


Fig. 11 The event consisting of a *yellow double crossing* and a *blue crossing* used as a first step for obtaining a blue mushroom event in the proof of Lemma 7.1. The *dashed crossing is blue* and the *dotted crossing is yellow*

crossings of the annulus associated with $\gamma_k^{\delta_k}$ are contained in the central sector, including at least one blue path π_{B}. As in the previous case, the left and right sectors can contain nested monochromatic excursions off $\partial B(v, \varepsilon)$ [and in this case also excursions off $\partial B(v, \varepsilon')$], but this time for every group of excursions, the outermost one is yellow in both sectors.

Now consider the annulus $B(v, \varepsilon/3) \setminus B(v, \varepsilon/8)$. We look for a yellow lateral crossing within the left sector from π_L to $\partial \hat{D}_k$ and a yellow lateral crossing within the right sector from π_R to $\partial \hat{D}_k$. Since the outermost excursions in both sectors are yellow, the conditional probability to find a yellow lateral crossing within the left sector from π_L to $\partial \hat{D}_k$ is bounded below by the probability to find a yellow circuit in an annulus with inner radius $\varepsilon/8$ and outer radius $\varepsilon/3$, and an analogous statement holds for the conditional probability to find a yellow lateral crossing within the right sector from π_R to $\partial \hat{D}_k$. Thus for any fixed $\varepsilon > 0$, by an application of the Russo–Seymour–Welsh lemma [26,30], the conditional probability to find both yellow lateral crossings remains bounded away from zero as $k \to \infty$ and $\varepsilon' \to 0$. But the presence of such yellow crossings would produce a (blue) mushroom event, leading to a contradiction with Lemma 7.4. □

Proof of Lemma 7.2. First of all, notice that

$$\bigcup_{v \in J_k \setminus J'_k} \mathcal{B}_k(v; \varepsilon, \varepsilon') \subset \bigcup_{v \in J_k \setminus J'_k} \mathcal{C}_k(v; \varepsilon, \varepsilon') \subset \bigcup_{v \in J_k \setminus J'_k} \mathcal{A}_k(v; \varepsilon, \varepsilon'). \tag{27}$$

By lemma 7.1, we only need consider events in $\bigcup_{v \in J_k \setminus J'_k} \{\mathcal{C}_k(v; \varepsilon, \varepsilon') \setminus \mathcal{B}_k(v; \varepsilon, \varepsilon')\}$, since

$$\lim_{\varepsilon' \to 0} \limsup_{k \to \infty} \mathbb{P}\left(\bigcup_{v \in J_k \setminus J'_k} \mathcal{B}_k(v; \varepsilon, \varepsilon') \right) = 0. \tag{28}$$

These are events such that $\gamma_k^{\delta_k}$ touches $\partial\hat{D}_k$ inside $B(v,\varepsilon)$, but it also contains a segment that stays within $B(v,\varepsilon)$, does not touch $\partial\hat{D}_k$ (or any segment that touches $\partial\hat{D}_k$ inside $B(v,\varepsilon)$), and has a double crossing of the annulus $B(v,\varepsilon)\setminus B(v,\varepsilon')$.

Let us assume by contradiction that, for some fixed $\hat{\varepsilon}>0$,

$$\limsup_{\varepsilon'\to 0}\limsup_{k\to\infty}\mathbb{P}\left(\bigcup_{v\in J_k\setminus J'_k}\mathcal{C}_k(v;\hat{\varepsilon},\varepsilon')\right)>0. \tag{29}$$

This implies (by using a coupling argument) that we can find a subsequence $\{n\}$ of $\{k\}$ and sequences $\delta_n\to 0$ and $\varepsilon_n\to 0$ such that (with strictly positive probability), as $n\to\infty$, $\gamma_n^{\delta_n}$ converges to a curve $\tilde{\gamma}$ that, for some $\bar{v}\in\partial\hat{D}$ (in fact, in $\overline{J\setminus J'}$), contains a segment that stays in $B(\bar{v},\hat{\varepsilon})$, touches $\partial\hat{D}$ at $\bar{v}$, and makes a double crossing of $B(\bar{v},\hat{\varepsilon})\setminus\{\bar{v}\}$. Since the events that we are considering are in $\bigcup_{v\in J_n\setminus J'_n}\{\mathcal{A}_n(v;\hat{\varepsilon},\varepsilon_n)\setminus\mathcal{B}_n(v;\hat{\varepsilon},\varepsilon_n)\}$, before the limit $n\to\infty$ is taken, $\gamma_n^{\delta_n}$ has a segment that stays in $B(\bar{v},\hat{\varepsilon})$, does not touch $\partial\hat{D}_n$ and makes a double crossing of the annulus $B(\bar{v}_n,\hat{\varepsilon})\setminus B(\bar{v}_n,\varepsilon_n)$ for some $\bar{v}_n\in\partial\hat{D}_n$ converging to $\bar{v}$. Moreover, $\gamma_n^{\delta_n}$ must touch $\partial\hat{D}_n$ inside $B(\bar{v}_n,\hat{\varepsilon})$.

