Probab. Theory Relat. Fields 136, 263–297 (2006)
Digital Object Identifier (DOI) 10.1007/s00440-005-0483-y

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Gaussian fluctuations of characters of symmetric groups and of Young diagrams

Received: 7 February 2005 / Revised version: 7 September 2005 /
Published online: 29 December 2005 –

Abstract. We study asymptotics of reducible representations of the symmetric groups S_q for large q. We decompose such a representation as a sum of irreducible components (or, alternatively, Young diagrams) and we ask what is the character of a randomly chosen component (or, what is the shape of a randomly chosen Young diagram). Our main result is that for a large class of representations the fluctuations of characters (and fluctuations of the shape of the Young diagrams) are asymptotically Gaussian; in this way we generalize Kerov's central limit theorem. The considered class consists of representations for which the characters almost factorize and this class includes, for example, the left-regular representation (Plancherel measure), irreducible representations and tensor representations. This class is also closed under induction, restriction, outer product and tensor product of representations. Our main tool in the proof is the method of genus expansion, well known from the random matrix theory.

1. Introduction

1.1. Representations of large symmetric groups

Irreducible representations of the symmetric groups S_q are indexed by Young diagrams and nearly all questions about them, such as values of the characters or decomposition into irreducible components of a restriction, induction, tensor product or outer product of representations can be answered by combinatorial algorithms such as Murnaghan-Nakayama formula or the Littlewood-Richardson rule. Unfortunately, these exact combinatorial tools become very complicated and cumbersome when the size of the symmetric group S_q tends to infinity. For example, a restriction of an irreducible representation consists typically of a very large number of Young diagrams and listing them all does not give much insight into their structure. In order to deal with such questions in the asymptotic region when $q \to \infty$ we should be more modest and ask questions of a more statistical flavor: what is the typical shape of a Young diagram contributing to a given representation? what are the fluctuations of the Young diagrams around the most probable shape?

In this article we are interested in the situation when—speaking informally—a typical Young diagram contributing to the considered representation of S_q has

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at most $O(\sqrt{q})$ rows and columns. The first results in this direction concerned the left-regular representation (or equivalently, Plancherel measure on Young diagrams): Vershik and Kerov [VK77] and Logan and Shepp [LS77] found the shape of a typical Young diagram which contributes to the left-regular representation and later Kerov [Ker93b] announced that fluctuations of Young diagrams contributing to the left-regular representation around their limit shape are Gaussian (for the complete proof together with a detailed history of the result we refer to the article by Ivanov and Olshanski [IO02]). A non-commutative version of this result was given by Hora [Hor02, Hor03]. Biane [Bia98, Bia01, Bia03] considered a more general case, namely representations with approximate factorization of characters and proved that the shape of the typical Young diagram contributing to such representations can be described by the means of Voiculescu's free probability theory [VDN92].

In this article we present a condition on factorization of characters which is strong enough to ensure Gaussian fluctuations of Young diagrams and of characters and which is weak enough to be very common among naturally arising representations. In this way we prove a generalization of Kerov's central limit theorem for a very wide class of representations.

1.2. Genus expansion, random matrices and free probability

Free probability of Voiculescu [VDN92] is a non-commutative probability theory which turned out to be very successful in describing random matrices. The combinatorial structure behind this theory is the lattice of non-crossing partitions [Kre72] and the corresponding notion of free cumulants [Spe97]. It was Biane [Bia98] who realized that the same structure describes the leading terms in the asymptotic description of representations of symmetric groups. In this way some notions concerning random matrices were matched (via free probability theory) to some notions concerning representations of symmetric groups.

However, the connection between the random matrix theory and the representations of symmetric groups is much deeper than just the connection to free probability and non-crossing partitions. Example of such a direct connection was given by Okounkov [Oko00] who showed that the joint distribution of the largest eigenvalues of a GUE random matrix coincides (after appropriate scaling) with the joint distribution of the longest rows of a Young diagram distributed according to the Plancherel measure. For proving such results it is not enough to consider the first-order approximation given by non-crossing partitions and one has to use exact formulas. Such formulas for the moments of large classes of random matrices were known for a long time and they can be viewed as series indexed by two-dimensional surfaces [Zvo97] and Okounkov [Oko00] found their counterpart for random Young diagrams distributed according to the Plancherel measure. The asymptotic behavior of a term in this expansion depends only on the topology of the surface and for this reason such formulas are called genus expansions. It became clear that the origin of the similarities between the random matrix theory and the theory of the representations of the symmetric groups is the common structure of the genus expansion.

In our recent work [Śni04] we pointed out that the genus expansion method can be applied not only to the Plancherel measure but to a wide class of representations. This method is also the main tool in the proofs in this article.

1.3. *Higher-order free probability*

As we mentioned above, the first order approximation is not sufficient to calculate the fluctuations of characters and of Young diagrams and therefore such fluctuations cannot be described in the framework of the (usual) free probability theory. On the other hand, in a series of articles [MS04, MŚS04, CMŚS05] it was demonstrated that by considering some more complicated versions of non-commutative probability spaces it is possible to describe fluctuations of random matrices in the framework of, so called, higher order free probability. In this theory the non-crossing partitions are replaced by a more general object, namely annular non-crossing partitions. Results presented in this article suggest that it should be possible to describe in this framework also the fluctuations of Young diagrams and we will deal with this problem in a forthcoming article.

1.4. *Factorization of characters*

Biane [Bia01] proved (under some mild technical assumptions) that the Young diagrams contributing to some finite-dimensional reducible representation of the symmetric group S_q will concentrate around some limit shape if and only if the normalized character of the representation

$$\chi(\pi) := \frac{\operatorname{Tr} \rho(\pi)}{\operatorname{Tr} \rho(e)}$$

approximately factorizes, i.e. informally speaking

$$\chi(\sigma_1 \cdots \sigma_n) \approx \chi(\sigma_1) \cdots \chi(\sigma_n) \tag{1}$$

for all permutations $\sigma_1, \dots, \sigma_n$ with disjoint supports (to be more precise: we consider a sequence of representations (ρ_q) where ρ_q is a representation of S_q and the approximate equality (1) should hold in the limit $q \to \infty$). The result of Biane can be viewed as an analogue of the law of large numbers while the results presented in this article are an analogue of the central limit theorem; one can ask therefore which condition should replace (1) in order to prove such stronger results.

We will not be very far from the truth when we say that for the results of Biane [Bia01] it is enough to assume some version of (1) for $n = 2$. If we treat permutations σ_1, σ_2 as random variables and the normalized character χ as an expectation $\mathbb{E}$ then this condition can be equivalently written as a condition for the covariance:

$$\operatorname{Cov}(\sigma_1, \sigma_2) = \mathbb{E}(\sigma_1\sigma_2) - \mathbb{E}(\sigma_1)\mathbb{E}(\sigma_2) \approx 0.$$

Covariance is a special case of a more general probabilistic notion of a cumulant (we will recall the necessary definitions in Section 2.3) therefore it is quite natural to expect that the correct condition for the factorization of characters should involve all cumulants and indeed in this article we prove that the following condition is sufficient for our purposes (Theorem and Definition 1): for any cycles $\sigma_1, \dots, \sigma_n$ with disjoint supports we assume that

$$k_n(\sigma_1, \dots, \sigma_n) = O\left(q^{-\frac{|\sigma_1|+\cdots+|\sigma_n|+2(n-1)}{2}}\right), \tag{2}$$

where $|\pi|$ denotes the minimal number of factors needed to write the permutation π as a product of transpositions.

It should be stressed that the decay of the cumulants in the condition (2) carries a strong resemblance to the decay of the cumulants of the entries of many interesting classes of random matrices [Col03] and, as we mentioned in Section 1.3, it is not an accident.

1.5. Fluctuations of Young diagrams

Every finite-dimensional representation ρ_q of the symmetric group S_q defines a canonical probability measure on the Young diagrams contributing to ρ_q given as follows: probability of a Young diagram λ should be proportional to the total dimension of all irreducible components of type $[\lambda]$. Our goal is to consider some interesting function f on the set of the Young diagrams with q boxes and to study the distribution of the random variable $f(\lambda)$. In principle, the information about the characters such as (2) should be sufficient to compute the distribution of the random variables $f(\lambda)$ for reasonable functions f, however this relation is not very direct. An analogue of this situation can be found in the random matrix theory, where the knowledge of the joint distribution of the entries of a random matrix should be enough to find the joint distribution of the eigenvalues, however the actual calculation might be quite involved.

In this article we will show that the joint distribution of the random variables of the form $f(\lambda)$ converges to a Gaussian one if f is the value of the corresponding irreducible character on a prescribed permutation or some functional describing the shape of λ.

1.6. Overview of this article

In Section 2 we present briefly all necessary notions needed to state the main result and its applications. In Section 3 we present the main result: Theorem and Definition 1 where four equivalent conditions are given which ensure Gaussian fluctuations of the characters and of the shape of the Young diagrams. We also show that the considered class of representations has many interesting examples and that it is closed under some natural operations such as induction, restriction, outer product and tensor product of representation. Section 4 contains the proof of the main result. Finally, in Section 5 we prove some technical results used in the proof of the main theorem.

2. Preliminaries

2.1. Normalized conjugacy class indicators

Let integer numbers $k_1, \dots, k_m \geq 1$ be given. We define the normalized conjugacy class indicator to be a central element in the group algebra $\mathbb{C}(S_q)$ given by [KO94, Bia03, Śni04]

$$\Sigma_{k_1,\dots,k_m} = \sum_a (a_{1,1}, a_{1,2}, \dots, a_{1,k_1}) \cdots (a_{m,1}, a_{m,2}, \dots, a_{m,k_m}), \tag{3}$$

where the sum runs over all one–to–one functions

$$a : \big\{ \{r, s\} : 1 \leq r \leq m,\ 1 \leq s \leq k_r \big\} \to \{1, \dots, q\}$$

and $(a_{1,1}, a_{1,2}, \dots, a_{1,k_1}) \cdots (a_{m,1}, a_{m,2}, \dots, a_{m,k_m})$ denotes the product of disjoint cycles. Of course, if $q < k_1 + \cdots + k_m$ then the above sum runs over the empty set and $\Sigma_{k_1,\dots,k_m} = 0$.

In other words, we consider a Young diagram with the rows of the lengths $k_1, \dots, k_m$ and all ways of filling it with the elements of the set $\{1, \dots, q\}$ in such a way that no element appears more than once. Each such a filling can be interpreted as a permutation when we treat the rows of the Young tableau as disjoint cycles.

It follows that $\Sigma_{k_1,\dots,k_m}$ is a linear combination of permutations which in the cycle decomposition have cycles of length $k_1, \dots, k_m$ (and, additionally, $q - (k_1 + \cdots + k_m)$ fix-points). Each such a permutation appears in $\Sigma_{k_1,\dots,k_m}$ with some positive integer multiplicity depending on the symmetry of the tuple $k_1, \dots, k_m$.

2.2. Disjoint product

Usually by a product of normalized conjugacy classes $\Sigma_{k_1,\dots,k_m}$ we understand their product as elements of the group algebra $\mathbb{C}(S_q)$. However, sometimes it is convenient to consider their disjoint product defined by

$$\Sigma_{k_1,\dots,k_m} \bullet \Sigma_{l_1,\dots,l_n} = \Sigma_{k_1,\dots,k_m,l_1,\dots,l_n}. \tag{4}$$

Remark 1. The readers familiar with the notion of partial permutations of Ivanov and Kerov [IK99] will see that (4) is compatible with the following definition of the product $\alpha_1 \bullet \alpha_2$, when α_1, α_2 are partial permutations:

$$\alpha_1 \bullet \alpha_2 = \begin{cases} \alpha_1 \alpha_2 & \text{if supports of } \alpha_1 \text{ and } \alpha_2 \text{ are disjoint,} \\ 0 & \text{otherwise.} \end{cases} \tag{5}$$

Further discussion can be found in Section 4.3.

