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On the Dirichlet problem for asymmetric zero-range process on increasing domains

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Abstract. In order to obtain hitting time estimates for the asymmetric zero-range process (AZRP) on $\mathbb{Z}^d$, in dimensions $d \geq 3$, we characterize the principal eigenvalue of the generator of the AZRP with Dirichlet boundary on special domains. We obtain a Donsker-Varadhan variational representation and show that the corresponding eigenfunction is unique in a natural class of functions.

1. Introduction

The AZRP models the conservative evolution of charged particles interacting over short range, in an electrical field. The process denoted by $\{\eta_t, t \geq 0\}$, lives on $\{\eta : \eta(i) \in \mathbb{N}, i \in \mathbb{Z}^d\}$, and evolves informally as follows. At time zero and at each site $i \in \mathbb{Z}^d$, we draw a number of particles $\eta(i) \in \mathbb{N}$. To each particle we attach the trajectory of an asymmetric random walk with transition kernel $\{p(i, j); i, j \in \mathbb{Z}^d\}$. Now, each site $i \in \mathbb{Z}^d$ has an independent exponential process, its *clock*, of intensity $g(\eta_t(i))$ at time t, where $g : \mathbb{N} \to [0, \infty)$ is increasing. At each site i and at each realization of its clock, say at time t, we choose a particle uniformly among the $\eta_t(i)$-ones and we move it to its next position along its attached trajectory. The conservation of the particles number imposes a one-parameter family of ergodic time-invariant measures $\{\nu_\rho, \ \rho > 0\}$, which consists of product measures [1, 14]. The name zero-range is justified since only particles at the same site can interact with each other. Note also that $g(k) = k$ corresponds to independent random walks.

A problem motivated by physics is to estimate the times of occurrence of spots with large densities of particles, say τ, when the gas is initially prepared with a homogeneous density. Thus, we consider a stationary process with respect to ν_ρ, and focus on hitting times of patterns of the type

$$\mathcal{A} := \{\eta : \sum_{i \in \mathcal{S}} \eta(i) > L\}, \quad \text{(and} \quad \tau := \inf\{t : \eta_t \in \mathcal{A}\}) \tag{1.1}$$

where $\mathcal{S}$, the support of $\mathcal{A}$, is a finite subset of $\mathbb{Z}^d$, and L a given integer.

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Though $\mathcal{L}$ is neither compact, irreducible, nor self-adjoint, its physical origin endows crucial monotonicity properties. Thus, the partial order $-\eta \prec \zeta$ meaning $\eta(i) \leq \zeta(i)$ for all $i \in \mathbb{Z}^d-$ is preserved under the evolution. A related feature is that the invariant measures $\{\nu_\rho,\ \rho > 0\}$ all satisfy FKG's inequality, i.e. for f and g increasing functions

$$\int fg d\nu_\rho \geq \int f d\nu_\rho \int g d\nu_\rho. \tag{1.2}$$

This was the setting of [2] whose relevant results we now recall. A simple subadditive argument yielded the asymptotical rough estimate

$$\lambda(\rho) = -\lim_{t\to\infty} \frac{1}{t} \log(P_{\nu_\rho}(\tau > t)). \tag{1.3}$$

When the drift, $\sum_i ip(0,i)$, is nonzero, $\lambda(\rho)$ is positive in any dimensions. Furthermore, if we denote by $\mathcal{L}^*$ the dual of $\mathcal{L}$ in $L^2(\nu_\rho)$, which corresponds to an AZRP with reversed drift, then when $d \geq 3$, there exist $u, u^* \in L^p(\nu_\rho)$ for any $p \geq 1$ in the domain of $\mathcal{L}$ and $\mathcal{L}^*$ respectively, with

$$\text{(i)} \quad 1_{\mathcal{A}^c}\mathcal{L}(u) + \lambda(\rho)u = 0, \text{ and (ii)} \quad 1_{\mathcal{A}^c}\mathcal{L}^*(u^*) + \lambda(\rho)u^* = 0. \tag{1.4}$$

However, and this was most unfortunate from a physical point of view, a link with finite dimensional dynamics was missing, as well as a variational representation for $\lambda(\rho)$. This is what we first establish in this paper. Then, we show uniqueness for u in some class of functions, which in turn yields an asymptotical estimate for the hitting time.

We have chosen to introduce some symbols intuitively so as to be able to state our main results postponing definitions and notations as much as possible to Section 2.

A way of defining the AZRP with initial law ν_ρ on $\mathbb{Z}^d$ is through a limit of irreducible processes, where particles evolve on $[-n,n]^d$ as a zero-range process with creation and annihilation at the boundary. Informally, if $\mathcal{F}_n$ is the σ-field generated by $\{\eta(i), i \in [-n,n]^d\}$, then we define

$$\mathcal{L}_n^\rho(\varphi) = E_{\nu_\rho}[\mathcal{L}(\varphi)|\mathcal{F}_n].$$

The generator $\mathcal{L}_n^\rho$ will be shown to inherit the same property of monotonicity as $\mathcal{L}$ and to have ν_ρ as invariant measure. Thus, its principal Dirichlet eigenvalue $\lambda_n(\rho)$ is obtained as in (1.3). We show, in Section 3.4, that $\mathcal{L}_n^\rho$ has a unique normalized eigenfunction $u_n \geq 0$ (in some cone), associated with $\lambda_n(\rho)$. Then, our main observation in Section 3.5 is the following.

Lemma 1.1. *For $\lambda(\rho)$ given by (1.3), and $\lambda_n(\rho)$ corresponding to $\mathcal{L}_n^\rho$, we have that $\{\lambda_n(\rho), n \in \mathbb{N}\}$ is a decreasing sequence with*

$$\lim_{n\to\infty} \lambda_n(\rho) = \lambda(\rho). \tag{1.5}$$

Moreover, we establish a link between finite and infinite volume eigenfunctions.

Theorem 1.2. *When $d \geq 3$, $\{u_n, n \in \mathbb{N}\}$ converges to a solution of (1.4(i)) in weak-$L^2(\nu_\rho)$.*

In [2], a solution of (1.4(i)) was obtained through another sequence, say $\{u_t, t \geq 0\}$ in which u_t was the density (w.r.t ν_ρ) of the marginal law at time t of the time-reversed process conditioned on $\{\tau > t\}$. The functions $\{u_t, t \geq 0\}$ where positive and decreasing on $\mathcal{A}^c$, and satisfied the following uniform bound: for site i large enough, if ϵ_i is the probability that a random walk starting on i with transition kernel $\{p(.,.)\}$ hits the support of $\mathcal{A}$, then when $d \geq 3$

$$0 \leq u_t(\eta) - u_t(\mathsf{A}_i^+\eta) \leq \epsilon_i u_t(\eta), \tag{1.6}$$

where $\mathsf{A}_i^+\eta$ is the configuration η with one more particle at site i.

We denote by $\mathcal{D}_\rho$ the convex set of non-negative decreasing functions of finite integral (w.r.t ν_ρ), satisfying (1.6). We denote by $\mathcal{D}_\rho^+$ the positive functions of $\mathcal{D}_\rho$. Finally, we define a *dual* space of probability measures, $\mathcal{M}_\rho$, absolutely continuous with respect to ν_ρ, and whose density satisfies a condition similar to (1.6).

Intuitively, a Donsker-Varadhan's type functional would read $\Gamma(\varphi, \mu) =$ "$\int \mathcal{L}(\varphi)/\varphi d\mu$" for $(\varphi, \mu) \in \mathcal{D}_\rho^+ \times \mathcal{M}_\rho$. One problem is that $\mathcal{L}$ cannot be defined on $\mathcal{D}_\rho^+$ as a convergent series. Thus, we define $\Gamma(\varphi, \mu)$, in Proposition 4.3 of Section 4, as the limit of the Cauchy sequence $\{\int \mathcal{L}_n^\rho(\varphi)/\varphi d\mu,\ n \in \mathbb{N}\}$ taking advantage of the gradient bounds (1.6) on φ and $d\mu/d\nu_\rho$ by an integration by parts formula.

We obtain in Section 4.2 a Donsker-Varadhan variational formula for the principal eigenvalue.

Theorem 1.3. *When $d \geq 3$, and $\mathcal{A}$ is increasing with bounded support, we have*

$$\lambda(\rho) = - \sup_{\mu \in \mathcal{M}_\rho} \inf_{\varphi \in \mathcal{D}_\rho^+} \Gamma(\varphi, \mu). \tag{1.7}$$

Obtaining (1.7) is linked with the issue of uniqueness of the principal eigenfunction, since the minimax theorem hidden behind Donsker-Varadhan's formula requires a convex functional $h \mapsto \Gamma(e^h, \mu)$ (over a convex set of functions regular enough). Note that $\mathcal{D}_\rho^+$ is all the more appropriate since when written for $h = \log(\varphi)$, with $\varphi \in \mathcal{D}_\rho^+$, condition (1.6) reads

$$h(\eta) \geq h(\mathsf{A}_i^+\eta) \geq h(\eta) + \log(1 - \epsilon_i) \quad (\text{when } \epsilon_i < 1), \tag{1.8}$$

and defines a convex set. Now, the main uniqueness result is the following.

Theorem 1.4. *When $d \geq 3$, there is a unique normalized Dirichlet eigenfunction in $\mathcal{D}_\rho$. This eigenfunction is positive ν_ρ-a.s. on $\mathcal{A}^c$.*

The proofs of Theorem 1.4 and Theorem 1.2 are written in Section 5. We sketch the simple intuitive steps behind the proof of uniqueness. Assume there exist $u, \tilde{u}$ solutions of (1.4(i)) in $\mathcal{D}_\rho$. Then, we show that they are positive (on $\mathcal{A}^c$), and satisfy

$$\forall \mu \in \mathcal{M}_\rho, \qquad \Gamma(u, \mu) = \Gamma(\tilde{u}, \mu) = -\lambda(\rho). \tag{1.9}$$

As already mentioned, if $u, \tilde{u} \in \mathcal{D}_\rho$ and $\gamma \in]0, 1[$, then $u_\gamma := u^\gamma \tilde{u}^{1-\gamma} \in \mathcal{D}_\rho$. Now, by convexity of $h \mapsto \Gamma(\exp(h), \mu)$

$$\forall \mu \in \mathcal{M}_\rho, \qquad -\lambda(\rho) = \gamma\Gamma(u, \mu) + (1 - \gamma)\Gamma(\tilde{u}, \mu) \geq \Gamma(u_\gamma, \mu) \tag{1.10}$$

We now choose a special μ so that equality obtains in (1.10). The space $\mathcal{M}_\rho$ is built so that if u^* is a positive solution of (1.4(ii)), then

$$d\mu^* := \frac{u_\gamma u^*}{\int u_\gamma u^* d\nu_\rho} d\nu_\rho \in \mathcal{M}_\rho. \tag{1.11}$$

Then, we show that it is legitimate to use the following formal duality

$$\Gamma(u_\gamma, \mu^*) = \text{“}\int \frac{\mathcal{L}(u_\gamma)}{u_\gamma} \frac{u_\gamma u^*}{\int u_\gamma u^* d\nu_\rho} d\nu_\rho\text{”} = \text{“}\int \frac{\mathcal{L}^*(u^*)}{u^*} \frac{u_\gamma u^*}{\int u_\gamma u^* d\nu_\rho} d\nu_\rho\text{”} = -\lambda(\rho). \tag{1.12}$$

Finally, the case of equality in (1.10) and the triviality of the σ-field of exchangeable events under ν_ρ imply that $\tilde{u}/u$ is ν_ρ-a.s. constant on $\mathcal{A}^c$.

As a consequence of Theorem 1.4, we obtain an asymptotical estimate for the hitting time of $\mathcal{A}$. To link this last result with those of [2], we recall Corollary 2.8 of [2] which was based on $L^p(\nu_\rho)$ estimates for u and u^*. When $d \geq 3$, there is a positive constant $\underline{c}$ such that for any $t \geq 0$,

$$\underline{c} \leq \exp(\lambda(\rho)t) P_{\nu_\rho}(\tau > t) \leq 1. \tag{1.13}$$

As a corollary of the uniqueness of the principal eigenfunction in $\mathcal{D}_\rho$, we obtain the following estimates whose proof makes up Section 6.

Theorem 1.5. *When $d \geq 3$, we have the following convergence in $L^1(\nu_\rho)$*

$$\frac{1}{t}\int_0^t e^{\lambda(\rho)s} P_\eta(\tau > s) ds \overset{t\to\infty}{\longrightarrow} \frac{u(\eta)}{\int u u^* d\nu_\rho}. \tag{1.14}$$

Also, after integrating (1.14), we obtain the tail asymptotics

$$\lim_{t\to\infty} \frac{1}{t}\int_0^t e^{\lambda(\rho)s} P_{\nu_\rho}(\tau > s) ds = \frac{1}{\int u u^* d\nu_\rho}. \tag{1.15}$$

2. Notations and preliminaries

We first recall in Section 2.1, the hypotheses needed to define the AZRP on $\mathbb{Z}^d$. Then, in Section 2.2, we describe the class of patterns we consider here. Section 2.3 contains the definition of all function spaces we use.

2.1. The zero-range process

The transition kernel $\{p(i,j),\ i,j \in \mathbb{Z}^d\}$ is associated with a single-particle trajectory and satisfies for all i, j in $\mathbb{Z}^d$

(i) $p(i,j) \geq 0, \quad p(i,i) = 0, \quad \sum_{i\in\mathbb{Z}^d} p(0,i) = 1.$
(ii) $p(i,j) = p(0, j-i)$ (translation invariance).
(iii) $p(i,j) = 0$ if $|i-j| > R$, for some fixed R (finite range).
(iv) If $p_s(i,j) = p(i,j) + p(j,i)$,

$$\text{then} \quad \forall i \in \mathbb{Z}^d,\ \exists n, \quad p_s^{(n)}(0,i) > 0 \quad \text{(irreducibility)}.$$
$$\text{(v)} \ \textstyle\sum_{i\in\mathbb{Z}^d} i p(0,i) \neq 0 \quad \text{(positive drift)}. \tag{2.1}$$

Note that by (i) and (ii), the transition kernel is doubly stochastic. Thus, we can introduce a *dual* transition kernel $\{p^*(i,j),\ i,j \in \mathbb{Z}^d\}$, with $p^*(i,j) = p(j,i)$.

We also need a particle dependent intensity g which satisfies

$$\begin{array}{rl} \text{(i)} & g : \mathbb{N} \to [0,\infty) \text{ is increasing.} \\ \text{(ii)} & g(0) = 0, \quad g(1) = 1 \quad \text{(normalization)}. \\ \text{(iii)} & \Delta := \sup_k (g(k+1) - g(k)) < \infty. \end{array} \tag{2.2}$$

For any $\gamma \in [0, \sup_k g(k)[$, we define a probability θ_γ on $\mathbb{N}$, by

$$\theta_\gamma(0) = 1/Z(\gamma), \qquad \text{and when } n \neq 0, \quad \theta_\gamma(n) = \frac{1}{Z(\gamma)} \frac{\gamma^n}{g(1)\dots g(n)}, \tag{2.3}$$

where $Z(\gamma)$ is the normalizing factor. If we set $\rho(\gamma) := \sum_{n=1}^{\infty} n\theta_\gamma(n)$, then $\rho : [0, \sup_k g(k)[\to [0,\infty[$ is increasing. Let $\gamma(.)$ be the inverse of $\rho(.)$, and for a constant density $\rho > 0$, let ν_ρ be the product probability with marginal law $\theta_{\gamma(\rho)}$. For notational simplicity, we call the intensity at site $i \in \mathbb{Z}^d$, $g_i(\eta) := g(\eta(i))$, and we have

$$\forall B \subset \mathbb{Z}^d, \quad \int \prod_{i\in B} \eta(i) d\nu_\rho = \rho^{|B|}, \quad \text{and}$$
$$\int g_i(\eta)\varphi(\mathsf{A}_i^- \eta) d\nu_\rho(\eta) = \gamma(\rho) \int \varphi d\nu_\rho, \tag{2.4}$$

where $\mathsf{A}_i^-\eta$ has one particle less than η at site i. Also, we will often use that

$$0 \le g(n) \le \Delta n, \quad \text{(by (ii) and (iii) of (2.2))}, \qquad \text{and}$$
$$\int g_i^p d\nu_\rho < \infty, \quad \text{for any } p \in \mathbb{N}. \tag{2.5}$$

Following [9], (see also [1] and [14] Section 2), let

$$\alpha(i) = \sum_{n=0}^{\infty} 2^{-n} p^n(i,0), \quad \text{and for } \eta, \zeta \in \mathbb{N}^{\mathbb{Z}^d}, \quad ||\eta - \zeta|| = \sum_{i\in\mathbb{Z}^d} |\eta(i) - \zeta(i)|\alpha(i).$$

Since the transition kernel is finite range (by 2.1(iii)), another possible choice is $\alpha(k) = \exp(-(|k_1| + \cdots + |k_d|))$ for any site $k = (k_1, \dots, k_d)$ (see [9]). Our state space is $\Omega = \{\eta : ||\eta|| < \infty\}$, and we call $\mathbb{L}$ the space of Lipshitz functions from $(\Omega, ||.||)$ to $(\mathbb{R}, |.|)$, and $\mathbb{L}_b$ the subspace of $\mathbb{L}$ consisting of bounded functions. For $\varphi \in \mathbb{L}$, we call

$$L(\varphi) := \sup\{\frac{|\varphi(\eta) - \varphi(\xi)|}{||\eta - \xi||} : ||\eta - \xi|| > 0,\ \eta, \xi \in \Omega\}. \tag{2.6}$$

In [1], it is shown that a semi-group can be constructed on $\mathbb{L}$ with formal generator

$$\mathcal{L}\varphi(\eta) := \sum_{i,j\in\mathbb{Z}^d} p(i,j)g(\eta(i))\left(\varphi(T^i_j\eta) - \varphi(\eta)\right), \tag{2.7}$$

where $T^i_j\eta(k) = \eta(k)$ if $k \notin \{i, j\}$, $T^i_j\eta(i) = \eta(i) - 1$, and $T^i_j\eta(j) = \eta(j) + 1$. If we set $\nabla^i_j\varphi = \varphi \circ T^i_j - \varphi$, we will often use that on $\{\eta(i) > 0\}$

$$\nabla^i_j\varphi = (\varphi \circ \mathsf{A}^+_j - \varphi \circ \mathsf{A}^+_i) \circ \mathsf{A}^-_i. \tag{2.8}$$

Thus, if we set $\Delta^j_i\varphi = \varphi \circ \mathsf{A}^+_j - \varphi \circ \mathsf{A}^+_i$, and use (2.5) and (2.8), we have the following integration by parts formula

$$\int g_i \nabla^i_j(\varphi) f d\nu_\rho = \gamma_\rho \int \Delta^j_i(\varphi)\, \mathsf{A}^+_i(f) d\nu_\rho. \tag{2.9}$$

Also, for convenience, we often write $\mathsf{A}^\pm_i\varphi$ for $\varphi \circ \mathsf{A}^\pm_i$.

