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# Universality of critical behaviour in a class of recurrent random walks

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**Abstract.** Let  $X_0 = 0, X_1, X_2, \dots$  be an aperiodic random walk generated by a sequence  $\xi_1, \xi_2, \dots$  of i.i.d. integer-valued random variables with common distribution  $p(\cdot)$  having zero mean and finite variance. For an  $N$ -step trajectory  $\mathbb{X} = (X_0, X_1, \dots, X_N)$  and a monotone convex function  $V: \mathbb{R}^+ \rightarrow \mathbb{R}^+$  with  $V(0) = 0$ , define  $\mathbb{V}(\mathbb{X}) = \sum_{j=1}^{N-1} V(|X_j|)$ . Further, let  $\mathcal{I}_{N,+}^{a,b}$  be the set of all non-negative paths  $\mathbb{X}$  compatible with the boundary conditions  $X_0 = a, X_N = b$ . We discuss asymptotic properties of  $\mathbb{X} \in \mathcal{I}_{N,+}^{a,b}$  under the probability distribution

$$P_{N,+,\lambda}^{a,b}(\mathbb{X}) = (Z_{N,+,\lambda}^{a,b})^{-1} \exp\{-\lambda \mathbb{V}(\mathbb{X})\} \prod_{i=0}^{N-1} p(X_{i+1} - X_i)$$

as  $N \rightarrow \infty$  and  $\lambda \rightarrow 0$ ,  $Z_{N,+,\lambda}^{a,b}$  being the corresponding normalization.

If  $V(\cdot)$  grows not faster than polynomially at infinity, define  $H(\lambda)$  to be the unique solution to the equation

$$\lambda H^2 V(H) = 1.$$

Our main result reads that as  $\lambda \rightarrow 0$ , the typical height of  $X_{[\alpha N]}$  scales as  $H(\lambda)$  and the correlations along  $\mathbb{X}$  decay exponentially on the scale  $H(\lambda)^2$ . Using a suitable blocking argument, we show that the distribution tails of the rescaled height decay exponentially with critical exponent  $3/2$ . In the particular case of linear potential  $V(\cdot)$ , the characteristic length  $H(\lambda)$  is proportional to  $\lambda^{-1/3}$  as  $\lambda \rightarrow 0$ .

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## 1. Introduction

In this work, we are interested in the path-wise behaviour of a general class of random walks on the integers, whose path measure is submitted to a special form of exponential perturbation, the physical motivation of which is discussed at the end of this section. More precisely, to each  $i \in \mathbb{Z}$ , we associate an integer<sup>1</sup> non-negative

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<sup>1</sup> Although in the sequel we'll discuss mainly integer-valued one-dimensional random walks, analogous results for real-valued walks can be obtained in a similar way.

value  $X_i$  and for any integer interval

$$\Delta_{l,r} = (l, r) \equiv \{l + 1, l + 2, \dots, r - 1\} \subset \mathbb{Z} \quad (1.1)$$

we denote by  $\mathcal{I}_{\Delta_{l,r},+}$  the set of all such trajectories in  $\Delta_{l,r}$ :

$$\mathcal{I}_{\Delta_{l,r},+} = \mathcal{I}_{(l,r),+} = \left\{ \mathbb{X} = (X_i)_{l < i < r} : X_i \geq 0 \right\}.$$

Let  $V : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  be a convex increasing continuous function with  $V(0) = 0$  and of bounded growth at infinity:

*There exists  $f : \mathbb{R}^+ \rightarrow \mathbb{R}^+$  such that for any  $\alpha > 0$  we have*

$$\limsup_{x \rightarrow \infty} \frac{V(\alpha x)}{V(x)} \leq f(\alpha) < \infty. \quad (1.2)$$

This property holds clearly for any (convex) polynomial function.

The probability of a trajectory  $\mathbb{X} \in \mathcal{I}_{(l,r),+}$  is defined then via

$$P_{(l,r),+,\lambda}^{a,b}(\mathbb{X}) = (Z_{(l,r),+,\lambda}^{a,b})^{-1} \exp \left\{ -\lambda \sum_{i=l+1}^{r-1} V(X_i) \right\} \prod_{i=l}^{r-1} p(X_{i+1} - X_i), \quad (1.3)$$

where the boundary conditions are given by  $X_l = a$  and  $X_r = b$ , the parameter  $\lambda$  is some strictly positive real number and  $p(\cdot)$  are the transition probabilities of a 1D integer-valued random walk with zero mean and finite second moment. We suppose that the random walk is *strictly aperiodic* in the sense that its  $n$ -step transition probabilities  $p^n(\cdot)$  possess the following property:

*there is  $A > 0$  such that*

$$\min \left\{ p^n(-1), p^n(0), p^n(1) \right\} > 0 \quad \text{for all } n \geq A. \quad (1.4)$$

When the boundary conditions are chosen such that  $a = b = 0$ , we omit them from the notation. We will denote by  $P_{\Delta_{l,r},\lambda}^{a,b}$  the analogue of the probability measure (1.3) without the positivity constraint, and by  $Z_{\Delta_{l,r},\lambda}^{a,b}$  the associated partition function. The special case of  $\Delta_{0,N} = (0, N)$  will be abbreviated to

$$\mathcal{I}_{N,+} = \left\{ \mathbb{X} = (X_i)_{1 \leq i \leq N-1} : X_i \geq 0 \right\} \quad (1.5)$$

and

$$P_{N,+, \lambda}^{a,b}(\mathbb{X}) = (Z_{N,+, \lambda}^{a,b})^{-1} \exp \left\{ -\lambda \sum_{i=1}^{N-1} V(X_i) \right\} \prod_{i=0}^{N-1} p(X_{i+1} - X_i) \quad (1.6)$$

respectively.

When  $\lambda = 0$  it is expected that, after a suitable rescaling, the law of the random walk converges to that of a Brownian excursion; at least, it is a direct corollary of the results in [4] and [7] that, in the diffusive scaling, the large- $N$  limit of the distribution  $P_{N,+,0}(\cdot)$  is non-trivial. In particular, the path de-localizes. When  $\lambda > 0$ , the

behaviour changes drastically: We'll prove below that the path remains localized, and that the correlations between positions  $X_i$  and  $X_j$  of the random walk decay exponentially with their separation  $|i - j|$ . Our main goal is to investigate how de-localization occurs as  $\lambda$  decreases to 0. The corresponding critical behaviour can be analyzed in quite some details and, most interestingly, under very weak assumptions on the original random walk. In this way, it is possible to probe its degree of universality.

The physical motivation for the path measure considered in this work is the phenomenon of critical prewetting. Consider a vessel containing the thermodynamically stable gaseous phase of some substance. When the boundary of the vessel displays a sufficiently strong preference towards the thermodynamically unstable liquid phase, there may be creation of a microscopic film of liquid phase coating the walls. As the system is brought closer and closer to liquid/gas phase coexistence, the layer of unstable phase starts to grow. For systems with short-range interactions, two kind of behaviours are possible: either there is an infinite sequence of first-order (so-called layering) phase transitions, at which the thickness increases by one mono-layer, or the growth occurs continuously; this is the case of critical prewetting, and it is typical in two-dimensional systems, as those modelled in the present work. We are thus interested in quantifying the growth as a function of the distance to phase coexistence. A natural parameter is the difference between the free energy densities of the stable and unstable phases. Choosing  $V(x) = |x|$ , we see that the perturbation  $\lambda \sum_i V(X_i)$  can be interpreted as the total excess free energy associated to the unstable layer, the parameter  $\lambda$  playing the role of the excess free energy density.

The problem of critical prewetting in continuous effective interface models in higher dimensions, as well as in the 2D Ising model, has been considered in [11]. The latter results are however restricted to a much smaller class of interactions, and take also a weaker form than those we obtain here. Notice that the thickness of the layer of the unstable phase in the 2D Ising model also grows with exponent  $1/3$ , showing (not surprisingly) that this model is in the same universality class as those considered in the present work; one can hope therefore that the finer estimates we obtain here have similar counterparts in the 2D Ising model. Results about critical prewetting for one-dimensional effective interface models have already been obtained in [1]; they are limited to the particular case of real-valued random walks with  $V(x) = |x|$  and  $p(x) \propto e^{-\beta|x|}$ , which turn out to be exactly solvable. The authors are thus able to extract more precise information than those we present here, including prefactors. It is not clear however to what extent these finer properties also have universal significance.

### 1.1. Main results

We recall that the potential function  $V(\cdot)$  in (1.6) is assumed to be continuous and increasing from 0 to  $\infty$  as  $x$  varies from 0 to  $\infty$ . Therefore, for any  $\gamma > 0$  there is a unique solution  $H > 0$  to the equation

$$\lambda H^2 V(2H) = 1. \quad (1.7)$$

The scale  $H = H(\lambda)$  will play an important role in our future considerations. As a simple corollary of the definition (1.7), note that  $\lambda^{1/3}H(\lambda)$  is a non-increasing function of  $\lambda$ . Indeed, let  $0 < \lambda_1 < \lambda_2$ . Since the function  $H \mapsto H^2V(2H)$  is increasing, it follows from (1.7) that  $H(\lambda_1) \leq H(\lambda_2)$ , and thus convexity of  $V(\cdot)$  implies that

$$V(2H(\lambda_1)) = V\left(2H(\lambda_2) \frac{H(\lambda_1)}{H(\lambda_2)}\right) \leq \frac{H(\lambda_1)}{H(\lambda_2)} V(2H(\lambda_2)),$$

which by (1.7) is equivalent to the desired monotonicity property.

Our first result says that the “average height” of the interface in the limit of small  $\lambda$  is of order  $H(\lambda)$ .

**Theorem 1.1.** *Let  $H = H(\lambda)$  be as defined in (1.7). There exist positive constants  $\delta_0$ ,  $\lambda_0$ , and  $C_1$ ,  $C_2$  such that the inequalities*

$$P_{N,+,\lambda}\left(\sum_{i=1}^N X_i \geq \delta^{-1}HN\right) \leq \frac{1}{C_1}e^{-C_1\delta^{-1}H^{-2}N}, \quad (1.8)$$

$$P_{N,+,\lambda}\left(\sum_{i=1}^N X_i \leq \delta HN\right) \leq \frac{1}{C_2}e^{-C_2\delta^{-2}H^{-2}N} \quad (1.9)$$

hold uniformly in  $0 < \lambda < \lambda_0$ ,  $0 < \delta \leq \delta_0$ , and  $N \geq H^2$ .

Our next result describes the tails of the point-wise height distribution. Although it is formulated for the height in the middle of a typical interface, the result holds for all points from  $[AH^2, N - AH^2]$ , with any fixed  $A > 0$ , i.e., lying sufficiently deep in  $[0, N]$ ; then, clearly,  $c_i = c_i(A) \rightarrow 0$ ,  $i = 1, 2$ , as  $A \rightarrow 0$ .

**Theorem 1.2.** *Let  $H = H(\lambda)$  be as defined in (1.7). There exist positive constants  $T_0$ ,  $K$ ,  $c_1$  and  $c_2$  such that for any  $T \geq T_0$  and all  $N \geq KH^2$  the inequalities*

$$\frac{1}{c_1} \exp\{-c_1T^{3/2}\} \leq P_{N,+,\lambda}(X_{[N/2]} > TH) \leq \frac{1}{c_2} \exp\{-c_2T^{3/2}\} \quad (1.10)$$

hold for all  $\lambda \in (0, \lambda_0]$ , where  $\lambda_0 = \lambda_0(T) > 0$ . In particular, these estimates are uniform on compact subsets of  $[0, \infty)$ .

*Remark 1.3.* Tail estimates for small  $\lambda$  uniform in  $T$  large enough can be obtained taking into account the tail behaviour of the original random-walk. For example, a variant of the argument below, actually even simpler, proves that for a Gaussian random walk, i.e.  $p(x) \propto e^{-c|x|^2}$ , the tail behaviour proved above holds with the same exponent for all  $\lambda \leq \lambda_0$  uniformly in large  $T$ . This behaviour is not universal however, as different behaviour of  $p(\cdot)$  can give rise to completely different tails.

Further, we describe the decay of correlations along the interface. Here again the horizontal scale  $H^2$  plays an important role.

**Theorem 1.4.** *Let  $H = H(\lambda)$  be as defined in (1.7). There exist positive constants  $C$ ,  $c$ , and  $\lambda_0$  such that for every  $\lambda \in (0, \lambda_0]$  and all  $i, j \in (0, N)$  we have*

$$\text{Cov}(X_i, X_j) \leq CH^{5/2} \exp\{-c|i-j|H^{-2}\}.$$

Thinking of  $N$  as of time parameter, our system can be described as a Markov chain on the positive half-line with certain attraction to the origin. Since this chain is positive recurrent, its distribution  $\mu_N^\lambda$  at time  $N$  approaches its stationary distribution  $\pi_\lambda$  exponentially fast on the horizontal time scale  $H^2$ :

**Theorem 1.5.** *Let  $H = H(\lambda)$  be as defined in (1.7). There exist positive constants  $C$ ,  $c$ , and  $\lambda_0$  such that for every  $\lambda \in (0, \lambda_0]$  we have*

$$\|\mu_N^\lambda - \pi_\lambda\|_{\text{TV}} \leq C \exp\{-cN H^{-2}\},$$

where  $\|\cdot\|_{\text{TV}}$  denotes the total variational distance between the probability measures.

*Remark 1.6.* The reader might wonder whether the appearance of the exponents  $1/3$ ,  $2/3$  and  $3/2$  in the case  $V(x) = |x|$  hints at a relationship between the critical behaviour of the model considered here and the much-studied Tracy-Widom distribution. We do not have a precise answer to this question; however at a heuristic level, the appearance of the same critical exponents can be understood by noticing the similarities between our model and the multi-layer PNG model introduced in [10], whose relation to the Tracy-Widom distribution has been studied in the latter work.

## 2. $L^1$ -bounds: Proof of Theorem 1.1

The proof goes along the following lines: We first establish a basic result allowing us to compare partition functions with and without the external field, see Lemma 2.1. This is used to extract the contribution of the external field and reduce the analysis to pure random-walk estimates. From this, one half of the theorem can be proved very easily (the upper bound on  $\sum_i X_i$ ), see Subsection 2.2. In Subsection 2.3, we then turn to the proof of the complementary inequality.

### 2.1. A basic comparison of partition functions

**Lemma 2.1.** *Let  $H=H(\lambda)$  be as defined in (1.7).*

*Then there exist positive constants  $\lambda_0$ ,  $c$ , and  $C$  such that, for any  $0 < \lambda \leq \lambda_0$ , every  $N \geq H^2$ , and all boundary conditions  $0 \leq a, b \leq H$ , one has*

$$c e^{-CNH^{-2}} e^{-\lambda V(2H)N} Z_{N,+,0}^{a,b} \leq Z_{N,+, \lambda}^{a,b} \leq Z_{N,+,0}^{a,b}. \quad (2.1)$$

*Remark 2.2.* Observe that

$$\frac{Z_{N,+, \lambda}^{a,b}}{Z_{N,+,0}^{a,b}} = E_{N,+,0}^{a,b} \left[ \exp \left\{ -\lambda \sum_{j=1}^{N-1} V(X_j) \right\} \right] \quad (2.2)$$

and thus the lemma states that for any  $\lambda > 0$  the exponential moment of the functional

$$\mathbb{V}(\mathbb{X}) = \sum_{j=1}^{N-1} V(X_j)$$

decays no faster than exponentially in  $N$  indicating that the typical value of  $V(X_j)$  (equivalently, the height  $X_j$  of the interface) is “bounded on average”. It is instructive to compare this property to the asymptotics

$$\mathbb{E}_{N,+}^{a,b} \left[ N^{-3/2} \sum_{j=1}^N X_j \right] \rightarrow \text{const} > 0$$

following from [4, 9].