2.3. Classical cumulants

The notion of cumulants (or semi-invariants) was introduced to describe the convolution of measures in the classical probability theory. For more details about cumulants in the classical and non-commutative probability theory we refer to overview articles [Hal00, Mat99, Leh04a].

If $X_1, \dots, X_n$ are random variables we define their (classical) cumulant to be an appropriate coefficient of the formal expansion of the logarithm of the multidimensional Fourier transform:

$$k(X_1, \dots, X_n) = \frac{\partial^n}{\partial t_1 \cdots \partial t_n}\bigg|_{t_1=\cdots=t_n=0} \log \mathbb{E} e^{t_1 X_1 + \cdots + t_n X_n}.$$

Cumulant is linear with respect to each of its arguments. If random variables $X_1, \dots, X_n$ can be split into two groups such that the random variables from the first class are independent with the random variables from the second class then their cumulant vanishes: $k(X_1, \dots, X_n) = 0$.

The first two cumulants

$$k(X) = \mathbb{E}X,$$
$$k(X_1, X_2) = \mathbb{E}(X_1 X_2) - \mathbb{E}(X_1)\mathbb{E}(X_2) = \operatorname{Cov}(X_1, X_2)$$

coincide with the mean value and the covariance.

2.4. Elements of the group algebra as random variables

Let us fix some finite-dimensional representation ρ_q of the symmetric group S_q. We can treat any commuting family of elements of the group algebra $\mathbb{C}(S_q)$ as a family of random variables equipped with the mean value given by the normalized character:

$$\mathbb{E}X := \chi^{\rho_q}(X) = \frac{\operatorname{Tr} \rho_q(X)}{\operatorname{Tr} \rho_q(1)}. \tag{6}$$

It should be stressed that in the general case we treat elements of $\mathbb{C}(S_q)$ as random variables only on a purely formal level; in particular we do not treat them as functions on some Kolmogorov probability space.

Usually by the product of such random variables we understand the natural product in the group algebra $\mathbb{C}(S_q)$ and we denote the resulting cumulants (called natural cumulants) by $k(X_1, \dots, X_n)$. However, sometimes it is more convenient to take as the product of random variables the disjoint product $\bullet$; we denote the resulting cumulants (called disjoint cumulants) by $k^\bullet(X_1, \dots, X_n)$.

2.5. Canonical probability measure on Young diagrams associated to a representation

There is a special case when it is possible to give a truly probabilistic interpretation to (6): it is when for the family of random variables we take the center of $\mathbb{C}(S_q)$ and the multiplication is the natural product in the group algebra $\mathbb{C}(S_q)$. The center of $\mathbb{C}(S_q)$ is isomorphic (via Fourier transform) to the algebra of functions on Young diagrams and the expected value (6) corresponds under this isomorphism to the probability measure on Young diagrams with q boxes such that the probability of λ is proportional to the total dimension of the irreducible components of type $[\lambda]$ in ρ_q.

2.6. Generalized Young diagrams

Let λ be a Young diagram. We assign to it a piecewise affine function $\omega^\lambda : \mathbb{R} \to \mathbb{R}$ with slopes ± 1, such that $\omega^\lambda(x) = |x|$ for large $|x|$ as it can be seen on the example from Figure 2. By comparing Figure 1 and Figure 2 one can easily see that the graph of ω^λ can be obtained from the graphical representation of the Young diagram by an appropriate mirror image, rotation and scaling by the factor $\sqrt{2}$. We call ω^λ the generalized Young diagram associated with the Young diagram λ [Ker93a, Ker98, Ker99].

The class of generalized Young diagrams consists of all functions $\omega : \mathbb{R} \to \mathbb{R}$ which are Lipschitz with constant 1 and such that $\omega(x) = |x|$ for large $|x|$ and

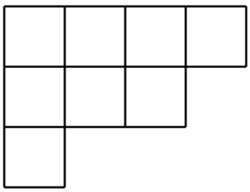

Fig. 1. Young diagram associated with a partition $8 = 4 + 3 + 1$

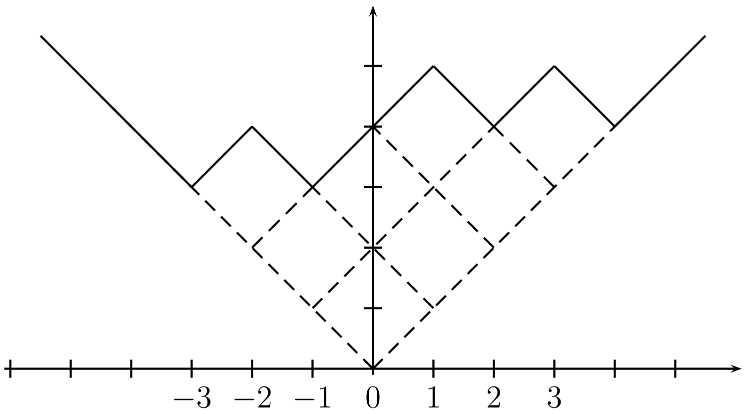

Fig. 2. Generalized Young diagram associated with a partition $8 = 4 + 3 + 1$

of course not every generalized Young diagram can be obtained by the above construction from some Young diagram λ.

The setup of generalized Young diagrams is very useful in the study of the asymptotic properties since it allows us to define easily various notions of convergence of the Young diagram shapes.

2.7. Functionals of the shape of Young diagrams

The main result of this article is that the fluctuations of the shape of some random Young diagrams converge (after some rescaling) to a Gaussian distribution. Since the space of (generalized) Young diagrams is infinite-dimensional therefore we need to be very cautious when dealing with such statements. In fact, we will consider a family of functionals on Young diagrams and we show that the joint distribution of each finite set of these functionals converges to the Gaussian distribution.

The functionals mentioned above are given as follows: for a Young diagram λ and the corresponding generalized Young diagram ω we denote $\sigma(x) = \frac{\omega(x)-|x|}{2}$ [Bia98, IO02] and consider the family of maps

$$\tilde{p}_n(\lambda) = \int_{\mathbb{R}} x^n \sigma''(x) dx. \tag{7}$$

Since σ'' makes sense as a distribution and σ is compactly supported hence the collection $\big(\tilde{p}_n(\lambda)\big)_n$ determines the Young diagram λ uniquely.

2.8. Transition measure of a Young diagram

To any generalized Young diagram ω we can assign the unique probability measure μ^ω on $\mathbb{R}$, called transition measure of ω, the Cauchy transform of which

$$G_{\mu^\omega}(z) = \int_{\mathbb{R}} \frac{1}{z-x} d\mu^\omega(x) \tag{8}$$

is given by

$$\log G_{\mu^\omega}(z) = -\frac{1}{2} \int_{\mathbb{R}} \log(z-x)\omega''(x)dx = -\frac{1}{2} \int_{\mathbb{R}} \frac{1}{z-x} \omega'(x) dx \tag{9}$$

for every $z \notin \mathbb{R}$. For a Young diagram λ we will write μ^λ as a short hand of μ^{ω^λ}. This definition may look artificial but it turns out [Ker93a, OV96, Bia98, Oko00] that it is equivalent to natural representation-theoretic definitions which arise by studying the irreducible representation ρ_q together with the inclusion $S_q \subset S_{q+1}$.

For $p > 0$ and a Young diagram λ we consider the rescaled (generalized) Young diagram $\omega^{p\lambda}$ given by $\omega^{p\lambda} : x \mapsto p\omega^\lambda\big(\frac{x}{p}\big)$. Informally speaking, the symbol $p\lambda$ corresponds to the shape of the Young diagram λ geometrically scaled by factor p (in particular, if λ has q boxes then $p\lambda$ has $p^2 q$ boxes). It is easy to see that (9) implies that the corresponding transition measure $\mu^{p\lambda}$ is a dilation of μ^λ:

$$\mu^{p\lambda} = D_p \mu^\lambda. \tag{10}$$

This nice behavior of the transition measure with respect to rescaling of Young diagrams makes it a perfect tool for the study of the asymptotics of symmetric groups S_q as $q \to \infty$.

2.9. Free cumulants of the transition measure

Cauchy transform of a compactly supported probability measure is given at the neighborhood of infinity by a power series

$$G_\mu(z) = \frac{1}{z} + \sum_{n \geq 1} M_n z^{-n-1},$$

where $M_n = \int_R x^n d\mu$ are the moments of the measure μ. It follows that on some neighborhood of infinity G_μ has a right inverse K_μ with respect to the composition of power series given by

$$K_\mu(z) = \frac{1}{z} + \sum_{n \geq 1} R_n z^{n-1}$$

convergent on some neighborhood of 0. The coefficients $R_i = R_i(\mu)$ are called free cumulants of measure μ. Free cumulants appeared implicitly in Voiculescu's R–transform [Voi86] and their combinatorial meaning was given by Speicher [Spe97].

Free cumulants are homogenous in the sense that if X is a random variable and c is some number then

$$R_i(cX) = c^i R_i(X)$$

and for this reason they are very useful in the study of asymptotic questions.

Each free cumulant R_n is a polynomial in the moments $M_1, M_2, \ldots, M_n$ of the measure and each moment M_n can be expressed as a polynomial in the free cumulants $R_1, \ldots, R_n$; in other words the sequence of moments $M_1, M_2, \ldots$ and the sequence of free cumulants $R_1, R_2, \ldots$ contain the same information about the probability measure. The functionals of Young diagrams considered in (7) have a nice geometric interpretation but they are not very convenient in actual calculations. For this reason we will prefer to describe the shape of a Young diagram by considering a family of functionals

$$\lambda \mapsto R_n(\mu^\lambda) \tag{11}$$

given by the free cumulants of the transition measure. Equation (9) shows that functionals $\tilde{p}_k$ from the family (7) can be expressed as polynomials in the functionals from the family (11) and vice versa.

Please note that the first two cumulants of a transition measure do not carry any interesting information since

$$R_1(\mu^\lambda) = M_1(\mu^\lambda) = 0,$$
$$R_2(\mu^\lambda) = M_2(\mu^\lambda) = q,$$

where q denotes the number of the boxes of the Young diagram λ.

Above we treated the free cumulant R_i as a function on Young diagrams, but it also can be viewed (via Fourier transform) as a central element in $\mathbb{C}(S_q)$.

3. Representations with character factorization property

3.1. Factorization of characters and Gaussian fluctuations

The following theorem is the main result of this article. In order not to scare the Reader we postpone its proof to Section 4.

Theorem and Definition 1. *For each $q \geq 1$ let ρ_q be a representation of S_q. We say that the sequence (ρ_q) has the character factorization property if it fulfills one (hence all) of the following equivalent conditions:*

– *for any cycles $\sigma_1, \dots, \sigma_n$ with disjoint supports*

$$k(\sigma_1, \dots, \sigma_n) q^{\frac{|\sigma_1|+\cdots+|\sigma_n|+2(n-1)}{2}} = O(1); \tag{12}$$

– *for any integers $l_1, \dots, l_n \geq 1$*

$$k^\bullet(\Sigma_{l_1}, \dots, \Sigma_{l_n}) q^{-\frac{l_1+\cdots+l_n-n+2}{2}} = O(1); \tag{13}$$

– *for any integers $l_1, \dots, l_n \geq 1$*

$$k(\Sigma_{l_1}, \dots, \Sigma_{l_n}) q^{-\frac{l_1+\cdots+l_n-n+2}{2}} = O(1); \tag{14}$$

– *for any integers $l_1, \dots, l_n \geq 2$*

$$k(R_{l_1}, \dots, R_{l_n}) q^{-\frac{l_1+\cdots+l_n-2(n-1)}{2}} = O(1). \tag{15}$$

Remark 2. In Corollary 19 we will prove that if conditions (13) and (14) hold true then they also hold true in a more general situation when conjugacy classes (Σ_{l_i}) with only one non-trivial cycle are replaced by general conjugacy classes $\Sigma_{l_{i,1},\dots,l_{i,m(i)}}$. Similarly one can show that if condition (12) holds true then it also holds true in a general situation when we do not assume that (σ_i) are cycles.