In [14] Section 2, $\mathcal{L}$ is extended to a generator, again called $\mathcal{L}$ for convenience, on $L^2(\nu_\rho)$ for any $\rho > 0$. It is also shown that $\mathbb{L}_b$ is a core for $\mathcal{L}$. Moreover, $\{\nu_\rho, \rho > 0\}$ are ergodic invariant measures for $\mathcal{L}$. We denote by $\mathcal{D}(\mathcal{L}, L^2(\nu_\rho))$ the domain of $\mathcal{L}$ in $L^2(\nu_\rho)$, and by $||.||_\nu$ the $L^2(\nu)$-norm, for any probability measure ν. Finally, we consider the adjoint (or time-reversed) of $\mathcal{L}$ in $L^2(\nu_\rho)$, acting on Lipshitz functions φ and ψ by

$$\int \mathcal{L}^*(\varphi)\psi d\nu_\rho := \int \varphi\mathcal{L}(\psi) d\nu_\rho. \tag{2.10}$$

With our hypothesis, $\mathcal{L}^*$ is again the generator of a zero-range process with transition kernel $p^*(.,.)$, and with the same function g. We denote by $\{S^*_t\}$ the associated semi-group, and by P^*_η the associated Markov process with initial configuration $\eta \in \Omega$.

2.2. *Special patterns*

We first recall that there is a partial order on Ω. For $\eta, \xi \in \Omega$, we say that $\eta \prec \xi$ if $\eta(i) \le \xi(i)$ for all $i \in \mathbb{Z}^d$. A function $f : \Omega \to \mathbb{R}$ is increasing if for $\eta \prec \xi$, $f(\eta) \le f(\xi)$. Also, we say that $A \subset \Omega$ is increasing if its indicator 1_A is increasing. Finally, for given probability measures ν, μ on Ω, we say that $\nu \prec \mu$ if $\int f d\nu \le \int f d\mu$ for every increasing function f. The zero-range process is a monotone process, i.e. there is a coupling such that $P_{\eta,\zeta}(\eta_t \prec \zeta_t, \forall t) = 1$ whenever $\eta \prec \zeta$.

We will be concerned with a pattern, $\mathcal{A}$, with the following properties dubbed ($\mathcal{C}$–$\mathcal{F}$) for connectedness and finiteness:

(i) $\mathcal{A}$ is non-empty, and its support $\mathcal{S}$ is bounded. Thus, $\nu_\rho(\mathcal{A}) > 0$.
(ii) $\mathcal{A}$ is increasing, and $0_\mathcal{S} := \{\eta : \eta(i) = 0, \forall i \in \mathcal{S}\} \not\subset \mathcal{A}$. Thus, $\nu_\rho(\mathcal{A}) < 1$.

(iii) Its complement, $\mathcal{A}^c$, is connected, and is partitioned into a finite number of cylinders with bases in $\mathcal{S}$, whose set we denote by $\Theta_{\mathcal{A}}$. In other words, for any cylinder $\theta \in \Theta_{\mathcal{A}}$, there is an integer n, a sequence $\theta_0, \dots, \theta_n \in \Theta_{\mathcal{A}}$, and $i_1, \dots, i_n \in \mathcal{S}$ such that

$$\theta_0 := 0_{\mathcal{S}}, \quad \theta_n = \theta, \quad \text{and} \quad \theta_k = \mathsf{A}^+_{i_k}\theta_{k-1} \quad \text{for } k = 1, \dots, n.$$

A typical example of patterns satisfying ($\mathcal{C}$–$\mathcal{F}$) is given in (1.1). Note also that if $\mathcal{A}$ satisfies ($\mathcal{C}$–$\mathcal{F}$), there is an integer L such that $\{\eta : \sum_{\mathcal{S}} \eta(i) > L\} \subset \mathcal{A}$.

We denote by $\bar{\mathcal{L}} := 1_{\mathcal{A}^c}\mathcal{L}$ and $\{\bar{S}_t, t \geq 0\}$, respectively the generator and associated semi-group for the process killed on $\mathcal{A}$.

2.3. Function spaces

The topology on $\{\eta : \eta(i) \in \mathbb{N}, i \in \mathbb{Z}^d\}$, is the product of discrete topology, so that $\{\eta_n, n \in \mathbb{N}\}$ converges to η, if for any site $i \in \mathbb{Z}^d$, there is n_0 such that for $n \geq n_0$, $\eta_n(i) = \eta(i)$.

Let $H_{\mathcal{S}} := \inf\{t \geq 0 : X_t \in \mathcal{S}\}$ for $\{X_t\}$ a random walk with transition kernel $\{p(i, j); i, j \in \mathbb{Z}^d\}$. When the dimension $d \geq 3$, $\epsilon_i := \mathbb{P}_i(H_{\mathcal{S}} < \infty) \to 0$ as $||i|| \to \infty$, (as well as ϵ_i^* corresponding to a reversed drift) and we have the classical results

$$\sum_{i \in \mathbb{Z}^d} \epsilon_i^2 + (\epsilon_i^*)^2 < \infty.$$

Let $\mathcal{A}$ satisfy ($\mathcal{C}$–$\mathcal{F}$). Choose n large enough so that $\mathcal{S} \subset \Lambda_n := [-n, n]^d$, and set $\Omega_n = \{\eta : \Lambda_n \to \mathbb{N}\}$, and $\mathcal{F}_n := \sigma(\{\eta(i), i \in \Lambda_n\})$. We often make the abuse of considering functions on Ω_n as defined also on Ω_m for $m \geq n$, but depending only on the sites of Λ_n.

2.3.1. Functions on Ω_n

Definition 2.1. A function φ on Λ_n with $\varphi|_{\mathcal{A}} \equiv 0$ belongs to $\mathcal{D}_n$ when

$$\begin{aligned}
&(0) \quad 0 \leq \varphi,\\
&(\text{i}) \quad \forall \eta, \zeta \in \Omega_n \backslash \mathcal{A}, \text{ if } \eta \prec \zeta \quad \text{then} \quad \varphi(\zeta) \leq \varphi(\eta),\\
&(\text{ii}) \quad \forall \eta \in \Omega_n \backslash \mathcal{A}, \ \forall i \in \Lambda_n \backslash \mathcal{S}, \qquad \varphi(\eta) - \varphi(\mathsf{A}_i^+\eta) \leq \varphi(\eta)\epsilon_i,\\
&(\text{iii}) \quad \int \varphi d\nu_\rho < \infty.
\end{aligned} \tag{2.11}$$

When ϵ_i^* replaces ϵ_i in (ii), we say that φ belong to $\mathcal{D}_n^*$. Also, we set $\mathcal{D}_n^+ := \mathcal{D}_n \cap \{\varphi$ positive on $\mathcal{A}^c\}$.

Lemma 2.2. *$\mathcal{D}_n$ is a convex subset of $\mathbb{L}_b$. When $d \geq 3$, if $\varphi \in \mathcal{D}_n^+$, then φ and $1_{\mathcal{A}^c}/\varphi$ are in $L^p(\nu_\rho)$ for any $p \geq 1$.*

Proof. If $\varphi \in \mathcal{D}_n$, note that φ is bounded since $0 \le \varphi(\eta) \le \varphi(0_{\Lambda_n})$, where 0_{Λ_n} is the empty configuration of Ω_n. Take $\eta, \zeta \in \Omega_n \backslash \mathcal{A}$, and let $\xi = \eta \vee \zeta - \eta \wedge \zeta$, and set $m = \sum_i \xi(i)$. Since φ is decreasing

$$|\varphi(\eta) - \varphi(\zeta)| \le \varphi(\eta \wedge \zeta) - \varphi(\eta \vee \zeta).$$

Now, let $\{\eta_i, i = 0, \dots, m\}$ be the ordered sequence with

$$\eta \wedge \zeta = \eta_0 \prec \eta_1 \prec \cdots \prec \eta_m = \eta \vee \zeta, \quad \text{with} \quad \eta_i = \mathsf{A}_{j_i}^{+} \eta_{i-1},$$

where $\{j_i, i = 1, \dots, m\}$ are the positions of the m particles of ξ. Then,

$$\varphi(\eta \wedge \zeta) - \varphi(\eta \vee \zeta) \le \sum_{i=0}^{n-1} \varphi(\eta_i) - \varphi(\eta_{i+1}) \le \sum_{i=1}^{n} \varphi(\eta_{i-1}) \epsilon_{j_i}.$$

We use that $\varphi(\eta_i) \le \varphi(0_{\Lambda_n})$, and that $\sum_i \epsilon_{j_i} = \sum_k \epsilon_k \xi(k)$. Thus,

$$|\varphi(\eta) - \varphi(\zeta)| \le \varphi(0_{\Lambda_n}) \sum_{k \in \Lambda_n} \epsilon_k \xi(k) \le \varphi(0_{\Lambda_n}) \sup_{k \in \Lambda_n} \left(\frac{\epsilon_k}{\alpha_k}\right) \sum_{k \in \Lambda_n} \xi(k) \alpha(k). \tag{2.12}$$

Now, if $\eta, \zeta \in \mathcal{A}$, then (2.12) holds. Assume that $\eta \in \Omega_n \backslash \mathcal{A}$ but $\zeta \in \mathcal{A}$. Inequality (2.12) follows once we notice that $||\eta - \zeta|| \ge \inf_{\mathcal{S}} \alpha > 0$. Thus, φ is a Lipshitz bounded function. Now, φ and $1_{\mathcal{A}^c}/\varphi$ are in $L^p(\nu_\rho)$ for any integer p by Lemmas 7.2 and 7.4 of the Appendix. □

Definition 2.3. A function h belongs to $\mathcal{E}_n$ if it satisfies

$$\begin{aligned} &\text{(i)} \ \ \forall \eta, \zeta \in \Omega_n \backslash \mathcal{A}, \ \text{if } \eta \prec \zeta \quad \text{then} \quad h(\zeta) \le h(\eta), \\ &\text{(ii)} \ \ \forall \eta \in \Omega_n \backslash \mathcal{A}, \ \forall i \in \Lambda_n \backslash \mathcal{S}, \qquad h(\mathsf{A}_i^{+} \eta) \ge h(\eta) + \log(1 - \epsilon_i), \\ &\text{(iii)} \ \int \exp(h) d\nu_\rho < \infty. \end{aligned} \tag{2.13}$$

Note that for any $\varphi \in \mathcal{D}_n^{+}$, its logarithm (on $\mathcal{A}^c$) belongs to $\mathcal{E}_n$. A key and simple observation is the following.

Lemma 2.4. *$\mathcal{E}_n$ is convex.*

Proof. Inequalities (2.13) (i) and (ii) are stable under convex combination. Also, for $\gamma \in]0, 1[$, and $h_1, h_2 \in \mathcal{E}_n$ by Hölder inequality

$$\int \exp(\gamma h_1 + (1-\gamma) h_2) d\nu_\rho \le \left(\int e^{h_1} d\nu_\rho \right)^{\gamma} \left(\int e^{h_2} d\nu_\rho \right)^{1-\gamma} < \infty. \tag{2.14}$$

□

We now define $\mathcal{M}_n$ a space of probability measures whose elements have a density with respect to ν_ρ, generically noted f satisfying: (i) f is decreasing on $\mathcal{A}^c$, $f|_{\mathcal{A}} \equiv 0$, and

$$\text{(ii)} \qquad \forall \eta \in \Omega_n \backslash \mathcal{A}, \ \forall i \notin \mathcal{S} \qquad f(\eta) - f(\mathsf{A}_i^{+} \eta) \le f(\eta)(\epsilon_i + \epsilon_i^*) \tag{2.15}$$

Lemma 2.5. *Assume that $d \geq 3$. $\mathcal{M}_n$ is convex and compact in the weak topology.*

Proof. The convexity of $\mathcal{M}_n$ is obvious. Consider the compact decreasing set

$$K_M = \{\eta \in \Omega_n : \eta(i) \leq M, \ \forall i \in \Lambda_n\}. \tag{2.16}$$

Note that $\mathcal{M}_n$ is tight, that is

$$\lim_{M\to\infty} \sup_{\mu\in\mathcal{M}_n} \mu(K_M^c) = 0.$$

Indeed, since $d\mu/d\nu_\rho$ is decreasing for any $\mu \in \mathcal{M}_n$, by FKG's inequality

$$\forall \mu \in \mathcal{M}_n, \qquad \mu(K_M^c) = \int \mathbf{1}_{K_M^c} \frac{d\mu}{d\nu_\rho} d\nu_\rho \leq \nu_\rho(K_M^c) \stackrel{M\to\infty}{\longrightarrow} 0.$$

Let $\{\mu_n, n \in \mathbb{N}\}$ be in $\mathcal{M}_n$, with densities $\{f_n := d\mu_n/d\nu_\rho\}$. Let $\{\mu_{n_k}\}$ a converging subsequence to μ. For any $\eta \in \Omega_n$, $\mathbf{1}_\eta$ is a bounded continuous function, so that

$$f_{n_k}(\eta)\nu_\rho(\eta) = \int \mathbf{1}_\eta d\mu_{n_k} \stackrel{k\to\infty}{\longrightarrow} \mu(\eta) = f(\eta)\nu_\rho(\eta). \tag{2.17}$$

Thus, f_{n_k} converges pointwise to f on Ω_n. It is clear that f satisfies (2.15) so that $\mu \in \mathcal{M}_n$. □

An important feature of $\mathcal{M}_n$ is the following.

Lemma 2.6. *Assume that $d \geq 3$. If $\varphi \in \mathcal{D}_n^+$ and $\varphi^* \in (\mathcal{D}_n^*)^+$, then*

$$d\mu = \frac{\varphi\varphi^* d\nu_\rho}{\int \varphi\varphi^* d\nu_\rho} \in \mathcal{M}_n. \tag{2.18}$$

Proof. First, by Lemma 7.2, $\int\varphi\varphi^* d\nu_\rho < \infty$. Also, note that $\varphi, \varphi^* > 0$ on $\mathcal{A}^c$ so that $\int \varphi\varphi^* d\nu_\rho > 0$. Thus, μ given in (2.18) is well defined. Now, since φ and φ^* are decreasing on $\mathcal{A}^c$ and positive, $d\mu/d\nu_\rho$ is decreasing on $\mathcal{A}^c$. Now, if $\zeta = \mathsf{A}_i^+\eta$, for $i \notin \mathcal{S}$

$$\begin{aligned} \varphi(\eta)\varphi^*(\eta) - \varphi(\zeta)\varphi^*(\zeta) &= \varphi^*(\eta)(\varphi(\eta) - \varphi(\zeta)) + \varphi(\zeta)(\varphi^*(\eta) - \varphi^*(\zeta)) \\ &\leq \varphi(\eta)\varphi^*(\eta)(\epsilon_i + \epsilon_i^*). \end{aligned} \tag{2.19}$$

Thus, μ satisfies (i) and (ii) of (2.15). □

2.3.2. Functions on Ω

We define $\mathcal{D}_\rho$ as the natural extention of $\mathcal{D}_n$ to functions of $L^2(\nu_\rho)$. Thus, functions in $\mathcal{D}_\rho$ satisfy the inequalities in (2.11(0)-(iii)) but almost surely with respect to ν_ρ. Also, $\mathcal{D}_\rho^+$ denotes the functions of $\mathcal{D}_\rho$ positive ν_ρ-a.s. on $\mathcal{A}^c$. Similarly, we extend $\mathcal{M}_n$ into $\mathcal{M}_\rho$, the space of probability measures absolutely continuous with respect to ν_ρ, whose densities satisfy ν_ρ-a.s. the same conditions as functions of $\mathcal{M}_n$, but extended on the whole of $\mathbb{Z}^d$. Note that by linearity of the conditional expectation, for $\varphi \in \mathcal{D}_\rho$, $E_{\nu_\rho}[\varphi|\mathcal{F}_n] \in \mathcal{D}_n$, and similarly if $\mu \in \mathcal{M}_\rho$ with density f, then the probability measure $E_{\nu_\rho}[f|\mathcal{F}_n]d\nu_\rho \in \mathcal{M}_n$.