As a straightforward application of the lemma above we get the following simple but quite useful fact.

**Corollary 2.3.** *Under the conditions of Lemma 2.1, for any collection  $\mathcal{A} \subset \mathcal{I}_{N,+}$  of trajectories we have:*

$$\mathbb{P}_{N,+,\lambda}^{a,b}(\mathcal{A}) \leq \frac{1}{c} \exp \left\{ CNH^{-2} + \lambda V(2H)N \right\} \mathbb{P}_{N,+0}^{a,b}(\mathcal{A}). \quad (2.3)$$

Similarly, for any  $\varepsilon > 0$ , we get

$$\begin{aligned} \mathbb{P}_{N,+,\lambda}^{a,b} \left( \mathcal{A}; \sum_{i=1}^N X_i \geq \varepsilon HN \right) \\ \leq \frac{1}{c} \exp \left\{ CNH^{-2} + \lambda V(2H)N - \lambda V(\varepsilon H)N \right\} \mathbb{P}_{N,+0}^{a,b}(\mathcal{A}) \end{aligned} \quad (2.4)$$

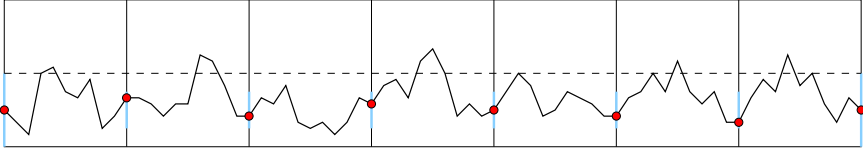
and for any constants  $A, B > 0$ ,

$$\begin{aligned} \mathbb{P}_{N,+,\lambda}^{a,b} \left( \mathcal{A}; \mathbb{V}(\mathbb{X}) \geq AV(BH)N \right) \\ \leq \frac{1}{c} \exp \left\{ CNH^{-2} + \lambda V(2H)N - \lambda AV(BH)N \right\} \mathbb{P}_{N,+0}^{a,b}(\mathcal{A}). \end{aligned} \quad (2.5)$$

*Proof.* Indeed, (2.4) follows immediately from convexity of  $V(\cdot)$ , as then

$$\mathbb{V}(\mathbb{X}) = \sum_{j=1}^N V(X_j) \geq NV \left( \sum_{j=1}^N X_j / N \right) \geq NV(\varepsilon H).$$

Other inequalities are obvious. □



**Fig. 1.** The event in the proof of Lemma 2.1.

*Proof of Lemma 2.1.* As the upper bound is obvious, one only needs to check the left inequality in (2.1).

We use a renormalisation argument. With  $H = \rho H(\lambda)$ , take positive  $\varepsilon$  small enough to satisfy  $\varepsilon \in (0, 1/4]$  and cut each trajectory of our RW into blocks of length<sup>2</sup>  $\Delta = \varepsilon H^2$ ; this generates  $n_\varepsilon = \lceil N/\varepsilon H^2 \rceil \geq 4$  such blocks (if there is a shorter piece left, we attach it to the last block). Further, denote  $n_1 = \Delta$ ,  $n_2 = (n_\varepsilon - 1)\Delta$  and consider the events

$$\mathcal{A} = \left\{ \forall j = 2, \dots, n_\varepsilon - 2 : \frac{1}{4}H \leq X_{j\Delta} \leq \frac{3}{4}H \right\},$$

$$\mathcal{B} = \left\{ \forall j = n_1 + 1, \dots, n_2 - 1 : 0 \leq X_j \leq 2H \right\},$$

$$\mathcal{C}_1 = \left\{ \forall j = 1, \dots, n_1 - 1 : 0 \leq X_j \leq 2H \right\} \cap \left\{ \frac{1}{4}H \leq X_{n_1} \leq \frac{3}{4}H \right\},$$

$$\mathcal{C}_2 = \left\{ \forall j = n_2 + 1, \dots, N - 1 : 0 \leq X_j \leq 2H \right\} \cap \left\{ \frac{1}{4}H \leq X_{n_2} \leq \frac{3}{4}H \right\}.$$

Using (2.2), we immediately get

$$\frac{Z_{N,+, \lambda}^{a,b}}{Z_{N,+, 0}^{a,b}} \geq \exp \left\{ -\lambda V(2H)N \right\} \mathbb{P}_{N,+, 0}^{a,b}(\mathcal{A} \cap \mathcal{B} \cap \mathcal{C}_1 \cap \mathcal{C}_2) \quad (2.6)$$

and it remains to bound below the probability of  $\mathcal{A} \cap \mathcal{B} \cap \mathcal{C}_1 \cap \mathcal{C}_2$ .

First, observe that in view of the Donsker invariance principle we have<sup>3</sup>

$$\min_{H/4 \leq x \leq 3H/4} \mathbb{P}_{\Delta, +}^x(H/4 \leq X_\Delta \leq 3H/4) \geq C_1,$$

$$\min_{H/4 \leq c, d \leq 3H/4} \mathbb{P}_{\Delta, +, 0}^{c,d}(\forall j = 1, \dots, \Delta - 1 : 0 \leq X_j \leq 2H) \geq C_2.$$

By the Markov property we thus estimate

$$\mathbb{P}_{N,+, 0}^{a,b}(\mathcal{A} \cap \mathcal{B} \mid \mathcal{C}_1 \cap \mathcal{C}_2) \geq (C_1 C_2)^{n_\varepsilon - 2} \geq c \exp \{-C n_\varepsilon\}$$

<sup>2</sup> both  $H$  and  $\Delta$  are assumed to be integer.

<sup>3</sup> Here and below,  $\mathbb{P}_{\Delta, +}^x(\cdot)$  denotes the conditional distribution of the walk  $X_0, X_1, \dots, X_\Delta$  subject to  $X_0 = x$  and the positivity constraint, i.e., the analogue of  $\mathbb{P}_{\Delta, +, 0}^{x, 6}$  without the condition  $X_\Delta = 6$ .

with some  $c, C > 0$ . On the other hand, by (conditional) independence of  $\mathcal{C}_1$  and  $\mathcal{C}_2$  it is sufficient to bound below the probabilities  $P_{N,+,0}^{a,b}(\mathcal{C}_1)$  and  $P_{N,+,0}^{a,b}(\mathcal{C}_2)$ . We shall estimate the former, the latter will follow in a similar way.

Combining the argument above with the conditional invariance principle due to Bolthausen [4] we get

$$\min_{a \in [H/4, H] \cup \{0\}} P_{\Delta,+}^a(\mathcal{C}_1) \geq C_3 > 0 \quad (2.7)$$

with some constant  $C_3 = C_3(\varepsilon)$ , uniformly in  $\lambda > 0$  small enough. Observe that the last bound holds also for any  $\tilde{\Delta}$ ,  $\Delta/2 \leq \tilde{\Delta} \leq \Delta$ , and perhaps smaller  $C_3 > 0$ . Consequently, it is enough to show that for some constant  $C_4 = C_4(\varepsilon) > 0$  we have, uniformly in sufficiently small  $\lambda > 0$ ,

$$\min_{0 < a < H/4} P_{\Delta,+}^a \left( \min \left\{ j \geq 0 : X_j \in [H/4, H] \right\} \leq \frac{\Delta}{4} \right) \geq C_4 > 0 \quad (2.8)$$

as then immediately

$$\min_{0 < a < H/4} P_{\Delta,+}^a(\mathcal{C}_1) \geq C_3 C_4 > 0. \quad (2.9)$$

To check (2.8), we fix an arbitrary integer  $a$ ,  $0 < a < H/4$ , and consider two independent trajectories  $X^a$  and  $X^0$  distributed according to  $P_{\Delta,+}^a(\cdot)$  and  $P_{\Delta,+}^0(\cdot)$  respectively. Let  $\mathcal{D} = \mathcal{D}(k)$  denote the “crossing event” at  $k$ ,

$$\mathcal{D}(k) = \left\{ \forall j = 0, 1, \dots, k-1 : X_j^a > X_j^0 \quad \text{and} \quad X_k^a \leq X_k^0 \right\},$$

and denote  $\mathcal{D}_\Delta = \cup_{k=1}^{n_3} \mathcal{D}(k)$ ,  $n_3 = [\Delta/4]$ . Our aim is to show that there exists a positive constant  $C_5 = C_5(\varepsilon)$  such that, uniformly in small enough  $\lambda > 0$ , one has (here and below,  $\mathcal{D}_\Delta^c$  stands for the complement of  $\mathcal{D}_\Delta$ )

$$P_{\Delta,+}^a(\mathcal{C}_1 \mid \mathcal{D}_\Delta) \geq C_5, \quad P_{\Delta,+}^a(\mathcal{C}_1 \mid \mathcal{D}_\Delta^c) \geq C_5. \quad (2.10)$$

Then the target inequality

$$P_{N,+,0}^{a,b}(\mathcal{C}_1) \geq C_6(\varepsilon) > 0$$

follows immediately.

The key observation towards (2.10) is the following. For an integer  $x \geq 0$ , the jump distribution  $p_x(\cdot)$  of our random walk from  $x$  is given by

$$p_x(k) = \frac{p(k)}{P(\xi \geq -x)} \mathbb{1}_{\{k \geq -x\}},$$

where  $\xi$  is a random variable with the unconstrained jump distribution  $p(\cdot)$ . Clearly, the mean  $e_x$  and the variance  $\sigma_x^2$  of  $p_x(\cdot)$  satisfy

$$e_x = \sum_k k p_x(k) \searrow e_\infty \equiv 0 \quad \text{as } x \nearrow \infty$$

$$\sigma_x^2 = \sum_k k^2 p_x(k) - (e_x)^2 \leq \frac{\sigma^2}{P(\xi \geq -x)} \quad (2.11)$$

with  $\sigma^2$  denoting the variance of  $p(\cdot)$ .

Now, suppose that the crossing event  $\mathcal{D}(k)$ ,  $k \in \{1, 2, \dots, n_3\}$ , takes place. In view of (2.11), with positive probability we have  $X_k^0 - X_k^a \leq M$ , where the constant  $M$  is independent of  $\lambda$ . Thanks to the analogue of the aperiodicity property (1.4) for the distributions  $p_x(\cdot)$  and  $p_y(\cdot)$ , where  $x = X_k^a$  and  $y = X_k^0$  (with the same lower bound (1.4) for all  $x, y \geq 0$ ), two independent trajectories started at  $x$  and  $y$  meet with positive probability within  $AM$  steps. Thus, the first inequality in (2.10) follows immediately from the standard independent coupling and the properties of  $P_{\Delta,+}^0(\cdot)$  mentioned above.

If  $\mathcal{D}_{\Delta}^c$  takes place, we have  $X_j^a > X_j^0$  for all  $j = 0, 1, \dots, n_3$ . Consequently, with positive probability the stopping time

$$\tau = \min\{j \geq 0 : X_j^a \geq H/4\}$$

satisfies  $\tau \leq n_3$ . Then the finite variance argument used above implies that, with positive probability, we have  $H/4 \leq X_{\tau}^a \leq H$  and the second inequality in (2.10) follows from a straightforward generalization of (2.7).  $\square$

A literal repetition of the argument above implies also the following result:

**Corollary 2.4.** *For positive  $\rho$  and  $\lambda$ , put  $H = 2\rho H(\lambda)$  and define the event*

$$\mathcal{B} = \mathcal{B}_{H,N} = \left\{ \forall j = 1, \dots, N : 0 \leq X_j \leq 2H \right\}. \quad (2.12)$$

*Then, for any  $\rho, \eta > 0$ , there exist positive constants  $\lambda_0, c$ , and  $C$  such that for all  $0 < \lambda \leq \lambda_0$ ,  $N \geq \eta H^2$  and  $0 \leq a, b \leq H$ , we have*

$$P_{N,+0}^{a,b}(\mathcal{B}) \geq c \exp\{-CNH^{-2}\}. \quad (2.13)$$

*Moreover, for any other event  $\mathcal{A}$  with  $P_{N,+0}^{a,b}(\mathcal{A} | \mathcal{B}) > 0$ , we get*

$$c e^{-CNH^{-2} - 2\lambda V(2H)N} \leq \frac{P_{N,+,\lambda}^{a,b}(\mathcal{A} | \mathcal{B})}{P_{N,+0}^{a,b}(\mathcal{A} | \mathcal{B})} \leq c^{-1} e^{CNH^{-2} + 2\lambda V(2H)N}, \quad (2.14)$$

*and thus, uniformly in bounded  $NH^{-2}$ , both conditional probabilities are positive simultaneously.*

*Proof.* Since  $V(\cdot)$  is non-negative and monotone, (2.12) implies that, for any event  $\mathcal{A}$ ,

$$e^{-\lambda V(2H)N} Z_{N,+0}^{a,b} P_{N,+0}^{a,b}(\mathcal{A}\mathcal{B}) \leq Z_{N,+,\lambda}^{a,b} P_{N,+,\lambda}^{a,b}(\mathcal{A}\mathcal{B}) \leq Z_{N,+0}^{a,b} P_{N,+0}^{a,b}(\mathcal{A}\mathcal{B}).$$

The inequality (2.14) now follows immediately from Lemma 2.1.  $\square$

*Remark 2.5.* Although the importance of the scale  $H = H_1$  (see (1.7)) should be clear from the proofs above, it is instructive to give another motivation for the definition (1.7). Clearly, each interface under consideration can be naturally decomposed into elementary excursions above the wall. Without external field (i.e., with  $\lambda = 0$ ) each such excursion of horizontal length  $l^2$  has typical height of order  $l$ . For  $\lambda > 0$  its energetic price is of order at most  $\lambda l^2 V(l)$  and thus the interaction with the field is negligible if  $\lambda l^2 V(l) \ll 1$ , that is if  $l = o(H)$ . In other words, the presence of the field  $\lambda$  is felt on the (vertical) scale  $H$  or larger.

## 2.2. The upper bound

The first half of Theorem 1.1, namely

$$P_{N,+, \lambda} \left[ N^{-1} \sum_{i=1}^N X_i \geq \delta^{-1} H \right] \leq \frac{1}{c} e^{-C \delta^{-1} N H^{-2}},$$

(with  $H = H(\lambda)$ , see (1.7)) follows directly from (2.4) and the inequality

$$V(\delta^{-1} H) \geq (2\delta)^{-1} V(2H)$$

valid for any  $H \geq 0$  and  $0 < 2\delta \leq 1$ :

$$P_{N,+, \lambda} \left[ N^{-1} \sum_{i=1}^N X_i \geq \delta^{-1} H \right] \leq \frac{1}{c} e^{C N H^{-2} - ((2\delta)^{-1} - 1) \lambda V(2H) N} \leq \frac{1}{c} e^{-C_\delta N H^{-2}};$$

here  $C_\delta = 1/(4\delta)$  and  $\delta$  is chosen small enough to satisfy

$$0 < \delta \leq \min \left( 1, \frac{1}{4(C+1)} \right).$$

□

## 2.3. The lower bound

Our proof of the lower bound (with  $H = H(\lambda)$ , see (1.7)),

$$P_{N,+, \lambda} \left[ N^{-1} \sum_{i=1}^N X_i \leq \delta H \right] \leq c e^{-C \delta^{-2} N H^{-2}}, \quad (2.15)$$

is based upon a certain renormalisation procedure. Namely, take  $\lambda > 0$  small enough and  $\varepsilon > 0$  to be chosen later (assuming, without loss of generality, that  $\varepsilon^2 \in (0, 1/12)$  and  $\varepsilon^2 H^2$  is an integer number larger than 1) and split every trajectory of the random walk under consideration into pieces of length  $4\varepsilon^2 H^2$  to be called blocks; clearly, there are exactly  $n_\varepsilon = \lceil N/(4\varepsilon^2 H^2) \rceil$  such blocks (and, perhaps, an additional piece of shorter length). Next, we split each block into four equal parts and use  $\mathcal{I}_m, m = 1, \dots, 4n_\varepsilon$  to denote all obtained sub-blocks. Further,

we fix a small enough  $\rho > 0$  and say that the trajectory under consideration is  $\rho$ -high in the  $k$ -th block, if

$$\max_{j \in \mathcal{I}_{4k+2}} X_j > \rho \varepsilon H, \quad \max_{j \in \mathcal{I}_{4k+4}} X_j > \rho \varepsilon H.$$

The main idea behind the argument below is as follows: for  $\rho > 0$  small enough, the number of  $\rho$ -high blocks in a typical trajectory is of order  $n_\varepsilon$ ; however, a typical contribution of a  $\rho$ -high block to the total area is of order at least  $\rho \varepsilon^3 H^3$ ; as a result, the typical area is bounded below by a quantity of order at least  $\rho \varepsilon^3 H^3 n_\varepsilon \asymp \rho \varepsilon H N$  and thus, for  $\delta > 0$  small enough, the event

$$\mathcal{A}_\delta \equiv \left\{ \sum_{i=1}^N X_i \leq \delta H N \right\}$$

falls into the large deviation region for the distribution under consideration. The target inequality (2.15) gives a quantitative estimate for this to happen.