Consider first whether there exist $0<\tilde{\varepsilon}<\hat{\varepsilon}$ and $n_0<\infty$ such that the point v_n, closest to $\bar{v}_n$, where $\gamma_n^{\delta_n}$ touches $\partial\hat{D}_n$ inside $B(\bar{v}_n,\hat{\varepsilon})$ is outside $B(\bar{v},\tilde{\varepsilon})$ for all $n\geq n_0$. If this occurred with strictly positive probability, it would imply that for all $\varepsilon<\tilde{\varepsilon}$,

$$\limsup_{n\to\infty}\mathbb{P}\left(\bigcup_{v\in J_n\setminus J'_n}\mathcal{B}_n(v;\varepsilon,\varepsilon_n)\right)>0, \tag{30}$$

which would contradict (14).

This leaves the case in which the point v_n, closest to $\bar{v}_n$, where $\gamma_n^{\delta_n}$ touches $\partial\hat{D}_n$ inside $B(\bar{v}_n,\hat{\varepsilon})$ converges to $\bar{v}$ (with strictly positive probability). Our assumption (29) implies that, as $n\to\infty$, the segment of $\gamma_n^{\delta_n}$ that stays in $B(\bar{v}_n,\hat{\varepsilon})$ and gets to within distance ε_n of $\partial\hat{D}_n$ without touching it, also does not touch the segment of $\gamma_n^{\delta_n}$ contained in $B(\bar{v}_n,\hat{\varepsilon})$ that touches $\partial\hat{D}_n$ at v_n. In that case, one can choose a subsequence $\{m\}$ of $\{n\}$ such that, for all m large enough, $|\bar{v}_m-v_m|\leq\varepsilon_m$ and there are five disjoint monochromatic crossings, not all of the same color, of the annulus $B(\bar{v}_m,\hat{\varepsilon}/2)\setminus B(\bar{v}_m,\varepsilon_m)$. Three crossings of alternating colors are associated with the segment of $\gamma_m^{\delta_m}$ contained in $B(\bar{v}_m,\hat{\varepsilon}/2)$ which does not touch $\partial\hat{D}_m$ and makes a double crossing of the annulus $B(\bar{v}_m,\hat{\varepsilon}/2)\setminus B(\bar{v}_m,\varepsilon_m)$, while two more crossings are associated with another segment which does touch $\partial\hat{D}_m$ and makes either a single or a double crossing of the annulus (see Figs. 12, 13, 14). This assures that $\gamma_m^{\delta_m}$ can have yet another segment that crosses the annulus $B(\bar{v}_m,\hat{\varepsilon}/2)\setminus B(\bar{v}_m,\varepsilon_m)$ (which would add at least one more disjoint monochromatic crossing to the existing five) for at most finitely many m's, since otherwise the standard six-arm bound proved in [17] would be violated.

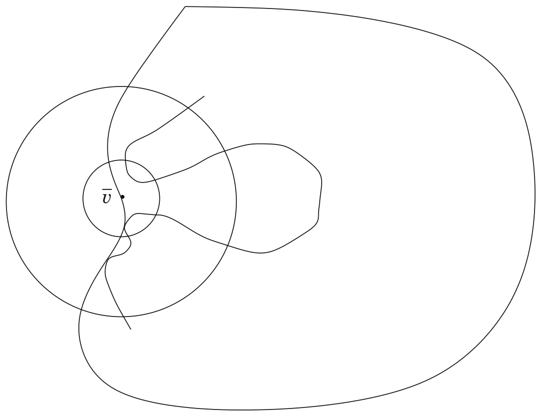


Fig. 12 One way in which a path inside a domain can have two double crossings of an annulus centered at $\bar{v}$, one which does not touch the boundary of the domain and one which does touch the boundary

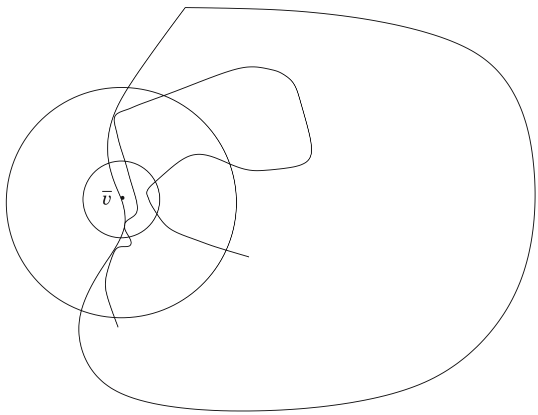


Fig. 13 The other (topologically distinct) way in which a path inside a domain can have two double crossings of an annulus centered at $\bar{v}$, one which does not touch the boundary of the domain and one which does touch the boundary

Therefore, we can choose $0 < \bar{\varepsilon} < \hat{\varepsilon}/2$ such that for all m large enough, $\gamma_m^{\delta_m}$ has at most two segments that stay in $B(\bar{v}_m, \bar{\varepsilon})$ and cross $B(\bar{v}_m, \bar{\varepsilon}) \setminus B(\bar{v}_m, \varepsilon_m)$, one of which does not touch $\partial \hat{D}_m$ and makes a double crossing, while the other touches $\partial \hat{D}_m$.

