Probab. Theory Relat. Fields (2007) 139:41–64
DOI 10.1007/s00440-006-0038-x

Simple transient random walks in one-dimensional random environment: the central limit theorem

Ilya Ya. Goldsheid

Received: 31 May 2006 / Revised: 27 September 2006 / Published online: 22 November 2006

Abstract We consider a simple random walk (dimension one, nearest neighbour jumps) in a quenched random environment. The goal of this work is to provide sufficient conditions, stated in terms of properties of the environment, under which the central limit theorem (CLT) holds for the position of the walk. Verifying these conditions leads to a complete solution of the problem in the case of independent identically distributed environments as well as in the case of uniformly ergodic (and thus also weakly mixing) environments.

Keywords RWRE · Simple random walks · Quenched random environments · Central limit theorem

Mathematics Subject Classification (2000) Primary: 60K37 · 60F05; Secondary: 60J05 · 82C44

1 Introduction

The study of the asymptotic behaviour of random walks (RW) in random environment (RWRE) has been started more than 30 years ago. The first mathematical results were obtained in the pioneering papers by M. Kozlov [5], Solomon [8], and Kesten et al. [4]. The asymptotic behaviour of a RW in annealed environments has been described in [4] in detail for all regimes except the recurrent one. Sinai [7] has completed this description by discovering the $\log^2$ law in the

I. Ya. Goldsheid (✉)
School of Mathematical Sciences, Queen Mary, University of London,
London E1 4NS, Great Britain
e-mail: I.Goldsheid@qmul.ac.uk

recurrent case. Recently, the results of [4] were extended by Mayer-Wolf et al. [6] to the case of Markovian environments.

But the question about the asymptotic behaviour of RW in a quenched (frozen) environment remains largely open. However, it has to be mentioned that Alili [1] proved the central limit theorem (CLT) for a random walk in a quasi-periodic environment with very special additional properties.

The aim of this work is to prove that under certain sufficient conditions (which are often also necessary) the CLT holds for simple (one-dimensional with nearest neighbour jumps) random walks in a typical quenched environment.

Traditionally, the simple random walk is characterized by two quantities: the hitting time $T(n)$ of site n and the position of the walk $X(t)$ at time t. One usually starts with the study of the asymptotic behaviour of $T(n)$ as $n \to \infty$ and then 'translates' the results of this study into results for the asymptotic behaviour of $X(t)$ as $t \to \infty$. Hitting times are easy to control due to the fact (used already in [8]) that, in this model, they can be presented as sums of independent random variables. In [1] the CLT for hitting times has been proved for RW's in ergodic environments. The proof of this fact is given below (Theorem 3) mainly because it is used in the proof of Theorem 4. Our main results are concerned with a less simple question about the position of the walk and are as follows.

Theorem 4 reduces the question about the CLT for $X(t)$ to a question about certain properties of the environment. It also offers a choice of two random centerings for $X(t)$ which are functions of the environment.

In Theorem 5, we prove that independent identically distributed (i.i.d.) random environments do have the properties allowing to apply Theorem 4 in this case. In fact it can be shown (though we do not do it here) that environments satisfying strong mixing conditions also have these properties.

Theorem 6 provides a very short and simple proof of the CLT for uniformly ergodic environments (in particular, quasi-periodic environments). It thus addresses the other side of the spectrum, as far as the mixing properties are concerned.

It should be emphasized that at present there is no proof of CLT for a position of the walk which would work in a general ergodic environment.

The CLT in annealed setting is not discussed in this paper. It can be derived from the quenched CLT in the case of environments with strong mixing properties but it should be stressed once gain that in the general ergodic setting even this question remains opened both for the hitting times and the position of the walk.

Apart of the above there is the following reason for appearance of this work. Our intention is to address the problem in the simplest case since this is where the ideas can be best explained and the proofs are short and transparent. The same approach, properly adapted, works in a much more general case of a RWRE on a strip but explaining it there is a much more technical matter.

Since the problems considered in this work stem directly from [4,8], we do not review the beautiful development that followed the appearance of these

papers. Relatively recent and modern introductions to the subject as well as comprehensive reviews can be found in [2,9,10].
The article is organized as follows.

We start by describing the models considered in this work. We then explain those results from [1,4,8] which are relevant to this work. This is followed by statement of our main results which are then proved in the next section. Appendix contains several technical results some of which may be new and some are unlikely to be new but are included mainly for the sake of completeness.

I am grateful to E. Bolthausen, A.-S. Sznitman, and O. Zeitouni for valuable discussions, suggestions, and support. I would like also to thank the referee whose careful and thoughtful reading of the paper had helped to improve it in a number of ways.

This paper was about to be submitted when I learned that O. Zeitouni and J. Peterson obtained a result which is similar to the one stated in Theorem 5.

1.1 Description of the model

Let $(\Omega, \mathcal{F}, \mathbb{P}, \mathcal{T})$ be a dynamical system with $\Omega = \{\omega\}$ denoting a set of elementary events, $\mathcal{F}$ being a σ-algebra of subsets of Ω, $\mathbb{P}$ denoting a probability measure on $(\Omega, \mathcal{F})$, and $\mathcal{T} : \Omega \mapsto \Omega$ being an invertible transformation of Ω preserving measure $\mathbb{P}$. Next, let $p : \Omega \mapsto (0,1)$ be a measurable real valued function on Ω such that $0 < p(\omega) < 1$ for all $\omega \in \Omega$.

Put $p_n \equiv p_n(\omega) = p(\mathcal{T}^n\omega), q_n = 1-p_n, -\infty < n < \infty$. For any such sequence p_n we shall now define a random walk $X(t;\omega,z)$ with discrete integer valued time $t \geq 0$. The phase space of the walk is a one-dimensional lattice $\mathbb{Z}$ and $p_n,\ q_n$ are its transition probabilities:

$$\mathcal{Q}_\omega(z_1, z_2) \stackrel{\text{def}}{=} \begin{cases} p_n & \text{if } z_1 = n,\ z_2 = n+1, \\ q_n & \text{if } z_1 = n,\ z_2 = n-1, \\ 0 & \text{otherwise.} \end{cases} \tag{1.1}$$

For any starting point $z \in \mathbb{Z}$ the probability law on the space of trajectories is denoted by $\mathrm{Pr}_{\omega,z}$ and is defined by its finite-dimensional distributions

$$\mathrm{Pr}_{\omega,z}\left(X(1) = z_1, \ldots, X(t) = z_t\right) \stackrel{\text{def}}{=} \mathcal{Q}_\omega(z, z_1)\mathcal{Q}_\omega(z_1, z_2)\cdots\mathcal{Q}_\omega(z_{t-1}, z_t). \tag{1.2}$$

We say that the sequence p_n (or, equivalently, the ω) is the *environment* or the *random environment* of the walk. The *annealed probability measure* on the product of the space of environments Ω and the space of trajectories $X(\cdot;\omega,z)$ starting form z is a semi-direct product of $\mathbb{P} \times \mathrm{Pr}_{\omega,z}$, defined by $\mathbb{P}(\mathrm{d}\omega)\mathrm{Pr}_{\omega,z}(\mathrm{d}X)$. We write Pr_ω for $\mathrm{Pr}_{\omega,z}$ and $X(\cdot)$ for $X(\cdot;\omega,z)$ when there is no danger of confusion. It is useful to remember that, unless explicitly stated otherwise, we always suppose that the environment is quenched (frozen).

The just described general class of models provides a natural setting for Theorems 3 and 4.

It has already been mentioned above that we shall consider two sub-classes of this model. The so-called *i.i.d. environments* form one of these sub-classes and arise when p_n is a sequence of independent identically distributed random variables.

A sub-class of random environments which we call *uniformly ergodic environments* is obtained when the dynamical system has very good ergodic properties which are usually combined with very week mixing properties. It is convenient to give the precise definition later but it is natural to mention here that a quasi-periodic environment is also a uniformly ergodic environment.