To start, we use (2.3) to remove the external field  $\lambda$ ,

$$\begin{aligned} \mathbb{P}_{N,+,\lambda}[\mathcal{A}_\delta] &\leq c^{-1} \exp\left\{C N H^{-2} + \lambda V(2H)N\right\} \mathbb{P}_{N,+,0}[\mathcal{A}_\delta] \\ &= c^{-1} \exp\left\{(C+1)N H^{-2}\right\} \mathbb{P}_{N,+,0}[\mathcal{A}_\delta]. \end{aligned}$$

Now, conditionally on the configuration  $\mathbb{X}$  in the blocks  $\mathcal{I}_{2m+1}$ ,  $m = 0, \dots, n_\varepsilon - 1$ , the events

$$\left\{ \max_{j \in \mathcal{I}_{2m}} X_j > \rho \varepsilon H \right\}$$

are mutually independent. Moreover, a straightforward generalization of the argument in Lemma 2.1 used to estimate below the probability of  $\mathcal{C}_1$ , shows that for some  $\eta > 0$  and  $\rho_0 > 0$  one has

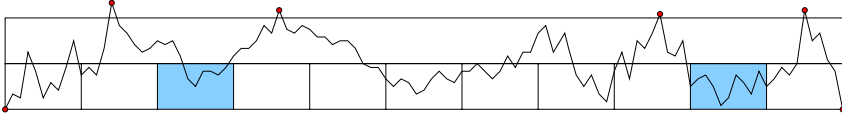
$$a_\rho \equiv \sup_m \sup_{0 \leq a, b \leq \rho \varepsilon H} \mathbb{P}_{\mathcal{I}_{2m},+,0}^{a,b} \left( \max_{j \in \mathcal{I}_{2m}} X_j \leq \rho \varepsilon H \right) \leq 1 - \eta$$

uniformly in  $0 < \rho \leq \rho_0$  and all  $\varepsilon H = \varepsilon H(\lambda)$  large enough; consequently, the events

$$\left\{ k\text{-th block is } \rho\text{-high} \right\},$$

which are also conditionally independent for fixed configuration in  $\mathcal{I}_{4k+1}$ ,  $k = 0, \dots, n_\varepsilon - 1$ , occur with probability at least  $(1 - a_\rho)^2 \geq \eta^2$ . As a result, the number  $n_\rho$  of  $\rho$ -high blocks for a typical trajectory is not less than  $\eta^2 n_\varepsilon / 2$ . More precisely, since the events under consideration are independent for individual blocks (conditionally on every fixed configuration in between), the standard large deviation bound implies

$$\mathbb{P}_{N,+,0} \left( n_\rho < \frac{\eta^2}{2} n_\varepsilon \right) \leq \exp\{-c n_\varepsilon\} \leq \exp\{-c' \varepsilon^{-2} N H^{-2}\}$$



**Fig. 2.** Two  $\rho$ -high blocks with oscillation

with some  $c' = c'(\rho) > 0$  not depending on  $\varepsilon$ . Thus, taking  $\varepsilon > 0$  small enough, we obtain

$$\mathbf{P}_{N,+, \lambda} \left( n_\rho < \frac{n^2}{4} n_\varepsilon \right) \leq \exp \{ -c_1 \varepsilon^{-2} N H^{-2} \}.$$

From now on, we shall restrict ourselves to the trajectories containing at least  $\eta^2 n_\varepsilon / 2$  blocks that are  $\rho$ -high. We shall say that a  $\rho$ -high block oscillates if

$$\min_{j \in \mathcal{I}_{4k+3}} X_j < \frac{\rho \varepsilon}{2} H$$

and observe that each  $\rho$ -high block without oscillation contributes an amount at least  $\rho \varepsilon^3 H^3 / 2$  to the total area. Our final step of the proof consists in evaluating the typical number of oscillating blocks.

Define

$$l_k = \min \left\{ j \in \mathcal{I}_{4k+2} : X_j > \rho \varepsilon H \right\}, \quad r_k = \max \left\{ j \in \mathcal{I}_{4k+4} : X_j > \rho \varepsilon H \right\}$$

and put  $a_k = X_{l_k}$ ,  $b_k = X_{r_k}$ ,  $L = 4k\varepsilon^2 H^2$ ,  $R = 4(k+1)\varepsilon^2 H^2$ . Then using essentially the same arguments as in the proof of Lemma 2.1, we deduce that

$$\begin{aligned} \min_{a,b} \mathbf{P}_{(L,R),+,0}^{a,b} (k\text{-th block does not oscillate}) \\ \geq \min \mathbf{P}_{r_k-l_k,+,0}^{a_k,b_k} \left( \min_j X_j > \rho \varepsilon H / 2 \right) \geq 1 - \bar{a}_\rho, \end{aligned}$$

where the bound  $\bar{a}_\rho < 1$  holds for any fixed  $\rho$  small enough. Arguing as before, we deduce that the number  $\bar{n}_{\text{osc}}$  of blocks without oscillation satisfies the estimate

$$\mathbf{P}_{N,+,0} \left( \bar{n}_{\text{osc}} < \frac{1 - \bar{a}_\rho}{2} n_\rho \right) \leq \exp \{ -c'' n_\rho \} = \exp \{ -c''' \varepsilon^{-2} N H^{-2} \}$$

with some  $c''' = c'''(\rho) > 0$  not depending on  $\varepsilon$ . Again, taking  $\varepsilon > 0$  small enough, we obtain

$$\mathbf{P}_{N,+, \lambda} \left( \bar{n}_{\text{osc}} < \frac{1 - \bar{a}_\rho}{2} n_\rho \right) \leq \exp \{ -c_2 \varepsilon^{-2} N H^{-2} \}.$$

However, on the complementary event we get

$$\bar{n}_{\text{osc}} \geq \frac{1 - \bar{a}_\rho}{2} n_\rho \geq \frac{1 - \bar{a}_\rho}{4} \eta^2 n_\varepsilon$$

for all  $\rho > 0$  small enough and thus the inequality

$$\sum_{i=1}^N X_i \geq \frac{\rho \varepsilon^3 H^3}{2} \bar{n}_{\text{osc}} > \frac{\rho(1 - \bar{a}_\rho)}{64} \eta^2 \varepsilon H N$$

renders the event  $\mathcal{A}_\delta$  impossible if  $\delta \leq \rho(1 - \bar{a}_\rho)\eta^2\varepsilon/64$ . As a result, for such  $\delta > 0$  we get

$$\begin{aligned} \mathbb{P}_{N,+, \lambda}[\mathcal{A}_\delta] &\leq \mathbb{P}_{N,+, \lambda}\left(n_\rho < \frac{\eta^2 n_\varepsilon}{2}\right) + \mathbb{P}_{N,+, \lambda}\left(\bar{n}_{\text{osc}} < \frac{1 - \bar{a}_\rho}{2} n_\rho\right) \\ &\leq 2 \exp\{-c_3 \delta^{-2} N H^{-2}\} \end{aligned}$$

with some  $c_3 = c_3(\rho) > 0$ .  $\square$

*Remark 2.6.* Obviously, the obtained lower  $L^1$ -bound on the total area implies immediately a simple lower bound for the height of the maximum of interfaces:

$$\begin{aligned} \mathbb{P}_{N,+, \lambda}\left(\max_{1 \leq k \leq N} X_k \leq \delta H\right) &\leq \mathbb{P}_{N,+, \lambda}\left(N^{-1} \sum_{i=1}^N X_i \leq \delta H\right) \\ &\leq c \exp\{-C \delta^{-2} N H^{-2}\}. \end{aligned} \quad (2.16)$$

We shall obtain a complementary bound after a more detailed analysis of the interfaces.

### 3. One-point distribution: Proof of Theorem 1.2

We treat the lower and the upper bounds in (1.10) separately. In Subsection 3.1, we establish several refinements of our comparison Lemma 2.1. The latter are used in Subsection 3.2 to prove the upper bound in (1.10); as a by-product of these considerations we obtain, in Corollary 3.5, upper bounds on the moments of the random variables  $X_i$  to be used later in the proof of Theorem 1.4. Finally, the lower bound in (1.10) is proved in Subsection 3.3.

#### 3.1. Two refinements of the basic comparison lemma

The following version of Lemma 2.1 gives a better bound than (2.1) for large values of  $\rho$ ,  $\rho \geq \rho_0(\eta) > 0$ . With  $H = \rho H(\lambda)$  and  $\eta \in (0, 1/2)$ , we put<sup>4</sup>

$$\tilde{H} = (1 - 2\eta)\rho H(\lambda), \quad \tilde{\Delta} = \varepsilon \tilde{H}^2. \quad (3.1)$$

**Lemma 3.1.** *Let  $c$  and  $C$  be as in Lemma 2.1. For  $H$  as above, there exists  $\lambda_0 > 0$  such that for any  $\eta \in (0, 1/2)$  and  $\zeta \in (0, 1/2)$  there is a constant  $\tilde{c} > 0$  such that for any  $0 < \lambda \leq \lambda_0$ , every  $N \geq H^2/(2\zeta)$ , and all boundary conditions  $0 \leq a, b \leq H$ , one has*

$$\tilde{c} c \exp\left\{-\frac{CN}{\tilde{H}^2} - \lambda \left[\zeta V(2H) + (1 - \zeta)V(2\tilde{H})\right]N\right\} \leq \frac{Z_{N,+, \lambda}^{a, b}}{Z_{N,+, 0}^{a, b}} \leq 1. \quad (3.2)$$

<sup>4</sup> assuming  $\tilde{H}$  and  $\tilde{\Delta}$  to be integer.

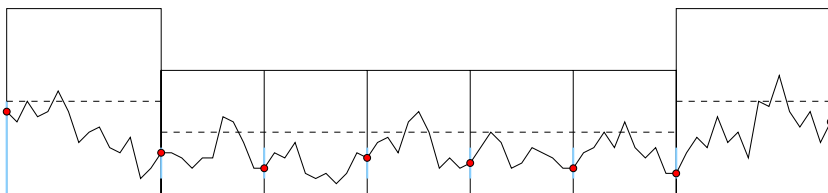


Fig. 3. Renormalization scheme in Lemma 3.1.

*Proof.* Let  $H = \rho H(\lambda)$ ,  $\varepsilon \in (0, 1/4]$  and  $\Delta = \zeta H^2/2$ . For  $\eta \in (0, 1/2)$  define

$$\tilde{n}_\varepsilon = \left\lceil \frac{(1-\zeta)N}{\tilde{\Delta}} \right\rceil \equiv \left\lceil \frac{(1-\zeta)N}{\varepsilon(1-2\eta)^2 H^2} \right\rceil \geq 2. \quad (3.3)$$

Further, let

$$J_0 = \left\{ \Delta + j\tilde{\Delta} : j = 0, 1, \dots, \tilde{n}_\varepsilon - 1 \right\} \cup \{N - \Delta\}$$

$$J_1 = \left\{ \Delta, \Delta + 1, \dots, N - \Delta \right\}, \quad J_2 = \left\{ 1, \dots, N \right\} \setminus J_1$$

and introduce the events (see Fig. 3):

$$\mathcal{A} = \left\{ \forall j \in J_0 : \frac{1}{4}\tilde{H} \leq X_j \leq \frac{3}{4}\tilde{H} \right\},$$

$$\mathcal{B} = \left\{ \forall j \in J_1 : 0 \leq X_j \leq 2\tilde{H} \right\} \cap \left\{ \forall j \in J_2 : 0 \leq X_j \leq 2H \right\}.$$

For trajectories belonging to  $\mathcal{A} \cap \mathcal{B}$  we have

$$\mathbb{V}(\mathbb{X}) = \sum_{j=1}^{N-1} V(X_j) \leq \zeta NV(2H) + (1-\zeta)NV(2\tilde{H})$$

and therefore,

$$\frac{Z_{N,+,\lambda}^{a,b}}{Z_{N,+,0}^{a,b}} \geq \exp \left\{ -\lambda \left[ \zeta V(2H) + (1-\zeta)V(2\tilde{H}) \right] N \right\} \mathbb{P}_{N,+,0}^{a,b}(\mathcal{A} \cap \mathcal{B}).$$

Using the assumption (3.3) and arguing as in the proof of Lemma 2.1 we get

$$\mathbb{P}_{N,+,0}^{a,b}(\mathcal{A} \cap \mathcal{B}) \geq ce^{-2\tilde{C}\tilde{n}_\varepsilon} \geq ce^{-CN\tilde{H}^{-2}}$$

with perhaps slightly smaller constant  $\lambda_0 > 0$ . □

Next, we present a short-droplet analogue of the previous lemma.

**Lemma 3.2.** *Let  $H = \rho H(\lambda)$  and  $N \leq K H^2(\lambda)$ . There exist positive constants  $\zeta$ ,  $\lambda_0$  and  $C$  such that uniformly in  $K/\rho^2 < \zeta$ , in  $\lambda \in (0, \lambda_0]$  and all boundary conditions  $0 \leq a, b \leq H$  one has*

$$C Z_{N,+,0}^{a,b} \leq Z_{N,+, \lambda}^{a,b} \leq Z_{N,+,0}^{a,b}.$$

Our argument is based upon the following small droplet bound to be verified in Appendix A below.

**Lemma 3.3.** *Let  $S_0 = 0$ ,  $S_k = \xi_1 + \dots + \xi_k$ ,  $k \geq 1$ , be the random walk generated by a sequence  $\xi_1, \xi_2, \dots$  of i.i.d. random variables such that  $\mathbf{E}\xi = 0$ ,  $\mathbf{E}\xi^2 = \sigma^2 < \infty$ . Let  $D > 0$  be an arbitrary fixed constant and, for any  $m \geq 1$ , let  $d_m$  satisfy  $\mathbf{P}(S_m = d_m) > 0$  and  $|d_m| \leq D$ . Then there exists  $\zeta > 0$  such that*

$$\mathbf{P}\left(\max_{0 < k < m} S_k > M \mid S_m = d_m\right) \leq \frac{1}{3}, \quad \text{as } M \rightarrow \infty, \quad (3.4)$$

uniformly in  $m/M^2 \leq \zeta$ .