Lemma 2.7. *$\mathcal{M}_\rho$ is compact in the weak topology.*

Proof. First, by Remark 7.3 of the Appendix, there is a constant $C(\rho, 2) > 0$ such that

$$\sup_{\mu \in \mathcal{M}_\rho} \int (\frac{d\mu}{d\nu_\rho})^2 d\nu_\rho \le C(\rho, 2).$$

Recall that by Banach-Alaoglu Theorem, $\{d\mu/d\nu_\rho,\ \mu \in \mathcal{M}_\rho\}$ is weak-$L^2(\nu_\rho)$ compact in $L^2(\nu_\rho)$. Secondly, recall that for any $\mu \in \mathcal{M}_\rho$ and integer n,

$$d\mu^{(n)} := E_{\nu_\rho}[\frac{d\mu}{d\nu_\rho}|\mathcal{F}_n] d\nu_\rho \in \mathcal{M}_n.$$

Now, let $\{\mu_k, k \in \mathbb{N}\}$ be in $\mathcal{M}_\rho$, and let μ_∞ be a weak-$L^2(\nu_\rho)$ limit along a subsequence, say $\{n_k\}$. Note that for each integer n, the following convergence holds in weak-$L^2(\nu_\rho)$

$$f_{n_k}^{(n)} := E_{\nu_\rho}[\frac{d\mu_{n_k}}{d\nu_\rho}|\mathcal{F}_n] \overset{k\to\infty}{\longrightarrow} f_\infty^{(n)} := E_{\nu_\rho}[\frac{d\mu_\infty}{d\nu_\rho}|\mathcal{F}_n].$$

Moreover, $f_\infty^{(n)} d\nu_\rho \in \mathcal{M}_n$, since $\mathcal{M}_n$ is compact by Lemma 2.5. Finally, the sequence $\{f_\infty^{(n)},\ n \in \mathbb{N}\}$ is a positive martingale which, by the martingale convergence Theorem, converges ν_ρ-a.s. to f_∞. Clearly, inequality (2.15) holds ν_ρ-a.s. for f_∞. □

Remark 2.8. With the same arguments, we obtain that $\mathcal{D}_\rho \cap \{\varphi : \int \varphi d\nu_\rho \le c\}$ is weak-$L^2(\nu_\rho)$ compact, for any constant $c > 0$.

Remark 2.9. We give now more details on how a solution u to (1.4(i)) was obtained in [2], and why u belongs to $\mathcal{D}_\rho$. We recall that for any probability μ, $\Phi(\mu)$ introduced in [8] was the invariant measure of the renewal process corresponding to $\{\eta_t\}$ started afresh from measure μ each time it hits $\mathcal{A}$. Also, for any integer k, the map $\Phi^{(k)}$ was the k-th iterates of Φ. It is shown in Theorem 2.4 of [2], that the densities of the Cesaro weak-$L^2(\nu_\rho)$ limits of $\{\Phi^{(k)}(\nu_\rho), k \in \mathbb{N}\}$ are solutions of (1.4(i)). There is actually a simple expression for $\Phi^{(k)}$. Since $\lambda(\rho) > 0$, we have $\int_0^\infty P_{\nu_\rho}(\tau > t) t^k dt < \infty$, and the following probability $dm_k(t)$ on $\{t \ge 0\}$ is well defined

$$dm_k(t) = \frac{P_{\nu_\rho}(\tau > t) t^k dt}{\int_0^\infty P_{\nu_\rho}(\tau > t) t^k dt} \quad \text{and} \quad \frac{d\Phi^{(k)}(\nu_\rho)}{d\nu_\rho}(\eta) = \int_0^\infty u_t(\eta) dm_k(t), \tag{2.20}$$

where u_t is mentioned in the paragraph preceding (1.6). Since, $u_t \in \mathcal{D}_\rho$, it is clear that for any integer k, $d\Phi^{(k)}(\nu_\rho)/d\nu_\rho$ belongs to $\mathcal{D}_\rho$ as well as its Cesaro mean, since $\mathcal{D}_\rho$ is convex. Now, since $\{\Phi^{(k)}(\nu_\rho), k \in \mathbb{N}\}$ are probability measures, Remark 2.8 implies that all their Cesaro limits are in $\mathcal{D}_\rho$. Thus, there exists a solution of (1.4(i)) in $\mathcal{D}_\rho$: we denote it by u. Notice also that our uniqueness result, Theorem 1.4, implies that $(\Phi^{(1)}(\nu_\rho) + \cdots + \Phi^{(n)}(\nu_\rho))/n$ converges to u, thus strengthening the results of [2].

3. From finite domains to $\mathbb{Z}^d$

3.1. Irreducible dynamics on Λ_n

Following the approach of [11], as in [1], we first consider, for any integers k and m, a finite-state Markov generator $\mathcal{L}^k_{(m)}$ on the hyper-surface

$$\Omega^k_{(m)} := \{\eta \in \mathbb{N}^{\Lambda_m} : \sum_{i\in\Lambda_m} \eta(i) = k\}.$$

For this purpose we introduce, for any integer n and for $i, j \in \Lambda_n$

$$p_n(i,j) := \begin{cases} p(i,j) & \text{if} \quad i \neq j \\ \sum_{k\notin\Lambda_n} p(i,k) & \text{if} \quad i = j \end{cases}, \quad \text{and}$$
$$p^*_n(i,j) := \begin{cases} p^*(i,j) & \text{if} \quad i \neq j \\ \sum_{k\notin\Lambda_n} p^*(i,k) & \text{if} \quad i = j. \end{cases} \tag{3.1}$$

Note that $\{p_n(i,j)\}$ is not doubly stochastic. We now can define

$$\mathcal{L}^k_{(m)}(\varphi)(\eta) = \sum_{i,j\in\Lambda_m} p_m(i,j) g_i(\eta)(\varphi(T^i_j\eta) - \varphi(\eta)), \qquad \forall \eta \in \Omega^k_{(m)}. \tag{3.2}$$

Now, we take $n < m - R$, where R is the range of the transition kernel $p(.,.)$, and for $\varphi \in \mathcal{D}_n$, we define

$$\mathcal{L}^\rho_n(\varphi) = \lim_{K\to\infty} \sum_{k=0}^{K} E_{\nu_\rho}[1_{\Omega^k_{(m)}} \mathcal{L}^k_{(m)}(\varphi)|\mathcal{F}_n]. \tag{3.3}$$

This limit is well define since $\mathcal{D}_n \subset \mathbb{L}_b$, and

$$p_m(i,j) g_i(\eta)(\varphi(T^i_j\eta) - \varphi(\eta)) \le L(\varphi) p_m(i,j) g_i(\eta)(\alpha(i) + \alpha(j)),$$

so that by Lemma 2.1 of [14], we have that

$$\sum_{k\ge 0} \int \left(\mathcal{L}^k_{(m)}(\varphi)\right)^2 1_{\Omega^k_{(m)}} d\nu_\rho < \infty.$$

Also, the expression $\mathcal{L}^k_{(m)}(\varphi)$, and the limit (3.3) are independent of m when $m > n + R$, and we called the latter $E_{\nu_\rho}[\mathcal{L}(\varphi)|\mathcal{F}_n]$ in the Introduction. Since $\{\mathcal{L}^k_{(m)}, k \in \mathbb{N}\}$ have the same expression, we henceforth drop the index k, as well as ρ in $\mathcal{L}^\rho_n$ since we work with a fixed density $\rho > 0$. Finally, a simple computation gives an expression for $\mathcal{L}_n$

$$\mathcal{L}_n(\varphi) = \mathcal{L}_{(n)}(\varphi) + \sum_{i\in\Lambda_n} p^*_n(i,i)\gamma_\rho(\varphi \circ \mathsf{A}^+_i - \varphi) + \sum_{i\in\Lambda_n} p_n(i,i) g_i(\varphi \circ \mathsf{A}^-_i - \varphi). \tag{3.4}$$

Note that by definition of $\mathcal{L}_n$, the product of measures $\theta_{\gamma(\rho)}$ over sites of Λ_n, which we denote either by $\nu_\rho^{\Lambda_n}$ or simply by ν_ρ, is the invariant measure for $\mathcal{L}_n$. Also, we have $\mathcal{L}_n^*(\varphi) = E_{\nu_\rho}[\mathcal{L}^*(\varphi)|\mathcal{F}_n]$. Finally, we omit the simple proof that $\mathcal{L}_n$ is a monotone irreducible process.

We denote by E_η^n (resp. $E_\eta^{(n)}$) the law of the Markov process generated by $\mathcal{L}_n$ (resp. $\mathcal{L}_{(n)}$) with initial configuration η. We denote by $\bar{\mathcal{L}}_n := 1_{\mathcal{A}^c}\mathcal{L}_n$ (resp. $\bar{\mathcal{L}}_{(n)} := 1_{\mathcal{A}^c}\mathcal{L}_{(n)}$) the process killed on $\mathcal{A}$, and by $\bar{S}_t^n$ (resp. $\bar{S}_t^{(n)}$) the associated semi-group. Note that for $\varphi|_\mathcal{A} \equiv 0$

$$\bar{S}_t^n(\varphi)(\eta) = E_\eta^n[\varphi(\eta_{t\wedge\tau})] = E_\eta^n[\varphi(\eta_t)1_{\tau>t}].$$

3.2. Approximating the killed process

The main approximation result is the following.

Lemma 3.1. *For any $\varphi \in \mathbb{L}_b$ with $\varphi|_\mathcal{A} \equiv 0$, we have*

$$\forall t > 0, \qquad \lim_{n\to\infty} \int |\bar{S}_t^n(\varphi) - \bar{S}_t(\varphi)|d\nu_\rho = 0.$$

Proof. We first approximate $\{\tau > t\}$ by $\{\eta(t_i) \notin \mathcal{A},\ i = 0, \dots, k\}$ where $\{t_i\}$ is a regular subdivision of $[0, t]$ of mesh t/k; we denote the latter event $\{\tau^k > t\}$. Thus, we show in Step 1 that for each $k > 0$, and $\varphi \in \mathbb{L}_b$ with $\varphi|_\mathcal{A} \equiv 0$

$$\lim_{n\to\infty} \int |E_\eta^n[1_{\{\tau^k>t\}}\varphi(\eta_t)] - E_\eta^{(n)}[1_{\{\tau^k>t\}}\varphi(\eta_t)]|d\nu_\rho = 0. \tag{3.5}$$

Since by Lemmas 2.3 and 2.6 of [1], we have the pointwise convergence

$$E_\eta^{(n)}[1_{\{\tau^k>t\}}\varphi(\eta_t)] = S_{t_1}^{(n)}\left(1_{\mathcal{A}^c}S_{t_2}^{(n)}\left(1_{\mathcal{A}^c}\dots S_{t_{k+1}}^{(n)}(\varphi)\right)\right)(\eta) \overset{n\to\infty}{\longrightarrow} E_\eta[1_{\{\tau^k>t\}}\varphi(\eta_t)], \tag{3.6}$$

we would conclude that

$$\lim_{n\to\infty} \int |E_\eta^n[1_{\{\tau^k>t\}}\varphi(\eta_t)] - E_\eta[1_{\{\tau^k>t\}}\varphi(\eta_t)]|d\nu_\rho = 0. \tag{3.7}$$

In Step 2, we show that there is a constant C independent of n such that

$$\int |E_\eta^n[1_{\{\tau^k>t\}}\varphi(\eta_t)] - E_\eta^n[1_{\{\tau>t\}}\varphi(\eta_t)]|d\nu_\rho \le C\frac{t}{k}, \tag{3.8}$$

and,

$$\lim_{k\to\infty} E_\eta[1_{\{\tau^k>t\}}\varphi(\eta_t)] = E_\eta[1_{\{\tau>t\}}\varphi(\eta_t)]. \tag{3.9}$$

The proof follows once we combine (3.7), (3.8) and (3.9).

Step 1.

First, we show by induction on k (the number of points in the subdivision of $[0,t]$) that there are two constants C_k, C'_k such that for $\eta \notin \mathcal{A}$ if we set $\delta_n(i) = (p_n(i,i) + p_n^*(i,i))\alpha(i)$

$$|E_\eta^n[1_{\{\tau^k>t\}}\varphi(\eta_t)] - E_\eta^{(n)}[1_{\{\tau^k>t\}}\varphi(\eta_t)]| \\ \le C_k \sum_{i\in\Lambda_n} \delta_n(i) \sum_{j=0}^{k-1} \int_0^t E_\eta^n[\eta_{s+s_j}(i) + C'_k]ds, \quad (3.10)$$

where $s_0 = 0$ and $s_j = t_1 + \cdots + t_j$.

For $k = 1$, we have $t_0 = 0$ and $t_1 = t$, so that (3.10) reduces to show that for $\eta \notin \mathcal{A}$, there are C_1, C'_1 such that

$$|S_t^n\varphi(\eta) - S_t^{(n)}\varphi(\eta)| \le C_1 \sum_{i\in\Lambda_n} \delta_n(i) \int_0^t E_\eta^n[\eta_s(i) + C'_1]ds, \quad (3.11)$$

To obtain (3.11), we use an integration by parts formula

$$S_t^n\varphi(\eta) - S_t^{(n)}\varphi(\eta) = \int_0^t S_{t-s}^n(\mathcal{L}_n - \mathcal{L}_{(n)})S_s^{(n)}\varphi(\eta)ds.$$

Since $\varphi \in \mathbb{L}_b$, Lemma 2.2 of [1] implies that for some constant C

$$L(S_s^{(n)}\varphi) \le e^{Cs}L(\varphi).$$

From (3.4) it is enough to bound terms of the form

$$|\mathsf{A}_i^\pm S_s^{(n)}\varphi(\eta) - S_s^{(n)}\varphi(\eta)| \le L(S_s^{(n)}\varphi)\alpha(i) \le L(\varphi)e^{Cs}\alpha(i). \quad (3.12)$$

Thus,

$$|S_t^n\varphi(\eta) - S_t^{(n)}\varphi(\eta)| \le L(\varphi) \sum_{i\in\Lambda_n} \delta_n(i) \int_0^t \left(S_s^n(g_i)(\eta) + \gamma_\rho\right) ds.$$

(3.11) follows after recalling that $g_i(\eta) \le \Delta\eta(i)$.

The induction step from k to $k+1$ follows with exactly the same arguments. First, we recall (3.6) and write similarly

$$E_\eta^n[1_{\{\tau^k>t\}}\varphi(\eta_t)] = S_{t_1}^n\left(1_{\mathcal{A}^c}S_{t_2}^n\left(1_{\mathcal{A}^c}\dots S_{t_{k+1}}^n(\varphi)\right)\right).$$

We call $\psi_2 := S_{t_2}^{(n)}(1_{\mathcal{A}^c}S_{t_3}^{(n)}(1_{\mathcal{A}^c}\dots))$, and recall that $\psi_2 \in \mathbb{L}_b$ by Lemma 2.3 of [1]. We now show that $1_{\mathcal{A}^c}\psi_2 \in \mathbb{L}_b$. Indeed, for $\eta, \zeta \in \Omega$

$$|\psi_2(\eta)1_{\eta\in\mathcal{A}^c} - \psi_2(\zeta)1_{\zeta\in\mathcal{A}^c}| \le 1_{\eta,\zeta\in\mathcal{A}^c}|\psi_2(\eta) - \psi_2(\zeta)| + 1_B(\eta,\zeta)|\psi_2|_\infty. \quad (3.13)$$

where we set $B := \mathcal{A}\times\mathcal{A}^c \cup \mathcal{A}^c\times\mathcal{A}$. Now, $(\eta,\zeta) \in B$ implies that $\sum_{\mathcal{S}}|\eta(i) - \zeta(i)| \ge 1$. Thus,

$$1_B(\eta,\zeta) \le \sum_{\mathcal{S}} |\eta(i)-\zeta(i)| \le \frac{\sum_{\mathcal{S}} |\eta(i)-\zeta(i)|\alpha(i)}{\inf_{\mathcal{S}} \alpha(i)} \le C\|\eta-\zeta\|. \quad (3.14)$$

Thus, combining (3.14) and (3.13) we obtain that $1_{\mathcal{A}^c}\psi \in \mathbb{L}_b$. Now,

$$\begin{aligned} E_\eta^n[1_{\{\tau^k>t\}}\varphi(\eta_t)] - E_\eta^{(n)}[1_{\{\tau^k>t\}}\varphi(\eta_t)] &= \Big(S_{t_1}^n(1_{\mathcal{A}^c}\psi_2) - S_{t_1}^{(n)}(1_{\mathcal{A}^c}\psi_2)\Big) \\ &\quad - S_{t_1}^n\left(1_{\mathcal{A}^c}\left(\psi_2 - S_{t_2}^n(1_{\mathcal{A}^c}S_{t_3}^n(1_{\mathcal{A}^c}\dots))\right)\right) \end{aligned} \quad (3.15)$$