1.2 Notations and assumptions

1. *Hitting times.* Let $T_k(n)$ be the hitting time of site n by a random walk $X(\cdot;\omega,k)$ starting from k:

$$T_k(n) \stackrel{\text{def}}{=} \inf\{t: \ X(t;\omega,k)=n\}.$$

The notation $T(n)$ is reserved for the case $k=0$. We put $\tau_k = T_k(k+1)$. The random variables τ_k are independent when ω is fixed (with their distributions depending on k and ω). As in [8], we shall make use of the following simple relation

$$T_k(n) = \sum_{j=k}^{n-1} \tau_j. \tag{1.3}$$

2. *Expectations.* Throughout the paper $\mathbb{E}$ denotes the expectation with respect to the measure $\mathbb{P}$. By $\mathcal{E}_{\omega,z}$ we denote the expectation with respect to the measure $\mathrm{Pr}_{\omega,z}$; in those cases when the starting point of the walk is clearly defined by the context we may use $\mathcal{E}_\omega$ for $\mathcal{E}_{\omega,z}$ (e.g. $\mathcal{E}_\omega\tau_k \equiv \mathcal{E}_{\omega,k-1}\tau_k$). The notation Var_ω will be used for the variance of a random variable calculated with respect to the measure $\mathrm{Pr}_{\omega,z}$, e.g. $\mathrm{Var}_\omega(\tau_k) = \mathcal{E}_\omega(\tau_k - \mathcal{E}_\omega\tau_k)^2$.
3. *Main assumptions.* The following set of assumptions is called Condition **C** and is supposed to be satisfied throughout the paper:
Condition C

C1 The dynamical system $(\Omega,\mathcal{F},\mathbb{P},\mathcal{T})$ is ergodic
C2 $\mathbb{E}\log p_k^{-1} < \infty, \ \mathbb{E}\log(1-p_k)^{-1} < \infty$
A set of stronger assumptions called Condition **C**$'$ consists of **C1**, **C3** and **C4**:
C3 There is a $\gamma > 2$ such that

$$\mathbb{E}p_k^{-\gamma} < \infty, \ \mathbb{E}(1-p_k)^{-\gamma} < \infty$$

C4 $\limsup_{n\to\infty} \left(\mathbb{E}\prod_{j=1}^n \left(\frac{q_j}{p_j}\right)^\gamma\right)^{\frac{1}{n}} < \infty$

Remark Obviously, **C4** follows from **C3** if the environment is i.i.d. This is not true in general ergodic setting.

1.3 Preliminaries: transience, recurrence, linear growth

We say that a random walk (in a fixed environment ω) is transient to the right (transient to the left) if

$$\lim_{t\to\infty} X(t) = \infty \quad \text{(correspondingly } \lim_{t\to\infty} X(t) = -\infty).$$

We shall now quote several statements from [8] in a form which suits us best. Let us put

$$A_j \stackrel{\text{def}}{=} \frac{q_j}{p_j} \quad \text{and} \quad \lambda \stackrel{\text{def}}{=} \mathbb{E} \ln A_j \tag{1.4}$$

(it is clear that λ does not depend on j).

The recurrence and transience criteria for our random walk are given by the following result from [8].

Theorem 1 *Suppose that Condition* **C** *is satisfied. Then*

(i) $\lambda < 0$ *implies for* $\mathbb{P}$*-a.e. environment* ω *that* X_ω *is transient to the right. Symmetrically,* $\lambda > 0$*, implies for* $\mathbb{P}$*-a.e. environment* ω *that* X_ω *is transient to the left.*

(ii) $\lambda = 0$ *if and only if* X_ω *is recurrent for* $\mathbb{P}$*-a.e.* ω*, that is*

$$\limsup_{t\to\infty} X(t) = +\infty, \quad \liminf_{t\to\infty} X(t) = -\infty \ \mathrm{Pr}_\omega\text{-almost surely}.$$

From now on we consider only those RWRE which are transient to the right, that is $\lambda < 0$. To state further results it is convenient to define a function $r(\kappa)$ depending on a parameter $\kappa \in [0, \gamma]$, where γ is the same as in **C3**, namely:

$$r(\kappa) \stackrel{\text{def}}{=} \limsup_{n\to\infty} \left(\mathbb{E} \prod_{j=1}^{n} A_j^\kappa \right)^{\frac{1}{n}}. \tag{1.5}$$

This function is a simple generalization of the one first considered in [4] (see also [6] where $\log r(\kappa)$ has been studied). If the p_n's are i.i.d. random variables then of course

$$r(\kappa) = \mathbb{E}\,(A_0)^\kappa\,.$$

As has been shown in [4,6], the asymptotic behaviour of the RWRE can be characterized in terms properties of $r(\kappa)$ which are well worth of being studied. However, for the purposes of this work, we only need the following simple

Lemma 1 *Suppose that Condition* $\mathbf{C}'$ *is satisfied. Then the function* $\ln r(\kappa)$ *is continuous and convex on* $[0, \gamma)$.

The proof of this Lemma is given in the Appendix.

Let us define a function of ω which plays a very important role in this paper (as it did already in [8]): for a fixed environment ω put

$$\mu_0(\omega) \stackrel{\text{def}}{=} 1 + 2A_0 + \cdots + 2A_0 A_{-1} \cdots A_{-j} + \cdots \equiv 1 + 2\sum_{j=0}^{\infty} \prod_{i=-j}^{0} A_i, \tag{1.6}$$

and

$$\mu_k(\omega) \stackrel{\text{def}}{=} \mu_0(\mathcal{T}^k \omega) \equiv 1 + 2\sum_{j=0}^{\infty} \prod_{i=k-j}^{k} A_i. \tag{1.7}$$

The probabilistic meaning of $\mu_k(\omega)$ is explained by the following

Lemma 2 [8,10] *If* $\lambda < 0$ *then* μ_k *is finite for* $\mathbb{P}$*-almost all* ω *and* $\mathcal{E}_\omega \tau_k = \mu_k$.

Let us note first that $\mu_k(\omega)$ has the following property:

$$\text{if } \kappa \le \gamma \text{ and } r(\kappa) < 1 \text{ then there is a } \delta > 0 \text{ such that } \mathbb{E}\mu_k^{\kappa+\delta} < \infty. \tag{1.8}$$

It is easy to see that property (1.8) holds for any κ, $0 < \kappa < \gamma$. But since we need it when $\kappa \ge 1$, it shall be explained only in this case (the other one is even simpler).

Namely, according to Lemma 1, $r(\cdot)$ is a continuous function. We thus can choose $\delta > 0$ and such that $r(\kappa + \delta) < 1$. Consider the Banach space $L_{\kappa+\delta}(\Omega)$ with $\|Y\|_{\kappa+\delta} \stackrel{\text{def}}{=} (\mathbb{E}|Y|^{\kappa+\delta})^{\frac{1}{\kappa+\delta}}$ for any function $Y \in L_{\kappa+\delta}(\Omega)$. We then have

$$\|\mu_0\|_{\kappa+\delta} \le 1 + 2\sum_{j=0}^{\infty} \left\| \prod_{i=-j}^{0} A_i \right\|_{\kappa+\delta} < \infty,$$

and this proves (1.8) (remember that $\|\mu_0\| = \|\mu_k\|$).

In particular if $r(1) < 1$ then

$$\mu \stackrel{\text{def}}{=} \mathbb{E}\mu_k \le \left(\mathbb{E}\mu_k^{1+\delta}\right)^{\frac{1}{1+\delta}} < \infty \quad \text{for some } \delta > 0. \tag{1.9}$$

The quenched Law of Large Numbers has been proved in [8] for i.i.d. environments and the same proof works in general ergodic setting (see [1] or [10] for more detailed explanations).

Theorem 2 *Suppose that Condition* $\mathbf{C}'$ *is satisfied and that* $\lambda < 0$. *Then:*

(i) $r(1) < 1$ *implies that for* $\mathbb{P}$*-a.e. environment* ω *with* Pr_ω*-probability* 1

$$\lim_{n\to\infty} \frac{T(n)}{n} = \mu < \infty \quad \textit{and} \quad \lim_{t\to\infty} \frac{X(t)}{t} = \mu^{-1} > 0, \tag{1.10}$$

(ii) $r(1) > 1$ *implies that for* $\mathbb{P}$*-a.e. environment* ω *with* Pr_ω*-probability* 1

$$\lim_{n\to\infty} \frac{T(n)}{n} = \infty \quad \textit{and} \quad \lim_{t\to\infty} \frac{X(t)}{t} = 0. \tag{1.11}$$

(iii) *If the environment is i.i.d. then* (1.11) *holds also for* $r(1) = 1$.