*Proof of Lemma 3.2.* As in Lemma 2.1, our argument is based on the bound (recall (2.6))

$$\frac{Z_{N,+, \lambda}^{a,b}}{Z_{N,+,0}^{a,b}} \geq \exp\left\{-\lambda V(5H)N\right\} \mathbf{P}_{N,+,0}^{a,b}(\mathcal{B}),$$

where

$$\mathcal{B} = \left\{ \mathbb{X} \in \mathcal{I}_{N,+} : \max X_j \leq 5H \right\};$$

because of (1.7), (1.2) and the condition  $N \leq K H^2(\lambda)$ , it remains to verify that the last probability is uniformly positive. Let  $\mathbb{X}$  be an arbitrary trajectory from  $\mathcal{I}_{N,+}$  (recall (1.5)). Then, either it belongs to the set

$$\mathcal{A}_1 = \left\{ \mathbb{X} : \forall j = 1, \dots, N-1 : 0 \leq X_j \leq 2H \right\}$$

or there exists a non-empty set  $[\ell', \tau'] \subset (0, N)$  such that

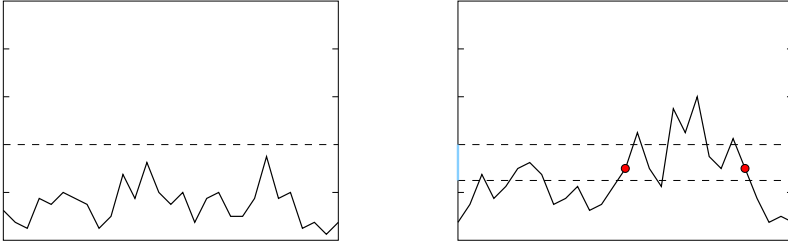
$$\ell' = \min\{j > 0 : X_j > 2H\} - 1, \quad \tau' = \max\{j < N : X_j > 2H\} + 1. \quad (3.5)$$

We shall write  $\Delta' = (\ell', \tau')$  and  $|\Delta'| = \tau' - \ell'$ . Fix any  $L \in (0, H)$  and denote

$$\mathcal{A}_2 = \left\{ \mathbb{X} : X_{\ell'} \geq 2H - L, X_{\tau'} \geq 2H - L \right\}. \quad (3.6)$$

According to the bounded variance estimate (2.11) and the Chebyshev inequality, one immediately gets

$$\mathbf{P}_{N,+,0}^{a,b}(\mathbb{X} \notin \mathcal{A}_1 \cup \mathcal{A}_2) \leq 2\tilde{\sigma}^2 L^{-2}, \quad (3.7)$$



**Fig. 4.** Trajectories from  $\mathcal{A}_1$  (left) and  $\mathcal{A}_2$  (right); black dots correspond to the decomposition in (3.9).

where

$$\tilde{\sigma}^2 = \max\left(\frac{\sigma^2}{\mathbb{P}(\xi \geq 0)}, \frac{\sigma^2}{\mathbb{P}(\xi \leq 0)}\right).$$

Taking  $L$  sufficiently large to have  $\mathbb{P}_{N,+,0}^{a,b}(\mathbb{X} \in \mathcal{A}_1 \cup \mathcal{A}_2) \geq 1/2$ , we shall restrict ourselves to trajectories  $\mathbb{X}$  belonging to  $\mathcal{A}_1 \cup \mathcal{A}_2$  only (see Fig. 4).

As  $\mathcal{A}_1 \subset \mathcal{B}$ , it remains to show that

$$\mathbb{P}_{N,+,0}^{a,b}(\mathcal{B} \mid \mathcal{A}_2) \geq c \quad (3.8)$$

for some constant  $c > 0$ . Indeed, once (3.8) is verified, we immediately get

$$\begin{aligned} \mathbb{P}_{N,+,0}^{a,b}(\mathcal{B}) &\geq \mathbb{P}_{N,+,0}^{a,b}(\mathcal{B} \mid \mathcal{A}_2) \mathbb{P}_{N,+,0}^{a,b}(\mathcal{A}_2) + \mathbb{P}_{N,+,0}^{a,b}(\mathcal{A}_1) \\ &\geq c \mathbb{P}_{N,+,0}^{a,b}(\mathcal{A}_1 \cup \mathcal{A}_2) \geq c/2 \end{aligned}$$

and therefore

$$\frac{Z_{N,+,0}^{a,b}}{Z_{N,+,0}^{a,b,\lambda}} \geq \exp\left\{-\lambda V(5H)N\right\} c/2 \geq \tilde{c}$$

uniformly in such  $N$  and  $\{a, b\} \subset [0, H]$ .

To prove (3.8), we rewrite

$$\begin{aligned} \mathbb{P}_{N,+,0}^{a,b}(\bar{\mathcal{B}} \mid \mathcal{A}_2) &= \sum_{l', r'} \mathbb{P}_{N,+,0}^{a,b}(l' = l', r' = r' \mid \mathcal{A}_2) \\ &\quad \times \sum_{2H-L \leq a', b' \leq 2H} \mathbb{P}_{(l', r'), +, 0}^{a', b'}(\bar{\mathcal{B}}) \\ &\quad \times \mathbb{P}_{N,+,0}^{a,b}(X_{l'} = a', X_{r'} = b' \mid l' = l', r' = r'). \quad (3.9) \end{aligned}$$

However,

$$\mathbb{P}_{(l', r'), +, 0}^{a', b'}(\bar{\mathcal{B}}) \leq \frac{\mathbb{P}_{|\Delta'|}(\max S_j > 3H \mid S_{|\Delta'|} = b' - a')}{1 - \mathbb{P}_{|\Delta'|}(\min S_j < -H \mid S_{|\Delta'|} = b' - a')}, \quad (3.10)$$

where  $\mathbb{P}_{|\Delta'|}$  refers to the distribution of  $|\Delta'|$ -step unconstrained random walk with the step distribution  $p(\cdot)$ , recall (1.3). Finally, using the small droplet bound (3.4) and taking  $|\Delta'|/H^2 \leq N/H^2$  sufficiently small, we can make the RHS above smaller than  $1 - c$ . This finishes the proof.  $\square$

### 3.2. The upper bound

We turn now to the proof of the upper bound in Theorem 1.2. Recall that due to the assumption (1.2) the function  $V(\cdot)$  does not grow too fast at infinity.

For  $\rho > 0$  and  $H = H(\lambda)$ , our canonical scale from (1.7), define (cf. (1.1))

$$\begin{aligned} \mathfrak{l}_\rho(M) &= \max \left\{ j < M : X_j \leq \rho H \right\}, \quad \mathfrak{r}_\rho(M) = \min \left\{ j > M : X_j \leq \rho H \right\}, \\ \Delta_\rho(M) &= \{ \mathfrak{l}_\rho + 1, \mathfrak{l}_\rho + 2, \dots, \mathfrak{r}_\rho - 1 \}, \quad |\Delta_\rho| = \mathfrak{r}_\rho - \mathfrak{l}_\rho - 1 \end{aligned} \quad (3.11)$$

and, for any integer interval  $\Delta$ ,

$$\mathcal{A}_\rho(\Delta) = \left\{ \forall j \in \Delta, X_j > \rho H \right\}. \quad (3.12)$$

Then, with  $\Delta$  and  $T$  being (large) natural numbers to be chosen later, we get (for  $N_2 = \lfloor N/2 \rfloor$  and  $\Delta_\rho = \Delta_\rho(N_2)$ )

$$\begin{aligned} \mathbb{P}_{N,+, \lambda} \left( X_{N_2} > TH \right) &\leq \mathbb{P}_{N,+, \lambda} \left( X_{N_2} > TH \mid |\Delta_\rho| < \Delta \right) \\ &\quad + \mathbb{P}_{N,+, \lambda} (|\Delta_\rho| \geq \Delta). \end{aligned} \quad (3.13)$$

To estimate the length of the droplet, rewrite

$$\begin{aligned} \mathbb{P}_{N,+, \lambda} (|\Delta_\rho| \geq \Delta) &= \sum_{l, r} \mathbb{P}_{N,+, \lambda} (\Delta_\rho = (l, r)) \\ &= \sum_{l, r} \sum_{0 \leq a, b \leq \rho H} \mathbb{P}_{N,+, \lambda} (X_l = a, X_r = b) \\ &\quad \times \mathbb{P}_{(l, r), +, \lambda}^{a, b} (\mathcal{A}_\rho(l, r)), \end{aligned} \quad (3.14)$$

where the first summation goes over all  $l, r$  satisfying

$$0 \leq l < N_2 < r \leq N, \quad r - l - 1 \geq \Delta. \quad (3.15)$$

Next, by convexity of  $V(\cdot)$  and the bounded growth assumption (1.2),

$$V(\rho H) \geq \frac{\rho}{2} V(2H) = \frac{\rho}{2\lambda H^2},$$

$$\zeta V(2H_\rho) + (1 - \zeta)V(2\tilde{H}) \leq \left[ \zeta f(2) + 2(1 - \zeta)(1 - 2\eta) \right] V(H_\rho) \leq \frac{1}{2} V(H_\rho)$$

where  $\tilde{H}$  is as in (3.1)  $H_\rho = \rho H(\lambda)$ ,  $\rho \geq 2$ , and the constants  $\zeta, \eta$  are chosen via

$$\zeta = \frac{1}{4f(2)}, \quad \eta = \frac{7}{16}.$$

Further, applying Lemma 3.1, we obtain (cf. Corollary 2.3)

$$\mathbb{P}_{(l, r), +, \lambda}^{a, b} (\mathcal{A}_\rho(l, r)) \leq C_1 \exp \left\{ \left[ \frac{C}{\rho^2(1 - 2\eta)^2} - \frac{\lambda V(H_\rho) H^2}{2} \right] \frac{r - l - 1}{H^2} \right\}$$

$$\begin{aligned} & \times \mathbf{P}_{(l,r),+,0}^{a,b}(\mathcal{A}_\rho(l,r)) \\ & \leq C_1 \exp\left\{-\left[1 - \frac{256C}{\rho^3}\right] \frac{\rho\Delta}{4H^2}\right\} \mathbf{P}_{(l,r),+,0}^{a,b}(\mathcal{A}_\rho(l,r)). \end{aligned} \quad (3.16)$$

With  $\Delta, \alpha \in (0, 1)$  and  $T > 0$  satisfying

$$\Delta = \sqrt{T}H^2, \quad \alpha T = \rho \geq \rho_0 = 8 \cdot C^{1/3}, \quad (3.17)$$

where  $C$  denotes the same constant as in Lemma 3.1, the last bound reads

$$\mathbf{P}_{(l,r),+,\lambda}^{a,b}(\mathcal{A}_\rho(l,r)) \leq C_1 \exp\left\{-\frac{\alpha}{8} T^{3/2}\right\} \mathbf{P}_{(l,r),+,0}^{a,b}(\mathcal{A}_\rho(l,r)).$$

Inserting it into (3.14) we immediately get

$$\mathbf{P}_{N,+,\lambda}(|\Delta_\rho| \geq \Delta) \leq C_1 \exp\left\{-\frac{\alpha}{8} T^{3/2}\right\}. \quad (3.18)$$

It remains to estimate the first term in the RHS of (3.13). Conditioning on the endpoints of the droplet of interest, we decompose

$$\begin{aligned} & \mathbf{P}_{N,+,\lambda}(X_{N_2} > TH \mid |\Delta_\rho| < \Delta) \\ & \leq \sum_{l,r} \mathbf{P}_{N,+,\lambda}(l_\rho = l, \mathfrak{r}_\rho = r \mid |\Delta_\rho| < \Delta) \\ & \quad \times \sum_{0 \leq a,b \leq H_\rho} \mathbf{P}_{N,+,\lambda}(X_l = a, X_r = b \mid l_\rho = l, \mathfrak{r}_\rho = r) \\ & \quad \times \mathbf{P}_{(l,r),+,\lambda}^{a,b}(X_{N_2} > TH \mid \mathcal{A}_\rho(l,r)), \end{aligned} \quad (3.19)$$

where the first summation goes over all  $l, r$  satisfying (cf. (3.15))

$$0 \leq l < N_2 < r \leq N, \quad r - l - 1 < \Delta.$$

To finish the proof of the lemma it remains to establish the following inequality

$$\mathbf{P}_{(l,r),+,\lambda}^{a,b}(X_{N_2} > TH \mid \mathcal{A}_\rho(l,r)) \leq \frac{1}{C_2} e^{-C_2 T^{3/2}}. \quad (3.20)$$

Notice that taking  $T_0$  large enough, we can achieve the bound

$$\frac{\Delta}{(\rho H)^2} \leq \frac{\sqrt{T}H^2}{(\alpha TH)^2} \leq \frac{1}{\alpha^2 T_0^{3/2}} \leq \zeta$$

for all  $T \geq T_0$ , where  $\zeta$  is the same constant as in Lemma 3.2, and thus can remove the field  $\lambda$  from our further considerations.

To prove (3.20), we shall proceed as in the proof of Lemma 3.2. Namely, defining  $\Delta' \subset (l, r)$  and  $\mathcal{A}_2$  similarly to (3.5) and (3.6), (see Fig. 5), we bound above

$$\mathbf{P}_{(l,r),+,0}^{a,b}(X_{N_2} > TH \mid \mathcal{A}_\rho(l,r)) \leq \mathbf{P}_{(l,r),+,0}^{a,b}(\overline{\mathcal{A}}_2) + \mathbf{P}_{(l,r),+,0}^{a,b}(X_{N_2} > TH \mid \mathcal{A}_2)$$

and decompose (cf. (3.9))

$$\begin{aligned}
 P_{(l,r),+,0}^{a,b}(X_{N_2} > TH \mid \mathcal{A}_2) &= \sum_{l \leq l' \leq r' \leq r} P_{(l,r),+,0}^{a,b}(l' = l', r' = r' \mid \mathcal{A}_2) \\
 &\quad \times \sum_{2H_\rho \leq a', b' \leq 2H_\rho + L} P_{(l',r'),+,0}^{a',b'}(X_{N_2} > TH) \\
 &\quad \times P_{(l',r'),+,0}^{a',b'}(X_{l'} = a', X_{r'} = b' \mid l' = l', r' = r').
 \end{aligned} \tag{3.9}$$

We shall estimate the probability  $P_{(l',r'),+,0}^{a',b'}(X_{N_2} > TH)$  depending on the length of the interval  $\Delta'$ . First, we observe that for any  $L > 0$  the Donsker invariance principle gives (recall (3.10))

$$\begin{aligned}
 P_{(l',r'),+,0}^{a',b'}(X_{N_2} > TH) &\leq \frac{P_{|\Delta'|}(\max S_j > (T - 2\rho)H \mid S_{|\Delta'|} = b' - a')}{1 - P_{|\Delta'|}(\min S_j < -\rho H \mid S_{|\Delta'|} = b' - a')} \\
 &\leq C_3 \exp\left\{-C_4 \frac{(T - 2\rho)^2}{\sqrt{T}}\right\} = C_3 e^{-C_4(1-2\alpha)^2 T^{3/2}}
 \end{aligned}$$

uniformly in  $H^{7/6} \leq |\Delta'| \leq \Delta = \sqrt{T}H^2$ , in  $\{a', b'\} \subset [2H_\rho, 2H_\rho + L]$  and all  $\lambda$  small enough.