Since $\{m\}$ is a subsequence of $\{k\}$, it follows from (29) that

$$\limsup_{m \to \infty} \mathbb{P}\left(\bigcup_{v \in J_m \setminus J'_m} \mathcal{C}_m(v; \bar{\varepsilon}, \varepsilon_m) \right) > 0. \tag{31}$$

When the event $\mathcal{C}_m(v; \bar{\varepsilon}, \varepsilon_m)$ happens, it follows from the above observations that $\gamma_m^{\delta_m}$ cannot touch $\partial \hat{D}_m$ inside $B(\bar{v}_m, \bar{\varepsilon})$ both before and after the segment that makes a double crossing of $B(\bar{v}_m, \bar{\varepsilon}) \setminus B(\bar{v}_m, \varepsilon_m)$ without touching $\partial \hat{D}_m$. We will assume that, following the exploration path $\gamma_m^{\delta_m}$ from a_m to the target region J'_m, with positive probability v_m is encountered after the segment that makes a double crossing of $B(\bar{v}_m, \bar{\varepsilon}) \setminus B(\bar{v}_m, \varepsilon_m)$ without touching $\partial \hat{D}_m$. To see

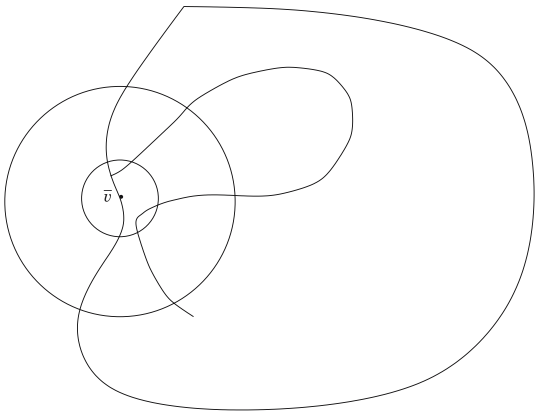


Fig. 14 A path in a domain making a double crossing of an annulus centered at $\bar{v}$, without touching the domain boundary, and a single crossing that ends on the boundary

that this assumption can be made without loss of generality, we note that we may consider the case where the target area J'_m is a single point b_m and then consider the "time-reversed" exploration path in $\hat{D}_m$ that starts at b_m and ends at a_m.

Now let $\gamma_m^{\delta_m}(t)$ be a parametrization of the exploration path $\gamma_m^{\delta_m}$ from a_m to J'_m and consider the stopping time T (where the dependence on m has been suppressed) defined as the first time such that there is a point v on J_m with the property that $\gamma_m^{\delta_m}$ up to time T has never touched $\partial\hat{D}_m \cap B(v,\bar{\varepsilon})$ and has a segment that makes a double crossing of $B(v,\bar{\varepsilon}) \setminus B(v,\varepsilon_m)$. It follows from (31) and the observation and assumption made right after it that such a T occurs with probability bounded away from zero as $m \to \infty$.

If this is the case, we can use the partial percolation configuration produced inside $B(v,\bar{\varepsilon}) \setminus B(v,\varepsilon_m)$ by the exploration process stopped at the stopping time T to construct a mushroom event with positive probability, as in the proof of Lemma 7.1. Notice, in fact, that the partial percolation configuration produced inside $B(v,\bar{\varepsilon}) \setminus B(v,\varepsilon_m)$ by the exploration process stopped at time T has exactly the same properties as the partial percolation configuration of the proof of Lemma 7.1 produced by the exploration process considered there inside the annulus $B(v,\varepsilon) \setminus B(v,\varepsilon')$—see the second part of the proof of Lemma 7.1. It follows that (31) implies positive probability of a mushroom event, contradicting Lemma 7.4. □

We are finally ready to prove the main result.

Theorem 5 *Let (D,a,b) be a Jordan domain with two distinct selected points on its boundary ∂D. Then, for Jordan sets D^δ from $\delta\mathcal{H}$ with two distinct selected e-vertices a^δ, b^δ on their boundaries ∂D^δ, such that $(D^\delta,a^\delta,b^\delta) \to (D,a,b)$ as $\delta \to 0$, the percolation exploration path $\gamma^\delta_{D,a,b}$ inside D^δ from a^δ to b^δ converges in distribution to the trace $\gamma_{D,a,b}$ of chordal SLE_6 inside D from a to b, as $\delta \to 0$.*

Proof It follows from [2] that $\gamma^\delta_{D,a,b}$ converges in distribution along subsequence limits $\delta_k \downarrow 0$. Since we have proved that the filling of any such subsequence limit

$\tilde{\gamma}$ satisfies the spatial Markov property (Theorem 4) and the hitting distribution of $\tilde{\gamma}$ is determined by Cardy's formula (Theorem 3), we can deduce from Theorem 2 that the limit is unique and that the law of $\gamma^{\delta}_{D,a,b}$ converges, as $\delta \to 0$, to the law of the trace $\gamma_{D,a,b}$ of chordal SLE_6 inside D from a to b. □