To the first term on the r.h.s we apply the estimates of the step $k=1$ of the induction. For the second term, the difference $\psi_2 - S_{t_2}^n(1_{\mathcal{A}^c}S_{t_3}^n(1_{\mathcal{A}^c}\dots))$ has k subdivision times, and we use our induction hypothesis to obtain (3.10) at order k; since $S_{t_1}^n$ is positive preserving, the inequality is preserved after applying $S_{t_1}^n$ and we obtain the desired (3.10) at order $k+1$. Now, to obtain (3.6), note that

$$\sum_{i\in\Lambda_n} \delta_n(i) \le C \sum_{i\in\Lambda_n\setminus\Lambda_{n-R}} \alpha(i) \stackrel{n\to\infty}{\longrightarrow} 0 \qquad (\text{since } \sum_{i\in\mathbb{Z}^d}\alpha(i)<\infty).$$

Step 2. Let $\sigma_{\mathcal{S}}$ be the first time a particle inside $\mathcal{S}$ escapes $\mathcal{S}$, and let θ_t be the time-translation by t. By the strong Markov property, for $\eta \notin \mathcal{A}$ and $\epsilon = t/k$

$$\begin{aligned} |P_\eta^n(\tau>t) - P_\eta^n(\tau^k>t)| &\le P_\eta^n\left(\bigcup_{i\le k}\{\tau\in]t_{i-1},t_i[, \sigma_{\mathcal{S}}\circ\theta_\tau<\epsilon\}\right) \\ &= \sum_{i=1}^k E_\eta^n[1_{\tau\in]t_{i-1},t_i[}P_{\eta_\tau}^n(\sigma_{\mathcal{S}}<\epsilon)]. \end{aligned} \quad (3.16)$$

We need now the uniform estimate $P_{\eta_\tau}^n(\sigma_{\mathcal{S}}<\epsilon) \le C\epsilon$. By the hypotheses made on $\mathcal{A}$, we know that at time τ, there is a bounded number of particles in $\mathcal{S}$. For the zero range process, it is routine to couple, from time τ onward, the motion of the particle inside $\mathcal{S}$ (at time τ) with a process containing only particles in $\mathcal{S}$ distributed as those of η_τ. Now, for this new process, at any site, the rate of jump is bounded (uniformly in η_τ, since the number of particles is uniformly bounded), and the probability of having a jump before time ϵ is smaller than $1-\exp(-\bar{c}\epsilon) \le \bar{c}\epsilon$. The limit (3.9) follows for the same reasons. This concludes Step 2. □

3.3. Donsker-Varadhan functionals in Λ_n

If $\varphi \in \mathcal{D}_n^+$, note that $\mathcal{L}_n(\varphi)/\varphi$ is a finite sum of terms belonging to $L^2(\nu_\rho)$ by (2.5) and Lemma 2.2. Thus, for $(\varphi,\mu) \in \mathcal{D}_n^+ \times \mathcal{M}_n$, the following expression is well defined

$$\Gamma_n(\varphi,\mu) := \int \frac{\mathcal{L}_n\varphi}{\varphi} d\mu. \quad (3.17)$$

The functional $\Gamma_n(\varphi,\mu)$ is useful if it has some regularity in μ and convexity in $\log(\varphi)$.

Lemma 3.2. *Assume $d \geq 3$. (i) For any $\varphi \in \mathcal{D}_n$, $\Gamma_n(\varphi, .) : \mathcal{M}_n \to \mathbb{R}$ is continuous. (ii) For any $\mu \in \mathcal{M}_n$, the map $\tilde{\Gamma}_n(., \mu) := \Gamma_n(\exp(.), \mu) : \mathcal{E}_n \to \mathbb{R}$ is convex.*

Proof. Since $\mathcal{L}_n(\varphi)/\varphi$ is not bounded, point (i) is not obvious. Let $\{\mu_k, k \in \mathbb{N}\}$ be in $\mathcal{M}_n$ converging weakly to μ. We show that for any $\varphi \in \mathcal{D}_n$, $\Gamma_n(\varphi, \mu_k)$ converges to $\Gamma_n(\varphi, \mu)$ as k tends to infinity. We recall the notation $\nabla^i_j = T^i_j - 1$,

$$\Gamma_n(\varphi, \mu_k) := \sum_{i,j \in \Lambda_n} p(i, j) \int g_i \frac{\nabla^i_j \varphi}{\varphi} d\mu_k + \sum_{i \in \Lambda_n} \int (p^*_n(i, i)\gamma_\rho \frac{\mathsf{A}^+_i \varphi - \varphi}{\varphi} + p_n(i, i) g_i \frac{\mathsf{A}^-_i \varphi - \varphi}{\varphi}) d\mu_k. \tag{3.18}$$

Let K_M be the compact set defined in (2.16). When integrating over K_M, the integrals on the r.h.s of (3.18) pose no problem since the integrand over K_M is bounded. When integrating over K^c_M, we recall first that for any integer p, φ, $1_{\mathcal{A}^c}/\varphi \in L^p(\nu_\rho)$ by Lemma 2.2, $g_i \in L^p(\nu_\rho)$ by (2.5) and $\{f_k := d\mu_k/d\nu_\rho, k \in \mathbb{N}\}$ are uniformely bounded in $L^p(\nu_\rho)$ by Remark 7.3. By Hölder's inequality for $p = 5$

$$\begin{aligned}
\int_{K^c_M} g_i \frac{T^i_j \varphi}{\varphi} d\mu_k &\leq \int_{K^c_M} g_i \frac{\varphi \circ \mathsf{A}^-_i}{\varphi} f_k d\nu_\rho \\
&\leq \Big(\int g_i \varphi^p \circ \mathsf{A}^-_i d\nu_\rho \int \frac{1_{\mathcal{A}^c}}{\varphi^p} d\nu_\rho \\
&\quad \int f^p_k d\nu_\rho \int g^{p-1}_i d\nu_\rho \nu_\rho(K^c_M) \Big)^{1/p} \\
&\leq \Big(\gamma_\rho \int \varphi^p d\nu_\rho \int \frac{1_{\mathcal{A}^c}}{\varphi^p} d\nu_\rho \\
&\quad \int f^p_k d\nu_\rho \int g^{p-1} d\nu_\rho \Big)^{1/p} \nu_\rho(K^c_M)^{1/p} \\
&\leq C \nu_\rho(K^c_M)^{1/p} \overset{M \to \infty}{\longrightarrow} 0.
\end{aligned} \tag{3.19}$$

The other terms of (3.18) are dealt with in the same way. To establish (ii), first note that by Lemma 2.4, $\mathcal{E}_n$ is convex. Then

$$\Gamma_n(e^h, \mu) = \sum_{i,j \in \Lambda_n} p(i, j) \int g_i (e^{\nabla^i_j h} - 1) d\mu + \sum_{i \in \Lambda_n} p^*_n(i, i)\gamma_\rho \int (e^{h \circ \mathsf{A}^+_i - h} - 1) + p_n(i, i) \int g_i (e^{h \circ \mathsf{A}^-_i - h} - 1) d\mu. \tag{3.20}$$

The convexity follows from the convexity of the exponential. □

3.4. A variational formula for $\lambda_n(\rho)$

Lemma 3.3. *For $d \geq 1$, there is $u_n \in \mathcal{D}_n$ and $\lambda_n(\rho) > 0$ such that*

$$1_{\mathcal{A}^c}\mathcal{L}_n(u_n) + \lambda_n(\rho)u_n = 0. \tag{3.21}$$

Moreover u_n is positive on $\mathcal{A}^c$.

Similarly, when $d \geq 1$, there is $u_n^* \in \mathcal{D}_n^*$, positive on $\mathcal{A}^c$, which satisfies $1_{\mathcal{A}^c}\mathcal{L}_n^* u_n^* + \lambda_n(\rho)u_n^* = 0$, and

$$-\lambda_n(\rho) = \lim_{t\to\infty} \frac{1}{t}\log(P_{\nu_\rho}^n(\tau > t)). \tag{3.22}$$

Proof. The proof follows the same lines as that of [2] (see also [8] and Remark 2.9). This is expected since $\mathcal{L}_n$ is a monotone operator with the same features as $\mathcal{L}$. Thus, (3.22) follows as simply as (1.3) by a subadditivity argument. Now, for $\eta \in \Omega_n$, we denote

$$u_{t,n}(\eta) = \frac{P_\eta^n(\tau > t)}{P_{\nu_\rho}^n(\tau > t)} = \frac{e^{t 1_{\mathcal{A}^c}\mathcal{L}_n}(1_{\mathcal{A}^c})(\eta)}{P_{\nu_\rho}^n(\tau > t)}, \text{ and } u_{t,n}^*(\eta) = \frac{e^{t 1_{\mathcal{A}^c}\mathcal{L}_n^*}(1_{\mathcal{A}^c})(\eta)}{P_{\nu_\rho}^n(\tau > t)} \tag{3.23}$$

and as in Step 1 of the proof of Lemma 2.6 of [2], $u_{t,n} \in \mathcal{D}_n$ and $u_{t,n}^* \in \mathcal{D}_n^*$. We focus now on $u_{t,n}$, though similar properties will hold for $u_{t,n}^*$. First, by Lemma 1.1, $\lambda_n(\rho) \geq \lambda(\rho) > 0$. Thus, for any k, $\int_0^\infty P_{\nu_\rho}^n(\tau > t)t^k dt < \infty$, and as in Remark 2.9 we define

$$dm_k(t) = \frac{P_{\nu_\rho}^n(\tau > t)t^k dt}{\int_0^\infty P_{\nu_\rho}^n(\tau > t)t^k dt} \quad \text{and} \quad \frac{d\Phi_n^{(k)}(\nu_\rho)}{d\nu_\rho}(\eta) = \int_0^\infty u_{t,n}(\eta)dm_k(t).$$

With identical arguments as in the proof of Theorem 2.4 of [2], the Cesaro weak-$L^2(\nu_\rho)$ limits of $\{\Phi_n^{(k)}(\nu_\rho), k \in \mathbb{N}\}$ are solutions of (3.21). Now, it is clear that $d\Phi_n^{(k)}(\nu_\rho)/d\nu_\rho \in \mathcal{D}_n$. Also, in the weak-$L^2(\nu_\rho)$ topology $\mathcal{D}_n$ is compact by Remark 2.8, and contain all the Cesaro weak limits of $\{\Phi_n^{(k)}(\nu_\rho), k \in \mathbb{N}\}$. Thus, there is a solution of (3.21) in $\mathcal{D}_n$: we denote it by u_n.

We now show that $u_n > 0$ on $\mathcal{A}^c$. By contradiction assume that for $\eta \in \Omega_n \backslash \mathcal{A}$, $u_n(\eta) = 0$. Then (3.22) implies that $\mathcal{L}_n(u_n)(\eta) = 0$. This, in turn, implies that

(i) For all $i, j \in \Lambda_n$ with $p(i, j) > 0$, we have $u_n(T_j^i \eta) = 0$.
(ii) For all $i \in \Lambda_n$ with $p_n^*(i, i) > 0$, we have $u_n(\mathsf{A}_i^+ \eta) = 0$.
(iii) For all $i \in \Lambda_n$ with $\eta(i)p_n(i, i) > 0$, we have $u_n(\mathsf{A}_i^- \eta) = 0$.

To conclude that $u_n \equiv 0$ on $\mathcal{A}^c$, it is enough to note that by the hypotheses ($\mathcal{C}$–$\mathcal{F}$) on $\mathcal{A}^c$, each $\eta \in \mathcal{A}^c$ can be transformed into O_{Λ_n} by a succession of actions $\{\mathsf{A}_i^-\}$ with $i \in \Lambda_n$, and $\{T_j^i\}$ with $i, j \in \Lambda_n$. The reverse operation is made through a succession of $\{\mathsf{A}_i^+\}$ with $i \in \Lambda_n$, and $\{T_j^i\}$ with $i, j \in \Lambda_n$. □

We now establish the Donsker-Varadhan representation for $\lambda_n(\rho)$.

Lemma 3.4. *Assume $d \geq 3$. If $\mathcal{A}$ satisfies ($\mathcal{C}$–$\mathcal{F}$) of Section 2.2, then $\lambda_n(\rho)$ is given by*

$$-\lambda_n(\rho) = \sup_{\mu \in \mathcal{M}_n} \inf_{\varphi \in \mathcal{D}_n^+} \int \frac{\mathcal{L}_n \varphi}{\varphi} d\mu. \tag{3.24}$$

Proof. Let us call γ_n the right hand side of (3.24). From Lemma 3.3, there is $u_n \in \mathcal{D}_n^+$ such that $\bar{\mathcal{L}}_n u_n + \lambda_n(\rho) u_n = 0$. This implies that $\gamma_n \leq -\lambda_n(\rho)$. We can use a classical minimax theorem [7], since we have that (i) for any fixed $\mu \in \mathcal{M}_n$, $h \mapsto \tilde{\Gamma}_n(h, \mu)$ is convex (by Lemma 3.2) on the convex set $\mathcal{E}_n$ (by Lemma 2.4), (ii) for any fixed $h \in \mathcal{E}_n$, $\mu \mapsto \tilde{\Gamma}_n(h, \mu)$ is continuous (by Lemma 3.2) on the compact set $\mathcal{M}_n$. Thus,

$$\gamma_n = \inf_{\varphi \in \mathcal{D}_n^+} \sup_{\mu \in \mathcal{M}_n} \int \frac{\mathcal{L}_n \varphi}{\varphi} d\mu. \tag{3.25}$$

Now, for any $\varphi \in \mathcal{D}_n^+$, $0 < \int \varphi u_n^* d\nu_\rho < \infty$, and we can define

$$d\mu^* = \frac{\varphi u_n^* d\nu_\rho}{\int \varphi u_n^* d\nu_\rho} \in \mathcal{M}_n \quad \text{(by Lemma 2.6)}.$$

Then, by duality

$$\int \frac{\mathcal{L}_n(\varphi)}{\varphi} d\mu^* = \int \frac{\mathcal{L}_n(\varphi)}{\varphi} \frac{\varphi u_n^*}{\int \varphi u_n^* d\nu_\rho} d\nu_\rho = \int \frac{\varphi}{\int \varphi u_n^* d\nu_\rho} \mathcal{L}_n^*(u_n^*) d\nu_\rho = -\lambda_n(\rho).$$

By (3.25), $\gamma_n \geq -\lambda_n(\rho)$, and the proof is concluded. □

In the following lemma, we establish the uniqueness of the principal Dirichlet eigenfunction.

Lemma 3.5. *Assume $d \geq 3$. There is a unique non-negative eigenfunction $u_n \in \mathcal{D}_n$ of $1_{\mathcal{A}^c} \mathcal{L}_n$ which satisfies $\int u_n d\nu_\rho = 1$.*

Proof. We know from Lemma 3.3 that there exists a positive eigenfunction u_n. Assume that $\tilde{u}$ is a non-negative Dirichlet eigenfunction with $\int \tilde{u} d\nu_\rho = 1$ and corresponding eigenvalue $\tilde{\lambda}$. By the same argument as in the proof of Lemma 3.3, we have that $\tilde{u}$ is positive on $\mathcal{A}^c$.

First, we show that $\tilde{\lambda} = \lambda_n$. Let u_n^* be the dual eigenfunction given in Lemma 3.3. We multiply equality (3.21) by u_n^*, integrate over ν_ρ and use duality

$$\int u_n^* \mathcal{L}_n(\tilde{u}) d\nu_\rho = -\tilde{\lambda} \int u_n^* \tilde{u} d\nu_\rho \Longrightarrow (\lambda_n(\rho) - \tilde{\lambda}) \int u_n^* \tilde{u} d\nu_\rho = 0. \tag{3.26}$$

Now, since u_n^* and $\tilde{u}$ are positive on $\mathcal{A}^c$ we conclude that $\tilde{\lambda} = \lambda_n(\rho)$.

Second, we show that $\tilde{u} = u_n$. Set $h := \log(u_n)$ and $\tilde{h} := \log(\tilde{u})$, on $\mathcal{A}^c$. For any $\mu \in \mathcal{M}_n$ and any $\gamma \in]0, 1[$, by the convexity of $\tilde{\Gamma}_n$

$$\gamma \tilde{\Gamma}_n(h, \mu) + (1 - \gamma) \tilde{\Gamma}_n(\tilde{h}, \mu) \geq \tilde{\Gamma}_n(\gamma h + (1 - \gamma) \tilde{h}, \mu). \tag{3.27}$$

Since u_n and $\tilde{u}$ are solution of (3.21), the left hand side of (3.27) is $-\lambda_n(\rho)$. We define $h_\gamma = \gamma h + (1-\gamma)\tilde{h} \in \mathcal{E}_n$ and we note that $0 < \int \exp(h_\gamma) u_n^* d\nu_\rho < \infty$. Now,

$$d\mu_\gamma = \frac{e^{h_\gamma} u_n^* d\nu_\rho}{\int e^{h_\gamma} u_n^* d\nu_\rho} \in \mathcal{M}_n, \qquad \text{and is such that}$$

$$\tilde{\Gamma}_n(h_\gamma, \mu_\gamma) = \Gamma_n(u_n^*, \mu_\gamma) = -\lambda_n(\rho).$$

Thus, we have equality in (3.27) with μ_γ. Since μ_γ gives a positive weight to any $\eta \in \Omega_n \backslash \mathcal{A}$, the following three conditions hold: (i) for all $i, j \in \Lambda_n$ with $g_i(\eta) p(i,j) > 0$, we have $\nabla_j^i \tilde{h} = \nabla_j^i h$; (ii) for all $j \in \Lambda_n$ with $p_n^*(j,j) > 0$, we have $(\mathsf{A}_j^+ - \mathbf{1})\tilde{h} = (\mathsf{A}_j^+ - \mathbf{1})h$; (iii) for all $j \in \Lambda_n$ with $g(\eta(j)) p_n(j,j) > 0$, we have $(\mathsf{A}_j^- - \mathbf{1})\tilde{h} = (\mathsf{A}_j^- - \mathbf{1})h$.