On the other hand, for  $|\Delta'| \leq H^{7/6}$ , the conditional Chebyshev inequality for the maximum (see Lemma A.1 in the appendix below) gives

$$P_{(l',r'),+,0}^{a',b'}(X_{N_2} > TH) \leq \frac{\text{const}_L |\Delta'|^{3/2} (T - 2\rho)^{-2} H^{-2}}{1 - \text{const}_L |\Delta'| \rho^{-2} H^{-2}} \leq C_5 \frac{\text{const}_L}{T^2 H^{1/4}}.$$

Now, combining the last two estimates with the bound (3.7), we obtain:

$$P_{(l,r),+,0}^{a,b}(X_{N_2} > TH \mid \mathcal{A}_2) \leq \frac{2\tilde{\sigma}^2}{L^2} + C_3 e^{-C_4(1-2\alpha)^2 T^{3/2}} + C_5 \frac{\text{const}_L}{T^2 H^{1/4}}. \tag{3.21}$$

To finish the proof, we first take  $L = \exp\left\{\frac{C_4}{2}(1-2\alpha)^2 T^{3/2}\right\}$  and then  $\lambda$  sufficiently small to make the last term smaller than the second. With this choice (3.21) reduces to (3.20).  $\square$

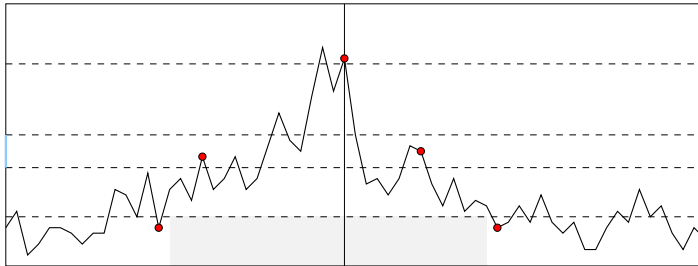


Fig. 5. Decompositions (3.19) and (3.9')

An obvious generalization of the argument above gives also the following bound.

**Corollary 3.4.** *Let  $\rho$  and  $\varepsilon$  denote some (small) positive constants and let the integers  $l_0, r_0, M \in [0, N]$  be such that*

$$l_0 < M < r_0 \quad \text{and} \quad \min(|M - l_0|, |M - r_0|) \geq \varepsilon H^2$$

*with  $H = H(\lambda)$  being our canonical scale from (1.7). Then there exists  $c_1 > 0$  such that for any  $T > 0$  large enough and all  $a_0, b_0 \in [0, 2\rho H]$  the inequality*

$$\mathbf{P}_{(l_0, r_0), +, \lambda}^{a_0, b_0}(X_M > TH \mid \mathcal{A}_{2\rho}(l_0, r_0)) \leq e^{-c_1 T^{3/2}}$$

*holds for all  $\lambda \in (0, \lambda_0]$ , where  $\lambda_0 = \lambda_0(T) > 0$ .*

As a straightforward modification of the proofs above one can show existence of moments of  $X_{N_2}$  to be used below.

**Corollary 3.5.** *There exist positive constants  $K$  and  $\lambda_0$  such that for all  $p, 1 < p < 21/8$ , we have*

$$\mathbf{P}_{N, +, \lambda}(X_{N_2})^{2p} \leq C(p) H^{2p+1},$$

*uniformly in  $\lambda \in (0, \lambda_0]$  and  $N \geq KH^2$ .*

*Proof.* Using the decomposition (3.14) with  $\rho = T/4$  and  $\Delta = H^2 T^{-9/10}$  in (3.17), we get the following analogue of (3.18):

$$\mathbf{P}_{N, +, \lambda}(|\Delta_\rho| \geq \Delta) \leq C_1 \exp\left\{-\frac{1}{32} T^{1/10}\right\}.$$

Next, we use the decomposition (3.9') and estimate above the height of the inner droplet,

$$\mathbf{P}_{(l', r'), +, 0}^{a', b'}(X_{N_2} > TH).$$

As in the proof of Lemma A.3 two cases to be considered separately,  $|r' - l'| \leq m_0$  and  $m_0 \leq |r' - l'| \leq \Delta$ . Clearly, w.l.o.g. we may and shall assume that  $m_0$  is chosen large enough to satisfy (cf. (A.2))

$$\mathbf{P}(S_m = d_m) \geq \frac{1}{2e\sqrt{2\pi\sigma^2 m}}$$

for all  $m \geq m_0$ .

Let  $|r' - l'| \leq m_0$ . Combining (A.6) and (A.3), we get

$$\mathbf{P}_{(l', r'), +, 0}^{a', b'}(X_{N_2} > TH) \leq \frac{4\sigma^4 m_0^4}{p(m_0, D)} (TH)^{-4}.$$

On the other hand, for  $(l', r')$  satisfying  $m_0 \leq |r' - l'| \leq \Delta$ , we apply Lemma A.2 to obtain

$$\mathbf{P}_{(l', r'), +, 0}^{a', b'}(X_{N_2} > TH) \leq \frac{C\sigma|r' - l'|^{5/2}}{(TH)^4}$$

with a numeric constant  $C \leq 18$ . As a result, for any  $T > 0$  we get

$$\mathbf{P}_{N,+,\lambda}(X_{N_2} > TH) \leq C_1 \exp\left\{-\frac{1}{32} T^{1/10}\right\} + \frac{C_2(m_0, D)}{(TH)^4} + \frac{C_3 H}{T^{25/4}},$$

and therefore, for  $1 < p < 21/8$ ,

$$\begin{aligned} \mathbf{P}_{N,+,\lambda}(X_{N_2})^{2p} &\leq H^{2p} \left(1 + \sum_{T>0} (T+1)^{2p} \mathbf{P}_{N,+,\lambda}[TH \leq X_i < (T+1)H]\right) \\ &\leq C_4(p) H^{2p+1}. \end{aligned}$$

□

### 3.3. The lower bound

Fix a (big) positive  $T$  and an integer  $\Delta = \sqrt{T}H^2$  and denote (recall the notation  $N_2 = \lfloor N/2 \rfloor$ )

$$l = N_2 - \Delta, \quad r = N_2 + \Delta.$$

It follows from the argument in Sect. 3.2 that for some constant  $T_0 > 0$  not depending on  $\lambda > 0$  we have

$$\mathbf{P}_{N,+,\lambda}(\{X_l, X_r\} \subset [0, T_0 H]) \geq \frac{1}{2} \quad (3.22)$$

(recall the running assumption that we omit the boundary conditions  $a = b = 0$  from the notation). We thus rewrite

$$\begin{aligned} \mathbf{P}_{N,+,\lambda}(X_{N_2} > TH) &\geq \sum_{0 \leq a, b \leq T_0 H} \mathbf{P}_{N,+,\lambda}(X_l = a, X_r = b) \\ &\quad \times \mathbf{P}_{(l,r),+,\lambda}^{a,b}(X_{N_2} > TH). \end{aligned} \quad (3.23)$$

Now, put

$$l' = l + H^2, \quad r' = r - H^2, \quad X_{l'} = a', \quad X_{r'} = b'$$

and estimate

$$\begin{aligned} \mathbf{P}_{(l,r),+,\lambda}^{a,b}(X_{N_2} > TH) &\geq \sum_{H/4 \leq a', b' \leq 3H/4} \mathbf{P}_{(l',r'),+,\lambda}^{a,b}(X_{l'} = a', X_{r'} = b') \\ &\quad \times \mathbf{P}_{(l',r'),+,\lambda}^{a',b'}(X_{N_2} > TH). \end{aligned} \quad (3.24)$$

By the Donsker invariance principle and the estimates for the maximum of the Brownian bridge, the last factor is bounded below by

$$\frac{1}{c} \exp\{-c(TH)^2/(r' - l')\} \geq \frac{1}{c'} \exp\{-c'T^{3/2}\}$$

uniformly in  $a'$  and  $b'$  under consideration, provided only  $\lambda > 0$  is small enough. Next, a literal repetition of the proof of (2.7) and (2.9) combined with the estimate (3.24) gives

$$P_{(l,r),+,\lambda}^{a,b}(X_{N_2} > TH) \geq \frac{1}{c''} \exp\{-c''T^{3/2}\} \quad (3.25)$$

uniformly in  $a, b$  from  $[0, T_0H]$  provided only  $\lambda > 0$  is sufficiently small. The lower bound in (1.10) now follows from (3.22), (3.23), and (3.25).  $\square$

#### 4. Refined asymptotics: Proofs of Theorems 1.4 and 1.5

In Subsection 4.1, we analyze in quite some details the quasirenewal structure of visits of the random-walk to a strip of width of order  $H(\lambda)$  along the wall, showing in particular that they occur with a positive frequency on length-scale of order  $H(\lambda)^2$ . In Subsection 4.2, we use these results to show that the paths of two independent copies of our process intersect with positive frequency on length-scale of order  $H(\lambda)^2$ . Finally, we deduce the claims of Theorems 1.4 and 1.5 in Subsections 4.3 and 4.4 respectively.

##### 4.1. Quasirenewal structure

The importance of the scale  $H = H(\lambda)$  demonstrated in the proofs of the previous sections is even more pronounced in the study of the refined behaviour of the interfaces under consideration. The aim of this section is to describe certain intrinsic renewal-type structure of the random walks distributed according to (1.3) that manifests itself in the diffusing scaling (i.e.,  $H(\lambda)^2$  in the horizontal direction and  $H(\lambda)$  in the vertical one).

For any  $\rho > 0$  and  $\lambda > 0$ , let  $S_\rho$  denote the horizontal strip of width  $4\rho H$ ,

$$S_\rho = \left\{ (x, y) \in \mathbb{Z}^2 : y \in [0, 4\rho H] \right\}.$$

In this section we shall establish certain quasirenewal property stating roughly that for all  $\lambda > 0$  small enough the “density” of visits of the RW under consideration to the strip  $S_\rho$  is “positive on the scale  $H(\lambda)^2$ ”.

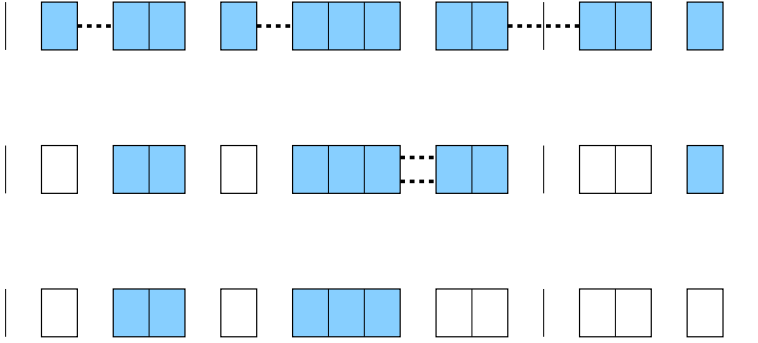
More precisely, for positive real  $\varepsilon, \rho, \lambda$  and integer  $K > 0$  we split our trajectories into  $K$ -blocks  $\mathcal{I}_m$  of length  $K\varepsilon H(\lambda)^2$  (assuming w.l.o.g.  $\varepsilon H(\lambda)^2$  to be integer) and introduce the random variables

$$Y_m = Y_m^\rho = \mathbb{1}_{\{\min_{j \in \mathcal{I}_m} X_j > 2\rho H\}}. \quad (4.1)$$

Let  $n_K = \lceil N/(K\varepsilon H(\lambda)^2) \rceil$  be the total number of such blocks and let  $n_Y$  be the total number of  $K$ -blocks labelled by ones:

$$n_Y = \left| \{m : Y_m = 1\} \right|.$$

Our first observation is that with high probability the total length  $n_Y K\varepsilon H(\lambda)^2$  of such  $K$ -blocks is small:



**Fig. 6.** Two-step selection procedure: from 7 clusters of total length 12 choose 2 clusters of total length 5; selected  $K$ -clusters are shadowed

**Lemma 4.1.** *Let  $f(\cdot)$  be defined as in (1.2),  $\varepsilon$  be a positive constant, and  $\alpha$  satisfy  $\alpha \in (0, 1)$ . For any  $\rho > 0$  there exist positive constants  $\lambda_0$ , and  $K_0$ , depending on  $\alpha$ ,  $\varepsilon$ , and  $\rho$  only, such that for any  $\lambda \in (0, \lambda_0)$ , any  $K \geq K_0$  and any  $N \geq 3K_0\varepsilon H(\lambda)^2$  we have*

$$\mathbb{P}_{N,+, \lambda}^{a,b}(n_Y \geq \alpha n_K) \leq \exp\left\{-\frac{\alpha}{16f(2/\rho)}NH(\lambda)^{-2}\right\} \quad (4.2)$$

uniformly in  $a, b \in (0, \rho H)$ .

*Proof of Lemma 4.1.* We use the blocking procedure described at the beginning of this section where, given  $\rho > 0$  and  $\varepsilon > 0$ , the constant  $K_0 \geq 8$  is chosen large enough.

The  $Y$ -labels defined in (4.1) with any  $K \geq K_0$  introduce a 0 – 1 encoding of each trajectory; using this encoding, we split the  $K$ -blocks labelled by ones into maximal “connected components” to be called  $K$ -clusters. Two neighbouring  $K$ -clusters are called connected if they are separated by exactly one  $K$ -block labelled by a zero. A  $K$ -cluster that is not connected to its neighbours is called isolated. Our next goal is to show that for any collection of  $K$ -clusters consisting of  $n_Y$   $K$ -blocks there is a sub-collection of isolated  $K$ -clusters of total length at least  $\lceil n_Y/4 \rceil$ . As soon as this is done, a simple reduction argument will imply the target estimate (4.2).

Our selection procedure consists of two steps. First, we split all  $K$ -clusters into subsequent pairs of neighbouring clusters and from each such pair we choose the longest cluster (or the left one if they are of equal length). Observe that each chosen cluster is either isolated or belongs to a pair of connected  $K$ -clusters. Next, we split all isolated  $K$ -clusters into subsequent pairs<sup>5</sup> and from each pair (connected or isolated) we choose the longest cluster (or the left one if they are of equal length). The obtained collection (together with the temporarily neglected  $K$ -cluster, if there was one) consists of isolated clusters which altogether contain at least  $n_Y/4$  of  $K$ -blocks, see Fig. 6.

<sup>5</sup> temporarily neglecting the very last  $K$ -cluster, if their total number is odd

Our second step relies upon a finer renormalisation, this time on the integer scale  $\varepsilon H(\lambda)^2$ . We split our trajectory into  $n_\varepsilon = \lceil N/\varepsilon H(\lambda)^2 \rceil$  blocks  $\mathcal{J}_l$  of length  $\varepsilon H(\lambda)^2$  each and similarly to (4.1) introduce the labels

$$Z_l = Z_l^\rho = \mathbb{1}_{\{\min_{j \in \mathcal{J}_l} X_j > 2\rho H\}}.$$

Of course, the natural (inclusion) correspondence between  $\varepsilon$ -blocks  $\mathcal{J}_l$  and  $K$ -blocks  $\mathcal{I}_m$ ,

$$\mathcal{J}_l \subset \mathcal{I}_m,$$

has the following property: if  $Y_m = 1$  and  $\mathcal{J}_l$  corresponds to  $\mathcal{I}_m$ , then  $Z_l = 1$ . As before, we split all  $\varepsilon$ -blocks labelled by ones into maximal connected components to be called  $\varepsilon$ -clusters. Clearly, as subsets of  $\{1, \dots, N\}$ , every  $K$ -cluster is included in the corresponding  $\varepsilon$ -cluster. Let  $\mathcal{E}$  be the collection of  $\varepsilon$ -clusters that correspond to the isolated  $K$ -clusters selected by applying a procedure as in Fig. 6. The following two properties of the collection  $\mathcal{E}$  will be important for our future application: 1) every  $\varepsilon$ -cluster from  $\mathcal{E}$  is bounded by two boundary  $\varepsilon$ -blocks labelled by zeroes; moreover, for different  $\varepsilon$ -clusters the boundary blocks are different; 2) the total length of  $\varepsilon$ -clusters from  $\mathcal{E}$  is at least  $Kn_Y \varepsilon H(\lambda)^2/4$ .