Acknowledgments We are grateful to Vincent Beffara, Greg Lawler, Oded Schramm, Yuri Suhov and Wendelin Werner for various interesting and useful conversations and to Stas Smirnov for communications about work in progress. We note that a discussion with Oded Schramm, about dependence of exploration paths with respect to small changes of domain boundaries, pointed us in a direction that eventually led to Lemmas 7.1–7.4. We are especially grateful to Vincent Beffara for pointing out a gap in a preliminary version of the proof of the main result. We thank Michael Aizenman, Oded Schramm, Vladas Sidoravicius, and Lai-Sang Young for comments about presentation, Alain-Sol Sznitman for his interest and encouragement, Steffen Rohde for extensive discussions about extensions of Corollary A.1, and an anonymous referee for useful comments. F. C. thanks Wendelin Werner for an invitation to Université Paris-Sud, and acknowledges the kind hospitality of the Courant Institute where part of this work was completed. C. M. N. acknowledges the kind hospitality of the Vrije Universiteit Amsterdam.

Appendix A: Sequences of conformal maps

In this appendix, we give some results about sequences of conformal maps. A standard reference with more details is [24]. There is a theorem attributed to both Courant [13] (see Theorem IX.14 of [35]) and to Radó [25] (see Theorem 2.11 of [24]) that provides conditions under which conformal maps ϕ_n from $\mathbb{D}$ onto Jordan domains G_n converge uniformly on all of $\overline{\mathbb{D}}$. One of the purposes of this appendix is to provide in Corollary A.1 an extension of the Courant-Radó theorem to admissible domains (which are not necessarily Jordan—see the definition in Sect. 4.1). Although this suffices here, it appears that there is a wider extension (S. Rohde, private communication) in which the domains G_n have boundaries given by continuous loops, without requiring admissibility.

To proceed, we need the next definition, in which a *continuum* denotes a compact connected set with more than one point.

Definition A.1 (Sect. 2.2 of [24]) *The closed set $A \subset \mathbb{C}$ is called locally connected if for every $\varepsilon > 0$ there is $\delta > 0$ such that, for any two points $a, b \in A$ with $|a - b| < \delta$, we can find a continuum B with diameter smaller than ε and with $a, b \in B \subset A$.*

We remark that every continuous curve (with more than one point) is a locally connected continuum (the converse is also true: every locally connected continuum is a curve). The concept of local connectedness gives a topological answer to the problem of continuous extension of a conformal map to the domain boundary, as follows.

Theorem 6 (Sect. 2.1 of [24]) *Let ϕ map the unit disk $\mathbb{D}$ conformally onto $G \subset \mathbb{C} \cup \{\infty\}$. Then ϕ has a continuous extension to $\overline{\mathbb{D}}$ if and only if ∂G is locally connected.*

When ϕ has a continuous extension to $\overline{\mathbb{D}}$, we do not distinguish between ϕ and its extension. This is the case for the conformal maps considered in this paper.

The problem of whether this extension is injective on $\overline{\mathbb{D}}$ has also a topological answer, as follows.

Theorem 7 (Sect. 2.1 of [24]) *In the notation of Theorem 6, the function ϕ has a continuous and injective extension if and only if ∂G is a Jordan curve.*

When considering sequences of domains whose boundaries are locally connected the following definition is useful.

Definition A.2 (Sect. 2.2 of [24]) *The closed sets $A_n \subset \mathbb{C}$ are uniformly locally connected if, for every $\varepsilon > 0$, there exists $\delta > 0$ independent of n such that any two points $a_n, b_n \in A_n$ with $|a_n - b_n| < \delta$ can be joined by continua $B_n \subset A_n$ of diameter smaller than ε.*

The convergence of domains used in this paper (i.e., $G_n \to G$ if $\partial G_n \to \partial G$ in the uniform metric (2) on continuous curves) allows us to use Carathéodory's kernel convergence theorem (Theorem 1.8 of [24]). However, we need uniform convergence in $\overline{\mathbb{D}}$. This is guaranteed by the Courant-Radó theorem in the case of Jordan domains; in the non-Jordan case, sufficient conditions to have uniform convergence are stated in the next theorem.

Theorem 8 (Corollary 2.4 of [24]) *Let $\{G_n\}$ be a sequence of bounded domains such that, for some $0 < r < R < \infty$, $B(0,r) \subset G_n \subset B(0,R)$ for all n and such that $\{\mathbb{C} \setminus G_n\}$ is uniformly locally connected. Let ϕ_n map $\mathbb{D}$ conformally onto G_n with $\phi_n(0) = 0$. If $\phi_n(z) \to \phi(z)$ as $n \to \infty$ for each $z \in \mathbb{D}$, then the convergence is uniform in $\overline{\mathbb{D}}$.*

To use Theorem 8 we need the following lemma. The definition of admissible and the related notion of convergence are given respectively in Sect. 4.1 and in Theorem 3.