To show that a given sequence of representations has the character factorization property usually it is the most convenient to verify condition (12) or condition (13). Then conditions (14) and (15) are important corollaries (for applications see Corollary 4 below).

Expressions appearing in conditions (12)–(15) are closely related to each other and knowledge of one of them allows us to compute the others (in fact this is how Theorem and Definition 1 will be proved). For general n these formulas are quite involved, however in the following we will need only such formulas for $n \in \{1, 2\}$ and these are provided by the following theorem.

Theorem 3. *Let (ρ_q) has the character factorization property. If the limit of one of the expressions* (12)–(15) *exists for $n \in \{1,2\}$ then the limits of all of the expressions* (12)–(15) *exist for $n \in \{1,2\}$.*

These limits fulfill

$$c_{l+1} := \lim_{q\to\infty} \mathbb{E}(\sigma) q^{\frac{l-1}{2}} = \lim_{q\to\infty} \mathbb{E}(\Sigma_l) q^{-\frac{l+1}{2}} = \lim_{q\to\infty} \mathbb{E}(R_{l+1}) q^{-\frac{l+1}{2}}, \tag{16}$$

where σ is a cycle of length l and

$$\begin{aligned}
&\lim_{q\to\infty} \operatorname{Cov}(R_{l_1+1}, R_{l_2+1}) q^{-\frac{l_1+l_2}{2}} \\
&\quad = \lim_{q\to\infty} \operatorname{Cov}(\Sigma_{l_1}, \Sigma_{l_2}) q^{-\frac{l_1+l_2}{2}} \\
&\quad = \lim_{q\to\infty} \operatorname{Cov}^{\bullet}(\Sigma_{l_1}, \Sigma_{l_2}) q^{-\frac{l_1+l_2}{2}} \\
&\qquad + \sum_{r\geq 1} \sum_{\substack{a_1,\dots,a_r\geq 1 \\ a_1+\cdots+a_r=l_1}} \sum_{\substack{b_1,\dots,b_r\geq 1 \\ b_1+\cdots+b_r=l_2}} \frac{l_1 l_2}{r} c_{a_1+b_1}\cdots c_{a_r+b_r} \\
&\quad = \lim_{q\to\infty} \operatorname{Cov}(\sigma_1, \sigma_2) q^{\frac{l_1+l_2}{2}} - l_1 l_2 c_{l_1+1} c_{l_2+1} \\
&\qquad + \sum_{r\geq 1} \sum_{\substack{a_1,\dots,a_r\geq 1 \\ a_1+\cdots+a_r=l_1}} \sum_{\substack{b_1,\dots,b_r\geq 1 \\ b_1+\cdots+b_r=l_2}} \frac{l_1 l_2}{r} c_{a_1+b_1}\cdots c_{a_r+b_r}
\end{aligned} \tag{17}$$

where σ_1, σ_2 are disjoint cycles of length l_1, l_2, respectively, and where the numbers c_i were defined in (16)*.*

Proof of this theorem is also postponed to Section 4; we will prove it together with Theorem 1. Identity (16) was proved by Biane [Bia98, Bia01] and we skip its proof; for Readers acquainted with the results of Section 4 and Section 5 it will be a simple exercise.

Corollary 4. *Let (ρ_q) be as in Theorem 3 and let λ be a random Young diagram distributed according to the canonical probability measure associated to ρ_q.*

1. (Gaussian fluctuations of free cumulants) *Then the joint distribution of the centered random variables*

$$r_i = q^{-\frac{i-2}{2}} (R_i - \mathbb{E} R_i)$$

converges to a Gaussian distribution in the weak topology of probability measures, where R_i denotes the free cumulant of the transition measure μ^λ.

2. (Gaussian fluctuations of characters) *Let σ_i denote a cycle of length i. Then the joint distribution of the centered random variables*

$$q^{\frac{|\sigma_i|+1}{2}} \left(\chi^\lambda(\sigma_i) - \mathbb{E}\chi^\lambda(\sigma_i)\right) \tag{18}$$

converges to a Gaussian distribution in the weak topology of probability measures.

3. (Gaussian fluctuations of the shape of the Young diagrams) *Then the joint distribution of the centered random variables*

$$q^{-\frac{i-2}{2}}(\tilde{p}_i - \mathbb{E}\tilde{p}_i)$$

converges to a Gaussian distribution in the weak topology of probability measures, where $\tilde{p}_i = \tilde{p}_i(\lambda)$ is the functional of the shape of the Young diagram defined in (7).

Proof. We will prove now point (1). Condition (15) implies that if $n \neq 2$ then

$$\lim_{q\to\infty} k(r_{i_1}, \dots, r_{i_n}) = 0$$

and therefore the family (r_i) converges in moments to a Gaussian distribution. Since Gaussian measures are uniquely determined by their moments it follows that the convergence holds true also in the weak topology of probability measures.

To prove point 2 we observe that asymptotically, as $q \to \infty$ random variables (18) have the same behavior as random variables

$$\lambda \mapsto q^{\frac{-|\sigma_i|+1}{2}} \left(q^{\underline{|\sigma_i|}} \chi^\lambda(\sigma_i) - \mathbb{E} q^{\underline{|\sigma_i|}} \chi^\lambda(\sigma_i) \right), \tag{19}$$

where

$$q^{\underline{k}} = q(q-1)\cdots(q-k+1) \tag{20}$$

denotes the falling power. Function on Young diagrams (19) corresponds (via Fourier transform) to the central function in $\mathbb{C}(S_q)$

$$q^{\frac{-|\sigma_i|+1}{2}} \left(\Sigma_i - \mathbb{E}\Sigma_i \right).$$

It follows that we may use (14) in the same way as in the above proof of point 1.

Ivanov and Olshanski [IO02] proved that point 1 implies point 3; their proof is a careful analysis of the fact that $\tilde{p}_i$ can be expressed as a polynomial in free cumulants. □

3.2. Examples

All examples presented in this section not only have the character factorization property but additionally are as in Theorem 3.

Example 5 (Left-regular representation). It is easy to check that if ρ_q is the left-regular representation of S_q then for any permutations $\sigma_1, \dots, \sigma_n$ with disjoint supports

$$k(\sigma_1, \dots, \sigma_n) q^{\frac{|\sigma_1|+\cdots+|\sigma_n|+2(n-1)}{2}} = \begin{cases} 1 & \text{if } n = 1 \text{ and } \sigma_1 = e, \\ 0 & \text{otherwise.} \end{cases}$$

It follows from condition (12) that the left-regular representation has the character factorization property and that the mean and the covariance of the free cumulants are given by

$$\lim_{q\to\infty} \mathbb{E}(R_{l+1})q^{-\frac{l+1}{2}} = \begin{cases} 1 & \text{if } l=1, \\ 0 & \text{if } l\geq 2, \end{cases} \tag{21}$$

and

$$\lim_{q\to\infty} \operatorname{Cov}(R_{l_1+1}, R_{l_2+1})q^{-\frac{l_1+l_2}{2}} = \begin{cases} l_1 & \text{if } l_1=l_2\geq 2, \\ 0 & \text{if } l_1=l_2=1, \\ 0 & \text{if } l_1\neq l_2. \end{cases} \tag{22}$$

By applying Corollary 4 we recover Kerov central limit theorem for the Plancherel measure [Ker93b, IO02].

Example 6 (Tensor representations). For some integer $d_q \geq 1$ let ρ_q be the representation of S_q acting on $(\mathbb{C}^{d_q})^{\otimes q}$ by permutation of factors. This representation appears naturally within Schur-Weyl duality. It is easy to check that for permutations $\sigma_1,\dots,\sigma_n$ with disjoint supports

$$k(\sigma_1,\dots,\sigma_n)q^{\frac{|\sigma_1|+\cdots+|\sigma_n|+2(n-1)}{2}} = \begin{cases} \left(\frac{\sqrt{q}}{d_q}\right)^{|\sigma_1|} & \text{if } n=1, \\ 0 & \text{otherwise,} \end{cases}$$

hence if the limit $p := \lim_{q\to\infty} \frac{\sqrt{q}}{d_q}$ exists then (condition (12)) the sequence (ρ_q) has the character factorization property and the mean and the covariance of the free cumulants are given by

$$\lim_{q\to\infty} \mathbb{E}(R_{l+1})q^{-\frac{l+1}{2}} = p^{l-1},$$

and

$$\lim_{q\to\infty} \operatorname{Cov}(R_{l_1+1}, R_{l_2+1})q^{-\frac{l_1+l_2}{2}} = \sum_{r\geq 2} \binom{l_1}{r}\binom{l_2}{r} r\, p^{l_1+l_2-2r}$$

for all integers $l, l_1, l_2 \geq 1$. Note that for $p=0$ we recover the fluctuations of the Plancherel measure.

Some asymptotic results for this representation were proved by Biane [Bia01].

Example 7 (Irreducible representations). Let $c > 0$ be a constant and let (λ_q) be a sequence of Young diagrams. We assume that λ_q has q boxes and it has at most $c\sqrt{q}$ rows and columns. Suppose that the shapes of rescaled Young diagrams $q^{-\frac{1}{2}}\lambda_q$ converge to some limit. The convergence of the shapes of Young diagrams implies convergence of the free cumulants and it follows (condition (15)) that the sequence (ρ^{λ_q}) of the corresponding irreducible representations has the characters factorization property.

In this example the cumulants (14) and (15) vanish for $n \geq 2$ since the Young diagrams are non-random and the corresponding limits for $n=1$ are determined by the limit of the shape of the Young diagrams.

The above three examples are the building blocks from which one can construct some more complex representations with the help of the operations on representations presented below.

Theorem 8 (Restriction of representations). *Suppose that the sequence of representations (ρ_q) has the character factorization property. Let a sequence of integers (r_q) be given, such that $r_q \geq q$ and the limit $p = \lim_{q\to\infty} \frac{q}{r_q}$ exists.*

Let ρ'_q denote the restriction of the representation ρ_{r_q} to the subgroup $S_q \subseteq S_{r_q}$. Then the sequence (ρ'_q) has the factorization property of characters. The fluctuations of the free cumulants are determined by

$$c'_{l+1} := \lim_{q\to\infty} \mathbb{E}(R'_{l+1}) q^{-\frac{l+1}{2}} = p^{\frac{l-1}{2}} \lim_{q\to\infty} \mathbb{E}(R_{l+1}) q^{-\frac{l+1}{2}} = p^{\frac{l-1}{2}} c_{l+1}, \quad (23)$$

$$\begin{aligned} &\lim_{q\to\infty} \operatorname{Cov}(R'_{l_1+1}, R'_{l_2+1}) q^{-\frac{l_1+l_2}{2}} \\ &\quad = p^{\frac{l_1+l_2}{2}} \Bigg[\lim_{q\to\infty} \operatorname{Cov}(R_{l_1+1}, R_{l_2+1}) q^{-\frac{l_1+l_2}{2}} - l_1 l_2 c_{l_1+1} c_{l_2+1} \left(p^{-1} - 1\right) \\ &\qquad + \sum_{r\geq 1} \sum_{\substack{a_1,\dots,a_r\geq 1\\ a_1+\dots+a_r=l_1}} \sum_{\substack{b_1,\dots,b_r\geq 1\\ b_1+\dots+b_r=l_2}} \frac{l_1 l_2}{r} c_{a_1+b_1} \cdots c_{a_r+b_r} \left(p^{-r} - 1\right) \Bigg] \end{aligned} \quad (24)$$

for all $l, l_1, l_2 \geq 1$, where the quantities R'_i, c'_i concern the representations (ρ'_q) while R_i, c_i concern the representations (ρ_q).

In particular, for $p = 0$ we recover the fluctuations of the Plancherel measure.