Our next aim is to establish certain one-droplet estimate from which the target bound (4.2) will follow immediately. Consider any  $\varepsilon$ -cluster from  $\mathcal{E}$  and denote its extremal  $\varepsilon$ -blocks (the first and the last one) by  $\mathcal{J}_{m_1}$  and  $\mathcal{J}_{m_2}$  respectively. Clearly, the length of this  $\varepsilon$ -cluster is  $n_\varepsilon^0 \varepsilon H^2 \equiv (m_2 - m_1 + 1) \varepsilon H^2$ ,  $n_\varepsilon^0 \geq K$ . Further, define

$$\begin{aligned} l &= \max \left\{ j \in \mathcal{I}_{m_1-1} : X_j < 2\rho H \right\}, & X_l &= a, \\ r &= \min \left\{ j \in \mathcal{I}_{m_2+1} : X_j < 2\rho H \right\}, & X_r &= b. \end{aligned}$$

Similarly, let

$$l_0 = m_1 \varepsilon H^2, \quad X_{l_0} = a_0, \quad r_0 = (m_2 - 1) \varepsilon H^2, \quad X_{r_0} = b_0. \quad (4.3)$$

Using the notation  $\mathcal{A}_2 = \mathcal{A}_{2\rho}(l_0, r_0)$  (recall (3.12)), let  $Z_{(l,r),+,\lambda}^{a,b}(\mathcal{A}_2)$  denote the partition function corresponding to this droplet, i.e. the restriction of the partition function  $Z_{(l,r),+,\lambda}^{a,b}$  to the set of trajectories belonging to  $\mathcal{A}_2$ . This restricted partition function can be bounded by

$$Z_{(l,r),+,\lambda}^{a,b}(\mathcal{A}_2) = \sum_{a_0, b_0 \geq 2\rho H} Z_{(l,l_0),+,\lambda}^{a,a_0} Z_{(l_0,r_0),+,\lambda}^{a_0,b_0}(\mathcal{A}_2) Z_{(r_0,r),+,\lambda}^{b_0,b}. \quad (4.4)$$

Clearly, the target inequality (4.2) follows immediately from the one-droplet bound

$$\sup_{\rho H \leq a_0, b_0 \leq 2\rho H} \frac{Z_{(l_0,r_0),+,\lambda}^{a_0,b_0}(\mathcal{A}_2)}{Z_{(l_0,r_0),+,\lambda}^{a_0,b_0}} \leq \exp \left\{ -\frac{\varepsilon n_\varepsilon^0}{2f(2/\rho)} \right\}, \quad (4.5)$$

the lower bound on the total length of  $\varepsilon$ -clusters from  $\mathcal{E}$ , provided only

$$K\varepsilon\alpha > 16f(2/\rho) \log 2,$$

to suppress the total number of  $0-1$  encodings (that is bounded above by  $2^{n_K}$ ).

Our proof of (4.5) will be based upon the decomposition (4.4) and the following two facts:

F1) there is  $\lambda_0 = \lambda_0(\varepsilon, \rho, \dots) > 0$  for which: for any  $\eta > 0$  there exists  $T > 0$  such that uniformly in  $\lambda \in (0, \lambda_0]$  one has (recall (4.3))

$$Z_{(l,r),+,\lambda}^{a,b}(\mathcal{A}_2, a_0 \leq 2TH, b_0 \leq 2TH) \geq (1 - \eta) Z_{(l,r),+,\lambda}^{a,b}(\mathcal{A}_2) \quad (4.6)$$

F2) for  $\varepsilon, \rho, T$  and  $\lambda_0$  as above there is a finite constant  $M > 0$  such that, uniformly in  $\lambda \in (0, \lambda_0]$ ,

$$\sup \frac{Z_{\Delta,+,\lambda}^{a,a_0}}{Z_{\Delta,+,\lambda}^{a,a'_0}} \leq M \quad (4.7)$$

with supremum taken over  $\varepsilon H^2 \leq \Delta \leq 2\varepsilon H^2$ ,  $\rho H \leq a \leq 2\rho H$ , and  $2\rho H \leq a_0 \leq (2T + \rho)H$ ,  $a'_0 \equiv a_0 - \rho H$ ; a similar estimate (with the same constant  $M$ ) holds for the ratio  $Z_{\Delta,+,\lambda}^{a_0,a}/Z_{\Delta,+,\lambda}^{a'_0,a}$ .

The inequality (4.5) follows easily from (4.6) and (4.7). Indeed, combining (4.4) and (4.6) we get

$$\begin{aligned} Z_{(l,r),+,\lambda}^{a,b}(\mathcal{A}_2) &\leq \frac{1}{1 - \eta} Z_{(l,r),+,\lambda}^{a,b}(\mathcal{A}_2, a_0 \leq 2TH, b_0 \leq 2TH) \\ &\leq \frac{1}{1 - \eta} \sum Z_{(l,l_0),+,\lambda}^{a,a_0} Z_{(l_0,r_0),+,\lambda}^{a_0,b_0}(\mathcal{A}_2) Z_{(r_0,r),+,\lambda}^{b_0,b} \end{aligned}$$

with the sum running over  $2\rho H \leq a_0, b_0 \leq (2T + \rho)H$ . Further, denoting  $\mathcal{A}_1 = \mathcal{A}_\rho(l_0, r_0)$ , taking  $\eta = 1/2$ , and using the estimate (4.7), the convexity of the function  $V(\cdot)$  and the reduction of the central part of the droplet as in Fig. 7, we bound the last expression by

$$\begin{aligned} 2M^2 e^{-\lambda n_\varepsilon^0 \varepsilon H^2 V(\rho H)} \sum Z_{(l,l_0),+,\lambda}^{a,a_0} Z_{(l_0,r_0),+,\lambda}^{a_0,b_0}(\mathcal{A}_1) Z_{(r_0,r),+,\lambda}^{b_0,b} \\ \leq 2M^2 e^{-\varepsilon(n_\varepsilon^0 - 2)/f(2/\rho)} Z_{(l,r),+,\lambda}^{a,b} \end{aligned}$$

with sum running over  $\rho H \leq a_0, b_0 \leq 2TH$ . Finally, taking  $K_0 \geq 8$  such that

$$2M^2 \leq \exp\{K_0 \varepsilon / [4f(2/\rho)]\} \quad (4.8)$$

we immediately get

$$Z_{(l,r),+,\lambda}^{a,b}(\mathcal{A}_2) \leq \exp\left\{-\frac{n_\varepsilon^0 \varepsilon}{2f(2/\rho)}\right\} Z_{(l,r),+,\lambda}^{a,b}$$

and thus (4.5). It remains to verify (4.6) and (4.7).

The proof of (4.6) follows the argument of Sect. 3.2. Clearly, it is enough to show that for some constant  $c_1 > 0$  one has

$$Z_{(l,r),+,\lambda}^{a,b}(\mathcal{A}_2, a_0 > 2TH) \leq \exp\{-c_1 T^{3/2}\} Z_{(l,r),+,\lambda}^{a,b}(\mathcal{A}_2).$$

In view of the definitions (3.11) and (3.12), the partition function in the LHS of the previous display is bounded above by

$$\sum Z_{(l,l'),+, \lambda}^{a,a'} Z_{(l',r'),+, \lambda}^{a',b'} (\mathcal{A}_{2\rho}(l', r'), a_0 > 2TH) Z_{(r',r),+, \lambda}^{b',b},$$

where the sum runs over  $0 \leq a', b' \leq 2\rho H$  and  $l', r'$  such that (recall (3.11))

$$(l_0, r_0) \subset (l', r') \subseteq (l, r) \quad \text{and} \quad \Delta_{2\rho} = (l', r').$$

Due to Corollary 3.4, the sum above can be further majorated by

$$e^{-c_1 T^{3/2}} \sum Z_{(l,l'),+, \lambda}^{a,a'} Z_{(l',r'),+, \lambda}^{a',b'} (\mathcal{A}_{2\rho}(l', r')) Z_{(r',r),+, \lambda}^{b',b} \leq e^{-c_1 T^{3/2}} Z_{(l,r),+, \lambda}^{a,b} (\mathcal{A}_2).$$

As a result,

$$Z_{(l,r),+, \lambda}^{a,b} (\mathcal{A}_2, a_0 \leq 2TH, b_0 \leq 2TH) \geq (1 - 2e^{-c_1 T^{3/2}}) Z_{(l,r),+, \lambda}^{a,b} (\mathcal{A}_2)$$

and it remains to choose  $T$  large enough. The estimate (4.6) follows.

Finally, we check (4.7). First, applying an obvious extension of Lemma 2.1, we remove the field  $\lambda$  (as above, we put  $b' = b - \rho H$ ): for some  $M_1 = M_1(\varepsilon, \rho, T)$ ,

$$\frac{Z_{\Delta,+, \lambda}^{a,b}}{Z_{\Delta,+, \lambda}^{a,b'}} \leq M_1 \frac{Z_{\Delta,+, 0}^{a,b}}{Z_{\Delta,+, 0}^{a,b'}} = M_1 \frac{\mathbb{P}_{\Delta,+, 0}^a(X_{\Delta} = b)}{\mathbb{P}_{\Delta,+, 0}^a(X_{\Delta} = b')} \quad (4.9)$$

uniformly in  $a, b, \Delta$  under consideration and all  $\lambda > 0$  small enough. Here and below,  $\mathbb{P}_{\Delta,+, 0}^a(\cdot)$  denotes the probability distribution of the  $\Delta$ -step random walk

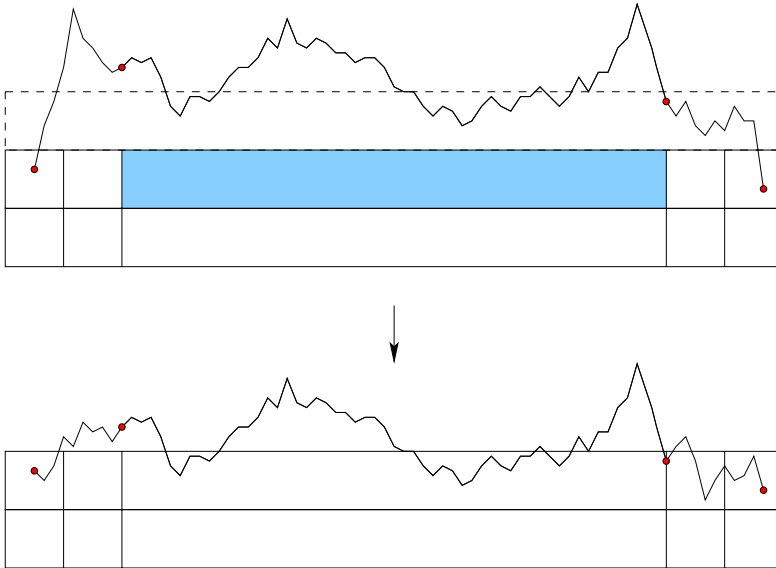


Fig. 7. Reduction of an  $\varepsilon$ -cluster

starting from  $a$  with transition probabilities  $p(\cdot)$  restricted to the set  $\mathcal{I}_{\Delta,+}$  of non-negative trajectories (recall (1.5)).

Now, denoting by  $\geq 0$  the wall constraint  $\mathbb{X} \in \mathcal{I}_{\Delta,+}$ , we rewrite the last ratio as

$$\frac{\mathbb{P}_{\Delta}^a(X_{\Delta} = b \mid \geq 0)}{\mathbb{P}_{\Delta}^a(X_{\Delta} = b' \mid \geq 0)} = \frac{\mathbb{P}_{\Delta}^a(X_{\Delta} = b, \geq 0)}{\mathbb{P}_{\Delta}^a(X_{\Delta} = b', \geq 0)} \quad (4.10)$$

and observe that uniformly in  $a, \Delta$  under consideration the  $\mathbb{P}_{\Delta}^a$ -probability of the event  $\geq 0$  is uniformly positive. Thus, applying the standard argument (see, e.g., [2, §11], [3, §9]) one deduces that, uniformly in  $a, b, \Delta$ , and  $\lambda$  under consideration, the ratio in (4.10) is bounded above by a positive constant  $M_2 = M_2(\varepsilon, \rho, \Delta, T)$ . The estimate (4.7) follows from (4.9) and (4.10).

The proof of the lemma is finished.  $\square$

Next, we fix  $\rho, K$ , and  $\varepsilon$  as in the proof above, use the  $K$ -blocks decomposition and introduce the labels (cf. (4.1))

$$U_m = U_m^{\rho}(X) = \mathbb{1}_{\{\max_{j \in \mathcal{I}_m} X_j < 4\rho H\}}. \quad (4.11)$$

Denote  $n_U = |\{m : U_m = 1\}|$ .

**Lemma 4.2.** *For any  $\rho > 0$  there exist positive constants  $\lambda_0, \gamma_0, K_0, c$ , and  $C$  such that for any  $\lambda \in (0, \lambda_0)$ ,  $\gamma \in (0, \gamma_0)$ ,  $K \geq K_0$  and all  $N \geq 3KH(\lambda)^2$  we have*

$$\mathbb{P}_{N,+, \lambda}^{a,b}(n_U \leq \gamma n_K) \leq C \exp\left\{-cNH(\lambda)^{-2}\right\} \quad (4.12)$$

uniformly in  $a, b \in (0, 2\rho H(\lambda))$ .

*Proof.* Our argument is similar to that of Sect 2.3. First, taking  $K = 3K_0$  in Lemma 4.1, we split all  $K$ -blocks into triples of consecutive blocks (neglecting the non-complete last triple if there is one) and call an index  $m$  regular if (recall (4.1))

$$Y_{3m+1} = Y_{3m+3} = 0.$$

Using the previous Lemma with  $\alpha = 1/9$ , we deduce that with probability not smaller than  $1 - \exp\{-N/(144f(2/\rho)H(\lambda)^2)\}$  there are at least  $2n_K/9$  regular indices  $m$ .