Remark If the environment is i.i.d. then a straightforward calculation leads to an explicit formula for μ (known since [8]): $\mu = \frac{1+r(1)}{1-r(1)}$.

We finish this section by defining uniformly ergodic environments.

Definition 1 *Let* $f : \Omega \mapsto \mathbb{R}$ *be an* $\mathcal{F}$*-measurable function on* Ω. *We say that the transformation* $\mathcal{T}$ *is* f*-uniformly ergodic if*

$$\left| n^{-1} \sum_{j=1}^{n} f(\mathcal{T}^j\omega) - \mathbb{E}f \right| \le \varepsilon_n \tag{1.12}$$

where the sequence ε_n *does not depend on* ω *and* $\lim_{n\to\infty} \varepsilon_n = 0$.

We say that a random environment is uniformly ergodic if $\mathcal{T}$ *is* μ_0*-uniformly ergodic.*

One of the simplest uniformly ergodic environments is generated by a quasi-periodic dynamical system with $\Omega = [0,1]$, $\mathcal{T}(\omega) = (\omega + \alpha)(\mathrm{mod} 1)$, where $\alpha \in [0,1]$ is an irrational number. If the function $p(\cdot)$ is continuous on $[0,1]$, $p(0) = p(1)$, and $\lambda = \int_0^1 \ln \frac{1-p(\omega)}{p(\omega)} \mathrm{d}\omega < 0$ then also $\mu_0(\omega)$ is a continuous function on $[0,1]$ and the uniform ergodicity of this environment follows.

Let us explain this statement in a more general setting. Suppose that $\mathcal{T}$ is a continuous homeomorphism of a compact metric space Ω and that $\mathbb{P}$ is its unique invariant measure. Suppose also that the function $p(\cdot)$ is continuous. It is then easy to see that $r(\kappa) = \mathrm{e}^{\kappa\lambda}$. Indeed,

$$\left(\mathbb{E} \prod_{j=1}^{n} A_j^\kappa \right)^{\frac{1}{n}} = \left(\mathbb{E}\mathrm{e}^{\kappa \sum_{j=1}^{n} \ln A_j} \right)^{\frac{1}{n}} = \left(\mathbb{E}\mathrm{e}^{\kappa n(\lambda + \varepsilon_n)} \right)^{\frac{1}{n}},$$

where $|\varepsilon_n(\omega)| \le \varepsilon_n$. Hence $\mathrm{e}^{\kappa(\lambda - \varepsilon_n)} \le r(\kappa) \le \mathrm{e}^{\kappa(\lambda + \varepsilon_n)}$ and the statement follows. If now $\lambda < 0$, then series (1.6) converges uniformly in $\omega \in \Omega$ and hence μ_0 is a continuous function on Ω. The latter in turn implies uniform ergodicity of the environment.

2 Main results

In order to state the CLT for $T(n)$ one has to know the variance of this random variable. It turns out that in the case of the simple walk an explicit expression for the variance can be found and the calculations are not complicated. Formula (2.2) has been obtained in [1] where branching processes are used for its derivation. We use a different approach which works also for more general models [3].

Lemma 3 *Suppose that $\lambda < 0$. Then for $\mathbb{P}$-almost every ω the variance of $T(n) \equiv T(n;\omega)$ is finite and is given by*

$$\mathrm{Var}_\omega(T(n)) = \sum_{k=0}^{n-1} \sigma_k^2(\omega), \tag{2.1}$$

where

$$\sigma_k^2(\omega) \stackrel{\mathrm{def}}{=} \mathrm{Var}_\omega(\tau_k) = \sum_{j=0}^{\infty} p_{k-j}^{-1}(\mu_{k-j-1}+1)^2 \prod_{i=k-j}^{k} A_i. \tag{2.2}$$

If in addition $r(2) < 1$, then

$$\sigma^2 \equiv \mathbb{E}\mathrm{Var}_\omega(\tau_k) < \infty. \tag{2.3}$$

If $r(2) < 1$ and the environment is i.i.d. then

$$\sigma^2 = \frac{4(r(1)+r(2))(1+r(1)^2)}{(1-r(1))^2(1-r(2))}. \tag{2.4}$$

The proof of Lemma 3 is given in Appendix. Denote

$$H(n,\omega) = \sum_{k=0}^{n-1} \mu_k \equiv \mathcal{E}_\omega(T(n)), \tag{2.5}$$

where the last equality follows from (1.3) and Lemma 2. We often write $H(n)$ for $H(n,\omega)$. It is clear from (2.5) that $H(n)$ is the natural centering in the CLT for $T(n)$. It turns out that centerings for $X(t)$ can too be expressed, with a varying degree of explicitness, in terms of the function $H(\cdot)$.

In the sequel we denote $\lfloor y \rfloor \stackrel{\mathrm{def}}{=}$ integer part of y, where y is any real number. We also use the following convention about summations. For any real numbers b_1, b_2 and a sequence d_k, $-\infty < k < \infty$,

$$\sum_{k=b_1}^{b_2} d_k \stackrel{\mathrm{def}}{=} \sum_{k=\lfloor b_1 \rfloor}^{\lfloor b_2 \rfloor} d_k \stackrel{\mathrm{def}}{=} -\sum_{k=\lfloor b_2 \rfloor}^{\lfloor b_1 \rfloor} d_k. \tag{2.6}$$

In particular for $y \geq 0$ we put $H(y) \stackrel{\mathrm{def}}{=} H(\lfloor y \rfloor)$.

Definition 2 *The function*

$$b(t;\omega) \stackrel{\text{def}}{=} 2\mu^{-1}t - \mu^{-1}H(\mu^{-1}t,\omega) \tag{2.7}$$

is said to be the explicit centering for $X(t)$. The integer valued function $\tilde{b}(t;\omega)$ such that

$$H(\tilde{b}(t;\omega)) \equiv \sum_{k=0}^{\tilde{b}(t;\omega)-1} \mu_k \le t < \sum_{k=0}^{\tilde{b}(t;\omega)} \mu_k \equiv H(\tilde{b}(t;\omega)+1). \tag{2.8}$$

is said to be the implicit centering for $X(t)$.

It is easy to see that

$$b(t;\omega) = \mu^{-1}t - \mu^{-1}\sum_{k=0}^{\mu^{-1}t-1}(\mu_k - \mu) + O_1(1), \tag{2.9}$$

where $O_1(1) = \mu^{-1}t - \lfloor \mu^{-1}t \rfloor < 1$.

For the rest of the paper we suppose that

Condition $\mathbf{C}'$ is satisfied and $r(2) < 1$.

Put $\Phi(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} \mathrm{e}^{-\frac{u^2}{2}}\mathrm{d}u$. We shall prove the following statements.

Theorem 3 [1] *For $\mathbb{P}$-almost every environment ω*

$$\Pr_\omega\left\{\frac{T(n)-H(n)}{\sqrt{n}\sigma} < x\right\} \to \Phi(x) \quad \textit{uniformly in } x \textit{ as } n \to \infty. \tag{2.10}$$

Remark Uniform convergence in (2.10) is used in the proof of Theorem 4.