Proof. Notice that for any permutations $\sigma_1, \dots, \sigma_n$ the value of the cumulant $k(\sigma_1, \dots, \sigma_n)$ is the same for the representation ρ'_q of S_q and for the representation ρ_{r_q} of S_{r_q}. It follows that

$$\begin{aligned} &\lim_{q\to\infty} k_{\rho'_q}(\sigma_1, \dots, \sigma_n) q^{\frac{|\sigma_1|+\dots+|\sigma_n|+2(n-1)}{2}} \\ &\quad = p^{\frac{|\sigma_1|+\dots+|\sigma_n|+2(n-1)}{2}} \lim_{q\to\infty} k_{\rho_{r_q}}(\sigma_1, \dots, \sigma_n) (r_q)^{\frac{|\sigma_1|+\dots+|\sigma_n|+2(n-1)}{2}}. \end{aligned}$$

Since the original representations (ρ_q) fulfill condition (12) hence restricted representations (ρ'_q) fulfill (12) as well. Equations (23) and (24) follow as special cases for $n = 1, 2$. □

Remark 9. Please notice that the above theorem concerns restrictions of the form $\rho_{r_q}\big\downarrow^{S_{r_q}}_{S_q}$ while it is even more interesting to ask about the asymptotics of the restrictions of the form $\rho''_q = \rho_{r_q}\big\downarrow^{S_{r_q}}_{S_q \times S_{r_q - q}}$. A typical question is the following one: let $[\lambda^{(1)}] \times [\lambda^{(2)}]$ be a random irreducible component of ρ''_q; is it true that the joint distribution of the free cumulants $\left(R_i(\lambda^{(r)})\right)_{r\in\{1,2\}; i\geq 2}$ converges after appropriate rescaling to a family of Gaussian variables? The answer for this question is positive and the Reader may easily prove it using the methods presented in Section 4.

For some interesting results concerning restrictions of irreducible representations corresponding to rectangular Young diagrams we refer to the work of Pittel and Romik [PR04].

Theorem 10 (Outer product of representations). *Suppose that for* $i \in \{1,2\}$ *the sequence of representations* $(\rho_q^{(i)})$ *has the character factorization property. Let sequences of positive integers* $r_q^{(i)}$ *be given, such that* $r_q^{(1)} + r_q^{(2)} = q$ *and the limits* $p^{(i)} := \lim_{q\to\infty} \frac{r^{(i)}}{q}$ *exist.*

Let $\rho'_q = \rho^{(1)}_{r_q^{(1)}} \circ \rho^{(2)}_{r_q^{(2)}}$ *denote the outer product of representations. Then the sequence* (ρ'_q) *has the factorization property of characters with*

$$c'_{l+1} := \lim_{q\to\infty} \mathbb{E}(R'_{l+1}) q^{-\frac{l+1}{2}} = \left(p^{(1)}\right)^{\frac{l+1}{2}} c^{(1)}_{l+1} + \left(p^{(2)}\right)^{\frac{l+1}{2}} c^{(2)}_{l+1}, \tag{25}$$

and with an explicit (but involved) covariance of free cumulants. The appropriate disjoint covariance is given by

$$\begin{aligned} &\lim_{q\to\infty} \mathrm{Cov}^{\bullet}_{\rho'_q}(\Sigma_{l_1}, \Sigma_{l_2}) q^{-\frac{l_1+l_2}{2}} \\ &\quad = (p^{(1)})^{\frac{l_1+l_2}{2}} \lim_{q\to\infty} \mathrm{Cov}^{\bullet}_{\rho_q^{(1)}}(\Sigma_{l_1}, \Sigma_{l_2}) q^{-\frac{l_1+l_2}{2}} \\ &\quad\quad + (p^{(2)})^{\frac{l_1+l_2}{2}} \lim_{q\to\infty} \mathrm{Cov}^{\bullet}_{\rho_q^{(2)}}(\Sigma_{l_1}, \Sigma_{l_2}) q^{-\frac{l_1+l_2}{2}}. \end{aligned}$$

Proof. By definition,

$$\rho'_q = \left(\rho^{(1)}_{r_q^{(1)}} \times \rho^{(2)}_{r_q^{(2)}}\right)\Big\uparrow^{S_q}_{S_{r_q^{(1)}} \times S_{r_q^{(2)}}}$$

and we can use the Frobenius reciprocity between the induction and restriction of representations. It follows that the corresponding normalized characters fulfill for all $l_1, \dots, l_n \geq 1$

$$\begin{aligned} \chi_{\rho'}(\Sigma_{l_1} \bullet \cdots \bullet \Sigma_{l_n}) = &\left(\chi_{\rho^{(1)}} \otimes \chi_{\rho^{(2)}}\right) \\ &\left((\Sigma_{l_1} \otimes 1 + 1 \otimes \Sigma_{l_1}) \bullet \cdots \bullet (\Sigma_{l_n} \otimes 1 + 1 \otimes \Sigma_{l_n})\right). \end{aligned}$$

We can treat the left-hand side as a mixed moment $\mathbb{E}(\Sigma_{l_1} \bullet \cdots \bullet \Sigma_{l_n})$ in the algebra of conjugacy classes equipped with the disjoint product; analogous interpretation is possible also for the right-hand side. Since equality holds for all $l_1, \dots, l_n \geq 2$ it follows that the joint distributions of the family of random variables $(\Sigma_i)_{i\geq 2}$ coincides with the joint distribution of random variables $(\Sigma_i \otimes 1 + 1 \otimes \Sigma_i)_{i\geq 2}$; in particular their cumulants are equal:

$$\begin{aligned} k^{\bullet}_{\rho'}(\Sigma_{l_1}, \dots, \Sigma_{l_n}) &= k^{\bullet}_{\rho^{(1)}\otimes\rho^{(2)}}(\Sigma_{l_1} \otimes 1 + 1 \otimes \Sigma_{l_1}, \dots, \Sigma_{l_n} \otimes 1 + 1 \otimes \Sigma_{l_n}) \\ &= k^{\bullet}_{\rho^{(1)}}(\Sigma_{l_1}, \dots, \Sigma_{l_n}) + k^{\bullet}_{\rho^{(2)}}(\Sigma_{l_1}, \dots, \Sigma_{l_n}). \end{aligned} \tag{26}$$

In the last equality we used that the cumulant is linear with respect to each of the arguments and that the mixed cumulant of independent random variables vanishes.

It follows that

$$\begin{aligned}
&\lim_{q\to\infty} k^{\bullet}_{\rho'_q}(\Sigma_{l_1},\dots,\Sigma_{l_n}) q^{-\frac{l_1+\cdots+l_n-n+2}{2}} \\
&\quad = (p^{(1)})^{\frac{l_1+\cdots+l_n-n+2}{2}} \lim_{q\to\infty} k^{\bullet}_{\rho^{(1)}_{r^{(1)}_q}}(\Sigma_{l_1},\dots,\Sigma_{l_n})(r^{(1)}_q)^{-\frac{l_1+\cdots+l_n-n+2}{2}} \\
&\qquad + (p^{(2)})^{\frac{l_1+\cdots+l_n-n+2}{2}} \lim_{q\to\infty} k^{\bullet}_{\rho^{(2)}_{r^{(2)}_q}}(\Sigma_{l_1},\dots,\Sigma_{l_n})(r^{(1)}_q)^{-\frac{l_1+\cdots+l_n-n+2}{2}}.
\end{aligned}$$

It follows that if the representations $\rho^{(i)}$ fulfill condition (13) then representations ρ'_q fulfill (13) as well. Equation (25) results as a special case of $n = 1$ and we leave it as a simple exercise to the Reader to study the case $n = 2$ and to find the covariance of the free cumulants, analogous to (23). □

Theorem 11 (Induction of representations). *Suppose that the sequence of representations (ρ_q) has character factorization property. Let a sequence of integers r_q be given, such that $r_q \le q$ and the limit $p = \lim_{q\to\infty} \frac{r_q}{q}$ exists.*

Let $\rho'_q = \rho_{r_q} \uparrow^{S_q}_{S_{r_q}}$ denote the induced representation. Then the sequence (ρ'_q) has the characters factorization property with

$$c'_{l+1} = \begin{cases} p^{\frac{l+1}{2}} c_{l+1} & \text{for } l \ge 2, \\ 1 & \text{for } l = 1, \end{cases}$$

and with an explicit (but involved) covariance of free cumulants.

Proof. It is enough to adapt the proof of Theorem 10. □

Theorem 12 (Tensor product of representations). *Suppose that for $i \in \{1,2\}$ the sequence of representations $(\rho^{(i)}_q)$ has character factorization property. Then the tensor product $\rho'_q = \rho^{(1)}_q \otimes \rho^{(2)}_q$ has the property of factorization of characters. Furthermore, the limit distribution and the fluctuations are the same as for the Plancherel measure* (21) *and* (22).

Proof. Since the normalized characters fulfill for any $\pi \in S_q$

$$\chi_{\rho'_q}(\pi) = \big(\chi_{\rho^{(1)}_q} \otimes \chi_{\rho^{(2)}_q}\big)(\pi \otimes \pi)$$

hence also the corresponding cumulants are equal:

$$k_{\rho'_q}(\sigma_1,\dots,\sigma_n) = k_{\rho^{(1)}_q \otimes \rho^{(2)}_q}(\sigma_1 \otimes \sigma_1, \dots, \sigma_n \otimes \sigma_n).$$

Theorem 16 of Leonov and Sirjaev can be used to calculate the right-hand side. Lemma 18 together with the condition (12) for $\rho^{(i)}$ show that the right-hand side is of order $O(q^{-(|\sigma_1|+\cdots+|\sigma_n|+n-1)})$ hence condition (12) is fulfilled for ρ'. It also follows that the limits in Theorem 3 are given by

$$c_{l+1} = \lim_{q\to\infty} \mathbb{E}(\sigma) q^{\frac{l-1}{2}} = \begin{cases} 1 & \text{if } l = 1, \\ 0 & \text{if } l \ge 2, \end{cases}$$

$$\lim_{q\to\infty} \operatorname{Cov}(\sigma_1,\sigma_2) q^{\frac{l_1+l_2}{2}} = 0,$$

where $\sigma, \sigma_1, \sigma_2$ are disjoint cycles with lengths, respectively, l, l_1, l_2. □

4. Proof of the main result

4.1. Toy example

Let us have a look on the main result (Theorem and Definition 1) for the simplest nontrivial case of $n = 2$. This will give us a heuristical insight into the problems which we shall encounter in the proof.

As we shall see in the following, the proof of the equivalence (14) and (15) is very easy, therefore the main difficulty is to show the equivalence of the conditions (12), (13), (14). For simplicity, we shall concentrate on the implications in only one direction (12) $\implies$ (13) $\implies$ (14).

The condition (14) for $n = 2$ requires that $k(\Sigma_{l_1}, \Sigma_{l_2})$ should not grow too fast. The identity

$$\begin{aligned} k(\Sigma_{l_1}, \Sigma_{l_2}) &= \chi(\Sigma_{l_1}\Sigma_{l_2}) - \chi(\Sigma_{l_1})\chi(\Sigma_{l_2}) \\ &= \chi(\Sigma_{l_1}\Sigma_{l_2} - \Sigma_{l_1} \bullet \Sigma_{l_2}) \\ &\quad + \left[\chi(\Sigma_{l_1} \bullet \Sigma_{l_2}) - \chi(\Sigma_{l_1})\chi(\Sigma_{l_2})\right] \end{aligned} \tag{27}$$

shows that there are three reasons why $k(\Sigma_{l_1}, \Sigma_{l_2})$ is non-zero:

1. the difference $\Sigma_{l_1}\Sigma_{l_2} - \Sigma_{l_1} \bullet \Sigma_{l_2}$ is non-zero. We need to estimate the conjugacy classes contributing to this difference.
2. there are $q^{\underline{l_1}} q^{\underline{l_2}}$ summands which contribute to $\chi(\Sigma_{l_1})\chi(\Sigma_{l_2})$ while there are only $q^{\underline{l_1+l_2}}$ summands which contribute to $\chi(\Sigma_{l_1} \bullet \Sigma_{l_2})$, where the falling powers $q^{\underline{k}}$ were defined in (20). Under the simplifying assumption that $\chi(\pi_1\pi_2) \approx \chi(\pi_1)\chi(\pi_2)$ the second summand in (27) is therefore of order
$$(q^{\underline{l_1+l_2}} - q^{\underline{l_1}} q^{\underline{l_2}})\chi(\pi_1\pi_2),$$
where π_1, π_2 are disjoint cycles of length l_1, l_2, respectively.
We need to find an estimate for $(q^{\underline{l_1+l_2}} - q^{\underline{l_1}} q^{\underline{l_2}})$.
3. every summand contributing to $\chi(\Sigma_{l_1})\chi(\Sigma_{l_2})$ is equal to $\chi(\pi_1)\chi(\pi_2)$ while every summand contributing to $\chi(\Sigma_{l_1,l_2})$ is equal to $\chi(\pi_1\pi_2)$. We need to find an estimate for $(\chi(\pi_1\pi_2) - \chi(\pi_1)\chi(\pi_2))$.