Lemma A.1 *Let $\{(G_n, a_n, c_n, d_n)\}$ be a sequence of domains admissible with respect to (a_n, c_n, d_n) and assume that, as $n \to \infty$, $(G_n, a_n, c_n, d_n) \to (G, a, c, d)$, where G is a domain admissible with respect to (a, c, d). Then the sequence of closed sets $\{\mathbb{C} \setminus G_n\}$ is uniformly locally connected.*

Proof If the conclusion of the theorem is not valid, then for some $\varepsilon > 0$, there are indices k (actually n_k, but we abuse notation a bit) and points $u_k, v_k \in \mathbb{C} \setminus G_k$ with $|u_k - v_k| \to 0$ that cannot be joined by a continuum of diameter $\leq \varepsilon$ in $\mathbb{C} \setminus G_k$. We assume this and search for a contradiction. By compactness, we may also assume that $u_k \to u, v_k \to v$, with $u = v$. There is an easy contradiction (using a small disc around $u = v$ as the connecting continuum) unless $u = v$ is on ∂G, and so we also assume that. Further, by considering points on ∂G_k near to u_k, v_k, we can also assume that $u_k, v_k \in \partial G_k$.

Splitting ∂G into three Jordan arcs, $J_1 = \overline{da}, J_2 = \overline{ac}, J_3 = \overline{cd}$, and ∂G_k into the corresponding $J_{1,k}, J_{2,k}, J_{3,k}$, we note that there is an easy contradiction (using arcs along ∂G_k as the continua) if u_k and v_k both belong to $J_{1,k} \cup J_{3,k}$ or both belong to $J_{2,k} \cup J_{3,k}$ for all k large enough, since the concatenation of $J_{1,k}$ with $J_{3,k}$ or of $J_{2,k}$ with $J_{3,k}$ is a Jordan arc. The above reasoning does not apply if $u = v$ is on both J_1 and J_2. But when $u = v = a$, one can paste together small Jordan arcs on $J_{1,k}$ and $J_{2,k}$ to get a suitable continuum leading again to a contradiction. The sole remaining case is when, for all k large enough, u_k belongs to the interior of $J_{1,k}$ and v_k belongs to the interior of $J_{2,k}$.

(Notice that we are ignoring the "degenerate" case in which $c = d$ coincides with the "last" [from a] double-point on ∂G, and J_3 is a simple loop. In that case u_k and v_k could converge to $u = v = c = d \in J_1 \cap J_2$ and u_k or v_k could still belong to $J_{3,k}$ for arbitrarily large k's. However, in that case one can find two distinct points on J_3, c' and d', such that D is admissible with respect to (a, c', d'), and points c'_k and d'_k on $J_{3,k}$ converging to c' and d' respectively, and define accordingly new Jordan arcs, J'_1, J'_2, J'_3 and $J'_{1,k}, J'_{2,k}, J'_{3,k}$, so that $u_k \in J'_{1,k}$ and $v_k \in J'_{2,k}$ for k large enough. We assume that this has been done if necessary, and for simplicity of notation drop the primes.)

In this case let $[u_k v_k]$ denote the closed straight line segment in the plane between u_k and v_k. Imagine that $[u_k v_k]$ is oriented from u_k to v_k and let v'_k be the first point of $J_{2,k}$ intersected by $[u_k v_k]$ and u'_k be the previous intersection of $[u_k v_k]$ with ∂G_k. Clearly, $u'_k \notin J_{2,k}$. For k large enough, u'_k cannot belong to $J_{3,k}$ either, or otherwise in the limit $k \to \infty$, J_3 would touch the interior of J_1 and J_2. We deduce that for all k large enough, $u'_k \in J_{1,k}$. Since $J_{1,k}$ and $J_{2,k}$ are continuous curves and therefore locally connected, u_k and u'_k belong to a continuum $B_{1,k}$ contained in $J_{1,k}$ whose diameter goes to zero as $k \to \infty$, and the same for v_k and v'_k (with $B_{1,k}$ and $J_{1,k}$ replaced by $B_{2,k}$ and $J_{2,k}$).

Since the interior of $[u'_k v'_k]$ does not intersect any portion of ∂G_k, it is either contained in G_k or in its complement $\mathbb{C} \setminus G_k$. If $[u'_k v'_k] \subset \mathbb{C} \setminus G_k$, we have a contradiction since the union of $[u'_k v'_k]$ with $B_{1,k}$ and $B_{2,k}$ is contained in $\mathbb{C} \setminus G_k$ and is a continuum containing u_k and v_k whose diameter goes to zero as $k \to \infty$.