Since u_n is positive on $\mathcal{A}^c$, we form $f = \tilde{u}/u_n$, and rewrite the conditions (i)-(iii) for f.

(i) For all $\eta \in \mathcal{A}^c$ and $i, j \in \Lambda_n$ with $\eta(i) p(i,j) > 0$, we have $f(T_j^i \eta) = f(\eta)$.
(ii) For all $i \in \Lambda_n$ with $p_n^*(i,i) > 0$, and $\mathsf{A}_i^+ \eta \in \mathcal{A}^c$, we have $f(\mathsf{A}_i^+ \eta) = f(\eta)$.
(iii) For all $\eta \in \mathcal{A}^c$ and $i \in \Lambda_n$ with $\eta(i) p_n(i,i) > 0$, we have $f(\mathsf{A}_i^- \eta) = f(\eta)$.

As in the proof of Lemma 3.3, we conclude that $\tilde{u} = u_n$. □

3.5. Approximating the principal eigenvalue

With an abuse of notations, we define for any finite domain U, $\mathcal{L}_U(\varphi) = E_{\nu_\rho}[\mathcal{L}(\varphi) | \mathcal{F}_U]$. We mean by $\mathcal{L}_U$ an expression like (3.4) where U replaces Λ_n: thus, a zero-range process on U with creations and annihilations on the boundaries of U. We denote by S_t^U the semi-group associated with $\mathcal{L}_U$ and by P_ν^U the corresponding Markov process with initial measure ν. We denote by $\bar{S}_t^U$ the semi-group killed on $\mathcal{A}$.

We first state an obvious corollary of Lemma 3.1 applied to $\varphi = 1_{\mathcal{A}^c}$.

Corollary 3.6. *When the pattern satisfies* $(\mathcal{C}$–$\mathcal{F})$, *we have*

$$\lim_{n\to\infty} P_{\nu_\rho}^n(\tau > t) = P_{\nu_\rho}(\tau > t).$$

Proof of Lemma 1.1. We divide the proof in two steps.
Step 1. We show that $n \mapsto P_{\nu_\rho}^n(\tau > t)$ is increasing.

Let U be a finite subset, $i \notin U$, and set $\tilde{U} = U \cup \{i\}$. Thus, it is enough to show that $\int (\bar{S}_t^{\tilde{U}} 1_{\mathcal{A}^c} - \bar{S}_t^U 1_{\mathcal{A}^c}) d\nu_\rho \geq 0$. Step 1 follows then by induction. Note that for φ $\mathcal{F}_U$-measurable and $j \in U$, we have $\varphi \circ T_j^i = \varphi \circ \mathsf{A}_j^+$, $\varphi \circ T_i^j = \varphi \circ \mathsf{A}_j^-$, $\varphi \circ \mathsf{A}_i^+ = \varphi$ and $\varphi \circ \mathsf{A}_i^- = \varphi$ so that

$$\begin{aligned}
(\bar{\mathcal{L}}_{\tilde{U}} - \bar{\mathcal{L}}_U)\varphi &= 1_{\mathcal{A}^c} \sum_{j\in U} \Big(p(j,i) g_j (\varphi \circ T_i^j - \varphi) + p(i,j) g_i (\varphi \circ T_j^i - \varphi) \Big) \\
&\quad - 1_{\mathcal{A}^c} \sum_{j\in U} \Big(p(j,i) g_j (\varphi \circ \mathsf{A}_j^- - \varphi) + p(i,j) \gamma_\rho (\varphi \circ \mathsf{A}_j^+ - \varphi) \Big) \\
&= 1_{\mathcal{A}^c} \sum_{j\in U} p(i,j)(g_i - \gamma_\rho) \Big(\varphi \circ \mathsf{A}_j^+ - \varphi \Big). \qquad (3.28)
\end{aligned}$$

Now, we set $\varphi_s := \bar{S}_s^U(1_{\mathcal{A}^c})$ and $\psi_s := (\bar{S}_s^{\tilde{U}})^*(1_{\mathcal{A}^c})$, and we use an integration by parts formula

$$\begin{aligned}&\int \bar{S}_t^{\tilde{U}}(1_{\mathcal{A}^c})d\nu_\rho - \int \bar{S}_t^U(1_{\mathcal{A}^c})d\nu_\rho\\&= \int\int_0^t \bar{S}_{t-s}^{\tilde{U}}(\bar{\mathcal{L}}_{\tilde{U}} - \bar{\mathcal{L}}_U)\bar{S}_s^U(1_{\mathcal{A}^c})dsd\nu_\rho\\&= \int\int_0^t (\bar{\mathcal{L}}_{\tilde{U}} - \bar{\mathcal{L}}_U)(\varphi_s)\psi_{t-s}dsd\nu_\rho.\end{aligned} \tag{3.29}$$

Thus, by (3.28)

$$\begin{aligned}&P_{\nu_\rho}^{\tilde{U}}(\tau > t) - P_{\nu_\rho}^U(\tau > t)\\&= \sum_{j\in U} p(i,j)\int\int_0^t (\mathsf{A}_j^+\varphi_s - \varphi_s)(g_i - \gamma_\rho)\psi_{t-s}dsd\nu_\rho^{\tilde{U}}\\&= \sum_{j\in U} p(i,j)\int\int_0^t (\mathsf{A}_j^+\varphi_s - \varphi_s)\int (g_i - \gamma_\rho)\psi_{t-s}d\nu_\rho^{\{i\}}dsd\nu_\rho^U.\end{aligned} \tag{3.30}$$

Note that for any s, $\eta \mapsto \psi_s(\eta)$ and $\eta \mapsto \varphi_s(\eta)$ is decreasing positive, whereas $\eta \mapsto g_i(\eta)$ is increasing and $\int g_i d\nu_\rho = \gamma_\rho$. Thus, by FKG inequality

$$\int (g_i - \gamma_\rho)\psi_{t-s}d\nu_\rho^{\{i\}} \le \int (g_i - \gamma_\rho)d\nu_\rho^{\{i\}}\int \psi_{t-s}d\nu_\rho^{\{i\}} = 0. \tag{3.31}$$

Thus, as $\varphi_s \circ \mathsf{A}_j^+ - \varphi_s \le 0$, the first step concludes. We call $\lambda_\infty(\rho)$ the limit of $\lambda_n(\rho)$.

Step 2. We show the following Lemma which allows us to conclude the proof of Lemma 1.1 readily.

Lemma 3.7. *Any subsequence of* $\{u_n\}$ *has a further subsequence converging, in weak-*$L^2(\nu_\rho)$*, to a solution* u *of (1.4(i)), and* $u \in \mathcal{D}_\rho$*. Moreover,* $\lambda_\infty(\rho) = \lambda(\rho)$.

Proof. For notational convenience, we write the proof for $\{u_n^*\}$. Recall that $\mathcal{D}_\rho^* \cap \{\varphi : \int \varphi d\nu_\rho = 1\}$ is compact in weak-$L^2(\nu_\rho)$ by Remark 2.8. Let $u^* \in \mathcal{D}_\rho^*$ be a (weak-$L^2(\nu_\rho)$) limit point of $\{u_n^*\}$ along a subsequence which for simplicity we still call $\{u_n^*\}$. For any $\varphi \in \mathbb{L}_b$, and any integer n

$$\int \bar{S}_t^n(\varphi)u_n^*d\nu_\rho = e^{-\lambda_n(\rho)t}\int \varphi u_n^* d\nu_\rho. \tag{3.32}$$

Then,

$$\begin{aligned}&\left|\int \bar{S}_t^n(\varphi)u_n^*d\nu_\rho - \int \bar{S}_t(\varphi)u^*d\nu_\rho\right|\\&= \left|\int \left(\bar{S}_t^n(\varphi) - \bar{S}_t(\varphi)\right)u_n^*d\nu_\rho\right| + \left|\int \bar{S}_t(\varphi)(u_n^* - u^*)d\nu_\rho\right|\\&\le \sup_n ||u_n^*||_{\nu_\rho}||\bar{S}_t^n(\varphi) - \bar{S}_t(\varphi)||_{\nu_\rho} + \left|\int \bar{S}_t(\varphi)(u_n^* - u^*)d\nu_\rho\right|.\end{aligned} \tag{3.33}$$

The $L^2(\nu_\rho)$ convergence of $\bar{S}_t^n(\varphi) - \bar{S}_t(\varphi)$ is equivalent to an $L^1(\nu_\rho)$ convergence, since φ is bounded and $\bar{S}_t$, $\bar{S}_t^n$ are contractions (in L^∞). Recalling Lemma 3.1, (3.32) and Step 1, and taking the limit n to infinity, we obtain

$$\int \bar{S}_t(\varphi)u^* d\nu_\rho = e^{-\lambda_\infty(\rho)t} \int \varphi u^* d\nu_\rho. \tag{3.34}$$

Now, since $\mathbb{L}_b$ is a dense set in $L^2(\nu_\rho)$, this implies that $u^* \in \mathcal{D}(\bar{\mathcal{L}}^*, L^2(\nu_\rho))$, and that (3.34) holds for any $\varphi \in L^2(\nu_\rho)$. Take $\varphi = u \in \mathcal{D}_\rho \subset L^2(\nu_\rho)$ solution of (1.4(i)), and use that

$$\bar{S}_t(u) = e^{-\lambda(\rho)t}u, \quad \nu_\rho - \text{a.s.} \Longrightarrow (e^{-\lambda_\infty(\rho)t} - e^{-\lambda(\rho)t}) \int uu^* d\nu_\rho = 0.$$

Now, since u and u^* are decreasing, and in $L^2(\nu_\rho)$, we have

$$\infty > ||u||_{\nu_\rho}||u^*||_{\nu_\rho} \geq \int uu^* d\nu_\rho \overset{\text{FKG}}{\geq} \int u d\nu_\rho \int u^* d\nu_\rho = 1.$$

Thus, $\lambda_\infty(\rho) = \lambda(\rho)$, and u^* satisfies (1.4(ii)). □

4. Donsker-Varadhan functionals on $\mathbb{Z}^d$

The main problem is that $\mathcal{L}(\varphi)$ does not make sense as a pointwise convergent series when $\varphi \in \mathcal{D}_\rho$. Indeed, even if φ were bounded, the naive bound $|\nabla_j^i\varphi| \leq |\varphi|_\infty(\epsilon_i + \epsilon_j)$ would be useless since $\sum_k \epsilon_k = \infty$. Thus, we show in this section how to properly define $\Gamma(\varphi, \mu)$.

4.1. Technical prerequisites

We first define a family of functionals, $\{\Gamma_n,\ n \in \mathbb{N}\}$, on $\mathcal{D}_\rho^+ \times \mathcal{M}_\rho$, whose limit when n tends to infinity is shown to exist.

Lemma 4.1. *Assume $d \geq 3$. For $\varphi \in \mathcal{D}_\rho^+$ and $\mu \in \mathcal{M}_\rho$, and any integer n, the functional $\Gamma_n(\varphi, \mu) := \int \mathcal{L}_n(\varphi)/\varphi d\mu$ is well defined. If we call $\tilde{\Gamma}_n(h, \mu) := \Gamma_n(\exp(h), \mu)$, then for any μ, the map $h \mapsto \tilde{\Gamma}_n(h, \mu)$ is convex on the convex set $\mathcal{E}_\rho$.*

Proof. The formal full expression of $\Gamma_n(\varphi, \mu)$ is

$$\Gamma_n(\varphi, \mu) = \sum_{i,j \in \Lambda_n} p(i, j) \int g_i \frac{\nabla_j^i \varphi}{\varphi} d\mu + \sum_{i \in \Lambda_n} \left(\gamma_\rho p_n^*(i, i) \int \frac{\nabla_i^+ \varphi}{\varphi} d\mu + p_n(i, i) \int g_i \frac{\nabla_i^- \varphi}{\varphi} d\mu \right). \tag{4.1}$$

Note that as $\varphi \in \mathcal{D}_\rho^+$, $T_j^i \varphi \leq \mathsf{A}_i^- \varphi$. Thus, (4.1) is defined if we bound $\int g_i \mathsf{A}_i^-(\varphi)/\varphi \, d\mu$ for each site $i \in \Lambda_n$. This is done as in (3.19).

From (4.1), an expression for $\tilde{\Gamma}_n(h,\mu)$ is as follows

$$\tilde{\Gamma}_n(h,\mu) = \sum_{i,j\in\Lambda_n} p(i,j)\int g_i\left(e^{\nabla^i_j h}-1\right)d\mu$$
$$+\sum_{i\in\Lambda_n}\left(\gamma_\rho p_n^*(i,i)\int(e^{\nabla_i^+ h}-1)d\mu + p_n(i,i)\int g_i(e^{\nabla_i^- h}-1)d\mu\right). \tag{4.2}$$

The convexity of $h\mapsto\tilde{\Gamma}_n(h,\mu)$ follows from the convexity of the exponential. □

We now express $\tilde{\Gamma}_n(h,\mu)$ in terms of gradients of h and μ.

Lemma 4.2. *For $h\in\mathcal{E}_\rho$ and $\mu\in\mathcal{M}_\rho$, we have with $f:=d\mu/d\nu_\rho$*

$$\tilde{\Gamma}_n(h,\mu) = \sum_{i,j\in\Lambda_n}\gamma_\rho p_n(i,j)$$
$$\left(\int\left(e^{\Delta_i^j h}-1\right)\nabla_i^+ f d\nu_\rho + \int\left(e^{\Delta_i^j h}-1-\Delta_i^j h\right)d\mu\right) + R_n(h,\mu), \tag{4.3}$$

with,

$$\lim_{n\to\infty}\sup_{h\in\mathcal{E}_\rho}\sup_{\mu\in\mathcal{M}_\rho}|R_n(h,\mu)| = 0. \tag{4.4}$$

Note also that

$$\int\mathcal{L}_n(\varphi)d\mu = \sum_{i,j\in\Lambda_n}\gamma_\rho p_n(i,j)\int\Delta_i^j\varphi\nabla_i^+ f d\nu_\rho$$
$$-\sum_{i\in\Lambda_n}\gamma_\rho p_n(i,i)\int\nabla_i^+\varphi\nabla_i^+ f d\nu_\rho. \tag{4.5}$$

Proof. First, we apply the integration by parts formula (2.9) to (4.2):

$$\tilde{\Gamma}_n(h,\mu) = \sum_{\substack{i,j\in\Lambda_n\\ i\neq j}} p(i,j)\int g_i\mathsf{A}_i^-\left(e^{\Delta_i^j h}-1\right)d\mu$$
$$+\sum_{i\in\Lambda_n}\left(p_n^*(i,i)\gamma_\rho\int(e^{\nabla_i^+ h}-1)d\mu + p_n(i,i)\int g_i\mathsf{A}_i^-(e^{-\nabla_i^+ h}-1)d\mu\right)$$
$$=\gamma_\rho\sum_{i,j\in\Lambda_n}p_n(i,j)$$
$$\left(\int\left(e^{\Delta_i^j h}-1\right)\nabla_i^+ f d\nu_\rho + \int\left(e^{\Delta_i^j h}-1-\Delta_i^j h\right)d\mu\right)$$
$$+R_n(h,\mu)+N(h,\mu), \tag{4.6}$$

with

$$R_n(h,\mu) := \sum_{i\in\Lambda_n} \gamma_\rho p_n(i,i) \int (e^{-\nabla_i^+ h} - 1)\nabla_i^+(f) d\nu_\rho$$
$$+\gamma_\rho \sum_{i\in\Lambda_n} p_n^*(i,i) \int (e^{\nabla_i^+ h} - 1 - \nabla_i^+ h) d\mu$$
$$+\gamma_\rho \sum_{i\in\Lambda_n} p_n(i,i) \int (e^{-\nabla_i^+ h} - 1 + \nabla_i^+ h) d\mu, \tag{4.7}$$

and,

$$N(h,\mu) := \sum_{i,j\in\Lambda_n} \gamma_\rho p(i,j) \int \Delta_i^j h d\mu + \sum_{i\in\Lambda_n} \gamma_\rho (p_n^*(i,i) - p_n(i,i)) \int (\nabla_i^+ h) d\mu. \tag{4.8}$$

To show that $N(h,\mu)$ vanishes, first write

$$\begin{aligned}\sum_{i,j\in\Lambda_n} p(i,j) \int (\nabla_j^+ h) d\mu &= \sum_{j\in\Lambda_n} \left(\sum_{i\in\Lambda_n} p(i,j) \right) \int (\nabla_j^+ h) d\mu \\ &= \sum_{j\in\Lambda_n} \left(1 - \sum_{i\notin\Lambda_n} p(i,j) \right) \int (\nabla_j^+ h) d\mu \\ &= \sum_{j\in\Lambda_n} (1 - p_n^*(j,j)) \int (\nabla_j^+ h) d\mu, \end{aligned} \tag{4.9}$$

and similarly,

$$\sum_{i,j\in\Lambda_n} p(i,j) \int (\nabla_i^+ h) d\mu = \sum_{i\in\Lambda_n} (1 - p_n(i,i)) \int (\nabla_i^+ h) d\mu.$$

It is thus clear that $N(h,\mu) = 0$.