Theorem 4 *Define $\sigma^{*2} = \mu^{-3}\sigma^2$. Suppose that at least one of the following two relations holds:*

$$\lim_{t\to\infty} t^{-\frac{1}{2}} \sum_{k=\mu^{-1}t}^{b(t)+\sqrt{t}\sigma^* x} (\mu_k - \mu) = 0 \quad \textit{with } \mathbb{P}\textit{-probability } 1 \textit{ for any real } x \tag{2.11}$$

$$\lim_{t\to\infty} t^{-\frac{1}{2}} \sum_{k=\tilde{b}(t)}^{\tilde{b}(t)+\sqrt{t}\sigma^* x} (\mu_k - \mu) = 0 \quad \textit{with } \mathbb{P}\textit{-probability } 1 \textit{ for any real } x. \tag{2.12}$$

Then for $\mathbb{P}$*-almost every environment* ω

$$\lim_{t\to\infty} \Pr_\omega \left\{ \frac{X(t)-\bar{b}(t)}{\sqrt{t\sigma^*}} \le x \right\} = \Phi(x), \tag{2.13}$$

with convergence in 2.13 *being uniform in* x *and*

$$\bar{b}(t) = \begin{cases} b(t) & \text{if (2.11) holds} \\ \tilde{b}(t) & \text{if (2.12) holds} \end{cases}$$

($\bar{b}(t)$ *can be equal to any of the two if both* (2.11) *and* (2.12) *hold).*

Remark 1. Relations (2.11) and (2.12) would look more natural if $\sigma^* x$ there would have been replaced by x. The only reason for keeping the factor σ^* there is that it arises in all applications and proofs.

2. Note that in (2.11) and in (2.12) the summation is carried out within random limits.
3. Relation (2.11) is a good approximation for (2.12) in the case of environments with sufficiently strong mixing properties.

We finish this section by stating two theorems which demonstrate the usefulness of conditions (2.11) and (2.12).

Theorem 5 *In the i.i.d. random environment* (2.11) *holds and thus for* $\mathbb{P}$*-almost every environment* ω

$$\lim_{t\to\infty} \Pr_\omega \left\{ \frac{X(t)-b(t)}{\sqrt{t\sigma^*}} \le x \right\} = \Phi(x) \tag{2.14}$$

with convergence in (2.14) *being uniform in* x.

Theorem 6 *Suppose that the environment is uniformly ergodic. Then* (2.12) *holds and thus for* $\mathbb{P}$*-almost every environment* ω

$$\lim_{t\to\infty} \Pr_\omega \left\{ \frac{X(t)-\tilde{b}(t)}{\sqrt{t\sigma^*}} \le x \right\} = \Phi(x) \tag{2.15}$$

and convergence in (2.15) *is uniform in* x.

3 Proofs

Proof of Theorem 3 Proving (2.10) essentially means proving a CLT for the sum of independent random variables τ_k. Indeed, since

$$\frac{T(n)-H(n,\omega)}{\sqrt{n\sigma}} = \frac{T(n)-\mathcal{E}_\omega(T(n))}{\sqrt{\mathrm{Var}_\omega(T(n))}} \sqrt{\frac{\mathrm{Var}_\omega(T(n))}{n\sigma^2}}, \tag{3.1}$$

it is enough to check that for $\mathbb{P}$-almost all ω

$$\frac{\mathrm{Var}_\omega(T(n))}{n\sigma^2} \to 1 \tag{3.2}$$

and that CLT holds for $T(n)$. Relation (3.2) follows from (2.1), (2.3), and the Birkhoff ergodic theorem. Next, for those ω for which (3.2) holds, also

$$\max_{0\le k\le n-1} \sigma_k^2/\mathrm{Var}_\omega(T(n)) \to 0 \quad \text{as } n \to \infty.$$

This in turn is well known to imply that the Lindeberg's conditions for the CLT for sums of non-identically distributed random variables τ_k holds. Theorem 3 is proved. □

Proof of Theorem 4 It is routine (whenever the proofs of CLT's are concerned) that the uniform in x convergence in (2.13) follows from the convergence for every fixed value of x. Thus, for the duration of the proof, x will be viewed as a fixed parameter. The proof of (2.13) will be split into three parts.

Part 1: approximating $X(t)$ by n_t. For any integer time t let n_t be a positive random integer such that

$$T(n_t) \le t < T(n_t + 1). \tag{3.3}$$

Remark This definition of n_t has been used already in [8] in the proof of the Law of Large Numbers cited above.

Since $|X(t)-X(t')| \le |t-t'|$ and since $X(T(n_t)) = n_t$ and $X(T(n_t+1)) = n_t+1$ (by the definition of $T(n)$), it follows that

$$|X(t) - n_t| = |X(t) - X(T(n_t))| \le t - T(n_t) < T(n_t + 1) - T(n_t) = \tau_{n_t}. \tag{3.4}$$

Hence

$$t^{-\frac{1}{2}}|X(t) - n_t| < t^{-\frac{1}{2}}\tau_{n_t}. \tag{3.5}$$

The sequence τ_k forms a stationary process in annealed environment and since $\mathbb{E}\left(\mathcal{E}_\omega\tau_k^2\right) < \infty$ we have that $t^{-\frac{1}{2}}\tau_{n_t} \to 0$ as $t \to \infty$ with $\mathbb{P} \times \mathrm{Pr}_\omega$-probability 1 which in turn implies that it holds for $\mathbb{P}$ -almost every ω with Pr_ω-probability 1. This implies that proving (2.13) is equivalent to proving that

$$\lim_{t\to\infty} \mathrm{Pr}_\omega\left\{\frac{n_t - \bar{b}(t)}{\sqrt{t}\sigma^*} \le x\right\} = \Phi(x). \tag{3.6}$$

Part 2: proof for the case when (2.11) *holds.* We need a simple (but very useful) identity. Namely, it follows from (3.3) and monotonicity of the function $T(\cdot)$ that for any $y \ge 0$ the following two events coincide:

$$\{n_t \le y\} = \{T(y+1) > t\}, \tag{3.7}$$

where as before $T(y+1) \equiv T(\lfloor y \rfloor + 1)$. This identity is a slight modification of the one which has been often used in the context of RWRE at least since the appearance of paper [4].

Hence, for sufficiently large values of t we can write

$$\begin{aligned}\Pr_\omega\left\{\frac{n_t - b(t)}{\sqrt{t}\sigma^*} \le x\right\} &= \Pr_\omega\{T(b(t) + \sqrt{t}\sigma^* x + 1) > t\} \\ &\equiv \Pr_\omega\{T(B(t)+1) > t\},\end{aligned} \tag{3.8}$$

where $B(t) \overset{\text{def}}{=} b(t) + \sqrt{t}\sigma^* x$. It is natural to use the fact that, for a typical fixed ω, $T(B(t)+1)$ is an asymptotically normal random variable. Let us take a closer look at the parameters of this random variable. First of all it follows from (2.7) and the Birkhoff ergodic theorem that

$$\lim_{t\to\infty} t^{-1}B(t) = \mu^{-1} \quad \text{for } \mathbb{P}\text{-almost every } \omega. \tag{3.9}$$

Relation (3.9) and the Birkhoff ergodic theorem imply that for $\mathbb{P}$-almost all ω

$$t^{-1}\mathrm{Var}_\omega(T(B(t)+1)) = t^{-1}\sum_{k=0}^{B(t)} \sigma_k^2(\omega) \to \mu^{-1}\sigma^2 \quad \text{as} \quad t\to\infty. \tag{3.10}$$

Finally,

$$\mathcal{E}_\omega\left(B(t)+1)\right) = \sum_{k=0}^{B(t)} \mu_k = B(t)\mu + \sum_{k=0}^{B(t)}(\mu_k - \mu) + O_2(1), \tag{3.11}$$

where $O_2(1) = \mu + \mu(\lfloor B(t)\rfloor - B(t)) \le \mu$. Returning to the original expression for $B(t)$ and simultaneously replacing $b(t)$ in the right hand side of (3.11) by its expression from (2.9) leads to

$$\begin{aligned}&\mathcal{E}_\omega\left(T(B(t)+1)\right) \\ &\quad= t + \sqrt{t}\sigma^* x\mu - \sum_{k=0}^{\mu^{-1}t}(\mu_k - \mu) + \sum_{k=0}^{b(t)+\sqrt{t}\sigma^* x}(\mu_k - \mu) + O_3(1) \\ &\quad= t + \sqrt{t}\sigma^* x\mu + \sum_{k=\mu^{-1}t}^{b(t)+\sqrt{t}\sigma^* x}(\mu_k - \mu) + O_3(1),\end{aligned} \tag{3.12}$$

where $O_3(1) = \mu(\mu^{-1}t - \lfloor \mu^{-1}t\rfloor) + O_2(1) \le 2\mu$. Putting

$$\mathcal{G}(t) = \frac{T(B(t)+1) - \mathcal{E}_\omega\left(T(B(t)+1)\right)}{\sqrt{t}\sigma^*\mu}$$

we can present the right hand side of (3.8) as

$$\begin{aligned}&\Pr_\omega\{T(b(t)+\sqrt{t}\sigma^* x+1)>t\}\\&\quad=\Pr_\omega\left\{\mathcal{G}(t)>-x-t^{-\frac12}(\sigma^*\mu)^{-1}\sum_{k=\mu^{-1}t}^{b(t)+\sqrt{t}\sigma^* x}(\mu_k-\mu)+O(t^{-\frac12})\right\}.\end{aligned} \tag{3.13}$$