If the interior of $[u'_k v'_k]$ is contained in G_k, let us consider a conformal map ϕ_k from $\mathbb{D}$ onto G_k. Since ∂G_k is locally connected, the conformal map ϕ_k extends continuously to the boundary of the unit disc. Let $u'_k = \phi_k(u^*_k)$, $v'_k = \phi_k(v^*_k)$, $a_k = \phi_k(a^*_k)$, $c_k = \phi_k(c^*_k)$ and $d_k = \phi_k(d^*_k)$. The points $c^*_k, d^*_k, u^*_k, a^*_k, v^*_k$ are in counterclockwise order on $\partial\mathbb{D}$, so that any curve in $\mathbb{D}$ from a^*_k to the counterclockwise arc $\overline{c^*_k d^*_k}$ must cross the curve from u^*_k to v^*_k whose image under ϕ_k is $[u'_k v'_k]$. This implies that any curve in G_k going from a_k to the counterclockwise arc $\overline{c_k d_k}$ of ∂G_k must cross the (interior of the) line segment $[u'_k v'_k]$. Then, in the limit $k \to \infty$, any curve in G from a to the counterclockwise arc $\overline{cd}$ must contain the limit point $u = \lim_{k\to\infty} u'_k = \lim_{k\to\infty} v'_k = v$. On the other hand, except for its starting and ending point, any such curve is completely contained in G, which implies that either $u = v = a$ or else that (in the limit $k \to \infty$) the counterclockwise arc $\overline{cd}$ is the single point at $u = v = c = d$. We have already dealt with the former case. In the latter case, one can paste together small Jordan arcs from u'_k to d_k, from d_k to c_k, and from c_k to v'_k, and take the union with $B_{1,k}$ and $B_{2,k}$ (defined above) to get a suitable continuum in $\mathbb{C} \setminus G_k$ containing u_k and v_k, leading to a contradiction. This concludes the proof. $\square$

Theorem 8, together with Carathéodory's kernel convergence theorem (Theorem 1.8 of [24]) and Lemma A.1, implies the following result.

Corollary A.1 *With the notation and assumptions of Lemma A.1 (and also assuming that G_n and G contain the origin), let ϕ_n map $\mathbb{D}$ conformally onto G_n with $\phi_n(0) = 0$ and $\phi_n'(0) > 0$, and ϕ map $\mathbb{D}$ conformally onto G with $\phi(0) = 0$ and $\phi'(0) > 0$. Then, as $n \to \infty$, $\phi_n \to \phi$ uniformly in $\overline{\mathbb{D}}$.*

Proof As already remarked, the convergence of ∂G_n to ∂G in the uniform metric (2) on continuous curves [which is part of the definition of $(G_n, a_n, c_n, d_n) \to (G, a, c, d)$] easily implies that the conditions in Carathéodory's kernel theorem (Theorem 8.1 of [24]) are satisfied and therefore that ϕ_n converges to ϕ locally uniformly in $\mathbb{D}$, as $n \to \infty$. By an application of Lemma A.1, the sequence $\{\mathbb{C} \setminus D_n\}$ is uniformly locally connected, so that we can apply Theorem 8 to conclude that, as $n \to \infty$, ϕ_n converges to ϕ uniformly in $\overline{\mathbb{D}}$. □

The next result is a corollary of the previous one and is used in the proof of Theorem 4.

Corollary A.2 *Let $\{(G_n, a_n, c_n, d_n)\}$ be a sequence of domains admissible with respect to (a_n, c_n, d_n) with b_n in the interior of $\overline{c_n d_n}$, and assume that, as $n \to \infty$, $(G_n, a_n, c_n, d_n) \to (G, a, c, d)$ and $b_n \to b$, where G is a domain admissible with respect to (a, c, d) and b is in the interior of $\overline{cd}$. Let f be a conformal map from $\mathbb{H}$ to G such that $f^{-1}(a) = 0$ and $f^{-1}(b) = \infty$. Then, there exists a sequence $\{f_n\}$ of conformal maps from $\mathbb{H}$ to G_n with $f_n^{-1}(a_n) = 0$ and $f_n^{-1}(b_n) = \infty$ and such that f_n converges to f uniformly in $\overline{\mathbb{H}}$.*

Proof The conformal transformation $f(\cdot)$ can be written as $\phi \circ \psi(\lambda \cdot)$, where λ is a positive constant, ϕ is the unique conformal transformation that maps $\mathbb{D}$ onto G with $\phi(0) = 0$ and $\phi'(0) > 0$ (we are assuming for simplicity that G contains the origin; if that is not the case, one can use a translated domain that does contain the origin), and

$$\psi(z) = \mathrm{e}^{i\theta_0} \left(\frac{(z+1) - z_0}{(z+1) - \overline{z_0}} \right) \tag{32}$$

maps $\overline{\mathbb{H}}$ onto $\overline{\mathbb{D}}$. θ_0 is chosen so that $\mathrm{e}^{i\theta_0} = \phi^{-1}(b)$ and z_0 so that $|1 - z_0| = 1$, $\mathrm{Im}(z_0) > 0$ and $\phi^{-1}(b)(\frac{1-z_0}{1-\overline{z_0}}) = \phi^{-1}(a)$, which implies that f^{-1} indeed maps a to 0 and b to ∞.