We now show (4.4). Note that for $i \notin \mathcal{S}$, $\epsilon_i < 1$. Also, for n large enough, if we define $\partial^R \Lambda_n := \Lambda_n \backslash \Lambda_{n-R}$, then $\partial^R \Lambda_n \cap \mathcal{S} = \emptyset$. Also, by (2.1) (iii), $p_n(i,i) = 0$ when $i \notin \partial^R \Lambda_n$. Thus, there is a constant $c_0 > 0$ such that for $i \in \partial^R \Lambda_n$, $|\nabla_i^+ h| \le -\log(1-\epsilon_i) \le c_0 \epsilon_i$, ν_ρ-a.s., and $|\nabla_i^+ f| \le (\epsilon_i + \epsilon_i^*) f$, ν_ρ-a.s. .Thus, there is a constant $c_1 > 0$ such that

$$\begin{aligned} p_n(i,i) \left| \int \left(e^{-\nabla_i^+ h} - 1 \right) \nabla_i^+ f d\nu_\rho \right| &\le \int (\exp(c_0\epsilon_i) - 1)(\epsilon_i + \epsilon_i^*) d\mu \\ &\le c_1 \epsilon_i (\epsilon_i + \epsilon_i^*), \end{aligned} \tag{4.10}$$

and by expanding to second order in $\nabla_i^+ h$

$$\begin{aligned} p_n^*(i,i) \left| \int (e^{\nabla_i^+ h} - 1 - \nabla_i^+ h) d\mu \right| &\le c_1 \epsilon_i^2, \quad \text{and} \\ p_n(i,i) \left| \int (e^{-\nabla_i^+ h} - 1 + \nabla_i^+ h) d\mu \right| &\le c_1 \epsilon_i^2, \end{aligned} \tag{4.11}$$

Combining (4.10) and (4.11), and summing over $i \in \partial^R \Lambda_n$, we obtain the desired asymptotics (4.4), since for dimension $d \geq 3$, $\sum \epsilon_i^2 < \infty$.

We obtain (4.5) from (4.3) by setting $h = \epsilon\varphi$ and expanding $\tilde{\Gamma}_n(h,\mu)$ to first order in ϵ. □

We are now ready for the key technical lemma of this section.

Proposition 4.3. *For* $(\varphi,\mu) \in \mathcal{D}_\rho^+ \times \mathcal{M}_\rho$, $\{\Gamma_n(\varphi,\mu), n \in \mathbb{N}\}$ *is a Cauchy sequence whose limit we denote by* $\Gamma(\varphi,\mu)$. *We have the following properties.*

(i) For $h \in \mathcal{E}_\rho$, $h \mapsto \tilde{\Gamma}(h,\mu) := \Gamma(e^h,\mu)$ *is convex.*
(ii) The Cauchy sequence is uniform in the following sense

$$\lim_{n\to\infty} \sup_{\varphi\in\mathcal{D}_\rho^+} \sup_{\mu\in\mathcal{M}_\rho} |\Gamma_n(\varphi,\mu) - \Gamma(\varphi,\mu)| = 0. \tag{4.12}$$

(iii) For any integer n, and any $\mu \in \mathcal{M}_\rho$ *we denote by* μ_n *the measure of* $\mathcal{M}_n$ *of density* $f_n := E_{\nu_\rho}[d\mu/d\nu_\rho|\mathcal{F}_n]$. *Then,*

$$\lim_{n\to\infty} \sup_{\mu\in\mathcal{M}_\rho} \sup_{\varphi_n\in\mathcal{D}_n} |\Gamma(\varphi_n,\mu_n) - \Gamma(\varphi_n,\mu)| = 0. \tag{4.13}$$

(iv) For $\varphi_n \in \mathcal{D}_n^+$ *and* $\mu_n \in \mathcal{M}_n$, *we have* $\Gamma(\varphi_n,\mu_n) = \Gamma_n(\varphi_n,\mu_n)$.

Proof. Step 1: We show that $\{\Gamma_n(\varphi,\mu), n \in \mathbb{N}\}$ is a Cauchy sequence and (4.12) holds.

By using the expression (4.3) of Lemma 4.2, we have for $m > n$

$$\tilde{\Gamma}_m(h,\mu) - \tilde{\Gamma}_n(h,\mu) = R_m(h,\mu) - R_n(h,\mu) + \sum_{\substack{(i,j)\in\Lambda_m^2\setminus\Lambda_n^2\\ i\neq j}} \gamma_\rho p(i,j)\left(\int (e^{\Delta_i^j h} - 1)\nabla_i^+ f d\nu_\rho + \int (e^{\Delta_i^j h} - 1 - \Delta_i^j h) d\mu\right) \tag{4.14}$$

Since $p(i,j) = 0$ when $|i-j| > R$, we can assume n and m so large that if $(i,j) \in \Lambda_m^2\setminus\Lambda_n^2$ with $p(i,j) > 0$, then $i, j \notin \mathcal{S}$. Thus, there is a positive constant c_0 such that ν_ρ-a.s.

$$\forall (i,j) \in \Lambda_m^2\setminus\Lambda_n^2 \text{ with } p(i,j) > 0, \qquad |\nabla_i^+ h| \leq -\log(1-\epsilon_i) \leq c_0\epsilon_i, \quad \text{and} \quad |\nabla_i^+ f| \leq (\epsilon_i + \epsilon_i^*) f. \tag{4.15}$$

Also, there is a positive constant c_1 such that

$$\begin{aligned} p(i,j)\left|\int (e^{\Delta_i^j h} - 1)\nabla_i^+ f d\nu_\rho\right| &\leq \int (e^{c_0(\epsilon_i+\epsilon_j)} - 1)(\epsilon_i + \epsilon_i^*) f d\nu_\rho \\ &\leq c_1(\epsilon_i + \epsilon_j)(\epsilon_i + \epsilon_i^*). \end{aligned} \tag{4.16}$$

Now, recalling that for $i \notin \mathcal{S}$, $\sum_j p(i,j)\epsilon_j = \epsilon_i$, and $\sum_j p(i,j) = 1$, we have

$$\sum_{(i,j)\in\Lambda_m^2\setminus\Lambda_n^2} c_1 p(i,j)(\epsilon_i + \epsilon_j)(\epsilon_i + \epsilon_i^*) \leq 2c_1 \sum_{i\in\Lambda_n^c\cup\partial^R\Lambda_n} \epsilon_i(\epsilon_i + \epsilon_i^*) \overset{n\to\infty}{\longrightarrow} 0, \tag{4.17}$$

since $\sum_i \epsilon_i^2 = \sum_i (\epsilon_i^*)^2 < \infty$ when $d \geq 3$. Similarly, the second integral in (4.14) will go to 0, after we perform a second order expansion and use (4.15). Now, from Lemma 4.2, $|R_m(h, \mu) - R_n(h, \mu)|$ converges to 0 uniformly in $\mathcal{E}_\rho$ and $\mathcal{M}_\rho$.
Step 2: The limit $h \mapsto \tilde{\Gamma}(h, \mu)$ is convex, since it is a pointwise limit of convex functions.
Step 3: We prove (4.13).

Let φ_n be in $\mathcal{D}_n$, and set $h_n = \log(\varphi_n)$. Note that for $i \notin \Lambda_n$, $\nabla_i^+ h_n = 0$. Also, for any function ψ, $\mathsf{A}_i^+ E_{\nu_\rho}[\psi|\mathcal{F}_n] = E_{\nu_\rho}[\mathsf{A}_i^+ \psi|\mathcal{F}_n]$. Thus, for $m > R + n$

$$\begin{aligned}
&\tilde{\Gamma}_m(h_n, \mu) = \tilde{\Gamma}_n(h_n, \mu) + R_m(h_n, \mu) - R_n(h_n, \mu)\\
&+ \sum_{\substack{i \in \Lambda_m \setminus \Lambda_n \\ j \in \Lambda_n}} \gamma_\rho p(i, j) \int \Big[(e^{\nabla_j^+ h_n} - 1)\nabla_i^+ f + (e^{\nabla_j^+ h_n} - \nabla_j^+ h_n - 1) f\Big] d\nu_\rho\\
&+ \sum_{\substack{j \in \Lambda_m \setminus \Lambda_n \\ i \in \Lambda_n}} \gamma_\rho p(i, j) \int \Big[(e^{-\nabla_i^+ h_n} - 1)\nabla_i^+ f + (e^{-\nabla_i^+ h_n} + \nabla_i^+ h_n - 1) f\Big] d\nu_\rho.
\end{aligned} \tag{4.18}$$

By observing that $\tilde{\Gamma}_n(h_n, \mu) = \tilde{\Gamma}_n(h_n, \mu_n)$, and that $R_m(h_n, \mu) = 0$ for $m > n + R$, we have

$$\begin{aligned}
&\tilde{\Gamma}_m(h_n, \mu) - \tilde{\Gamma}_n(h_n, \mu_n) = \sum_{\substack{i \in \Lambda_m \setminus \Lambda_n \\ j \in \partial^R \Lambda_n}} \gamma_\rho p(i, j) \int (e^{\nabla_j^+ h_n} - 1)\nabla_i^+ f d\nu_\rho\\
&\sum_{j \in \partial^R \Lambda_n} \gamma_\rho p_n(j, j) \left(\int (e^{-\nabla_j^+ h_n} - 1)\nabla_j^+ (f) d\nu_\rho + \int (e^{-\nabla_j^+ h_n} + \nabla_j^+ h_n - 1) f d\nu_\rho \right)\\
&\sum_{j \in \partial^R \Lambda_n} \gamma_\rho p_n^*(j, j) \int (e^{\nabla_j^+ h_n} - \nabla_j^+ h_n - 1) f d\nu_\rho - R_n(h_n, \mu).
\end{aligned} \tag{4.19}$$

Now, using again that for $j \in \partial^R \Lambda_n$, $|\nabla_j^+ h_n| \leq c_0 \epsilon_j$, and for $i \in \partial^R \Lambda_n \cup \Lambda_n^c$, $|\nabla_i^+ f| \leq (\epsilon_i + \epsilon_i^*) f$, we have a constant C_1 such that

$$\begin{aligned}
|\tilde{\Gamma}_m(h_n, \mu) - \tilde{\Gamma}_n(h_n, \mu_n)| &= |R_n(h_n, \mu)| + C_1 \sum_{\substack{i \in \Lambda_m \setminus \Lambda_n \\ j \in \Lambda_n}} \partial(i, j) \epsilon_j (\epsilon_i + \epsilon_i^*)\\
&\quad + 2C_1 \sum_{j \in \partial^R \Lambda_n} \epsilon_j^2 + C_1 \sum_{j \in \partial^R \Lambda_n} \epsilon_j (\epsilon_j + \epsilon_j^*)\\
&\leq |R_n(h_n, \mu)| + 2C_1 \sum_{i \notin \Lambda_n} \epsilon_i^2 + (\epsilon_i^*)^2\\
&\quad + C_1 \sum_{j \in \partial^R \Lambda_n} (4\epsilon_j^2 + (\epsilon_j^*)^2)
\end{aligned} \tag{4.20}$$

Equation (4.13) follows after we take the limit m to infinity in (4.20) and use (4.4) of Lemma 4.2.

Step 4: We show that $\Gamma(\varphi_n, \mu_n) = \Gamma_n(\varphi_n, \mu_n)$. Indeed, for $m > R+n$, $\mathcal{L}_{(m)}(\varphi_n) = \mathcal{L}(\varphi_n)$ so that

$$\Gamma_m(\varphi_n, \mu_n) = \int E_{\nu_\rho}\left[\frac{\mathcal{L}_{(m)}(\varphi_n)}{\varphi_n} f_n \Big| \mathcal{F}_m\right] d\nu_\rho$$
$$= \int \frac{E_{\nu_\rho}[\mathcal{L}_{(m)}(\varphi_n)|\mathcal{F}_n]}{\varphi_n} f_n d\nu_\rho = \Gamma_n(\varphi_n, \mu_n). \tag{4.21}$$

□

Now, a minimax theorem for Γ will be a corollary of Lemma 2.7.

Proposition 4.4. *A minimax theorem holds for* Γ. *In other words,*

$$\sup_{\mu \in \mathcal{M}_\rho} \inf_{\varphi \in \mathcal{D}_\rho} \Gamma(\varphi, \mu) = \inf_{\varphi \in \mathcal{D}_\rho} \sup_{\mu \in \mathcal{M}_\rho} \Gamma(\varphi, \mu). \tag{4.22}$$

Proof. We need to check that for any $\varphi \in \mathcal{D}_\rho$, the map $\mu \mapsto \Gamma(\varphi, \mu)$ on $\mathcal{M}_\rho$ is continuous on the compact space $\mathcal{M}_\rho$. Let $\{\mu_k, k \in \mathbb{N}\}$ be in $\mathcal{M}_\rho$, converging weakly to $\mu \in \mathcal{M}_\rho$. By Lemma 7.2, all densities $f_k = d\mu_k/d\nu_\rho$ are uniformly bounded in $L^2(\nu_\rho)$. Thus f_k converges in weak-$L^2(\nu_\rho)$ to $d\mu/d\nu_\rho$. Now, for $\varphi \in \mathcal{D}_\rho^+$, as in (3.19), $g_i(\varphi \circ T_j^i)/\varphi \in L^2(\nu_\rho)$, so that for $i, j \in \Lambda_n$

$$\int g_i \frac{\varphi \circ T_j^i}{\varphi} d\mu_k \overset{k\to\infty}{\longrightarrow} \int g_i \frac{\varphi \circ T_j^i}{\varphi} d\mu.$$

Thus, $\Gamma_n(\varphi, \mu_k) \to \Gamma_n(\varphi, \mu)$ as $k \to \infty$. Now, the uniform Cauchy property (4.12) implies that $\Gamma(\varphi, \mu_k) \to \Gamma(\varphi, \mu)$ as $k \to \infty$. □

4.2. Proof of Theorem 1.3

If u_n is the principal normalized eigenfunction of $\mathcal{L}_n$, then for any n and any $\mu_n \in \mathcal{M}_n$, we have by Proposition 4.3 (iv)

$$\Gamma(u_n, \mu_n) = -\lambda_n(\rho). \tag{4.23}$$

Now, by (4.13) of Proposition 4.3, for any $\epsilon > 0$, there is n_0 such that for any $n \geq n_0$

$$\sup_{\mu \in \mathcal{M}_\rho} |\Gamma(u_n, \mu) - \Gamma(u_n, \mu_n)| \leq \epsilon, \quad \text{where } d\mu_n := E[\frac{d\mu}{d\nu_\rho}|\mathcal{F}_n] d\nu_\rho. \tag{4.24}$$

Thus, for any $\mu \in \mathcal{M}_\rho$ and $n \geq n_0$

$$\Gamma(u_n, \mu) \leq -\lambda_n(\rho) + \epsilon \Longrightarrow \inf_{\varphi \in \mathcal{D}_\rho} \Gamma(\varphi, \mu) \leq -\lambda_n(\rho) + \epsilon \qquad (\text{since } \mathcal{D}_n \subset \mathcal{D}_\rho). \tag{4.25}$$

Recalling Lemma 1.1, and taking the limit $n \to \infty$, we obtain

$$\sup_{\mu \in \mathcal{M}_\rho} \inf_{\varphi \in \mathcal{D}_\rho} \Gamma(\varphi, \mu) \leq -\lambda(\rho). \tag{4.26}$$

Conversely, if $h_n = E_{\nu_\rho}[h|\mathcal{F}_n]$ for $h \in \mathcal{E}_\rho$, we show by a convexity argument that

$$\forall \mu_n \in \mathcal{M}_n, \qquad \tilde{\Gamma}(h, \mu_n) \geq \tilde{\Gamma}(h_n, \mu_n) - \epsilon_n \quad \text{with} \quad \lim_{n\to\infty} \epsilon_n = 0. \tag{4.27}$$

Indeed, take $m > n$ and in expression (4.3), we break down the gradient $\nabla_i^+ f_n$ so as to obtain

$$\tilde{\Gamma}_m(h, \mu_n) = \sum_{i,j\in\Lambda_m} \gamma_\rho p(i, j) \int \Big[(e^{\Delta_i^j h} - 1)\mathsf{A}_i^+ f_n - (\Delta_i^j h) f_n\Big] d\nu_\rho + R_m(h, \mu_n), \tag{4.28}$$

where $R_m(h, \mu_n)$ is given in (4.7). We further divide the sum over Λ_m into a sum over Λ_n and a small remainder

$$\sum_{i,j\in\Lambda_n} \gamma_\rho p(i, j) \int \Big[(e^{\Delta_i^j h} - 1)\mathsf{A}_i^+ f_n - \Delta_i^j h) f_n\Big] d\nu_\rho + Q_n^m(h, \mu_n). \tag{4.29}$$