But, according to (2.11), we have for $\mathbb{P}$-almost every ω:

$$\lim_{t\to\infty} t^{-\frac12}\sum_{k=\mu^{-1}t}^{b(t)+\sqrt{t}\sigma^* x}(\mu_k-\mu)=0, \tag{3.14}$$

and also, because of (2.10), we have that for $\mathbb{P}$-almost every ω the sequence $\mathcal{G}(t)$ converges in distribution to a standard normal random variable. Hence

$$\lim_{t\to\infty}\Pr_\omega\left\{\mathcal{G}(t)>-x-t^{-\frac12}(\sigma^*\mu)^{-1}\sum_{k=\mu^{-1}t}^{b(t)+\sqrt{t}\sigma^* x}(\mu_k-\mu)+O(t^{-\frac12})\right\}=\Phi(x)$$

which proves (2.13).

Part 3: proof in the case when (2.12) *holds.* The proof goes along the same lines as in Part 2 with the natural replacement of $b(t)$ by $\tilde b(t)$. On the other hand, subtle differences appear at the end of the proof; this is why it may be useful to give a brief outline of it here.

As in Part 2, it follows from (3.7) that

$$\Pr_\omega\left\{\frac{n_t-\tilde b(t)}{\sqrt{t}\sigma^*}\le x\right\}=\Pr_\omega\{T(\tilde B(t)+1)>t\}, \tag{3.15}$$

where $\tilde B(t)\stackrel{\text{def}}{=}\tilde b(t)+\sqrt{t}\sigma^* x$. Also for $\mathbb{P}$-almost all ω

$$\lim_{t\to\infty} t^{-1}B(t)=\mu^{-1}\quad\text{and}\quad\lim_{t\to\infty} t^{-1}\mathrm{Var}_\omega(T(B(t)+1))=\mu^{-1}\sigma^2. \tag{3.16}$$

Next

$$\begin{aligned}\mathcal{E}_\omega(T(\tilde B(t)+1))&=\sum_{k=0}^{\tilde b(t)+\sqrt{t}\sigma^* x}\mu_k=\sum_{k=0}^{\tilde b(t)-1}\mu_k+\sum_{k=\tilde b(t)}^{\tilde b(t)+\sqrt{t}\sigma^* x}\mu_k t+\sqrt{t}\sigma^* x\mu\\&\quad+\sum_{k=\tilde b(t)}^{\tilde b(t)+\sqrt{t}\sigma^* x}(\mu_k-\mu)+O_4(1),\end{aligned} \tag{3.17}$$

where the second line in (3.17) follows from the definition of $\tilde{b}(t)$ (see (2.8)) and

$$O_4(1) = t - \sum_{k=0}^{\tilde{b}(t)-1} \mu_k + \mu(1 + \sqrt{t}\sigma^* x - \lfloor\sqrt{t}\sigma^* x\rfloor) \leq \mu_{\tilde{b}(t)} + 2\mu.$$

Let us denote $\tilde{\mathcal{G}}(t) = (\sqrt{t}\sigma^*\mu)^{-1}(T(\tilde{B}(t)+1) - \mathcal{E}_\omega(T(\tilde{B}(t)+1)))$. We can then right that

$$\begin{aligned} &\Pr_\omega\left\{T(\tilde{b}(t) + \sqrt{t}\sigma^* x + 1) \geq t\right\} \\ &= \Pr_\omega\left\{\tilde{\mathcal{G}}(t) \geq -x - t^{-\frac{1}{2}}(\sigma^*\mu)^{-1} \sum_{k=\tilde{b}(t)}^{\tilde{b}(t)+\sqrt{t}\sigma^* x} (\mu_k - \mu) + t^{-\frac{1}{2}} O_5(1)\right\}, \end{aligned} \tag{3.18}$$

where $O_5(1)$ is proportional to $O_4(1)$. We note that $t^{-\frac{1}{2}}\mu_{\tilde{b}(t)} \to 0$ with $\mathbb{P}$-probability 1 because μ_k^2 is a stationary sequence with $\mathbb{E}\mu_k^2 < \infty$ (see (1.8)). This together with (2.12) and the asymptotic normality of $\tilde{\mathcal{G}}(t)$ finishes the proof. □

Proof of Theorem 5 According to Theorem 4 we only have to check that for i.i.d. environments (2.11) holds true. In fact, we shall prove that i.i.d. environments satisfy (3.24) which is slightly stronger than (2.11). To explain the last statement let us put

$$\mathcal{H}(n,\omega) = \sum_{j=0}^{n-1}(\mu_j - \mu), \;\; \mathcal{H}^*(n,\omega) = \max_{0\leq s\leq n-1}\sum_{j=0}^{s}(\mu_j - \mu) \tag{3.19}$$

We shall use the following notations. If Y is a random variable then $\|Y\|$ is its usual norm in $L_{2+2\delta}(\Omega)$, that is

$$\|Y\| = (\mathbb{E}|Y|^{2+2\delta})^{\frac{1}{2+2\delta}}, \tag{3.20}$$

where $\delta > 0$ is such that $r(2+2\delta) < 1$.

Lemma 4 *In the i.i.d. environment with $r(2+2\delta) < 1$ the following relations hold:*

$$\|\mathcal{H}^*(n)\| \leq Cn^{\frac{1}{2}} \quad \textit{and} \tag{3.21}$$

$$\lim_{n\to\infty} n^{-\frac{1+c}{2}}\mathcal{H}(n,\omega) = 0 \quad \textit{with } \mathbb{P} \textit{ probability } 1 \textit{ for any } c > 0, \tag{3.22}$$

The constant C in (3.21) depends only on δ and the distribution of the environment.

Remark Even though the random variables μ_j are not independent, a statement which is stronger than (3.22) can be proved. We do not do this because (3.22) is sufficient for our purposes.

The proof of Lemma 4 will be given at the end of this section. We now continue the proof of the theorem.

If ω is such that (3.22) holds then there is $t(\omega)$ such that

$$|b(t,\omega) + \sqrt{t}\sigma^* x - \mu^{-1}t| \equiv | - \mu^{-1}\mathcal{H}(\mu^{-1}t,\omega) + \sqrt{t}\sigma^* x| < t^{\frac{1+c}{2}} \quad \text{if } t > t(\omega) \tag{3.23}$$

(see (2.7)). Hence the following Lemma implies the result we want:

Lemma 5 *For a sufficiently small* $c > 0$

$$\lim_{n\to\infty} n^{-\frac{1}{2}} \max_{|s|\le n^{\frac{1+c}{2}}} \left| \sum_{k=n}^{n+s} (\mu_k - \mu) \right| = 0 \quad \textit{with } \mathbb{P} \textit{ probability } 1 \tag{3.24}$$

We put $n = \mu^{-1}t$ in (3.23); σ^* and x which are present in (2.11) disappear here because of (3.23) and the presence of the small c under the max sign.