We now take $f_n(\cdot)$ of the form $\phi_n \circ \psi_n(\lambda \cdot)$, where ϕ_n is the unique conformal transformation that maps $\mathbb{D}$ onto G_n with $\phi_n(0) = 0$ and $\phi_n'(0) > 0$ (the assumption that G contains the origin implies that G_n also contains the origin, for large n),and

$$\psi_n(z) = \mathrm{e}^{i\theta_n} \left(\frac{(z+1) - z_n}{(z+1) - \overline{z_n}} \right) \tag{33}$$

maps $\overline{\mathbb{H}}$ onto $\overline{\mathbb{D}}$. Note that the same λ is used as appeared in the expression $\phi \circ \psi(\lambda \cdot)$ for $f(\cdot)$. θ_n is chosen so that $\mathrm{e}^{i\theta_n} = \phi_n^{-1}(b_n)$ and z_n is chosen so that

$|1-z_n|=1$, $\mathrm{Im}(z_n)>0$ and $\phi_n^{-1}(b_n)(\frac{1-z_n}{1-\bar{z}_n})=\phi_n^{-1}(a_n)$, which implies that f_n^{-1} indeed maps a_n to 0 and b_n to ∞.

Corollary A.1 implies that, as $n\to\infty$, ϕ_n converges to ϕ uniformly in $\overline{\mathbb{D}}$. This, together with the convergence of a_n to a and b_n to b, implies that $\phi_n^{-1}(a_n)$ converges to $\phi^{-1}(a)$ and $\phi_n^{-1}(b_n)$ to $\phi^{-1}(b)$. Therefore, we also have the convergence of ψ_n to ψ uniformly in $\overline{\mathbb{H}}$, which implies that f_n converges to f uniformly in $\overline{\mathbb{H}}$. □

We conclude this appendix with a simple lemma, used in the proof of Theorem 3, about the continuity of Cardy's formula with respect to the shape of the domain and the positions of the four points on the boundary.

Lemma A.2 *For $\{(D_n,a_n,c_n,b_n,d_n)\}$ and (D,a,c,b,d) as in Theorem 3, let Φ_n denote Cardy's formula (see (6)) for a crossing inside D_n from the counterclockwise segment $\overline{a_nc_n}$ of ∂D_n to the counterclockwise segment $\overline{b_nd_n}$ of ∂D_n and Φ the corresponding Cardy's formula for the limiting domain D. Then, as $n\to\infty$, $\Phi_n\to\Phi$.*

Proof Let ϕ_n be the conformal map that takes $\mathbb{D}$ onto D_n with $\phi_n(0)=0$ and $\phi_n'(0)>0$, and let ϕ denote the conformal map from $\mathbb{D}$ onto D with $\phi(0)=0$ and $\phi'(0)>0$; let $z_1=\phi^{-1}(a)$, $z_2=\phi^{-1}(c)$, $z_3=\phi^{-1}(b)$, $z_4=\phi^{-1}(d)$, $z_1^n=\phi_n^{-1}(a_n)$, $z_2^n=\phi_n^{-1}(c_n)$, $z_3^n=\phi_n^{-1}(b_n)$, and $z_4^n=\phi_n^{-1}(d_n)$. We can apply Corollary A.1 to conclude that, as $n\to\infty$, ϕ_n converges to ϕ uniformly in $\overline{\mathbb{D}}$. This, in turn, implies that, as $n\to\infty$, $z_1^n\to z_1$, $z_2^n\to z_2$, $z_3^n\to z_3$ and $z_4^n\to z_4$.

Cardy's formula for a crossing inside D_n from the counterclockwise segment $\overline{a_nc_n}$ of ∂D_n to the counterclockwise segment $\overline{b_nd_n}$ of ∂D_n is given by

$$\Phi_n=\frac{\Gamma(2/3)}{\Gamma(4/3)\Gamma(1/3)}\eta_n^{1/3}{}_2F_1(1/3,2/3;4/3;\eta_n), \tag{34}$$

where

$$\eta_n=\frac{(z_1^n-z_2^n)(z_3^n-z_4^n)}{(z_1^n-z_3^n)(z_2^n-z_4^n)}. \tag{35}$$

Because of the continuity of η_n in z_1^n, z_2^n, z_3^n, z_4^n, and the continuity of Cardy's formula (34) in η_n, the convergence of $z_1^n\to z_1$, $z_2^n\to z_2$, $z_3^n\to z_3$ and $z_4^n\to z_4$ immediately implies the convergence of Φ_n to Φ. □

References

1. Aizenman, M.: Scaling limit for the incipient spanning clusters. In: Mathematics of Multiscale Materials; the IMA Volumes in Mathematics and its Applications. Golden, K., Grimmett, G., James, R., Milton, G., Sen, P.,(eds.) Springer, Berlin Heidelberg New York (1998)
2. Aizenman, M., Burchard, A.: Hölder regularity and dimension bounds for random curves. Duke Math. J. **99**, 419–453 (1999)
3. Aizenman, M., Duplantier, B., Aharony, A.: Connectivity exponents and the external perimeter in 2D independent percolation. Phys. Rev. Lett. **83**, 1359–1362 (1999)
4. Beffara, V.: Hausdorff dimensions for SLE_6. Ann. Probab. **32**, 2606–2629 (2004)