Indeed, since $Q_n^m(h, \mu_n)$ contains the sum over $(i, j) \in \Lambda_m^2 \backslash \Lambda_n^2$, estimates similar to those showing that $R_n(h, \mu)$ goes to 0 when n tends to infinity uniformly in h and μ, in the proof of Proposition 4.3, establish that $Q_n^m(h, \mu)$ vanishes as n and m tend to infinity. Using that for any function ψ and $i \in \Lambda_n$, $\mathsf{A}_i^+ E_{\nu_\rho}[\psi|\mathcal{F}_n] = E_{\nu_\rho}[\mathsf{A}_i^+ \psi|\mathcal{F}_n]$, we have by Jensen's inequality for the conditional expectation

$$\begin{aligned}
\tilde{\Gamma}_m(h, \mu_n) = &\sum_{i,j\in\Lambda_n} \gamma_\rho p(i, j) \int E_{\nu_\rho}[e^{\Delta_i^j h} - 1|\mathcal{F}_n]\mathsf{A}_i^+ f_n - E_{\nu_\rho}[\Delta_i^j h|\mathcal{F}_n] f_n d\nu_\rho \\
&+ Q_n^m(h, \mu_n) + R_m(h, \mu_n) \\
&\geq \sum_{i,j\in\Lambda_n} \gamma_\rho p(i, j) \int (e^{\Delta_i^j h_n} - 1)\mathsf{A}_i^+ f_n - \Delta_i^j (h_n) f_n d\nu_\rho + Q_n^m(h, \mu_n) \\
&+ R_m(h, \mu_n) \\
&\geq \tilde{\Gamma}_n(h_n, \mu_n) - R_n(h_n, \mu_n) + R_m(h, \mu_n) + Q_n^m(h, \mu_n) \\
&= \tilde{\Gamma}(h_n, \mu_n) - R_n(h_n, \mu_n) + R_m(h, \mu_n) + Q_n^m(h, \mu_n).
\end{aligned} \tag{4.30}$$

Thus, by taking the limit as m tends to infinity, we obtain (4.27) with

$$\epsilon_n := \lim_{m\to\infty} \sup_{h,\mu}(|Q_n^m(h, \mu)| + |R_n(h_n, \mu_n)| + |R_m(h, \mu_n)|) \overset{n\to\infty}{\longrightarrow} 0. \tag{4.31}$$

Now, for any $h \in \mathcal{E}_\rho$, $h_n := E[h|\mathcal{F}_n] \in \mathcal{E}_n$ and $\infty > \int \exp(h_n) u_n^* d\nu_\rho > 0$, so that we can define

$$\frac{d\mu_n^*}{d\nu_\rho} = \frac{e^{h_n} u_n^*}{\int e^{h_n} u_n^* d\nu_\rho}.$$

Thus, by duality $\tilde{\Gamma}_n(h_n, \mu_n^*) = \Gamma_n^*(u_n^*, \mu_n^*) = -\lambda_n(\rho)$. and,

$$\sup_{\mu\in\mathcal{M}_\rho} \tilde{\Gamma}(h, \mu) \geq \tilde{\Gamma}(h, \mu_n^*) \geq \tilde{\Gamma}(h_n, \mu_n^*) - \epsilon_n = -\lambda_n(\rho) - \epsilon_n. \tag{4.32}$$

Thus, by taking the limit n to infinity, and using Lemma 1.1, we obtain

$$\sup_{\mu\in\mathcal{M}_\rho} \Gamma(e^h,\mu) \geq -\lambda(\rho) \Longrightarrow \inf_{\varphi\in\mathcal{D}_\rho^+} \sup_{\mu\in\mathcal{M}_\rho} \Gamma(\varphi,\mu) \geq -\lambda(\rho). \tag{4.33}$$

Now, since by Proposition 4.4, the minimax Theorem holds for Γ the proof concludes.

5. Uniqueness: Proofs of Theorems 1.4 and 1.2

The proofs of Theorem 1.4 and Theorem 1.2 will follow from three observations, which we have written as separate lemmas. First, any limit point of $\{u_n\}$ solves (1.4(i)) and belongs to $\mathcal{D}_\rho^+$: this is shown in Lemmas 3.7 and 5.1. Second, solutions of (1.4(i)) in $\mathcal{D}_\rho^+$ satisfy $\Gamma(u,\mu)+\lambda(\rho)=0$ for any $\mu\in\mathcal{M}_\rho$: this is shown in Lemma 5.2. Third, by convexity of $h\mapsto\Gamma(\exp(h),\mu)$ shown in Proposition 4.3, there is a unique solution of $\Gamma(u,\mu)+\lambda(\rho)=0$ for any $\mu\in\mathcal{M}_\rho$: this is shown in Lemma 5.3.

Lemma 5.1. *If $u\in\mathcal{D}_\rho$, $\int u d\nu_\rho = 1$, and u satisfies (1.4(i)), then u is positive ν_ρ-a.s. on $\mathcal{A}^c$.*

Proof. We denote by $\mathcal{B} := \{\eta :\ u(\eta)=0\}$. Since $u\in\mathcal{D}_\rho$, we have for $i\notin\mathcal{S}$ and η ν_ρ-a.s.,

$$u(\eta)\geq u(\mathsf{A}_i^+\eta) \quad \text{and} \quad u(\mathsf{A}_i^+\eta)\geq \frac{1}{1-\epsilon_i}u(\eta).$$

Thus, for $i\notin\mathcal{S}$, $\mathcal{B}=(\mathsf{A}_i^+)^{-1}(\mathcal{B})$ ν_ρ-a.s.. For any cylinder θ with base in $\mathbb{N}^{\mathcal{S}}\backslash\mathcal{A}$, we will consider $\mathcal{B}_\theta := \mathcal{B}\cap\theta$. If $T^{i,j}$ denotes the exchange operator at site $i,j\in\mathbb{Z}^d$, then

$$\mathcal{B}_\theta \overset{\nu_\rho-\text{a.s}}{=} (\mathsf{A}_i^+)^{-1}(\mathcal{B}_\theta), \quad \forall i\notin\mathcal{S} \Longrightarrow \mathcal{B}_\theta \overset{\nu_\rho-\text{a.s}}{=} (T^{i,j})^{-1}(\mathcal{B}_\theta), \quad \forall i,j\notin\mathcal{S}.$$

Indeed,

$$\mathcal{B}_\theta \overset{\nu_\rho-\text{a.s}}{=} \bigcup_{k,l\in\mathbb{N}} \mathcal{B}_\theta\cap\{\eta(i)=k,\eta(j)=l\},$$

so that we can go from

$$\mathcal{B}_\theta\cap\{\eta(i)=k,\eta(j)=l\} \quad \text{to} \quad \mathcal{B}_\theta\cap\{\eta(i)=l,\eta(j)=k\}=(T^{i,j})^{-1}$$
$$(\mathcal{B}_\theta\cap\{\eta(i)=k,\eta(j)=l\}$$

by a finite succession of creation and annihilation of particles. Now, by Hewitt-Savage 0-1 law on the lattice $\mathbb{Z}^d\backslash\mathcal{S}$, we conclude that $\nu_\rho(\mathcal{B}_\theta)\in\{0,1\}$. Assume that for some cylinder θ_0, $\nu_\rho(\mathcal{B}_{\theta_0})=1$. Since u satisfies (1.4(i)) and $1_{\theta_0}\in\mathbb{L}_b$, we have

$$\int u\mathcal{L}^*(1_{\theta_0})d\nu_\rho = 0 \Longrightarrow \sum_{i,j\in\mathbb{Z}^d} p^*(i,j)\int g_i(\eta)u(\eta)1_{\theta_0}(T_j^i\eta)d\nu_\rho = 0. \tag{5.1}$$

Now,

$$(T_j^i)^{-1}(\theta_0) = \begin{cases} T_i^j(\theta_0) \text{ if } \theta_0(j) > 0, \text{ and } \emptyset \text{ if } \theta_0(j) = 0 \text{ when } i, j \in \mathcal{S} \\ \mathsf{A}_i^+(\theta_0) \text{ when } i \in \mathcal{S}, \ j \notin \mathcal{S} \\ \mathsf{A}_j^-(\theta_0) \text{ if } \theta_0(j) > 0, \text{ and } \emptyset \text{ if } \theta_0(j)=0 \text{ when } i \notin \mathcal{S}, \ j \in \mathcal{S} \\ \theta_0 \text{ when } i, j \notin \mathcal{S} \end{cases} \tag{5.2}$$

Since the moves on the right hand side generates all cylinders with base in $\mathbb{N}^{\mathcal{S}} \backslash \mathcal{A}$, we obtain

$$\forall \theta \in \mathbb{N}^{\mathcal{S}} \backslash \mathcal{A}, \quad \int_\theta u d\nu_\rho = 0, \tag{5.3}$$

which is absurd since $\int u d\nu_\rho = 1$. Thus, $\nu_\rho(\mathcal{B}) = 0$ and the proof is concluded. □

Lemma 5.2. *If $u \in \mathcal{D}_\rho$ satisfies (1.4(i)) then $\Gamma(u, \mu) = -\lambda(\rho)$, for any $\mu \in \mathcal{M}_\rho$.*

Proof. Let u satisfies (1.4(i)). By Lemma 5.1, $u \in \mathcal{D}_\rho^+$. For any $\varphi_n \in \mathbb{L}_b$ with φ_n $\mathcal{F}_n$-measurable, we write (1.4(i)) as

$$\int \mathcal{L}^*(\varphi_n) u d\nu_\rho + \lambda(\rho) \int \varphi_n u d\nu_\rho = 0. \tag{5.4}$$

We make the standard integration by parts and use cancellations as in (4.8) to obtain

$$\begin{aligned} \int \mathcal{L}^*(\varphi_n) u d\nu_\rho = & \sum_{i,j \in \Lambda_n} \gamma_\rho p^*(i, j) \int \Delta_i^j \varphi_n \mathsf{A}_i^+ u d\nu_\rho \\ & - \sum_{i \in \Lambda_n} \sum_{j \notin \Lambda_n} \gamma_\rho p^*(i, j) \int \nabla_i^+ \varphi_n \mathsf{A}_i^+ u d\nu_\rho \\ & + \sum_{i \notin \Lambda_n} \sum_{j \in \Lambda_n} \gamma_\rho p^*(i, j) \int \nabla_j^+ \varphi_n \mathsf{A}_i^+ u d\nu_\rho \\ = & \sum_{i,j \in \Lambda_n} \gamma_\rho p^*(i, j) \int \Delta_i^j (\varphi_n) \nabla_i^+ (u) d\nu_\rho + \tilde{R}_n(\varphi_n), \end{aligned} \tag{5.5}$$

where

$$\begin{aligned} \tilde{R}_n(\varphi_n) = & - \sum_{i \in \Lambda_n} \gamma_\rho p_n^*(i, i) \int \nabla_i^+ \varphi_n \nabla_i^+ u d\nu_\rho \\ & + \sum_{i \notin \Lambda_n} \sum_{j \in \Lambda_n} \gamma_\rho p_n(j, i) \int \nabla_j^+ \varphi_n \nabla_i^+ u d\nu_\rho. \end{aligned} \tag{5.6}$$

Now, for any $\mu \in \mathcal{M}_\rho$ with density f, it is easy to note that for a fix large integer M,

$$\varphi_n^{(M)} := E_{\nu_\rho}[\frac{f}{u} \wedge M | \mathcal{F}_n] \in \mathbb{L}_b,$$

and if we set $\varphi = f/u$ and $\varphi^{(M)} = (f/u) \wedge M$, then both $\varphi^{(M)}$ and φ are in $L^p(\nu_\rho)$ for any integer p, and are such that for i large enough $|\nabla_i^+(\psi)| \le 2\psi(\epsilon_i + \epsilon_i^*)$. Indeed, for $i \notin \mathcal{S}$

$$u \ge \mathsf{A}_i^+ u \ge u(1-\epsilon_i), \quad \text{and} \quad f \ge \mathsf{A}_i^+ f \ge f(1-\epsilon_i-\epsilon_i^*). \tag{5.7}$$

Thus, if i is such that $1-\epsilon_i-\epsilon_i^* > 0$,

$$\frac{f}{u}(1-\epsilon_i-\epsilon_i^*-1) \le \nabla_i^+(\frac{f}{u}) \le \frac{f}{u}(\frac{1}{1-\epsilon_i}-1). \tag{5.8}$$

Thus, for i large enough $|\nabla_i^+(\varphi)| \le 2\varphi(\epsilon_i + \epsilon_i^*)$. Also, since $f \in L^p(\nu_\rho)$ and $1_{\mathcal{A}^c}/u \in L^p(\nu_\rho)$ for any integer p by Lemma 7.2, we obtain that $\varphi \in L^p(\nu_\rho)$ for any p. The same is true for $\varphi^{(M)}$ after a simple algebra.

By a reasoning by now standard, since $\varphi_n^{(M)}$ satisfies a bound like (5.8)

$$\begin{aligned} |\tilde{R}_n(\varphi_n^{(M)})| &\le c_1 \sum_{i\in\partial^R\Lambda_n} (\epsilon_i^2 + (\epsilon_i^*)^2) \int \varphi_n^{(M)} u d\nu_\rho \\ &\le c_1 ||\varphi||_{\nu_\rho} ||u||_{\nu_\rho} \sum_{i\in\partial^R\Lambda_n} \epsilon_i^2 + (\epsilon_i^*)^2 \stackrel{n\to\infty}{\longrightarrow} 0. \end{aligned} \tag{5.9}$$

Now, since $\varphi^{(M)} \in L^2(\nu_\rho)$, and $\{\varphi_n^{(M)},\ n \in \mathbb{N}\}$ is a positive martingale, we have that $\{\varphi_n^{(M)}\}$ converges to $\varphi^{(M)}$ in $L^2(\nu_\rho)$ and a.s. . Also, since for $i \in \Lambda_n$, $\mathsf{A}_i^+\varphi_n^{(M)} = E_{\nu_\rho}[\mathsf{A}_i^+\varphi^{(M)}|\mathcal{F}_n]$, we have for any $\psi \in L^2(\nu_\rho)$, and $i, j \in \Lambda_n$

$$\begin{aligned} \lim_{n\to\infty} \int \varphi_n^{(M)}\psi d\nu_\rho = \int \varphi^{(M)}\psi d\nu_\rho, \quad \text{and} \quad &\lim_{n\to\infty} \int \nabla_j^+\varphi_n^{(M)}\nabla_i^+\psi d\nu_\rho \\ &= \int \nabla_j^+\varphi^{(M)}\nabla_i^+\psi d\nu_\rho. \end{aligned} \tag{5.10}$$

Thus, combining (5.5), (5.9) and (5.10) we obtain (the series being absolutely convergent)

$$\sum_{i,j\in\mathbb{Z}^d} \gamma_\rho p^*(i,j) \int \Delta_i^j(\frac{f}{u} \wedge M)\nabla_i^+ u d\nu_\rho + \lambda(\rho)\int (\frac{f}{u}\wedge M) u d\nu_\rho = 0. \tag{5.11}$$

An identical expression to (5.11) is also valid for f/u as we take the limit M to infinity.

We will now show that $\Gamma(u,\mu)$ has the same expression as the first term of (5.11). Now, by taking the limit n to infinity in expression (4.5), we obtain

$$\lim_{n\to\infty}\int \mathcal{L}_n(u)\frac{f}{u}d\nu_\rho = \sum_{i,j\in\mathbb{Z}^d}\gamma_\rho p^*(i,j)\int \Delta_i^j\frac{f}{u}\nabla_i^+ u d\nu_\rho. \tag{5.12}$$

Indeed, (4.5) only requires that $f/u \in L^p(\nu_\rho)$ and that for i large enough $|\nabla_i^+(\frac{f}{u})| \le 2\frac{f}{u}(\epsilon_i+\epsilon_i^*)$. Finally, since $\Gamma(u,\mu)=\lim_{n\to\infty}\Gamma_n(u,\mu)$, (5.12) concludes the proof. □

Lemma 5.3. *If $u,\tilde{u}\in\mathcal{D}_\rho^+$, and for any $\mu\in\mathcal{M}_\rho$, $\Gamma(u,\mu)=\Gamma(\tilde{u},\mu)=-\lambda(\rho)$, and $\int u d\nu_\rho=\int\tilde{u}d\nu_\rho$, then $u=\tilde{u}$ ν_ρ-a.s. .*

Proof. We can define

$$h:=\log(u),\quad\text{and}\quad \tilde{h}:=\log(\tilde{u}),\qquad\text{with}\qquad h,\tilde{h}\in\mathcal{E}_\rho.$$

Now, for $\gamma\in]0,1[$, we form $h_\gamma=\gamma h+(1-\gamma)\tilde{h}$, and by convexity of $\tilde{\Gamma}_n$, for any $\mu\in\mathcal{M}_\rho$,

$$0\le a_n(\mu) := \gamma\tilde{\Gamma}_n(h,\mu)+(1-\gamma)\tilde{\Gamma}_n(\tilde{h},\mu)-\tilde{\Gamma}_n(h_\gamma,\mu)\overset{n\to\infty}{\longrightarrow}-\lambda(\rho)-\tilde{\Gamma}(h_\gamma,\mu), \tag{5.13}$$

where we used Lemma 5.2. Now, Lemma 5.2 is also valid for any u^* limit point of u_n^*, the principal eigenfunction of $\mathcal{L}_n^*$. Note that since $u,\tilde{u}\in L^2(\nu_\rho)$, we have that $\exp(h_\gamma)\in L^2(\nu_\rho)$ and $\int\exp(h_\gamma)u^*d\nu_\rho<\infty$. Finally, Lemma 5.1 implies that $u^*|_{\mathcal{A}^c}>0$ ν_ρ-a.s. , so that $\int u^*\exp(h_\gamma)d\nu_\rho>0$, and we can define

$$d\mu^* := \frac{e^{h_\gamma}u^*d\nu_\rho}{\int e^{h_\gamma}u^*d\nu_\rho}\in\mathcal{M}_\rho. \tag{5.14}$$

Now, by duality, and Lemma 5.2 applied to $\mathcal{L}^*$

$$\tilde{\Gamma}_n(h_\gamma,\mu^*)=\int\frac{\mathcal{L}_n^*(u^*)}{u^*}d\mu^*=\Gamma_n^*(u^*,\mu^*)\overset{n\to\infty}{\longrightarrow}-\lambda(\rho)=\tilde{\Gamma}(h_\gamma,\mu^*). \tag{5.15}$$

Thus, $a_n(\mu^*)$ vanishes as n tends to ∞. However, for any $i,j\in\mathbb{Z}^d$ and n large enough

$$a_n(\mu^*)\ge p(i,j)\int g_iA_{i,j}d\mu^*,\quad\text{with}\quad 0\le A_{i,j}:=\gamma e^{\nabla_j^i h}+(1-\gamma)e^{\nabla_j^i\tilde{h}} - e^{(\gamma\nabla_j^i h+(1-\gamma)\nabla_j^i\tilde{h})}. \tag{5.16}$$

Now $a_n(\mu^*)\to 0$, $e^{h_\gamma}u^*>0$ ν_ρ-a.s. on $\mathcal{A}^c$, and (5.16) imply that $p(i,j)g_iA_{i,j}=0$ ν_ρ-a.s. on $\mathcal{A}^c$. This in turn, implies that for $\eta(i)p(i,j)>0$, ν_ρ-a.s., we have $\nabla_j^i h=\nabla_j^i\tilde{h}$ in $\mathcal{A}^c$. Let us denote $f:=\tilde{u}/u$ on $\mathcal{A}^c$. Since, $p(.,.)$ is irreducible, we obtain

$$\forall i,j\quad\text{with }\eta(i)p(i,j)>0\qquad f(T_j^i\eta)=f(\eta),\quad\nu_\rho-\text{a.s.}\,. \tag{5.17}$$

This in turn, implies that for $i, j \notin \mathcal{S}$, $f(T^{i,j}\eta) = f(\eta)$ ν_ρ-a.s. , so that by Hewitt-Savage 0-1 law for exchangeable events, we conclude that f is ν_ρ-a.s. constant on each cylinder θ with base in $\mathbb{N}^{\mathcal{S}}\backslash\mathcal{A}$, say $c_\theta := f|_\theta$.