The difficulty caused by 3 can be very easily overcome: it is basically the condition (12). Our proof of the main result will be therefore divided into two parts; each devoted to one of the remaining difficulties.

Note that the second summand on the right-hand side in (27) can be written as

$$k^\bullet(\Sigma_{l_1}, \Sigma_{l_2}) = \chi(\Sigma_{l_1} \bullet \Sigma_{l_2}) - \chi(\Sigma_{l_1})\chi(\Sigma_{l_2});$$

in other words the proof of the implication (12) $\implies$ (13) is equivalent to overcoming the difficulty 2 and the proof of the implication (13) $\implies$ (14) is equivalent to overcoming the difficulty 1.

4.2. Partitions

We recall that $\pi = \{\pi_1, \dots, \pi_r\}$ is a partition of a finite ordered set X if sets $\pi_1, \dots, \pi_r$ are nonempty and disjoint and if $\pi_1 \cup \cdots \cup \pi_r = X$.

We say that a partition π is smaller (or finer) than a partition ρ of the same set if every block of π is a subset of some block of ρ and we denote it by $\pi \le \rho$. For partitions π_1, π_2 of the same set we denote by $\pi_1 \vee \pi_2$ the minimal partition which is greater or equal than both π_1 and π_2. The set of all partitions of some given set X has a maximal element, namely the partition with only one block equal to X.

4.3. Algebra of conjugacy classes and its filtration

Ivanov and Kerov [IK99] defined a partial permutation of a set X as a pair $\alpha = (d, w)$, where d (called support of α) is any subset of X and $w : X \to X$ is a bijection which is equal to identity outside of d. The usual product of partial permutations is given by

$$(d_1, w_1)(d_2, w_2) = (d_1 \cup d_2, w_1 w_2)$$

and their disjoint product $\bullet$ was defined in (5). Partial permutations behave like the usual permutations (to a partial permutation we can canonically associate the usual permutation w) except that we can distinguish two kinds of fix-points x for a partial permutations: true fix-points (i.e. $x \notin d$) and cycles of length one ($x \in d$, $w(x) = x$). Partial permutations form a semigroup; in this article we are interested also in the corresponding semigroup algebra which should be regarded as an analogue of the permutation group algebra $\mathbb{C}(S_q)$ equipped with some additional structure.

In fact, to define correctly the notion of the disjoint product $\bullet$ from Section 2.2 we must use the semigroup algebra corresponding to partial permutations and not the group algebra $\mathbb{C}(S_q)$. The reason for this is that we must distinguish two kinds of fix-points since, for example, $\Sigma_{1,1}$ and Σ_1 represent multiples of each other in $\mathbb{C}(S_q)$, but the disjoint product treats them differently.

One can show [IK99, Śni04] that the family of normalized conjugacy classes $(\Sigma_{k_1,\dots,k_n})$ and the family of free cumulants (R_i) generate the same filtered algebra, called algebra of conjugacy classes, when for the degrees of the generators we take

$$\deg \Sigma_{k_1,\dots,k_n} = (k_1 + 1) + \cdots + (k_n + 1),$$

$$\deg R_i = i.$$

The above statement holds true both when as the product we take the usual product of partial permutations (in this case we denote the resulting algebra by $\mathfrak{A}$) and when as the product we take the disjoint product (in this case we denote the resulting algebra by $\mathfrak{A}^\bullet$).

One can also show [IK99, Śni04] that in the scaling considered in this article (i.e. a typical Young diagram has at most $O(\sqrt{q})$ rows and columns) so defined degree determines the asymptotic behavior of an element, namely

$$\mathbb{E}X = O\big(q^{\frac{\deg X}{2}}\big)$$

for any element X of this algebra. Furthermore, the generators fulfill

$$R_i = \Sigma_{i-1} + (\text{terms of degree at most } i-2). \tag{28}$$

4.4. Three probability spaces

Commutative algebras $\mathfrak{A}$ and $\mathfrak{A}^\bullet$ can be regarded as algebras of random variables on a purely formal level (usually it is not possible to represent them as algebras of functions on some Kolmogorov probability space).

Algebras $\mathfrak{A}$ and $\mathfrak{A}^\bullet$ are trivially isomorphic as vector spaces; we will denote by $\mathbb{E}^{\mathrm{id}} : \mathfrak{A} \to \mathfrak{A}^\bullet$ the identity map between them, in other words

$$\mathbb{E}^{\mathrm{id}}(x) = x.$$

One can think that $\mathbb{E}^{\mathrm{id}}$ is a kind of a 'conditional expectation'.

If ρ is a representation of S_q we consider maps $\mathbb{E} : \mathfrak{A} \to \mathbb{C}$ and $\mathbb{E}^\bullet : \mathfrak{A}^\bullet \to \mathbb{C}$ given by

$$\mathbb{E}(x) = \mathbb{E}^\bullet(x) = \chi_\rho(x).$$

In this way the following diagram commutes:

$$\begin{tikzcd} \mathfrak{A} \arrow[r, "\mathbb{E}^{\mathrm{id}}"'] \arrow[rr, bend left, "\mathbb{E}"] & \mathfrak{A}^\bullet \arrow[r, "\mathbb{E}^\bullet"'] & \mathbb{C} \end{tikzcd} . \tag{29}$$

and we may consider three different probability structures:

- algebra $\mathfrak{A}$ equipped with the expectation $\mathbb{E}$ (which gives rise to the natural cumulants k),
- algebra $\mathfrak{A}^\bullet$ equipped with the expectation $\mathbb{E}^\bullet$ (which gives rise to the disjoint cumulants $k^\bullet$),
- algebra $\mathfrak{A}$ equipped with the 'conditional' expectation $\mathbb{E}^{\mathrm{id}} : \mathfrak{A} \to \mathfrak{A}^\bullet$ (corresponding 'conditional' cumulants belong to $\mathfrak{A}^\bullet$ and will be denoted by k^{id}).

The commutativity of the diagram (29) implies that the relation between the corresponding three cumulants is given by the following formula of Brillinger [Bri69] (see also [Leh04b]).

Proposition 13. *For $x_1, \dots, x_n \in \mathfrak{A}$*

$$k(x_1, \dots, x_n) = \sum_{\pi} k^\bullet\big[k^{\mathrm{id}}(x_i : i \in \pi_j) : j = 1, 2, \dots\big], \tag{30}$$

where the sum runs over all partitions $\pi = \{\pi_1, \pi_2, \dots\}$ of the set $\{1, \dots, n\}$.

Example 14. Let us consider $n = 2$: there are two partitions of $\{1, 2\}$, namely $\{\{1\}, \{2\}\}$ and $\{\{1, 2\}\}$ therefore

$$k(x_1, x_2) = k^\bullet\big(k^{\mathrm{id}}(x_1), k^{\mathrm{id}}(x_2)\big) + k^\bullet\big(k^{\mathrm{id}}(x_1, x_2)\big).$$

Similarly, for $n = 3$

$$\begin{aligned} k(x_1, x_2, x_3) = {} & k^\bullet\big(k^{\mathrm{id}}(x_1), k^{\mathrm{id}}(x_2), k^{\mathrm{id}}(x_2)\big) + k^\bullet\big(k^{\mathrm{id}}(x_1, x_2), k^{\mathrm{id}}(x_3)\big) \\ & + k^\bullet\big(k^{\mathrm{id}}(x_1, x_3), k^{\mathrm{id}}(x_2)\big) + k^\bullet\big(k^{\mathrm{id}}(x_2, x_3), k^{\mathrm{id}}(x_1)\big) \\ & + k^\bullet\big(k^{\mathrm{id}}(x_1, x_2, x_3)\big). \end{aligned}$$

The following result will be of great importance in this article.

Theorem 15. *For any* $x_1, \dots, x_n \in \mathfrak{A}$

$$\deg k^{\mathrm{id}}(x_1, \dots, x_n) \leq \deg x_1 + \cdots + \deg x_n - 2(n - 1). \tag{31}$$

Furthermore, the highest-order term of the second cumulant is given by

$$\begin{aligned} k^{\mathrm{id}}(\Sigma_{l_1}, \Sigma_{l_2}) = \sum_{r \geq 1} \sum_{\substack{a_1, \dots, a_r \geq 1 \\ a_1 + \cdots + a_r = l_1}} \sum_{\substack{b_1, \dots, b_r \geq 1 \\ b_1 + \cdots + b_r = l_2}} \frac{l_1 l_2}{r} \Sigma_{(a_1+b_1-1), \dots, (a_r+b_r-1)} \\ \textit{terms of degree at most } (l_1 + l_2 - 2). \end{aligned} \tag{32}$$

We postpone its proof to Section 5.

4.5. Multiplicative extension of cumulants

If $\rho = \{\rho_1, \dots, \rho_k\}$ is a partition of the set $\{1, \dots, n\}$ with blocks $\rho_i = \{\rho_{i,1}, \dots, \rho_{i,m(i)}\}$ we define partition–indexed cumulants given by a multiplicative extension of the usual cumulants:

$$k_\rho(X_1, \dots, X_n) = \prod_i k(X_{\rho_{i,1}}, X_{\rho_{i,2}}, \dots, X_{\rho_{i,m(i)}}); \tag{33}$$

for example

$$k_{\{1,3,4\},\{2,5\}}(X_1, X_2, X_3, X_4, X_5) = k(X_1, X_3, X_4)\, k(X_2, X_5).$$

In this article we will use the following property of cumulants: it turns out that the cumulants are implicitly determined by a sequence of relations

$$\mathbb{E}(X_1 \cdots X_n) = \sum_\mu k_\mu(X_1, \dots, X_n), \tag{34}$$

where the sum runs over all partitions μ of the set $\{1, \dots, n\}$.

4.6. Cumulants of products

The following formula for cumulants of products of random variables was proved by Leonov and Sirjaev [LS59].

Theorem 16. *Let $i_1 < i_2 < \cdots < i_{n+1}$ be integers and let $X_{i_1+1}, X_{i_1+2}, \dots, X_{i_{n+1}}$ be a family of random variables; then*

$$k\Big(\prod_{i_1+1 \le j \le i_2} X_j, \dots, \prod_{i_n+1 \le j \le i_{n+1}} X_j \Big) = \sum_{\pi} k_\pi (X_{i_1+1}, X_{i_1+2}, \dots, X_{i_{n+1}}), \tag{35}$$

where the sum runs over all partitions π of the set $\{i_1+1, i_1+2, \dots, i_{n+1}\}$ with the additional property that

$$\pi \vee \big\{ \{i_1+1, i_1+2, \dots, i_2\}, \dots, \{i_n+1, i_n+2, \dots, i_{n+1}\} \big\}$$

is the maximal partition with only one block.