Proof of Lemma 5. Put

$$R(n,c,\omega) = n^{-\frac{1}{2}} \max_{|s|\le n^{\frac{1+c}{2}}} \left| \sum_{k=n}^{n+s} (\mu_k - \mu) \right| \tag{3.25}$$

Note first that if (3.24) holds for a subsequence $R(n^2,\tilde{c},\omega)$ then (3.24) holds for the whole sequence $R(n,c,\omega)$, where $c = 0.5\tilde{c}$. Indeed, suppose that ω and $\tilde{c} > 0$ are such that (3.24) holds for the subsequence $R(n^2,\tilde{c},\omega)$. Then, for $i \in [1, 2n]$, we have

$$R(n^2+i,c,\omega) = (n^2+i)^{-\frac{1}{2}} \max_{|s|\le (n^2+i)^{\frac{1+c}{2}}} \left| \sum_{k=n^2+i}^{n^2+i+s} (\mu_k - \mu) \right|. \tag{3.26}$$

But

$$\left| \sum_{k=n^2+i}^{n^2+i+s} (\mu_k - \mu) \right| \le \left| \sum_{k=n^2}^{n^2+i+s} (\mu_k - \mu) \right| + \left| \sum_{k=n^2}^{n^2+i-1} (\mu_k - \mu) \right|. \tag{3.27}$$

We note next that, since $c < \tilde{c}$, the inequality $i + |s| < n^{1+\tilde{c}}$ holds for sufficiently large values of n. This together with (3.27) implies that for sufficiently large n

$$R(n^2+i,c,\omega) \le 2R(n^2,\tilde{c},\omega). \tag{3.28}$$

It remains to prove that if $\tilde{c} > 0$ is small enough then

$$\lim_{n\to\infty} n^{-1} \max_{|s|\le n^{1+\tilde{c}}} \left| \sum_{k=n^2}^{n^2+s} (\mu_k - \mu) \right| = 0 \quad \text{with } \mathbb{P} \text{ probability } 1. \tag{3.29}$$

Note that (3.21) is equivalent to saying that the sequence μ_j has the following property: for any m

$$\mathbb{E}\left(\max_{0\le s\le m} \left| \sum_{k=0}^{s} (\mu_k - \mu) \right| \right)^{2+2\delta} \le Cm^{1+\delta}, \tag{3.30}$$

where C is a constant (related to the previous C in an obvious way). Using (3.30) and stationarity of μ_j we obtain that

$$\mathbb{E}(R(n^2,\tilde{c},\omega))^{2+2\delta} \equiv \mathbb{E}\left(n^{-1} \max_{|s|\le n^{1+\tilde{c}}} \left| \sum_{k=n^2}^{n^2+s} (\mu_k - \mu) \right| \right)^{2+2\delta} \le Cn^{(1+\tilde{c})(1+\delta)-2-2\delta}. \tag{3.31}$$

It is now obvious that if $\tilde{c} < (1+\delta)^{-1}\delta$ then

$$\sum_{n=1}^{\infty} \mathbb{E}(R(n^2,\tilde{c},\omega))^{2+2\delta} < \infty \tag{3.32}$$

and the latter in particular implies that $\lim_{n\to\infty} R(n^2,\tilde{c},\omega) = 0$ for almost all ω. Lemma 5 and thus also Theorem 5 is proved. □

Proof of Theorem 6 In order to check that (2.12) holds we note that μ_0-uniform ergodicity (see Definition 1) implies that

$$\left| n^{-1} \sum_{j=k+1}^{k+n} (\mu_0(\mathcal{T}^j\omega) - \mu) \right| \equiv \left| n^{-1} \sum_{j=k+1}^{k+n} (\mu_j(\omega) - \mu) \right| \le \varepsilon_n. \tag{3.33}$$

This is due to the fact that, since ω in (1.12) is arbitrary, it can be replaced by $\mathcal{T}^k\omega$. In particular the left hand side in (2.12) can be estimated as

$$t^{-\frac{1}{2}} \left| \sum_{k=\tilde{b}(t)}^{\tilde{b}(t)+\sqrt{t}\sigma^* x-1} (\mu_k - \mu) \right| \le (\sigma^*|x|+1)\varepsilon_{\sqrt{t}\sigma^*|x|}. \tag{3.34}$$

The proof of (2.12) is finished. □

Remark We suppose, without loss of generality, that ε_n in (3.33) is a monotonically decaying function of n and $\varepsilon_y \equiv \varepsilon_{\lfloor y \rfloor}$ for any $y \ge 0$.

Proof of Lemma 4 It follows from (1.7) that

$$\mu_j - \mu = \sum_{i=1}^{\infty} B(i,j), \quad \text{where } B(i,j) = 2(A_j \dots A_{j-i+1} - r(1)^i) \tag{3.35}$$

Let us put $\beta = r(2+2\delta)^{\frac{1}{2+2\delta}}$. Since $\|A_j \dots A_{j-i+1}\| = \beta^i$ and $r(1) \le \beta$ by Jensen's inequality, we have

$$\|B(i,j)\| \le 2\|A_j \dots A_{j-i+1}\| + 2r(1)^i \le 4\beta^i. \tag{3.36}$$

The $\mathcal{H}(n,\omega)$ can be presented as

$$\mathcal{H}(n,\omega) = \sum_{j=0}^{n-1}\sum_{i=1}^{\infty} B(i,j) = \sum_{i=1}^{\infty}\sum_{j=0}^{n-1} B(i,j) = \sum_{i=1}^{l}\sum_{j=0}^{n-1} B(i,j) + \sum_{i=l+1}^{\infty}\sum_{j=0}^{n-1} B(i,j), \tag{3.37}$$

where $1 \ll l \ll n$ will be chosen later. Denote

$$B_n(i) = \sum_{j=0}^{n-1} B(i,j) \quad \text{and} \quad B_n^*(i) = \max_{0\le s\le n-1} \sum_{j=0}^{s} B(i,j). \tag{3.38}$$

It is then clear that

$$\mathcal{H}^*(n,\omega) \le \sum_{i=1}^{l} B_n^*(i) + \sum_{i=l+1}^{\infty}\sum_{j=0}^{n-1} |B(i,j)| \tag{3.39}$$

and hence

$$\|\mathcal{H}^*(n)\| \le \sum_{i=1}^{l} \|B_n^*(i)\| + \sum_{i=l+1}^{\infty}\sum_{j=0}^{n-1} \|B(i,j)\| \le \sum_{i=1}^{l} \|B_n^*(i)\| + 4n\frac{\beta^{l+1}}{1-\beta}, \tag{3.40}$$

where the last step is due to (3.36). To estimate $\|B_n^*(i)\|$ we note that

$$B_n(i) = \sum_{j=0}^{n-1} B(i,j) = \sum_{k=0}^{i-1}\sum_{j=0}^{s_k} B(i,k+ij), \quad \text{where } s_k = \left\lfloor \frac{n-1-k}{i} \right\rfloor.$$

Each $D_n(i,k) \stackrel{\text{def}}{=} \sum_{j=0}^{s_k} B(i,k+ij)$ is a sum of i.i.d. random variables. We put

$$D_n^*(i,k) \stackrel{\text{def}}{=} \max_{0\le s\le s_k} \left| \sum_{j=0}^{s} B(i,k+ij) \right|$$

By Doob's inequality

$$\|D_n^*(i,k)\| \le \frac{2+2\delta}{1+2\delta}\|D_n(i,k)\|$$

and then by Marcinkiewicz–Zygmund inequality

$$\|D_n^*(i,k)\| \le \frac{2+2\delta}{1+2\delta}C_\delta\left(\frac{n}{i}\right)^{\frac{1}{2}}\|B(i,0)\| \le C_1\left(\frac{n}{i}\right)^{\frac{1}{2}}\beta^i,$$

where C_δ depends only on δ and $C_1 = 4\frac{2+2\delta}{1+2\delta}C_\delta$. But since

$$B_n^*(i) \le \sum_{k=0}^{i-1} D_n^*(i,k)$$

we have that

$$\|B_n^*(i)\| \le \sum_{k=0}^{i-1}\|D_n^*(i,k)\| \le C_1\,(ni)^{\frac{1}{2}}\,\beta^i.$$