We now show that the constants $\{c_\theta\}$ are the same. Assume $\theta, \theta' \in \mathbb{N}^{\mathcal{S}}\backslash\mathcal{A}$ with $T^i_j\theta = \theta'$. If we denote $X_\theta := f^{-1}(\{c_\theta\})$, then

$$\theta \subset X_\theta, \quad \theta' = T^i_j\theta \subset (T^j_i)^{-1}(X_\theta), \text{ and by (5.17)} \quad (T^j_i)^{-1}(X_\theta) \overset{\nu_\rho \text{a.s}}{=} X_\theta. \tag{5.18}$$

This yields $c_\theta = c_{\theta'}$. Assume now that for $j \in \mathcal{S}$ with

$$\sum_{i \notin \mathcal{S}} p(i,j) > 0, \quad \text{we have} \quad \theta' = \mathsf{A}^+_j\theta.$$

Take $i \notin \mathcal{S}$ with $p(i,j) > 0$, and note that

$$\theta' = \mathsf{A}^+_j\theta \subset (T^j_i)^{-1}(X_\theta) \overset{\nu_\rho \text{a.s}}{=} X_\theta \quad \text{[by (5.17)]} \tag{5.19}$$

Thus, $c_\theta = c_{\theta'}$ in this case also. Now, we have assumed that $\mathbb{N}^{\mathcal{S}}\backslash\mathcal{A}$ was a connected set containing $0_{\mathcal{S}}$. Thus, by a succession of moves T^j_i and A^+_j applied to $0_{\mathcal{S}}$, we cover all of $\mathbb{N}^{\mathcal{S}}\backslash\mathcal{A}$, and conclude that f is constant ν_ρ-a.s. . □

6. Hitting time: Proof of Theorem 1.5

Let u (resp. u^*) be the principal Dirichlet eigenfunction of $\mathcal{L}$ (resp. $\mathcal{L}^*$) in $\mathcal{D}_\rho$ (resp. $\mathcal{D}^*_\rho$). By Lemma 5.1, u and u^* are ν_ρ-a.s. positive on $\mathcal{A}^c$. Thus, we define a Markov semi-group on $\mathcal{A}^c$, (known as the h-process, see e.g. Chapter 4.1 of [13])

$$\forall \eta \in \mathcal{A}^c, \qquad S^u_t(\varphi)(\eta) := e^{\lambda(\rho)t}\frac{\bar{S}_t(u\varphi)}{u(\eta)}. \tag{6.1}$$

This semi-group is stationary with respect to

$$d\hat{\mu}_\rho = \frac{uu^* d\nu_\rho}{\int uu^* d\nu_\rho}. \tag{6.2}$$

Note that since $1_{\mathcal{A}^c}/u \in L^2(\hat{\mu}_\rho)$ by Lemma 7.4, we have by definition, for all $\eta \in \mathcal{A}^c$,

$$S^u_t(\frac{1_{\mathcal{A}^c}}{u})(\eta) = c_t\frac{u_t(\eta)}{u(\eta)} \text{ with } c_t = e^{\lambda(\rho)t}P_{\nu_\rho}(\tau > t), \text{ and } u_t := \frac{P_\eta(\tau > t)}{P_{\nu_\rho}(\tau > t)}. \tag{6.3}$$

From (1.13), $c_t \in [\underline{c}, 1]$, whereas $u_t \in \mathcal{D}_\rho$ from inequality (4.7) of [2]. It is then easy to check directly, using the convexity of $\mathcal{D}_\rho$, that for any $t > 0$

$$uS^u_t(\frac{1_{\mathcal{A}^c}}{u}) \in \mathcal{D}_\rho \Longrightarrow \text{if} \quad \psi_t := \frac{1}{t}\int_0^t S^u_s(\frac{1_{\mathcal{A}^c}}{u})ds, \quad \text{then} \quad u\psi_t \in \mathcal{D}_\rho. \tag{6.4}$$

Now, by Jensen's inequality, $\{S_t^u, t > 0\}$ is a contraction semi-group on $L^2(\mathcal{A}^c, \hat{\mu}_\rho)$. Thus, by von Neumann's mean ergodic theorem in Hilbert space (see e.g. [12] Th.1.2 page 24), we obtain

$$\psi_t \stackrel{t\to\infty}{\longrightarrow} \psi \quad \text{in} \quad L^2(\hat{\mu}_\rho), \text{ and for any } t \geq 0 \quad S_t^u(\psi) = \psi, \ \hat{\mu}_\rho - \text{a.s..} \tag{6.5}$$

If $\{\psi_t\}$ converges to ψ in $L^2(\hat{\mu}_\rho)$, then $\{\psi_t u\}$ converges in $L^1(\nu_\rho)$ towards ψu. Indeed,

$$\begin{aligned}\int |u(\psi_t - \psi)| d\nu_\rho &\leq \left(\int uu^* d\nu_\rho\right) \|\frac{1}{u^*}\|_{\hat{\mu}_\rho} \|\psi_t - \psi\|_{\hat{\mu}_\rho} \\ &\leq \left(\int uu^* d\nu_\rho \int \frac{u}{u^*} d\nu_\rho\right)^{1/2} \|\psi_t - \psi\|_{\hat{\mu}_\rho}. \end{aligned} \tag{6.6}$$

It remains now to show that ψ is contant ν_ρ-a.s.. Since $\int \psi_t u d\nu_\rho = \int_0^t c_s ds/t \leq 1$, the Remark 2.8 yields that $u\psi \in \mathcal{D}_\rho$. Finally, since $\hat{\mu}_\rho$ and ν_ρ are equivalent in $\mathcal{A}^c$, (6.5) implies that for any $t \geq 0$,

$$\bar{S}_t(\psi u) = e^{-\lambda(\rho)t} \psi u \text{ on } \mathcal{A}^c \ \nu_\rho - \text{a.s..} \tag{6.7}$$

Thus, by differentiating (6.7) at $t = 0$, we obtain that ψu is a Dirichlet principal eigenfunction in $\mathcal{D}_\rho^+$, with $\underline{c} \leq \int u\psi d\nu_\rho \leq 1$. By Theorem 1.4, this means that ψ is constant. To find the value of ψ, integrate (6.5) against $1_{\mathcal{A}^c}$.

$$\psi \equiv \int \psi d\hat{\mu}_\rho = \lim_{t\to\infty} \int \frac{1}{t} \int_0^t S_s^u(\frac{1}{u}) ds \, d\hat{\mu}_\rho = \int \frac{1}{u} d\hat{\mu}_\rho = \frac{1}{\int uu^* d\nu_\rho}. \tag{6.8}$$

This concludes the proof. □

7. Appendix

We have often used Lemma 7.2 below to obtain regularity of probability densities satisfying a gradient bound (1.6) [3, 5, 2, 4]. For ease of reading, we recall its simple proof. Then, in Lemma 7.4, we show how related arguments yield the regularity of $1_{\mathcal{A}^c}/\varphi$ for $\varphi \in \mathcal{D}_\rho^+$.

We recall that $\{\theta \in \Theta_{\mathcal{A}}\}$ is a partition of $\mathcal{A}$ into cylinders with bases in $\mathcal{S}$. Also, recalling the notations used in the definition of ν_ρ (see (2.3)), let ν_ϵ be the product measure

$$d\nu_\epsilon(\eta) = \prod_{i\in\mathcal{S}} d\theta_{\gamma(\rho)}(\eta_i) \prod_{i\notin\mathcal{S}} d\theta_{(1-\epsilon_i)\gamma(\rho)}(\eta_i)\,.$$

We showed in [2] that when $d \geq 3$, ν_ϵ is absolutely continuous with respect to ν_ρ, and that if $\psi_\epsilon := d\nu_\epsilon/d\nu_\rho$, then for any integer p

$$\int \psi_\epsilon^p d\nu_\rho < \infty, \text{ and } \int \frac{1}{\psi_\epsilon^p} d\nu_\rho < \infty. \tag{7.1}$$

Remark 7.1. The purpose of introducing ψ_ϵ was that for any $i \notin \mathcal{S}$, $\mathsf{A}_i^+\psi_\epsilon = (1-\epsilon_i)\psi_\epsilon$. Thus, if $\varphi \in \mathcal{D}_\rho$, then φ/ψ_ϵ is increasing outside $\mathcal{S}$. Indeed, using (2.11)(ii),

$$\forall i \notin \mathcal{S}, \qquad \mathsf{A}_i^+(\varphi/\psi_\epsilon) \geq \varphi/\psi_\epsilon.$$

Lemma 7.2. *We assume that $d \geq 3$. For any integer n, any $\theta \in \Theta_{\mathcal{A}}$ and $\varphi \in \mathcal{D}_\rho$*

$$\int_\theta \varphi^n d\nu_\rho \leq \left(\frac{\int_\theta \varphi d\nu_\rho}{\nu_\epsilon(\theta)}\right)^n \int_\theta \psi_\epsilon^n d\nu_\rho. \tag{7.2}$$

Also,

$$\int \varphi^n d\nu_\rho \leq C_n \left(\int \varphi d\nu_\rho\right)^n \qquad \text{with} \qquad C_n := \frac{\int \psi_\epsilon^n d\nu_\rho}{\nu_\epsilon(0_{\mathcal{S}})^{n+1}} < \infty. \tag{7.3}$$

Proof. We define the measure $d\mu = \varphi d\nu_\rho$, and for $\theta \in \Theta_{\mathcal{A}}$, we define two probability measures $d\mu_\theta = 1_\theta d\mu/\mu(\theta)$ and $d\nu_\theta = 1_\theta d\nu_\epsilon/\nu_\epsilon(\theta)$. Note that on θ, the probability measure ν_ϵ satisfies Holley's condition (see Theorem 2.9, p.75 in [10]) which implies that it satisfies FKG's inequality.

Step 1. We first show that for any ϕ decreasing on θ,

$$\int \phi d\mu_\theta \leq \int \phi d\nu_\theta. \tag{7.4}$$

By the Remark 7.1, $d\mu_\theta/d\nu_\theta$ is increasing in θ. We apply FKG's inequality on θ

$$\int \phi d\mu_\theta = \int \phi \frac{d\mu_\theta}{d\nu_\theta} d\nu_\theta \leq \int \phi d\nu_\theta. \tag{7.5}$$

Step 2. First, note that φ and $\psi_\epsilon = d\nu_\epsilon/d\nu_\rho$ are non-negative decreasing on θ. So is $\varphi^i\psi_\epsilon^j$ for any integers i, j. We apply (7.4) to $\phi := \varphi^i\psi_\epsilon^j$ and obtain

$$\int_\theta \varphi^{i+1}\psi_\epsilon^j \frac{d\nu_\rho}{\mu(\theta)} = \int \varphi^i\psi_\epsilon^j d\mu_\theta \leq \int \varphi^i\psi_\epsilon^j d\nu_\theta = \int_\theta \varphi^i\psi_\epsilon^{j+1}\frac{d\nu_\rho}{\nu_\epsilon(\theta)}. \tag{7.6}$$

By induction, we obtain (7.2) for any integer n. Now, (7.3) obtains after taking $\theta = 0_{\mathcal{S}}$ and using FKG's inequality once more. Indeed, since φ and $1_{0_{\mathcal{S}}}$ are both decreasing

$$\int \varphi^n d\nu_\rho \leq \frac{\int \varphi^n 1_{0_{\mathcal{S}}} d\nu_\rho}{\nu_\rho(0_{\mathcal{S}})}. \tag{7.7}$$

□

Remark 7.3. Actually if $\mu \in \mathcal{M}_\rho$, then its density $f := d\mu/d\nu_\rho$ satisfies an inequality like (7.3) but with $U := \{i : (\epsilon_i + \epsilon_i^*) \geq 1\}$ replacing $\mathcal{S}$ which was the domain where $\epsilon_i = 1$. Since U is bounded, $\nu_\epsilon(0_U) > 0$ will replace $\nu_\epsilon(0_{\mathcal{S}})$ in (7.3).

Lemma 7.4. *We assume that $d \geq 3$. Let $\varphi \in \mathcal{D}_\rho^+$ and $\theta \in \Theta_{\mathcal{A}}$. Then, for any integer n*

$$\int_\theta \varphi^n d\nu_\rho \int_\theta \frac{1}{\varphi^n} d\nu_\rho \leq \int_\theta \psi_\epsilon^n d\nu_\rho \int_\theta \frac{1}{\psi_\epsilon^n} d\nu_\rho. \tag{7.8}$$

Furthermore,

$$\int_{\mathcal{A}^c} \frac{1}{\varphi^n} d\nu_\rho \leq c_{\varphi,n} \int \psi_\epsilon^n d\nu_\rho \int_{\mathcal{A}^c} \frac{1}{\psi_\epsilon^n} d\nu_\rho, \quad \textit{with}$$

$$c_{\varphi,n} := \sup_{\theta \in \Theta_{\mathcal{A}}} \{\nu_\rho(\theta) \left(\int_\theta \varphi \frac{d\nu_\rho}{\nu_\rho(\theta)} \right)^{-n} \} < \infty. \tag{7.9}$$

Proof. Recall that φ/ψ_ϵ is increasing on $\mathcal{A}^c$ whereas for any integer n, φ^n is decreasing. Thus, for any cylinder $\theta \in \Theta_{\mathcal{A}}$, if we denote $d\tilde{\nu}_\rho = 1_\theta d\nu_\rho/\nu_\rho(\theta)$, then, by FKG's inequality

$$\int_\theta \frac{1}{\psi_\epsilon^n} d\tilde{\nu}_\rho = \int_\theta \left(\frac{\varphi}{\psi_\epsilon}\right)^n \frac{1}{\varphi^n} d\tilde{\nu}_\rho \geq \int_\theta \left(\frac{\varphi}{\psi_\epsilon}\right)^n d\tilde{\nu}_\rho \int_\theta \frac{1}{\varphi^n} d\tilde{\nu}_\rho. \tag{7.10}$$

Also, since ψ_ϵ^n is decreasing

$$\int_\theta \left(\frac{\varphi}{\psi_\epsilon}\right)^n d\tilde{\nu}_\rho \int_\theta \psi_\epsilon^n d\tilde{\nu}_\rho \geq \int_\theta \left(\frac{\varphi}{\psi_\epsilon}\right)^n \psi_\epsilon^n d\tilde{\nu}_\rho = \int_\theta \varphi^n d\tilde{\nu}_\rho \tag{7.11}$$

Multiplying (7.10) by $\int_\theta \psi_\epsilon^n d\tilde{\nu}_\rho$, using (7.11), and simplifying by $\int_\theta (\varphi/\psi_\epsilon)^n d\tilde{\nu}_\rho > 0$, (since $\varphi > 0$ ν_ρ-a.s.) we obtain (7.8). Note that $c_{\varphi,n} < \infty$ since there is a finite number of elements in $\Theta_{\mathcal{A}}$, on each of which $\int_\theta \varphi d\nu_\rho > 0$. Finally, (7.9) is obtained by summing over all $\theta \in \Theta_{\mathcal{A}}$, and applying Hölder's inequality to $\int \varphi^n d\tilde{\nu}_\rho$. □

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