For each such  $m$  we define

$$\begin{aligned} l &= \max\left\{j \in \mathcal{I}_{3m+1} : X_j \leq 2\rho H(\lambda)\right\}, & X_l &= a, \\ r &= \min\left\{j \in \mathcal{I}_{3m+3} : X_j \leq 2\rho H(\lambda)\right\}, & X_r &= b. \end{aligned}$$

Now,  $K\varepsilon H(\lambda)^2 = 3K_0\varepsilon H(\lambda)^2 \leq r - l \leq 3K\varepsilon H(\lambda)^2$  and thus, using Corollary 2.4 we get

$$\begin{aligned} \mathbb{P}_{(l,r),+, \lambda}^{a,b}(U_{3m+2} = 1) &\geq ce^{-C(r-l)/(2\rho H(\lambda))^2} \\ &\geq ce^{-3CK\varepsilon/(4\rho^2)} =: p_{\rho, K\varepsilon} > 0. \end{aligned} \quad (4.13)$$

Therefore, on average there are at least  $2p_{\rho, K\varepsilon}n_K/9$  indices  $m$  whose labels satisfy  $U_{3m+2} = 1$ . By a standard large deviation bound we get

$$\mathbb{P}_{N,+, \lambda} \left( n_U < \frac{p_{\rho, K\varepsilon}}{9} n_K \right) \leq C e^{-c n_K}$$

and the lemma is proved.  $\square$

#### 4.2. Coupling

With  $\rho > 0$  fixed as above and  $\lambda > 0$  denote (recall (1.7))

$$H = H(\rho, \lambda) := 4\rho H(\lambda). \quad (4.14)$$

Let  $X, Y$  be two independent trajectories of our process and let  $\mathbb{P}_{N,+, \lambda}^{a_x, b_x, a_y, b_y}(\cdot, \cdot)$  denote their joint distribution (recall (1.5)–(1.6)),

$$\mathbb{P}_{N,+, \lambda}^{a_x, b_x, a_y, b_y}(\cdot, \cdot) = \mathbb{P}_{N,+, \lambda}^{a_x, b_x}(\cdot) \otimes \mathbb{P}_{N,+, \lambda}^{a_y, b_y}(\cdot)$$

with the shorthand notation  $\mathbb{P}_{N,+, \lambda}(\cdot)$  if  $a_x = b_x = a_y = b_y = 0$ . Everywhere in this section we shall consider only boundary conditions satisfying  $0 \leq a_x, b_x, a_y, b_y \leq H$ . For a set of indices  $A \subseteq [0, N] \cap \mathbb{Z}$ , let

$$\mathcal{N}_A = \mathcal{N}_A(X, Y) = \left\{ \forall j \in A, X_j \neq Y_j \right\}$$

be the event “trajectories  $X$  and  $Y$  do not intersect within the set  $A$ ”. Our main observation is that, with probability going to one, any two independent trajectories of our RW meet within a time interval of order at most  $H^2 = O(H(\lambda)^2)$ :

**Lemma 4.3.** *There exist positive constants  $\lambda_0, C, c$ , and  $\rho_0$  such that the inequality*

$$\mathbb{P}_{N,+, \lambda}(\mathcal{N}_{(0, N)}) \leq C e^{-cN/H^2} \quad (4.15)$$

*holds uniformly in  $0 < \lambda \leq \lambda_0, 0 < \rho \leq \rho_0$  and  $N \geq H^2$ .*

*Proof.* Consider the decomposition into  $K$ -blocks  $\mathcal{I}_m$  of length  $K\varepsilon H(\lambda)^2$  described in the previous section and denote

$$n_U^2 = \left| \left\{ m : U_m(X) = U_m(Y) = 1 \right\} \right|$$

with labels  $U_m(\cdot)$  defined as in (4.11). Following the proof of Lemma 4.2 with  $\alpha = 1/9$  we deduce that with probability not smaller than

$$1 - 2 \exp \left\{ - \frac{N}{144 f(2/\rho) H(\lambda)^2} \right\}$$

there are on average at least  $(1/3 - 2\alpha)n_K = n_K/9$  regular indices  $m$  that are common for both  $X$  and  $Y$ . As a result (recall (4.13)),

$$\mathbb{P}_{N,+, \lambda} \left( n_U^2 < \frac{(p_{\rho, K})^2}{18} n_K \right) \leq C_1 e^{-c_1 n_K} \leq C_1 e^{-c_2 N/H^2}, \quad (4.16)$$

that is, with high probability there is a positive fraction of blocks  $\mathcal{I}_m$  for which the event

$$D_m = \left\{ U_m(X) = U_m(Y) = 1 \right\} = \left\{ \forall j \in \mathcal{I}_m, 0 \leq X_j, Y_j \leq H \right\} \quad (4.17)$$

is realized. By taking each second such  $K$ -block we construct a disjoint collection  $\mathcal{K}$  of  $K$ -blocks possessing property (4.17). The collection  $\mathcal{K}$  has the following important properties to be used in the sequel:

- 1) with probability at least  $1 - C_1 \exp\{-c_2 N H^{-2}\}$ , there are no less than  $p_{\rho,K}^2 n_K / 36$  blocks in  $\mathcal{K}$ ;
- 2) conditioned on  $\{U_m\}$  and on the configuration in the complement of  $K$ -blocks from  $\mathcal{K}$ , distributions inside individual  $K$ -blocks are independent.

Another important ingredient of our argument is the following observation:

**Lemma 4.4.** *Let  $H = H(\rho, \lambda)$  be as in (4.14) and let  $D = D_N(X) \cap D_N(Y)$ , where*

$$D_N(Z) = \left\{ 0 \leq Z_j \leq H, \forall j = 1, \dots, N-1 \right\}. \quad (4.18)$$

*Then there exist positive constants  $\lambda_0$  and  $c_3$  such that the estimate*

$$\max_{a_x, b_x, a_y, b_y} \mathbb{P}_{N,+,0}^{a_x, b_x, a_y, b_y} \left( \mathcal{N}_{(0,N)} \mid D \right) \leq e^{-c_3 N H^{-2}}, \quad (4.19)$$

*holds uniformly in  $0 < \lambda \leq \lambda_0$ , in  $N \geq H^2$  and in boundary conditions  $0 \leq a_x, b_x, a_y, b_y \leq H$ .*

We postpone the proof of Lemma 4.4 till the end of this section and deduce our main estimate (4.15) first. Combining Corollary 2.4 with the inequality (4.19) and using the estimate (with  $2\rho \leq 1$ )

$$\lambda V(H)N \leq \frac{32\rho^3 N}{H^2}$$

we obtain the uniform bound (similarly as in (2.14))

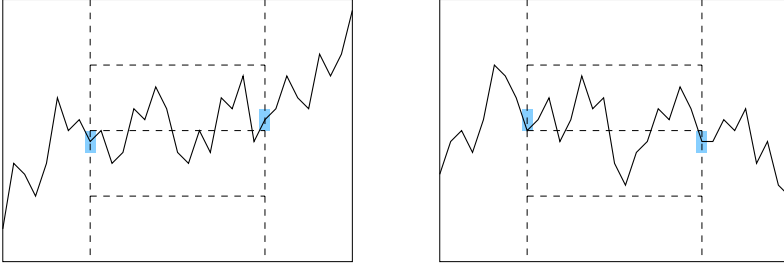
$$\max_{a_x, b_x, a_y, b_y} \mathbb{P}_{N,+, \lambda}^{a_x, b_x, a_y, b_y} \left( \mathcal{N}_{(0,N)} \mid D \right) \leq e^{-(c_3 - 64\rho^3) N H^{-2}} \leq e^{-c_3 N / 2H^2} \quad (4.20)$$

provided only  $0 < \rho \leq \rho_0$  with  $128\rho_0^2 < c_3$ . Now, using the bound

$$\begin{aligned} \mathbb{P}_{N,+, \lambda} \left( \mathcal{N}_{(0,N)} \right) &\leq \mathbb{P}_{N,+, \lambda} \left( n_U^2 < \frac{(p_{\rho,K})^2}{18} n_K \right) \\ &\quad + \mathbb{P}_{N,+, \lambda} \left( \mathcal{N}_{(0,N)} \mid |\mathcal{K}| \geq \frac{(p_{\rho,K})^2}{36} n_K \right), \end{aligned}$$

“freezing” the joint configuration  $(X, Y)_{\mathcal{K}^c}$  in all blocks that do not belong to the collection  $\mathcal{K}$  and using the estimate (4.20) for all blocks from  $\mathcal{K}$ , we bound the last term by

$$\max_{(X,Y)_{\mathcal{K}^c}} \mathbb{P}_{N,+, \lambda} \left( \mathcal{N}_{(0,N)} \mid (X, Y)_{\mathcal{K}^c}, |\mathcal{K}| \geq \frac{(p_{\rho,K})^2}{36} n_K \right)$$



**Fig. 8.** Crossing events  $\mathcal{A}_m^{\nearrow}$  (left) and  $\mathcal{A}_m^{\searrow}$  (right)

$$\leq \exp\left\{-\frac{c_3|\mathcal{I}_m|}{2H^2} \cdot \frac{(p_{\rho,K})^2}{36} n_K\right\} \leq e^{-c_4 NH^{-2}}. \quad (4.21)$$

Averaging this inequality over  $(X, Y)_{K^c}$  and combining the result with (4.16), we obtain the target estimate (4.15).  $\square$

*Proof of Lemma 4.4.* We use again a blocking argument. Fix two positive constants  $\varepsilon$  and  $\Delta$  such that  $\varepsilon \leq 1$ ,  $\Delta \leq 1/8$  and assume w.l.o.g. that  $\varepsilon H^2$  is integer. Split the interval  $[0, N]$  into blocks  $\mathcal{I}_m$  of length  $4\varepsilon H^2$ ,  $m = 1, 2, \dots, \lfloor N/(4\varepsilon H^2) \rfloor =: n_\varepsilon$ , and for each such block introduce the “crossing event”

$$\mathcal{A}_m(X, Y) = \mathcal{A}_m^{\nearrow}(X) \cap \mathcal{A}_m^{\searrow}(Y),$$

where (see Fig. 8)

$$\begin{aligned} \mathcal{A}_m^{\nearrow}(Z) &= D_m(Z) \cap \widehat{D}_m^o(Z) \cap \mathcal{B}_m^{\nearrow}(Z), \\ \mathcal{A}_m^{\searrow}(Z) &= D_m(Z) \cap \widehat{D}_m^o(Z) \cap \mathcal{B}_m^{\searrow}(Z) \end{aligned} \quad (4.22)$$

with  $D_m = D_{\mathcal{I}_m}$  defined as in (4.17),  $\widehat{D}_m^o = \widehat{D}_{\mathcal{I}_m^o}$  given by

$$\begin{aligned} \widehat{D}_J(Z) &= \left\{ \frac{1}{4}H \leq Z_j \leq \frac{3}{4}H, \forall j \in J \right\}, \quad \forall J \subseteq [0, N], \\ \mathcal{I}_m^o &= \left\{ j \in \mathcal{I}_m : (4m-3)\varepsilon H^2 < j < (4m-1)\varepsilon H^2 \right\} \end{aligned}$$

and, finally,

$$\begin{aligned} \mathcal{B}_m^{\nearrow}(Z) &= \left\{ Z_{(4m-3)\varepsilon H^2} \in C_{\Delta}^-, Z_{(4m-1)\varepsilon H^2} \in C_{\Delta}^+ \right\}, \\ \mathcal{B}_m^{\searrow}(Z) &= \left\{ Z_{(4m-3)\varepsilon H^2} \in C_{\Delta}^+, Z_{(4m-1)\varepsilon H^2} \in C_{\Delta}^- \right\} \end{aligned}$$

with

$$C_{\Delta}^+ = \left[ \frac{1}{2}H, \frac{1+\Delta}{2}H \right], \quad C_{\Delta}^- = \left[ \frac{1-\Delta}{2}H, \frac{1}{2}H \right].$$

Our argument below is based upon the following three facts to be checked below:

F1) conditioned on the “boundary” values

$$\{(X_j, Y_j), j = 4m\varepsilon H^2, m = 1, 2, \dots, n_\varepsilon\},$$

the trajectories  $X$  and  $Y$  behave independently in different blocks;

F2) for all  $\mathcal{I}_m$ , uniformly in “boundary” values, the crossing event  $\mathcal{A}_m(X, Y)$  occurs with positive probability:

$$\min_{a_x, b_x, a_y, b_y} \mathbb{P}_{\mathcal{I}_m, +, 0}^{a_x, b_x, a_y, b_y}(\mathcal{A}_m(X, Y) \mid D_m) \geq p_1 > 0 \quad (4.23)$$

F3) conditioned on  $\mathcal{A}_m(X, Y)$ , the trajectories  $X$  and  $Y$  intersect in  $\mathcal{I}_m^o$  with positive probability:

$$\min_{a_x, b_x, a_y, b_y} \mathbb{P}_{\mathcal{I}_m, +, 0}^{a_x, b_x, a_y, b_y}(\overline{\mathcal{N}}_{\mathcal{I}_m^o} \mid \mathcal{A}_m(X, Y)) \geq p_2 > 0 \quad (4.24)$$

where  $\overline{\mathcal{N}}_{\mathcal{I}_m^o}$  denotes the intersection event in the central part of the block  $\mathcal{I}_m$ :

$$\overline{\mathcal{N}}_{\mathcal{I}_m^o} = \left\{ \exists j \in \mathcal{I}_m^o : X_j = Y_j \in \left[ \frac{1}{8}H, \frac{7}{8}H \right] \right\}.$$

Indeed, let  $\mathcal{A}$  be the collection of indices  $m$  for which the event  $\mathcal{A}_m(X, Y)$  occurs. Denoting by  $n_{\text{cross}}$  the cardinality of  $|\mathcal{A}|$ , we use the Markov property and the standard large deviation bound to get

$$\max_{a_x, b_x, a_y, b_y} \mathbb{P}_{N, +, 0}^{a_x, b_x, a_y, b_y} \left( n_{\text{cross}} < \frac{p_1}{2} n_\varepsilon \mid D_m \right) \leq e^{-c_5 n_\varepsilon} \quad (4.25)$$

with some  $c_5 > 0$ ; thus, it remains to consider the case  $n_{\text{cross}} = |\mathcal{A}| \geq p_1 n_\varepsilon / 2$ . Now, the bound (4.24) and the conditional independence of blocks  $\mathcal{I}_m$  imply that

$$\begin{aligned} \mathbb{P}_{N, +, 0}^{a_x, b_x, a_y, b_y}(\mathcal{N}_{(0, N)} \mid \mathcal{A}, D) &\leq \prod_{m \in \mathcal{A}} \mathbb{P}_{\mathcal{I}_m, +, 0}^{a_x^m, b_x^m, a_y^m, b_y^m}(\mathcal{N}_{\mathcal{I}_m^o} \mid \mathcal{A}_m(X, Y), D_m) \\ &\leq (1 - p_2)^{|\mathcal{A}|} \end{aligned}$$

uniformly in all such  $\mathcal{A}$  and boundary conditions  $a_x^m, b_x^m, a_y^m, b_y^m$  thus giving the uniform upper bound

$$(1 - p_2)^{p_1 n_\varepsilon / 2} \leq e^{-c_6 n_\varepsilon}. \quad (4.26)$$

Finally, (4.19) follows immediately from (4.25) and (4.26).

Thus, it remains to check the properties F1)–F3). The Markov property F1) being obvious, we need only to prove the inequalities (4.23) and (4.24).

To check (4.23) we proceed as in the proof of Lemma 2.1. By conditional independence, it is enough to show that for some  $p_3 > 0$  one has

$$\begin{aligned} \min_{a_x, b_x} \mathbb{P}_{\mathcal{I}_m, +, 0}^{a_x, b_x}(\mathcal{A}_m^{\nearrow}(X) \mid D_m) &\geq p_3, \\ \min_{a_y, b_y} \mathbb{P}_{\mathcal{I}_m, +, 0}^{a_y, b_y}(\mathcal{A}_m^{\searrow}(Y) \mid D_m) &\geq p_3. \end{aligned}$$

We shall verify the first of these inequalities, the second follows from analogous considerations.

First, for any boundary conditions  $a, b$  we rewrite (recall (4.22))

$$\begin{aligned} P_{\mathcal{I}_m, +, 0}^{a, b}(\mathcal{A}_m^\nearrow \mid D_m) &= P_{\mathcal{I}_m, +, 0}^{a, b}(\widehat{D}_m^o \cap \mathcal{B}_m^\nearrow \mid D_m) \\ &= P_{\mathcal{I}_m, +, 0}^{a, b}(\widehat{D}_m^o \mid \mathcal{B}_m^\nearrow \cap D_m) \cdot P_{\mathcal{I}_m, +, 0}^{a, b}(\mathcal{B}_m^\nearrow \mid D_m). \end{aligned}$$

Next, using the Markov property and the usual invariance principle (for unconditioned RWs), we get

$$\begin{aligned} &P_{\mathcal{I}_m, +, 0}^{a, b}(\widehat{D}_m^o \mid \mathcal{B}_m^\nearrow \cap D_m) \\ &\geq \min_{x \in C_\Delta^-, y \in C_\Delta^+} \left[ P_{2\varepsilon H^2}^{x, y}(\widehat{D}_m^o) - P_{2\varepsilon H^2}^{x, y}(\overline{D}_{\mathcal{I}_m^o}) \right] \\ &\geq \min_{x \in C_\Delta^-, y \in C_\Delta^+} P_{2\varepsilon H^2}^{x, y} \left( \max |X_j - x| \leq (1 - 2\Delta)H/4 \right) \\ &\quad - \max_{x \in C_\Delta^-, y \in C_\Delta^+} P_{2\varepsilon H^2}^{x, y} \left( \max |X_j - x| > (1 - \Delta)H/2 \right) \geq p_4 \end{aligned}$$

where,  $\overline{D}_{\mathcal{I}_m^o}$  denotes the complement of the event  $D_{\mathcal{I}_m^o}$  (cf. (4.17)),

$$D_{\mathcal{I}_m^o} = \left\{ \forall j \in \mathcal{I}_m^o, 0 \leq X_j, Y_j < 4\rho H \right\},$$

and for fixed  $\Delta \in (0, 1/8]$ , the bound  $p_4 = p_4(\varepsilon, \Delta)$  is positive uniformly in all  $\lambda$  and  $\varepsilon$  small enough. On the other hand, a slight modification of the proof of Lemma 2.1 implies the inequality

$$P_{\mathcal{I}_m, +, 0}^{a, b}(\mathcal{B}_m^\nearrow \mid D_m) \geq p_5$$

with some  $p_5 = p_5(\varepsilon, \Delta) > 0$  uniformly in all  $\lambda$  small enough. The estimate (4.23) follows.