Corollary 17. *Let permutations $\sigma_1, \dots, \sigma_n$ be disjoint cycles of length $l_1, \dots, l_n$. Then*

$$\begin{aligned} &k^\bullet(\Sigma_{l_1}, \dots, \Sigma_{l_n}) \\ &\quad = \sum_{\pi^{(1)}, \pi^{(2)}} k_{\pi^{(1)}}(\sigma_1, \dots, \sigma_n)\, k^\bullet_{\pi^{(2)}}(\Sigma_{\underbrace{1, \dots, 1}_{l_1 \text{ times}}}, \dots, \Sigma_{\underbrace{1, \dots, 1}_{l_n \text{ times}}}), \end{aligned} \tag{36}$$

where the sum runs over all partitions $\pi^{(1)}, \pi^{(2)}$ of the set $\{1, \dots, n\}$ such that $\pi^{(1)} \vee \pi^{(2)} = \big\{ \{1, \dots, n\} \big\}$.

Proof. We consider the algebra $\mathbb{C}(S_q) \otimes \mathfrak{A}^\bullet$ with the usual product on the first factor and the disjoint product on the second one: $(a_1 \otimes b_1)(a_2 \otimes b_2) = (a_1 a_2) \otimes (b_1 \bullet b_2)$. We equip it with the expected value $\mathbb{E}(a \otimes b) = \chi_\rho(a) \chi_\rho(b)$.

For all integers $l_1, \dots, l_n \ge 1$ and disjoint cycles $\sigma_1, \dots, \sigma_n$ with appropriate lengths we clearly have

$$\mathbb{E}\big(\Sigma_{l_1} \bullet \cdots \bullet \Sigma_{l_n}\big) = \mathbb{E}\big[(\sigma_1 \otimes \Sigma_{\underbrace{1, \dots, 1}_{l_1 \text{ times}}}) \cdots (\sigma_n \otimes \Sigma_{\underbrace{1, \dots, 1}_{l_n \text{ times}}})\big],$$

where the product on the left-hand side is taken in $\mathfrak{A}^\bullet$ and the product on the right-hand side is taken in $\mathbb{C}(S_q) \otimes \mathfrak{A}^\bullet$. In other words: the mixed moments of the family of random variables (Σ_{l_i}) coincide with the mixed moments of the family $(\sigma_i \otimes \Sigma_{\underbrace{1, \dots, 1}_{l_i \text{ times}}})$. It follows that the corresponding cumulants are equal:

$$k^\bullet\big(\Sigma_{l_1}, \dots, \Sigma_{l_n}\big) = k(\sigma_1 \otimes \Sigma_{\underbrace{1, \dots, 1}_{l_1 \text{ times}}}, \dots, \sigma_n \otimes \Sigma_{\underbrace{1, \dots, 1}_{l_n \text{ times}}}).$$

We use now Theorem 16 to compute the right-hand side which finishes the proof. □

In applications of this result we will find useful the following lemma.

Lemma 18. *Let $\pi = \{\pi_1, \dots, \pi_m\}$ be a partition which contributes to the right-hand side of* (35). *Then*

$$\sum_i \big(|\pi_i| - 1\big) \geq n - 1.$$

Let $\pi^{(k)} = \{\pi_1^{(k)}, \dots, \pi_{m^{(k)}}^{(k)}\}$ for $k \in \{1, 2\}$ be partitions which contribute to the right-hand side of (36). *Then*

$$\sum_i \big(|\pi_i^{(1)}| - 1\big) + \sum_i \big(|\pi_i^{(2)}| - 1\big) \geq n - 1.$$

Proof. Partition π can be obtained from the trivial partition (every block consists of exactly one element) by performing $\sum_i \big(|\pi_i| - 1\big)$ times the following operation: we select two blocks and merge them into a single one. Clearly, the number of blocks of partition

$$\pi \vee \big\{\{i_1 + 1, i_1 + 2, \dots, i_2\}, \dots, \{i_n + 1, i_n + 2, \dots, i_{n+1}\}\big\}$$

either decreases by one or stays the same in each step. The initial number of blocks is equal to n and the final number of blocks is equal to 1, which finishes the proof.

The proof of the second statement is analogous and we skip it. □

Corollary 19. *Let X be a set of generators of the algebra of conjugacy classes $\mathfrak{A}$. Suppose that*

$$k(a_1, \dots, a_n) q^{-\frac{\deg a_1 + \cdots + \deg a_n - 2(n-1)}{2}} = O(1) \tag{37}$$

holds true for all $n \geq 1$ and $a_1, \dots, a_n \in X$. Then (37) *holds true for all $n \geq 1$ and $a_1, \dots, a_n \in \mathfrak{A}$.*

Proof. Clearly, it is enough to consider the case when $a_1, \dots, a_n$ are monomials in elements of X. In order to estimate each summand on the right-hand side of (35) we apply Lemma 18. □

Remark 20. Note that the above Corollary holds true also when the cumulants k are replaced by $k^\bullet$. In particular, this Corollary can be applied for (13) and (14).

4.7. Proof of the main theorem: equivalence (13) $\iff$ (14)

Proof (Proof of the implication (13) $\implies$ (14)*).* Our goal is to use Proposition 13 in order to express $k^\bullet(\Sigma_{l_1}, \dots, \Sigma_{l_n})$ in terms of the cumulants k^{id} and $k^\bullet$. To estimate a summand on the right-hand side of (30) corresponding to a partition $\pi = \{\pi_1, \dots, \pi_m\}$ of $\{1, \dots, n\}$ we use Theorem 15 and get

$$\deg k^{\mathrm{id}}(x_i : i \in \pi_j) \leq \Big(\sum_{i \in \pi_j} \deg x_i\Big) - 2(|\pi_j| - 1).$$

Thus

$$\Big(\sum_{1\leq j\leq m} \deg k^{\mathrm{id}}(x_i : i\in\pi_j)\Big) - 2(m-1) \leq \Big(\sum_i \deg x_i\Big) - 2(n-1). \quad (38)$$

Assumption (13) and Corollary 19 show that

$$k^{\bullet}\big[k^{\mathrm{id}}(x_i : i\in\pi_j) : j=1,2,\dots\big] q^{\frac{-\sum_j \deg k^{\mathrm{id}}(x_i : i\in\pi_j)+2(m-1)}{2}} = O(1).$$

Now (38) finishes the proof. □

As a byproduct, in a similar way we obtain a proof of the identity

$$\lim_{q\to\infty} \mathrm{Cov}(\Sigma_{l_1},\Sigma_{l_2}) q^{-\frac{l_1+l_2}{2}} = \lim_{q\to\infty} \mathrm{Cov}^{\bullet}(\Sigma_{l_1},\Sigma_{l_2}) q^{-\frac{l_1+l_2}{2}} + \sum_{r\geq 1} \sum_{\substack{a_1,\dots,a_r\geq 1\\ a_1+\cdots+a_r=l_1}} \sum_{\substack{b_1,\dots,b_r\geq 1\\ b_1+\cdots+b_r=l_2}} \frac{l_1 l_2}{r} c_{a_1+b_1}\cdots c_{a_r+b_r} \quad (39)$$

which is a part of Theorem 3.

Proof (Proof of the implication (14) $\Longrightarrow$ (13)*).* We will use induction with respect to ($\deg\Sigma_{l_1}+\cdots+\deg\Sigma_{l_n}$). Equation (30) can be written in the form

$$k^{\bullet}(\Sigma_{l_1},\dots,\Sigma_{l_n}) = k(\Sigma_{l_1},\dots,\Sigma_{l_n}) - \sum_{\pi\neq\{\{1,\dots,n\}\}} k^{\bullet}\big[k^{\mathrm{id}}(x_i : i\in\pi_j) : j=1,2,\dots\big].$$

The inductive hypothesis can be used to estimate the right-hand side in the same way as in the proof of the opposite implication above. □

4.8. *Proof of the main theorem: equivalence* (12) $\Longleftrightarrow$ (13)

4.8.1. Cumulants of falling factorials

Element $\Sigma_{\underbrace{1,\dots,1}_{k \text{ times}}} \in \mathbb{C}(S_q)$ is equal to $q(q-1)\cdots(q+1-k)$, the multiple of identity, therefore no matter which representation we consider we always have

$$\mathbb{E}\Sigma_{\underbrace{1,\dots,1}_{k \text{ times}}} = q(q-1)\cdots(q+1-k).$$

However, it should be stressed that if we consider the algebra of conjugacy classes equipped with the disjoint product then this element is not longer a multiple of identity and

$$\Sigma_{\underbrace{1,\dots,1}_{k_1 \text{ times}}} \bullet \Sigma_{\underbrace{1,\dots,1}_{k_2 \text{ times}}} = \Sigma_{\underbrace{1,\dots,1}_{k_1+k_2 \text{ times}}}.$$

Lemma 21. *For any integers $l_1, \dots, l_n \geq 1$*
$$k^{\bullet}(\Sigma_{\underbrace{1,\dots,1}_{l_1 \text{ times}}}, \dots, \Sigma_{\underbrace{1,\dots,1}_{l_n \text{ times}}}) = O(q^{l_1+\cdots+l_n+1-n}).$$

Proof. Since $\Sigma_{\underbrace{1,\dots,1}_{k \text{ times}}} \in \mathbb{C}(S_q)$ is a multiple of identity therefore

$$k(\Sigma_{\underbrace{1,\dots,1}_{l_1 \text{ times}}}, \dots, \Sigma_{\underbrace{1,\dots,1}_{l_n \text{ times}}}) = \begin{cases} q(q-1)\cdots(q+1-l_1) & \text{if } n=1, \\ 0 & \text{if } n \geq 2, \end{cases}$$

hence in particular
$$k(\Sigma_{\underbrace{1,\dots,1}_{l_1 \text{ times}}}, \dots, \Sigma_{\underbrace{1,\dots,1}_{l_n \text{ times}}}) q^{-(l_1+\cdots+l_n+1-n)} = O(1).$$

We leave it as a simple exercise to the reader to check that the presented above proof of the implication (14) $\Longrightarrow$ (13) can be applied here. □

Proof (Proof of the implication (12) $\Longrightarrow$ (13)*).* Our goal is to estimate the right-hand side of (36). The assumption (12) implies that
$$k_{\pi_{(1)}}(\sigma_1, \dots, \sigma_n) = O\Big(q^{-\frac{|\sigma_1|+\cdots+|\sigma_n|+2n-2(\text{number of blocks of } \pi_{(1)})}{2}}\Big);$$
Lemma 21 implies that
$$k^{\bullet}_{\pi_{(2)}}(\Sigma_{\underbrace{1,\dots,1}_{l_1 \text{ times}}}, \dots, \Sigma_{\underbrace{1,\dots,1}_{l_n \text{ times}}}) = O\Big(q^{l_1+\cdots+l_n+(\text{number of blocks of } \pi_{(2)})-n}\Big).$$
Now it is enough to use Lemma 18. □

We leave the proof of the identity
$$\lim_{q\to\infty} \operatorname{Cov}^{\bullet}(\Sigma_{l_1}, \Sigma_{l_2}) q^{-\frac{l_1+l_2}{2}} = \lim_{q\to\infty} \operatorname{Cov}(\sigma_1, \sigma_2) q^{\frac{l_1+l_2}{2}} - l_1 l_2 c_{l_1+1} c_{l_2+1}$$
which is a part of Theorem 3 as a simple exercise.