Substituting this estimate in (3.40), we obtain

$$\|\mathcal{H}^*(n)\| \le C_1\sum_{i=1}^{l}(ni)^{\frac{1}{2}}\,\beta^i + 4n\frac{\beta^{l+1}}{1-\beta} = C(n,l)n^{\frac{1}{2}}, \tag{3.41}$$

where $C(n,l) = C_1\sum_{i=1}^{l} i^{\frac{1}{2}}\beta^i + 4n^{\frac{1}{2}}\frac{\beta^{l+1}}{1-\beta}$. Put $l = n^{\frac{1}{2}}$; then $\sup_n C(n, n^{\frac{1}{2}}) \le C$ for some constant C. This proves (3.21). The proof of (3.22) follows immediately from Lemma 6 (see Appendix) whose conditions are satisfied because of (3.21) and because μ_j is a stationary sequence. □

4 Appendix

4.1 Proof of Lemma 1

Put

$$r_n(\kappa) \stackrel{\text{def}}{=} \left(\mathbb{E}\prod_{j=1}^{n} A_j^\kappa\right)^{\frac{1}{n}} \quad \text{and} \quad f_{n,m}(\kappa) \stackrel{\text{def}}{=} \max_{0\le s\le m}\log r_{n+s}(\kappa).$$

By Jensen's inequality $\log r_n(\kappa) \ge \kappa\mathbb{E}\log A_1$ and by the same inequality $r_n(\kappa) \le (\mathbb{E}\prod_{j=1}^{n} A_j^\gamma)^{\frac{1}{n\gamma}} = r_n(\gamma)^{\frac{1}{\gamma}}$. Condition $\mathbf{C}'$ thus implies that the set of functions

$\{r_n(\cdot)\}$ is uniformly bounded on $[0,\gamma]$ and hence also the set of functions $\{f_{n,m}(\cdot)\}$ is uniformly bounded on $[0,\gamma]$. Since functions $r_n(\cdot)$ are convex on $[0,\gamma]$, the functions $f_{n,m}(\cdot)$ are convex on $[0,\gamma]$ too. Next, $f_n(\kappa) \stackrel{\text{def}}{=} \lim_{m\to\infty} f_{n,m}(\kappa)$ is a limit of functions which converge uniformly on $[0,\gamma-\varepsilon]$, where $\varepsilon > 0$ is small enough. This happens because of (a) monotonicity in m of the sequence under the limit sign, (b) convexity, and (c) existence of bounded right derivatives $f'_{n,m}(0)$. But then also the monotonically decaying sequence $f_n(\cdot)$ converges uniformly on $[0,\gamma-\varepsilon]$ (because of the same reasons). Finally, since $r(\kappa) = \lim_{n\to\infty} f_n(\kappa)$, the lemma is proved. □

4.2 Sequences of random variables satisfying the maximal inequality

Let $Y_1(\omega), Y_2(\omega), \ldots$ be a sequence of random variables on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$. We put

$$S^*_{k,n}(\omega) = \max_{0\le s\le n-1} \left| \sum_{j=k}^{k+s} Y_j \right|, \quad S_n(\omega) = \left| \sum_{j=1}^{n} Y_j \right|.$$

Lemma 6 *Suppose that for some constant C the inequality $\|S^*_{k,n}\| \le Cn^{\frac{1}{2}}$ holds for all k, n. Then*

$$\lim_{n\to\infty} n^{-\frac{1+c}{2}} S_n = 0 \quad \textit{with } \mathbb{P} \textit{ probability } 1 \textit{ for any } c > 0. \tag{4.1}$$

Proof The condition of the lemma implies that

$$\mathbb{E}\big(n^{-\frac{1+c}{2}} S_n\big)^{2+2\delta} = n^{-(1+c)(1+\delta)} \|S_n\|^{2+2\delta} \le C_1 n^{-c(1+\delta)},$$

where $C_1 = C^{2+2\delta}$. If an integer m is such that $c(1+\delta)m > 1$, then

$$\sum_{n=1}^{\infty} \mathbb{E}\big(n^{-\frac{1+c}{2}m} S_{n^m}\big)^{2+2\delta} \le C_1 \sum_{n=1}^{\infty} n^{-c(1+\delta)m} < \infty.$$

This proves (4.1) for the subsequence n^m. To control the rest of the sequence, we shall show that

$$V(n,l) \equiv \left|(n^m + l)^{-\frac{1+c}{2}} S_{n^m+l} - n^{-\frac{1+c}{2}m} S_{n^m}\right| \to 0 \quad \text{as } n\to\infty$$

uniformly in $l \in [1, (n+1)^m - n^m]$ with $\mathbb{P}$-probability 1. To this end note that

$$\begin{aligned} V(n,l) &= \left|(n^m+l)^{-\frac{1+c}{2}}(S_{n^m+l} - S_{n^m}) - \big(n^{-\frac{1+c}{2}m} - (n^m+l)^{-\frac{1+c}{2}}\big) S_{n^m}\right| \\ &\le I_1(n,l) + I_2(n), \end{aligned}$$

where $I_1(n,l) = n^{-m\frac{1+c}{2}}|S_{n^m+l} - S_{n^m}|$ and $I_2(n) = n^{-m\frac{1+c}{2}}S_{n^m}$. We have just proved that $I_2(n) \to 0$ as $n \to \infty$. To estimate $I_1(n,l)$ note that

$$I_1(n,l) \le n^{-m\frac{1+c}{2}} \left| \sum_{j=n^m+1}^{n^m+l} Y_j \right| \le n^{-m\frac{1+c}{2}} S^*_{n^m,(n+1)^m-n^m} \equiv I_3(n).$$

But then

$$\begin{aligned}\mathbb{E}(I_3(n))^{2+2\delta} &= n^{-m(1+c)(1+\delta)} \left\| S^*_{n^m,(n+1)^m-n^m} \right\|^{2+2\delta} \\ &\le C_1 n^{-m(1+c)(1+\delta)}((n+1)^m - n^m)^{1+\delta} \\ &\le C_2 n^{-m(1+c)(1+\delta)+(m-1)(1+\delta)} = C_2 n^{-(1+\delta)(mc+1)},\end{aligned}$$

where the choice of C_2 is obvious. It is now clear that

$$\sum_{n=1}^{\infty} \mathbb{E}(I_3(n))^{2+2\delta} < \infty$$

and hence $\lim_{n\to\infty} I_3(n) = 0$ with $\mathbb{P}$-probability 1. This implies that $I_1(n,l) \to 0$ and thus also $V(n,l) \to 0$ as $n \to \infty$ uniformly in $l \in [1,(n+1)^m - n^m]$ with $\mathbb{P}$-probability 1. The lemma is proved. □

4.3 General equations for $\mathcal{E}_x T_x$ and $\mathrm{Var}_x(T_x)$

We shall make use of two general equations. One is the well known equation for the expectations of hitting times (Eq. (4.2)). It can be found in any textbook on Markov chains. The other (Eq. (4.3)) establishes relationbetween the expectation and the variance of a hitting time of a random walk. It is equally elementary but it seems that it is easier to derive it than to find a proper reference. Since the proof of (4.3) naturally includes the derivation of (4.2) both relations are proved here.

Consider a connected Markov chain with a discrete phase space S and a transition kernel $k(x,y)$, and let $\mathcal{B}$ be a proper subset of S. For $x \in S \setminus \mathcal{B}$ denote by T_x the first moment at which the random walk starting from x hits $\mathcal{B}$. Put

$$e(x) \stackrel{\text{def}}{=} \mathcal{E}_x(T_x), \quad v(x) \stackrel{\text{def}}{=} \mathcal{E}_x(T_x - e(x))^2 \equiv \mathrm{Var}_x(T_x),$$

where $\mathcal{E}_x$ is the usual expectation with respect to the measure on the space of trajectories starting from x. All expectations considered in this section are supposed to be finite.

Lemma 7 *The functions $e(x)$ and $v(x)$ satisfy the following systems of equations:*

$$\begin{cases} e(x) = \sum_y k(x,y)e(y) + 1 & \text{if } x \in S \setminus \mathcal{B}, \\ e(x) = 0 & \text{if } x \in \mathcal{B}, \end{cases} \tag{4.2}$$

$$\begin{cases} v(x) = \sum_y k(x,y)v(y) + f(x) & \text{if } x \in S \setminus \mathcal{B}, \\ v(x) = 0 & \text{if } x \in \mathcal{B}, \end{cases} \tag{4.3}$$

where $f(x) = \sum_y k(x,y)(e(y) - e(x) + 1)^2$.