Finally, we verify the inequality (4.24). Morally, our argument is based upon the following observation: on the event  $\mathcal{A}_m(X, Y)$  there is  $j \in \mathcal{I}_m^o$  satisfying

$$X_j - Y_j \leq 0 \leq X_{j+1} - Y_{j+1}; \quad (4.27)$$

since  $X$  and  $Y$  are independent processes whose jumps have the same distribution of finite variance  $\sigma^2 > 0$ , the bound

$$\min \left\{ |X_j - Y_j|, |X_{j+1} - Y_{j+1}| \right\} \leq R\sigma \quad (4.28)$$

holds with positive probability provided only the absolute constant  $R > 0$  is chosen large enough; finally, thanks to the aperiodicity property (1.4), conditioned on the event (4.28), the trajectories  $X$  and  $Y$  meet with positive probability within  $A[R\sigma]$  steps.

We sketch the main steps of the argument. In view of the Markov property, it is sufficient to show that

$$\min_{x \in C_\Delta^-, y \in C_\Delta^+} P_{\mathcal{I}_m^o}^{x, y} \left( \exists j \in \mathcal{I}_m^o, |X_j - Y_j| \leq R\sigma \mid \widehat{D}_m^o \right)$$

is uniformly positive for all  $\lambda$  small enough. Clearly, the minimum above is bounded below by the expression

$$\Pr\left(|\xi - \eta| \leq R\sigma\right) - 2 \max \mathbb{P}_{\mathcal{I}_m^o}^{x,y} \left( \max_{j \in \mathcal{I}_m^o} |X_j - x| > \frac{1 - 2\Delta}{4} H \right),$$

where  $\xi$  and  $\eta$  are i.i.d. r.v. with the basic distribution  $p(\cdot)$  and max is taken over all  $x \in C_{\Delta}^-, y \in C_{\Delta}^+$ . As  $R \rightarrow \infty$ , the first term approaches 1, whereas the second vanishes asymptotically as  $\varepsilon \rightarrow 0$ , uniformly in  $\Delta \in (0, 1/8]$  and in all  $\lambda > 0$  small enough. Thus, for some  $p_6 = p_6(\varepsilon, \Delta, R) > 0$  we get

$$\min_{a_x^m, b_x^m, a_y^m, b_y^m} \mathbb{P}_{\mathcal{I}_m, +, 0}^{a_x^m, b_x^m, a_y^m, b_y^m} \left( \exists j \in \mathcal{I}_m^o, |X_j - Y_j| \leq R\sigma \mid \widehat{D}_m^o \right) \geq p_6$$

and thanks to the aperiodicity property (1.4) the trajectories  $X$  and  $Y$  have a positive probability to meet within the time interval  $J_0 = [j_0, j_0 + A \lceil R\sigma \rceil]$ :

$$\mathbb{P}_{\mathcal{I}_m, +, 0}^{a_x^m, b_x^m, a_y^m, b_y^m} \left( \exists j \in J_0, X_j = Y_j \in [0, H] \mid \mathcal{A}_m(X, Y), |X_{j_0} - Y_{j_0}| \leq R\sigma \right) \geq p_7$$

with some  $p_7 > 0$ , uniformly in boundary conditions  $0 \leq a_x^m, b_x^m, a_y^m, b_y^m \leq H$  and in positive  $\lambda$  small enough. This implies the estimate (4.24).

The proof of the lemma is complete.  $\square$

#### 4.3. Inverse correlation length: Proof of Theorem 1.4

Let  $X$  be our RW and  $Y$  its independent copy; we have:

$$\text{Cov}(X_i, X_j) = \frac{1}{2} \mathbb{E}_{+, \lambda} [(X_i - Y_i)(X_j - Y_j)], \quad (4.29)$$

where  $\mathbb{E}_{+, \lambda}$  is the expectation w.r.t. the limiting measure and  $\text{Cov}$  is the corresponding covariance; denote by  $A$  the event that both RW's  $X$  and  $Y$  intersect between  $i$  and  $j$ . According to the above, the probability of the complement  $\bar{A}$  of  $A$  is bounded above by the RHS of (4.15):

$$\mathbb{P}_{+, \lambda}(\bar{A}) \leq C \exp\{-c|i - j|H^{-2}\}.$$

Moreover, by symmetry of the RHS of (4.29) on the event  $A$ , we have

$$\mathbb{E}_{+, \lambda} [(X_i - Y_i)(X_j - Y_j)\mathbb{1}_A] = 0.$$

Consequently, for any  $p > 1$ ,

$$\begin{aligned} 2\text{Cov}(X_i, X_j) &= \mathbb{E}_{+, \lambda} [(X_i - Y_i)(X_j - Y_j)\mathbb{1}_{\bar{A}}] \\ &\leq \mathbb{E}_{+, \lambda} [(X_i X_j + Y_i Y_j)\mathbb{1}_{\bar{A}}] \\ &\leq 2(\mathbb{E}_{+, \lambda} [(X_i X_j)^p])^{1/p} (\mathbb{P}_{+, \lambda}(\bar{A}))^{(p-1)/p}. \end{aligned}$$

However, for any  $p$ ,  $1 < p < 21/8$  we get (recall Corollary 3.5)

$$\mathbb{E}_{+, \lambda} (X_{N_2})^{2p} \leq C(p) H^{2p+1},$$

and thus, by the Cauchy-Schwarz inequality,

$$\mathbf{E}_{+, \lambda}[(X_i X_j)^p] \leq \left[ \mathbf{E}_{+, \lambda}((X_i)^{2p}) \mathbf{E}_{+, \lambda}((X_j)^{2p}) \right]^{1/2} \leq C H^{2p+1}$$

leading to

$$\text{Cov}(X_i, X_j) \leq C H(\lambda)^{2+1/p} \exp\{-c|i-j|H^{-2}\}.$$

Finally, take  $p = 2$ . □

#### 4.4. Relaxation to equilibrium: Proof of Theorem 1.5

It is an immediate corollary of (4.15): just consider the initial RW and another one started at equilibrium. By the coupling inequality [8, pg. 12] the total variance distance between the distribution of our RW after  $N$  steps and the equilibrium measure is bounded above by the LHS of (4.15).

#### A. Small droplet bound

Our aim here is to prove the small droplet bound—Lemma 3.3. The key step of our argument will be based upon the following, having an independent interest, conditional Chebyshev inequality for maximum.

**Lemma A.1.** *Let  $S_0 = 0$ ,  $S_k = \xi_1 + \dots + \xi_k$ ,  $k \geq 1$ , be the random walk generated by a sequence  $\xi_1, \xi_2, \dots$  of i.i.d. random variables such that  $\mathbf{E}\xi = 0$ ,  $\mathbf{E}\xi^2 = \sigma^2 < \infty$ . Let  $D > 0$  be an arbitrary constant and, for any  $m \geq 1$ , let  $d_m$  satisfy  $\mathbf{P}(S_m = d_m) > 0$  and  $|d_m| \leq D$ . Then there exists a positive constant  $c = c(D)$  such that the inequality*

$$\mathbf{P}\left(\max_{0 \leq k \leq m} S_k > M \mid S_m = d_m\right) \leq c \frac{m^{3/2}}{M^2} \quad (\text{A.1})$$

holds for all  $m \geq 2$ .

*Proof.* Since  $D > 0$  is a finite constant, the local limit theorem [6] implies that for some  $c_1 = c_1(D) > 0$

$$\mathbf{P}(S_m = d_m) \geq \frac{c_1}{\sqrt{2\pi\sigma^2 m}} \exp\left\{-\frac{(d_m)^2}{2\sigma^2 m}\right\} \quad (\text{A.2})$$

uniformly in  $m \geq 1$  and  $|d_m| \leq D$ .

On the other hand, by the Etemadi (see, e.g., [3, pg. 256]) and Chebyshev inequalities,

$$\mathbf{P}\left(\max_{0 \leq k \leq m} S_k > M\right) \leq 3 \max_{0 \leq k \leq m} \mathbf{P}(S_k > M/3) \leq 3 \max_{0 \leq k \leq m} \frac{\sigma^2 k}{(M/3)^2} = \frac{27\sigma^2 m}{M^2}.$$

The target bound (A.1) follows immediately from the last two displays and the assumption  $|d_m| \leq D$ . □

*Proof of Lemma 3.3.* Let first  $m \leq M^{7/6}$ . Then, the Conditional Chebyshev inequality (A.1) gives

$$\mathbf{P}\left(\max_{0 < k < m} S_k > M \mid S_m = d_m\right) \leq C_1 \frac{m^{3/2}}{M^2} \leq \frac{C_1}{M^{1/4}}.$$

Let now  $m$  satisfy  $M^{7/6} \leq m \leq \zeta M^2$  (and thus  $m \rightarrow \infty$ ). Since  $D > 0$  is finite, it follows from the main result in [7] that

$$\mathbf{P}\left(\max_{0 < k < m} S_k > M \mid S_m = d_m\right) \leq C_2 \exp\left\{-C_3 \frac{M^2}{m}\right\} \leq C_2 \exp\left\{-\frac{C_3}{\zeta}\right\}$$

if only  $M$  is large enough,  $M \geq M_0$ . The small droplet bound (3.4) follows, provided  $\zeta > 0$  is chosen small enough.  $\square$

Next, we present a simple one-point analogue of Lemma A.1.

**Lemma A.2.** *Under the conditions of Lemma A.1, there is a positive constant  $\bar{c}$  depending on  $D$  and the distribution of  $\xi$  only, such that*

$$\max_{0 < k < m} \mathbf{P}(S_k > M + D \mid S_m = d_m) \leq \bar{c} \frac{m^{5/2}}{M^4}$$

for all  $m \geq 2$ .

*Proof.* Using the independence of jumps and the Chebyshev inequality, we get

$$\begin{aligned} \mathbf{P}(S_k > M + D, S_m = d_m) &\leq \mathbf{P}(S_k > M) \mathbf{P}(S_m - S_k < -M) \\ &\leq \frac{k(m-k)}{M^4} \sigma^4 \leq \frac{m^2}{4M^4} \sigma^4. \end{aligned} \quad (\text{A.3})$$

Combining this estimate with the lower bound (A.2), we deduce the result.  $\square$

Finally, we present a stronger (for large  $m$ ) version of the previous claim.

**Lemma A.3.** *Under the conditions of Lemma A.1, there is a positive constant  $\bar{c}$  depending on  $D$  and the distribution of  $\xi$  only, such that*

$$\max_{0 < k < m} \mathbf{P}(S_k > M + D \mid S_m = d_m) \leq \bar{c} \frac{m}{M^2}.$$

*Proof.* We start by observing that if  $\xi_1, \xi_2, \dots$ , are i.i.d. random variables and  $d_m$  is chosen such that  $\mathbf{P}(S_m = d_m) > 0$ , then the variables  $\eta_j$  defined via  $\eta_j = (\xi_j \mid S_m = d_m)$  are exchangeable. As a result [2, §24], for any  $k$ ,  $1 \leq k \leq m$ ,

$$\begin{aligned} \mathbf{E}(S_k \mid S_m = d_m) &= k \mathbf{E}(\xi_1 \mid S_m = d_m) = \frac{k d_m}{m}, \\ \mathbf{Var}(S_k \mid S_m = d_m) &= \frac{k(m-k)}{m-1} \mathbf{Var}(\xi_1 \mid S_m = d_m). \end{aligned} \quad (\text{A.4})$$

Our next observation formalizes an intuitively obvious fact that for large  $m$  the variable  $\xi_k$  becomes asymptotically independent of  $S_m$  and thus the variances of  $\eta_1$

and  $\xi_1$  are close to each other. We shall restrict ourselves to the case of integer-valued variables  $\xi$  having zero mean and the variance  $\mathbf{E} \xi^2 = \sigma^2$ .

For any finite  $D > 0$  there exists  $m_0$ , depending only on  $D$  and the distribution of  $\xi$ , such that the inequality

$$\mathbf{E}(\xi_1^2 \mid S_m = d) \leq 4\mathbf{E} \xi_1^2 = 4\sigma^2 \quad (\text{A.5})$$

holds uniformly in  $m \geq m_0$  and  $|d| \leq D$ .

To check (A.5), we observe that the characteristic function of  $\eta = (\xi \mid S_m = d)$  equals

$$\mathbf{E}(e^{is\xi} \mid S_m = d) = \frac{\int \phi(t+s)\phi^{m-1}(t)e^{-itd} dt}{\int \phi^m(t)e^{-itd} dt},$$

where  $\phi(t)$  is the unconditional characteristic function of  $\xi$ ,  $\phi(t) = \mathbf{E}e^{it\xi}$ , and the integration goes over an interval of periodicity of  $\phi(t)$ . Consequently,

$$\mathbf{E}(\xi^2 \mid S_m = d) = \frac{-\int \phi''(t)\phi^{m-1}(t)e^{-itd} dt}{\int \phi^m(t)e^{-itd} dt}.$$

According to (A.2), we have

$$\int \phi^m(t)e^{-itd} dt = \mathbf{P}(S_m = d) \geq \frac{1}{2\sqrt{2\pi\sigma^2 m}} \exp\left\{-\frac{d^2}{2\sigma^2 m}\right\}$$

uniformly in  $|d| \leq D$  and all  $m \geq m_1$  with  $m_1$  large enough. Analogously, applying the standard Laplace method to the integral in the numerator (see, e.g., [5, 12]), we get

$$\left| \int \phi''(t)\phi^{m-1}(t)e^{-itd} dt \right| \leq \frac{2}{\sqrt{2\pi\sigma^2 m}} \exp\left\{-\frac{d^2}{2\sigma^2 m}\right\} \mathbf{E} \xi^2,$$

uniformly in  $|d| \leq D$  and all  $m \geq m_2$  with  $m_2$  large enough. The bound (A.5) follows from the last two displays.

Next, we combine (A.4) and (A.5) to deduce that, uniformly in  $|d| \leq D$  and all  $m \geq m_0$  with  $m_0$  large enough, the inequality

$$\mathbf{E}(S_k^2 \mid S_m = d) \leq \left(\frac{kd}{m}\right)^2 + \frac{k(m-k)}{m-1} 4\mathbf{E} \xi^2 \leq d^2 + \frac{m^2}{m-1} \mathbf{E} \xi^2$$

holds for all  $k \in \{1, \dots, m\}$ . By Chebyshev,

$$\mathbf{P}(S_k > M + D \mid S_m = d_m) \leq \frac{C_1 m}{M^2}$$

for all such  $m$ .

It remains to consider  $m \leq m_0$ . Denoting

$$p(m_0, D) = \min_{m \leq m_0, |d| \leq D} \left\{ \mathbf{P}(S_m = d) : \mathbf{P}(S_m = d) > 0 \right\} > 0, \quad (\text{A.6})$$

we immediately get, via Chebyshev,

$$\mathbf{P}(S_k > M + D \mid S_m = d_m) \leq \frac{\mathbf{P}(S_k > M + D)}{p(m_0, D)} \leq \frac{C_2 m}{M^2}.$$

The proof is finished.  $\square$

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