Proof (Proof of the implication (13) $\Longrightarrow$ (12)*).* We use the induction over n. We let us split the sum in equation (36) into the sum of terms where $\pi_{(1)}$ is the maximal partition with only one block and the sum of all the other terms; therefore
$$\begin{aligned} k(\sigma_1, \dots, \sigma_n) \mathbb{E}\big(\Sigma_{\underbrace{1,\dots,1}_{l_1 \text{ times}}} \bullet \cdots \bullet \Sigma_{\underbrace{1,\dots,1}_{l_n \text{ times}}}\big) \\ = k^{\bullet}(\Sigma_{l_1}, \dots, \Sigma_{l_n}) \\ - \sum_{\pi_{(1)}, \pi_{(2)}} k_{\pi_{(1)}}(\sigma_1, \dots, \sigma_n)\, k^{\bullet}_{\pi_{(2)}}(\Sigma_{\underbrace{1,\dots,1}_{l_1 \text{ times}}}, \dots, \Sigma_{\underbrace{1,\dots,1}_{l_n \text{ times}}}), \end{aligned}$$
where the sum runs over partitions $\pi_{(1)}, \pi_{(2)}$ such as in (36) with the additional constraint that $\pi_{(1)}$ is not equal to the maximal partition with only one block. We use the inductive hypothesis and estimate the summands on the right-hand side in the same way as in the proof of the opposite implication above. □

4.9. Proof of the main theorem: equivalence (14) $\iff$ (15)

Proof (Proof of equivalence (14) $\iff$ (15)*).* It is enough to apply Corollary 19. □

The remaining part of Theorem 3, namely

$$\lim_{q\to\infty} \operatorname{Cov}(R_{l_1+1}, R_{l_2+1}) q^{-\frac{l_1+l_2}{2}} = \lim_{q\to\infty} \operatorname{Cov}(\Sigma_{l_1}, \Sigma_{l_2}) q^{-\frac{l_1+l_2}{2}}$$

follows from the relation (28) between free cumulants and conjugacy classes.

5. Proof of Theorem 15

This section is devoted to the proof of Theorem 15 which is the only missing component in the proof of the main theorem. We will use some tools presented in our recent work [Śni04].

5.1. Partition–indexed conjugacy classes

In the following we present some constructions on partitions of the set $X = \{1, 2, \dots, N\}$. However, it should be understood that by a change of labels these constructions can be performed for any finite ordered set X.

We consider a matrix J, the entries of which belong to $\mathbb{C}(S_q)$, the symmetric group algebra:

$$J = \begin{bmatrix} 0 & (1,2) & \dots & (1,q) & 1 \\ (2,1) & 0 & \dots & (2,q) & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ (q,1) & (q,2) & \dots & 0 & 1 \\ 1 & 1 & \dots & 1 & 0 \end{bmatrix} \in \mathcal{M}_{q+1}(\mathbb{C}) \otimes \mathbb{C}(S_q).$$

Except for the last row, the last column and the diagonal, the entry in the i-th row and the j-th column is equal to the transposition interchanging i and j.

Let $p = (p_1, \dots, p_l)$ be a sequence with $p_1, \dots, p_l \in \{1, \dots, q+1\}$ and let π be a partition of the set $\{1, \dots, l\}$. We say that $p \sim \pi$ if for any $1 \le i, j \le l$ the equality $p_i = p_j$ holds if and only if i and j belong to the same block of the partition π. We define [Śni04]

$$\Sigma_\pi = \sum_{\substack{p\sim\pi \\ p_l=q+1}} J_{p_1p_2} J_{p_2p_3} \cdots J_{p_{l-1}p_l} J_{p_lp_1} \in \mathbb{C}(S_q). \tag{40}$$

We will treat each summand as a partial permutation with a support $\{p_1, \dots, p_l\} \setminus \{q+1\}$. Some partial results concerning expressions of this form were obtained by Biane [Bia98]. We can show [Śni04] that $\Sigma_\pi = \Sigma_{k_1,\dots,k_t}$ for some integers $k_1, \dots, k_t \ge 1$ which will be presented explicitly in Section 5.6. For this reason we call Σ_π a partition-indexed conjugacy class.

5.2. Products of conjugacy classes

Theorem 22. *Let $i_1 < \cdots < i_{n+1}$ be integers and for each $1 \leq s \leq n$ let π_s be a partition of the set $\rho_s = \{i_s + 1, i_s + 2, \dots, i_{s+1}\}$. We denote*

$$\pi_1 \bullet \cdots \bullet \pi_n = (\pi_1 \cup \cdots \cup \pi_n) \vee \big\{\{i_2, i_3, \dots, i_{n+1}\}\big\}.$$

Then

$$\Sigma_{\pi_1} \bullet \cdots \bullet \Sigma_{\pi_n} = \Sigma_{(\pi_1 \bullet \cdots \bullet \pi_n)}. \tag{41}$$

Furthermore,

$$\Sigma_{\pi_1} \cdots \Sigma_{\pi_n} = \sum_{\sigma} \Sigma_{\sigma}, \tag{42}$$

where the sum on the right-hand side runs over all partitions σ of the set $\{i_1 + 1, i_1 + 2, \dots, i_{n+1}\}$ such that

1. *for any $a, b \in \rho_s$, $1 \leq s \leq n$ we have that a and b are connected by σ if and only if they are connected by π_s,*
2. *elements $i_2, i_3, \dots, i_{n+1}$ belong to the same block of σ.*

Furthermore,

$$k^{\mathrm{id}}(\Sigma_{\pi_1}, \dots, \Sigma_{\pi_n}) = \sum_{\sigma} \Sigma_{\sigma}, \tag{43}$$

where the sum on the right-hand side runs over all partitions σ of the set $\{i_1 + 1, i_1 + 2, \dots, i_{n+1}\}$ such that

1. *for any $a, b \in \rho_s$, $1 \leq s \leq n$ we have that a and b are connected by σ if and only if they are connected by π_s,*
2. *elements $i_2, i_3, \dots, i_{n+1}$ belong to the same block of σ; we will denote this block by σ_1,*
3. *$(\sigma \setminus \sigma_1) \vee \{\rho_1, \dots, \rho_n\}$ is the maximal partition with only one block.*

Proof. Equations (41) and (42) follow immediately from the definition (40).

We shall treat (43) as a definition of the left-hand side and we shall verify that so defined cumulants fulfill the defining relation of cumulants (34). Before we do this we need to compute the corresponding partition-indexed cumulants. Let μ be a partition of $\{1, \dots, n\}$; we denote by $\tilde{\mu}$ a partition of the set $\{i_1+1, i_1+2, \dots, i_{n+1}\}$ such that $a \in \rho_s$ and $b \in \rho_t$ belong to the same block of $\tilde{\mu}$ if and only if s and t belong to the same block of μ. Then (43) implies

$$k^{\mathrm{id}}_{\mu}(\Sigma_{\pi_1}, \dots, \Sigma_{\pi_n}) = \sum_{\sigma} \Sigma_{\sigma}, \tag{44}$$

where the sum on the right-hand side runs over all partitions σ of the set $\{i_1 + 1, i_1 + 2, \dots, i_{n+1}\}$ such that

1. for any $a, b \in \rho_s$, $1 \leq s \leq n$ we have that a and b are connected by σ if and only if they are connected by π_s,

2. elements $i_2, i_3, \dots, i_{n+1}$ belong to the same block of σ; we will denote this block by σ_1,
3. $(\sigma \setminus \sigma_1) \vee \{\rho_1, \dots, \rho_n\} = \tilde{\mu}$.

Equation (44) implies immediately (34). □

For example, for

$$\begin{aligned} \rho_1 &= \{1,2,3,4\},\ \rho_2 = \{5,6,7,8\}, \\ \pi_1 &= \big\{\{1,3\},\{2,4\}\big\}, \pi_2 = \big\{\{5,7\},\{6\},\{8\}\big\}, \end{aligned} \tag{45}$$

the above theorem states that

$$\begin{aligned} \Sigma_{\pi_1}\Sigma_{\pi_2} = {} & \Sigma_{\left\{\{1,3\},\{2,4,8\},\{5,7\},\{6\}\right\}} + \Sigma_{\left\{\{1,3,6\},\{2,4,8\},\{5,7\}\right\}} \\ & + \Sigma_{\left\{\{1,3,5,7\},\{2,4,8\},\{6\}\right\}} \end{aligned}$$

and

$$k^{\mathrm{id}}(\Sigma_{\pi_1}, \Sigma_{\pi_2}) = \Sigma_{\left\{\{1,3,6\},\{2,4,8\},\{5,7\}\right\}} + \Sigma_{\left\{\{1,3,5,7\},\{2,4,8\},\{6\}\right\}};$$

the readers acquainted with the results of Section 5.6 may check that it is equivalent to

$$\Sigma_1 \Sigma_{1,1} = \Sigma_{1,1,1} + \Sigma_{1,1} + \Sigma_{1,1}$$

and

$$k^{\mathrm{id}}(\Sigma_1, \Sigma_{1,1}) = \Sigma_{1,1} + \Sigma_{1,1}.$$

5.3. Geometric interpretation of the degree of Σ_π

It is very useful to represent partitions graphically by arranging the elements of the set $X = \{1, \dots, n\}$ counterclockwise on a circle and joining elements of the same block by a line, as it can be seen on Figure 3.

We consider a large sphere with a small circular hole. The boundary of this hole is the circle mentioned above. Let us draw the blocks of the partition π with a fat pen; in this way each block becomes a disc glued to the boundary of the hole, cf Figure 4.

After gluing this first collection of discs, our sphere becomes a surface with a number of holes. The boundary of each hole is a circle and we shall glue this hole with a disc from the second collection. Thus we obtained an orientable surface without a boundary. We call the genus of this surface the genus of the partition π and denote it by genus_π.

The following result was proved in our previous work [Śni04].

Proposition 23. *For any partition π of an n–element set*

$$\deg \Sigma_\pi = n - 2 \operatorname{genus}_\pi . \tag{46}$$

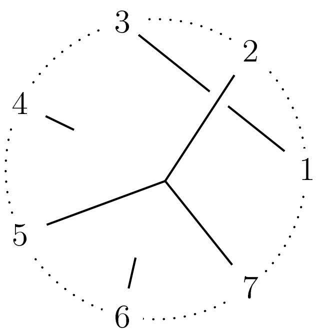

Fig. 3. Graphical representation of a partition $\{\{1,3\},\{2,5,7\},\{4\},\{6\}\}$

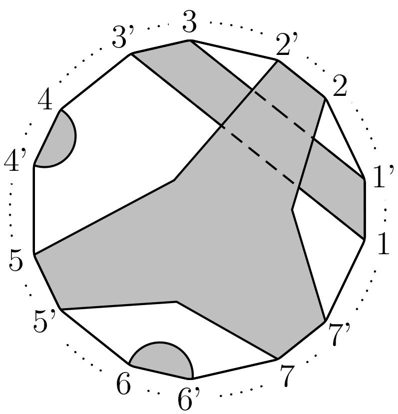

Fig. 4. The first collection of discs for partition π from Figure 3

5.4. Geometric interpretation of Theorem 22

We will use the notations of Theorem 22. On the surface of a large sphere we draw a small circle on which we mark counterclockwise points $i_1+1, i_1+2, \dots, i_{r+1}$. Inside the circle we cut r holes; for any $1 \leq s \leq r$ the corresponding hole has a shape of a disc, the boundary of which passes through the points from the block ρ_s. For every $1 \leq s \leq r$ the partition π_s connects some points on the boundary of the hole ρ_s and this situation corresponds exactly to the case we considered in Section 5.3. We shall glue to the hole ρ_s only the first collection of discs that we considered in Section 5.3, i.e. the discs which correspond to the blocks of the partition π_s. Thus we obtained a number of holes with a collection of glued discs (cf Figure 5).