Proof Denote by $\chi_{x,y}$ the indicator function of the event

{the first step of a random walk starting from x is to y}.

Obviously $1 = \sum_y \chi_{x,y}$ and hence

$$T_x = \sum_y \chi_{x,y} T_x = \sum_y \chi_{x,y}(T_y + 1). \tag{4.4}$$

Since $\mathcal{E}_x(\chi_{x,y}(T_y + 1)) = k(x,y)(\mathcal{E}_y T_y + 1)$, applying $\mathcal{E}_x$ to both parts of (4.4) leads to (4.2).

Similarly

$$(T_x - e(x))^2 = \sum_y \chi_{x,y}(T_x - e(x))^2 = \sum_y \chi_{x,y}(T_y + 1 - e(x))^2 \tag{4.5}$$

and applying $\mathcal{E}_x$ to both parts of (4.5) leads to

$$v(x) = \sum_y k(x,y)\mathcal{E}_y(T_y + 1 - e(x))^2. \tag{4.6}$$

In order to obtain the first equation of (4.3) it remains to observe that

$$\mathcal{E}_y(T_y + 1 - e(x))^2 = \mathcal{E}_y(T_y - e(y))^2 + (e(y) - e(x) + 1)^2 = v(y) + (e(y) - e(x) + 1)^2$$

and to substitute the last relation into (4.6).

Finally, the second equation in (4.2) and (4.3) is obvious. □

4.4 Proof of Lemma 3

To prove Lemma 3 we shall use the results of the previous subsection in the case when $S = \mathbb{Z}$ is a line and $\mathcal{B} \equiv \mathcal{B}_n$ is a semi-line of integers which are $\geq n$. Technically, equations (4.2) are a particular case of (4.3) and it makes sense to solve that latter for a general function $f(x)$. Note first that (4.3) can be re-written in terms of parameters p_k, $-\infty < k < \infty$, as follows:

$$\begin{cases} g_k = p_k g_{k+1} + q_k g_{k-1} + f_k & \text{if } k < n, \\ g_n = 0, \end{cases} \tag{4.7}$$

where the meaning of g_k depends on the choice of f. Solving (4.7) is a relatively simple and well studied matter. The following lemma is included into this work for the sake of completeness. As before, $A_j = p_j q_j^{-1} \equiv p_j(1-p_j)^{-1}$; the sequence $\omega = (p_j)_{-\infty<j<\infty}$ is fixed throughout this section.

Lemma 8 *Suppose that*

(i) $\sum_{j=0}^{\infty} \prod_{i=0}^{j} A_{-i} < \infty$ *and*
(ii) f_k *is such that* $\sum_{j=0}^{\infty} |f_{-j}| \prod_{i=0}^{j} A_{-i} < \infty$.

Then the solution (g_k), $-\infty < k \le n-1$, *to* (4.7) *is given by*

$$g_k = \sum_{j=k}^{n-1} d_j, \quad \text{where} \quad d_j = \sum_{i=0}^{\infty} A_j \dots A_{j-i+1} p_{j-i}^{-1} f_{j-i}. \tag{4.8}$$

This solution can be obtained as $g_k = \lim_{a\to-\infty} h_k$, *where* h_k *is a solutions to*

$$\begin{cases} h_k = p_k h_{k+1} + q_k h_{k-1} + f_k & \text{if } a < k < n, \\ h_a = h_n = 0, \end{cases} \tag{4.9}$$

Proof To solve (4.9), present h_k as

$$h_k = \varphi_k h_{k+1} + \tilde{d}_k, \quad k \ge a. \tag{4.10}$$

If we put $\varphi_a = 0$ and $\tilde{d}_a = 0$, then an easy induction argument (involving (4.9)) leads to the following formulae:

$$\varphi_k = (1 - q_k\varphi_{k-1})^{-1} p_k, \quad k \ge a+1, \tag{4.11}$$

$$\tilde{d}_k = A_k \tilde{d}_{k-1} + w_k, \; k \ge a+1, \quad \text{where } w_k = (1 - q_k\varphi_{k-1})^{-1} f_k. \tag{4.12}$$

Iterating (4.10) and (4.12) leads to

$$h_k = \tilde{d}_k + \varphi_k \tilde{d}_{k+1} + \dots + \varphi_k\varphi_{k+1}\dots\varphi_{n-1}\tilde{d}_{n-1}$$

and

$$\tilde{d}_k = w_k + A_k w_{k-1} + \dots + A_k \cdots A_{a+2} w_{a+1}.$$

It follows from (4.11) that $0 \leq \varphi_k < 1$ and (direct calculation) $1 - \varphi_k = q_k(1 - q_k\varphi_{k-1})^{-1}(1 - \varphi_{k-1})$. Hence

$$1 - \varphi_k \leq A_k(1 - \varphi_{k-1}) \leq A_k A_{k-1} \dots A_{a+1} \to 0 \quad \text{as} \quad a \to -\infty,$$

where the last relation follows from condition (i) of the Lemma. In other word, $\lim_{a\to-\infty} \varphi_k = 1$, and condition (ii) now implies that $\lim_{a\to-\infty} \tilde{d}_k = d_k$ and hence $\lim_{a\to-\infty} \tilde{h}_k = g_k$. □

We shall now prove Lemma 3. To this end note first that if we substitute $f_k \equiv 1$ into (4.8) and (4.9), then, according to (4.2) we obtain formulae for $e_k \equiv \mathcal{E}_\omega T_k(n)$ and thus also for $\mu_k = e_{k+1} - e_k$ (see Lemma 2). Next, to find $v_k \equiv \mathrm{Var}_\omega T_k(n)$ we have to put

$$f_k = p_k(e_{k+1} - e_k + 1)^2 + q_k(e_{k-1} - e_k + 1)^2 = p_k(\mu_k + 1)^2 + q_k(1 - \mu_{k-1})^2.$$

The main equation in (4.8) can be rewritten as

$$p_k(g_k - g_{k+1}) = q_k(g_{k-1} - g_k) + f_k \quad \text{and thus}$$
$$(g_k - g_{k+1}) = A_k(g_{k-1} - g_k) + p_k^{-1} f_k.$$

In particular this leads to the following relations:

$$\mu_k = A_k \mu_{k-1} + p_k^{-1}.$$

To see now that d_j in (4.8) turns into (2.2) is a matter of very simple calculation.

Relation (2.3) follows now from the condition $r(2) < 1$. Finally, the explicit expression (2.4) is again a matter of simple calculation. Lemma 3 is proved. □

References

1. Alili, S.: Asymptotic behaviour for random walks in random environments. J. Appl. Probab. **36**, 334–349 (1999)
2. Bolthausen, E., Sznitman, A.: Ten lectures on Random Media, DMV-Lectures, vol. 32, Birkhäuser, Basel (2002)
3. Goldsheid, I.Ya.: Linear and sub-linear growth and the CLT for hitting times of a random walk in random environment on a Strip (in preparation)
4. Kesten, H., Kozlov, M.V., Spitzer, F.: Limit law for random walk in a random environment. Composito Mathematica **30**, 145–168 (1975)
5. Kozlov, M.V.: A random walk on a line with stochastic structure (in Russian). Probab. Theory Appl. **18**, 406–408 (1973)
6. Mayer-Wolf, E., Roitershtein, A., Zeitouni, O.: Limit theorems for one-dimensional random walks in Markov random environments, Arxiv preprint math.0308154, 2003—arxiv.org
7. Sinai, Ya.G.: The limiting behavior of a one-dimensional random walk in a random medium. Theory Probab. Appl. **27**, 256–268 (1982)

8. Solomon, F.: Random walks in a random environment. Ann. Probab. **3**, 1–31 (1975)
9. Sznitman, A.-S.: Topics in random walks in random environment. In: School and Conference on Probability Theory, ICTP Lecture Notes Series, Trieste, vol. 17, pp. 203–266 (2004)
10. Zeitouni, O.: Random walks in random environment, XXXI Summer school in Probability, St. Flour (2001). Lecture Notes in Mathematics, vol. 1837, pp. 193–312. Springer, Berlin Heidelberg New York (2004)