5. Beffara, V.: Cardy's formula on the triangular lattice, the easy way, available at http://www.umpa.ens-lyon.fr/~vbeffara/index.php (2005)
6. Billingsley, P.: Weak Convergence of Measures: Applications in Probability, Society for Industrial and Applied Mathematics, Philadelphia (1971)
7. Camia, F., Newman, C.M.: Continuum nonsimple loops and 2D critical percolation. J. Stat. Phys. **116**, 157–173 (2004)
8. Camia, F., Newman, C.M.: The full scaling limit of two-dimensional critical percolation, available at arXiv:math.PR/0504036 (2005), including a long appendix roughly corresponding to the present paper
9. Camia, F., Newman, C.M.: Two-dimensional critical percolation: the full scaling limit. Comm. Math. Phys. **268**, 1–38 (2006)
10. Camia, F., Newman, C.M.: SLE_6 and CLE_6 from critical percolation, submitted, preprint arXiv:math.PR/0611116 (2006)
11. Camia, F., Newman, C.M., Sidoravicius, V.: Cardy's formula for some dependent percolation models. Bull. Brazilian Math. Soc. **33**, 147–156 (2002)
12. Cardy, J.L.: Critical percolation in finite geometries. J. Phys. A **25**, L201–L206 (1992)
13. Courant, R.: Über eine Eigenschaft der Abbildungsfunktionen bei konformer Abbildung. Göttinger Nachr., pp.101–109 (1914) (Erratum: Göttinger Nachr., 69-70 (1922))
14. Dubédat, J.: Excursion decompositions for SLE and Watts' crossing formula. Probab. Theory Relat. Fields **134**, 453–488 (2006)
15. Kager, W., Nienhuis, B.: A guide to Stochastic Löwner Evolution and its applications. J. Phys. A **115**, 1149–1229 (2004)
16. Kesten, H.: Scaling relations for 2D-percolation. Comm. Math. Phys. **105**, 109–156 (1987)
17. Kesten, H., Sidoravicius, V., Zhang, Y.: Almost all words are seen in critical site percolation on the triangular lattice. Electr. J. Probab. **3**, paper no. 10 (1998)
18. Lawler, G.: Conformally invariant processes. In: Lecture notes for the 2002 ICTP School and Conference on Probability, ICTP Lecture Notes Series, vol. XVII (2004)
19. Lawler, G.F.:Conformally Invariant Processes in the Plane, Mathematical Surveys and Monographs, vol. 114. American Mathematical Society, RI (2005)
20. Lawler, G., Schramm, O., Werner, W.: Values of Brownian intersection exponents I: Half-plane exponents. Acta Math. **187**, 237–273 (2001)
21. Lawler, G., Schramm, O., Werner, W.: One arm exponent for critical 2D percolation. Electron. J. Probab. **7**(2) (2002)
22. Lawler, G., Schramm, O., Werner, W.: Conformal invariance of planar loop-erased random walk and uniform spanning trees. Ann. Probab. **32**, 939–995 (2004)
23. Lawler, G., Schramm, O., Werner, W.: Conformal restriction: the chordal case. J. Amer. Math. Soc. **16**, 917–955 (2003)
24. Pommerenke, Ch.: Boundary Behaviour of Conformal Maps. Springer, Berlin, Heidelberg, New York (1992)
25. Radó, T.: Sur la représentation conforme de domaines variables. Acta Sci. Math. (Szeged) **1**, 180–186 (1923)
26. Russo, L.: A note on percolation. Z. Wahrsch. Ver. Geb. **43**, 39–48 (1978)
27. Schramm, O.: Scaling limits of loop-erased random walks and uniform spanning trees. Israel J. Math. **118**, 221–288 (2000)
28. Schramm, O.: A percolation formula. Elect. Comm. Probab. **6**, 115–120 (2001)
29. Schramm, O., Sheffield, S.: The harmonic explorer and its convergence to SLE(4). Ann. Probab. **33**, 2127-2148 (2005)
30. Seymour, P.D., Welsh, D.J.A.: Percolation probabilities on the square lattice. In: Bollobás, B.(ed.) Advances in Graph Theory , Annals of Discrete Mathematics vol. 3, pp. 227–245, North-Holland, Amsterdam (1978)
31. Smirnov, S.: Critical percolation in the plane: Conformal invariance, Cardy's formula, scaling limits. C.R.Acad. Sci.Paris **333**, 239–244 (2001)
32. Smirnov, S.:Critical percolation in the plane. I. Conformal invariance and Cardy's formula. II. Continuum scaling limit. (long version of [31], dated Nov. 15, 2001), available at http://www.math.kth.se/~stas/papers/index.html.
33. Smirnov, S.:Towards conformal invariance of 2*D* lattice models. In: Proceedings Madrid ICM, Europ. Math. Soc. (2006)

34. Smirnov, S., Werner, W.: Critical exponents for two-dimensional percolation. Math. Rev. Lett. **8**, 729–744 (2001)
35. Tsuji, M.: Potential Theory in Modern Function Theory. Maruzen, Tokyo (1959)
36. Werner, W.:Critical exponents, conformal invariance and planar Brownian motion. In: Europ. Congress of Math., vol. II. Prog. Math. **202**, 87–103 (2001)
37. Werner, W.: Random planar curves and Schramm–Loewner Evolutions. In: Lectures on Probability Theory and Statistics, Lecture Notes in Math., vol. 1840, pp. 107-195, Springer, Berlin, Heidelberg, NewYork (2004)