When we inflate the original small holes inside the circle we may think about this picture alternatively: instead of r small holes we have a big one (in the shape of the circle) but some arcs on its boundary are glued by an additional disc (on Figure 6 drawn in black) glued to vertices $i_2, i_3, \dots, i_{r+1}$. Furthermore we still have a collection of all discs (on Figure 6 drawn in gray) corresponding to

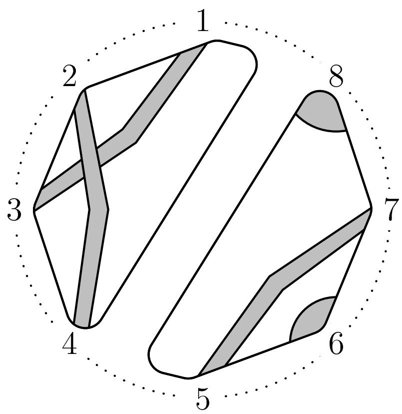

Fig. 5. Graphical representation of example (45)

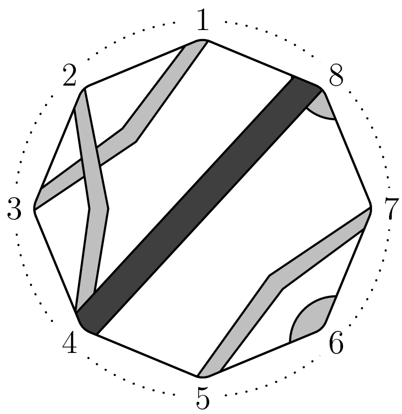

Fig. 6. Figure 5 after inflating small holes

partitions π_s. We merge the additional black disc to a gray disc from this collection if they touch the same vertex. After this merging the collection of discs corresponds to the partition $\pi_1 \bullet \cdots \bullet \pi_r = \big\{\{i_2, i_3, \dots, i_{r+1}\}\big\} \vee (\pi_1 \cup \cdots \cup \pi_r)$ which appears in the formula (41) for the disjoint product.

The last step is to consider all ways of merging of the discs (or equivalently: all partitions $\sigma \geq \pi_1 \bullet \cdots \bullet \pi_r$) with the property that any two vertices that were lying on the boundary of the same small hole ρ_s if were not connected by a disc from the collection π_s then they also cannot be connected after all mergings. In this way we obtain all partitions which contribute to (42).

By splitting the holes (we recall that each hole corresponds to some set ρ_i) into some new holes we can view the surface associated to the partition ρ as a sphere with a number of new holes glued in pairs by handles and the number of these handles is equal to the genus of the surface. In this way we obtain a graph G_σ the vertices of which correspond to the old holes (or, equivalently, sets ρ_i) and the edges correspond to the handles between new holes. Of course the above construction can be sometimes performed in many different ways but we do not mind it. Note that multiple connections between vertices are allowed. Also, a vertex can be connected with itself and the number of such loops is equal to the genus of the corresponding partitions π_i. The genus of σ is equal to the total number of edges in G_σ.

In order to obtain all partitions σ which contribute to (43) we should restrict our attention only to σ such that the graph G_σ is connected. A connected graph with n vertices has at least $n-1$ non-loop edges therefore

$$\text{genus}_\sigma \geq \text{genus}_{\pi_1} + \cdots + \text{genus}_{\pi_n} + (n-1). \tag{47}$$

Proof (Proof of Theorem 15). In order to prove (31) for the special case when $x_i = \Sigma_{\pi_i}$ it is enough to apply Proposition 23 and (47).

It remains now to prove (32) and we shall do it in the following. We use the notations of Theorem 22 if $n = 2$ and π_1, π_2 are trivial partitions (every blocks consists of a single element). Let σ be a partition which contributes to (43) with the minimal possible genus, namely $\text{genus}_\sigma = 1$. It follows that one of the blocks of σ is equal to $\{i_2, i_3\}$. Secondly, some of the elements of the set $\{i_1+1, \ldots, i_2-1\}$ (but at least one of them) are paired with some of the elements of the set $\{i_2+1, \ldots, i_1+i_3-1\}$; this pairing however is not arbitrary. Let us travel counterclockwise along the boundary of the hole corresponding to the block π_1; in other words we visit the vertices in the order $i_1+1 \to i_1+2 \to \cdots \to i_2-1 \to i_1+1$. Some of these vertices are paired by σ with some of the elements of $\{i_2+1, \ldots, i_3-1\}$ corresponding to the other hole; let us have a look in which order these counterparts appear during our walk. From the very definition of the genus of a partition we know that it is possible to draw the blocks of $\sigma \setminus \big\{\{i_2, i_3\}\big\}$ on the surface of the handle in such a way that lines do not cross. It follows that these counterparts will be visited in the clockwise order $i_3-1 \to i_3-2 \to \cdots \to i_2+1 \to i_3-1$, cf Figure 7.

Let $a_1, \ldots, a_r$ denote the distances (counted cyclically counterclockwise) between consecutive elements of the pairs in the first hole and let $b_1, \ldots, b_r$ denote the distances (counted cyclically counterclockwise) between consecutive elements of the pairs in the second hole. Readers acquainted with the results of Section 5.6 will see that

$$\Sigma_\sigma = \Sigma_{a_1+b_1-1, \ldots, a_r+b_r-1}.$$

It is easy to see that $a_1+\cdots+a_r = i_2-i_1$ and $b_1+\cdots+b_r = i_3-i_2$. Such sequences (a_i) and (b_i) uniquely determine σ once we specify the first element in each cycle (there are i_2-i_1-1 choices for the first one and i_3-i_2-1 choices for the second one). Partition σ can be represented like this in r different ways which correspond to the cyclic rotations of the sequences (a_i) and (b_i). Since $\Sigma_{\pi_1} = \Sigma_{i_2-i_1-1}$ and $\Sigma_{\pi_2} = \Sigma_{i_3-i_2-1}$ this finishes the proof of (32). □

5.5. *Fat partitions*

Let $\pi = \{\pi_1, \ldots, \pi_r\}$ be a partition of the set $\{1, \ldots, n\}$. For every $1 \leq s \leq r$ let $\pi_s = \{\pi_{s,1}, \ldots, \pi_{s,l_s}\}$ with $\pi_{s,1} < \cdots < \pi_{s,l_s}$. We define π_{fat}, called fat partition

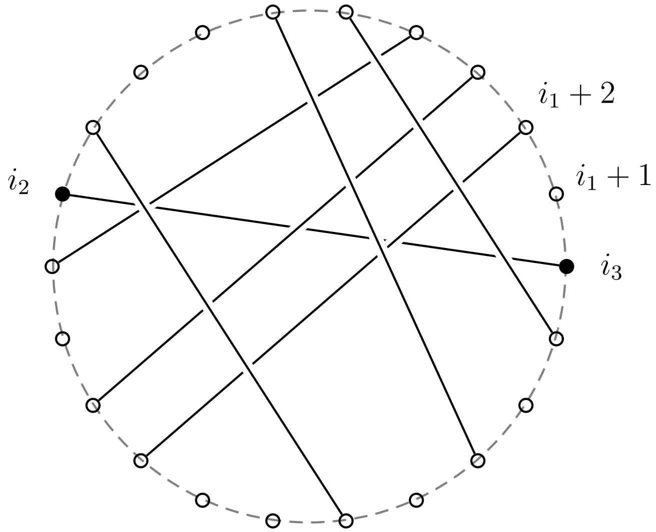

Fig. 7. Partitions σ with genus 1 which contribute to (43)

of π, to be a pair partition of the $2n$–element ordered set $\{1, 1', 2, 2', \dots, n, n'\}$ given by

$$\pi_{\text{fat}} = \Big\{ \{\pi'_{s,t}, \pi_{s,t+1}\} : 1 \leq s \leq r \text{ and } 1 \leq t \leq l_s \Big\},$$

where it should be understood that $\pi_{s,l_s+1} = \pi_{s,1}$.

This operation can be easily described graphically as follows: we draw the blocks of the partition with a fat pen and take the boundary of each block, as it can be seen on Figure 8. This boundary is a collection of lines hence it is a pair

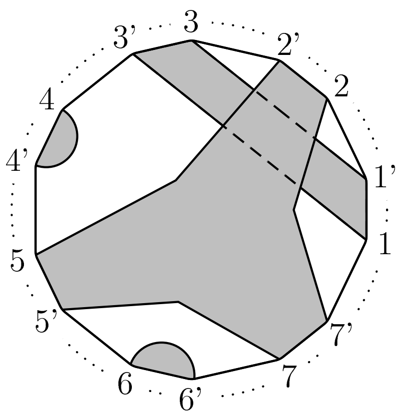

Fig. 8. The fat partition π_{fat} corresponding to the partition π from Figure 3

partition. However, every vertex $k \in \{1, \dots, n\}$ of the original partition π has to be replaced by its 'right' and 'left' copy (denoted respectively by k and k').

5.6. *Explicit form of the partition–indexed conjugacy class indicator* Σ_π

Let π be a partition of the set $\{1, \dots, n\}$. Since the fat partition π_{fat} connects every element of the set $\{1', 2', \dots, n'\}$ with exactly one element of the set $\{1, 2, \dots, n\}$, we can view π_{fat} as a bijection $\pi_{\text{fat}} : \{1', 2', \dots, n'\} \to \{1, 2, \dots, n\}$. We also consider a bijection $c : \{1, 2, \dots, n\} \to \{1', 2', \dots, n'\}$ given by $\dots, 3 \mapsto 2', 2 \mapsto 1', 1 \mapsto n', n \mapsto (n-1)', \dots$. Finally, we consider a permutation $\pi_{\text{fat}} \circ c$ of the set $\{1, 2, \dots, n\}$.

For example, for the partition π given by Figure 3 the composition $\pi_{\text{fat}} \circ c$ has a cycle decomposition $(1, 2, 3, 5, 4)(6, 7)$, as it can be seen from Figure 9.

We decompose the permutation

$$\pi_{\text{fat}} \circ c = (b_{1,1}, b_{1,2}, \dots, b_{1,j_1}) \cdots (b_{t,1}, \dots, b_{t,j_t})$$

as a product of disjoint cycles. Every cycle $b_s = (b_{s,1}, \dots, b_{s,j_s})$ can be viewed as a closed clockwise path on a circle and therefore one can compute how many times it winds around the circle, cf Figure 10.

To a cycle b_s we assign the number

$$\begin{aligned} k_s = {} & (\text{number of elements in a cycle } b_s) \\ & -(\text{number of clockwise winds of } b_s). \end{aligned}$$

In the above example we have $b_1 = (1, 2, 3, 5, 4)$, $b_2 = (6, 7)$ and $k_1 = 2$, $k_2 = 1$, as it can be seen from Figure 10, where all lines clockwise wind around the central disc.

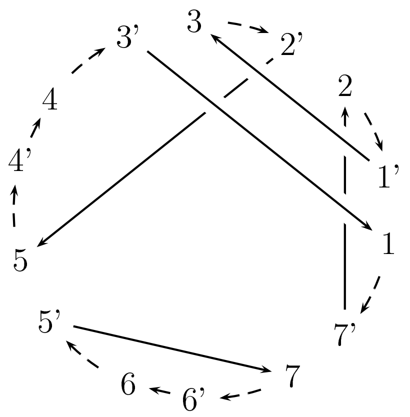

Fig. 9. Bijection corresponding to the partition π_{fat} from Figure 8 plotted with a solid line and the bijection c plotted with a dashed line

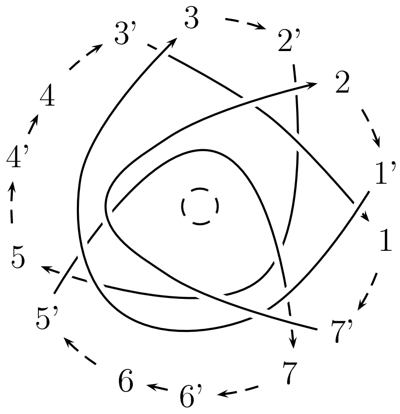

Fig. 10. A version of Figure 9 in which all lines wind clockwise around the central disc

In our recent work [Śni04] we proved that

$$\Sigma_\pi = \Sigma_{k_1,\dots,k_t},$$

where $\Sigma_{k_1,\dots,k_t}$ on the right–hand side should be understood as in Section 2.1.

Acknowledgements. I thank Akihito Hora and Philippe Biane for many discussions and remarks.

Research supported by State Committee for Scientific Research (Komitet Badań Naukowych) grant No. 2P03A00723; by EU Network “QP-applications”, contract HPRN-CT-2002-00729; by KBN-DAAD project 36/2003/2004.

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