Probab. Theory Relat. Fields 129, 340–380 (2004)
Digital Object Identifier (DOI) 10.1007/s00440-004-0343-1

Dapeng Zhan

Stochastic Loewner evolution in doubly connected domains

Received: 12 October 2003 / Revised version: 19 January 2004 /
Published online: 25 March 2004 – © Springer-Verlag 2004

Abstract. This paper introduces the annulus SLE_κ processes in doubly connected domains. Annulus SLE_6 has the same law as stopped radial SLE_6, up to a time-change. For $\kappa \neq 6$, some weak equivalence relation exists between annulus SLE_κ and radial SLE_κ. Annulus SLE_2 is the scaling limit of the corresponding loop-erased conditional random walk, which implies that a certain form of SLE_2 satisfies the reversibility property. We also consider the disc SLE_κ process defined as a limiting case of the annulus SLE's. Disc SLE_6 has the same law as stopped full plane SLE_6, up to a time-change. Disc SLE_2 is the scaling limit of loop-erased random walk, and is the reversal of radial SLE_2.

1. Introduction

Stochastic Loewner evolution (SLE), introduced by O. Schramm in [16], is a family of random growth processes of plane sets in simply connected domains. The evolution is described by the classical Loewner differential equation with the driving term being a one-dimensional Brownian motion. SLE depends on a parameter $\kappa > 0$, the speed of the Brownian motion, and behaves differently for different value of κ. See [15] by S. Rohde and O. Schramm for the basic fundamental properties of SLE.

Schramm's processes turned out to be very useful. On the one hand, they are amenable to computations, on the other hand, they are related with some statistical physics models. In a series of papers [5]-[9], G. F. Lawler, O. Schramm and W. Werner used SLE to determine the Brownian motion intersection exponents in the plane, identified SLE_2 and SLE_8 with the scaling limits of LERW and UST Peano curve, respectively, and conjectured that $SLE_{8/3}$ is the scaling limit of SAW. S. Smirnov proved in [17] that SLE_6 is the scaling limit of critical site percolation on the triangular lattice.

For various reasons, a similar theory should also exist for multiply connected domains and even for general Riemann surfaces. We expect that the definition and some study of general SLE will give us better understanding of SLE itself and its physics background. The definition of SLE in simply connected domains uses the fact that the complement of SLE stopped at a finite time in a simply connected domain other than $\mathbb{C}$ is still simply connected, so it is conformally equivalent to

D. Zhan: Department of Mathematics, Mail code: 253-37, California Institute of Technology, Pasadena, CA 91125, USA. e-mail: dapeng@its.caltech.edu

the whole domain. But this property does not hold for general domains. That is the main difficulty in our definition of general SLE.

As a start, we consider SLE in the most simple non-simply connected domains: doubly connected domains. We show that the corresponding processes, the annulus SLE_κ, have features similar to those in the simply connected case. More specifically, we prove that annulus SLE_6 has locality property; and for all $\kappa > 0$, annulus SLE_κ is equivalent to radial SLE_κ. We also justify this definition by proving that annulus SLE_2 is the scaling limit of the corresponding loop-erased conditional random walk.

After these, we define disc SLE in simply connected domains, which is the limit case of annulus SLE. Disc SLE_6 also has locality property, so its final hull has the same law as the hull generated by a plane Brownian motion stopped on hitting the boundary. Disc SLE_2 is the scaling limit of the corresponding loop-erased random walk. It then follows that disc SLE_2 is the reversal of radial SLE_2 started from a random point on the boundary with harmonic measure.

1.1. SLE in simply connected domains

For $\kappa \geq 0$, the standard radial SLE_κ is obtained by solving the Loewner differential equations:

$$\partial_t \varphi_t(z) = \varphi_t(z) \frac{1 + \varphi_t(z)/\chi_t}{1 - \varphi_t(z)/\chi_t}, \quad 0 \leq t < \infty, \quad \varphi_0(z) = z$$

where

$$\chi_t = \exp(i B(\kappa t)),$$

and $B(t)$ is a standard Brownian motion on $\mathbb{R}$ started from 0. Let K_t be the set of points z in $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ such that the solution $\varphi_s(z)$ blows up before or at time t. Then $D_t := \mathbb{D} \setminus K_t$ is a simply connected domain, $0 \in D_t$, and φ_t maps D_t conformally onto $\mathbb{D}$ with $\varphi_t(0) = 0$ and $\varphi_t'(0) = e^t$. The family of hulls $(K_t, 0 \leq t < \infty)$ grows in $\mathbb{D}$ from 1 to 0, and is called the standard radial SLE_κ. If Ω is a simply connected domain (other than $\mathbb{C}$), a a prime end, $b \in \Omega$, then $\mathrm{SLE}_\kappa(\Omega; a \to b)$, radial SLE_κ in Ω from a to b, is defined as the image of the standard radial SLE_κ under the conformal map $(\mathbb{D}; 1, 0) \to (\Omega, a, b)$. By construction, radial SLE is conformally invariant.

Suppose (K_t) is a radial $\mathrm{SLE}_\kappa(\Omega; a \to b)$. Then for any fixed $s \geq 0$, the law of a certain conformal image of $(K_{s+t} \setminus K_s)$ is the same as the law of (K_t), and is independent of $(K_r)_{0 \leq r \leq s}$. In other words, radial SLE_κ has "i.i.d." increments, in the sense of conformal equivalence. This property, together with the symmetry of the law in $(\mathbb{D}; 1, 0)$ w.r.t. complex conjugation, characterizes radial SLE up to κ.

Chordal SLE_κ processes are defined in a similar way. In this case, the family of hulls (K_t) grows in a simply connected domain from one boundary point (prime end) to another. Once again, the properties of conformal invariance, "i.i.d." increments, and the corresponding symmetry property determine a one-parameter family of such processes.

Radial SLE and chordal SLE are equivalent in the following sense. Suppose Ω is a simply connected domain, a and c are two distinct prime ends, and $b \in \Omega$. For a fixed $\kappa > 0$, let (K_t) be a radial $\mathrm{SLE}_\kappa(\Omega; a \to b)$ and (L_s) a chordal $\mathrm{SLE}_\kappa(\Omega; a \to c)$. Let T be the first time that K_t swallows c, S the first time that L_s swallows b. We set T or S to be ∞ by convention if the corresponding hitting time does not exist. If $\kappa = 6$, up to a time-change, the law of $(K_t)_{0\le t\le T}$ is the same as the law of $(L_s)_{0\le s\le S}$. If $\kappa \neq 6$, there exist two sequences of stopping times $\{T_n\}$ and $\{S_n\}$ such that $T = \vee_n T_n$, $S = \vee_n S_n$, and for each $n \in \mathbb{N}$, up to a time-change, the laws of $(K_t)_{0\le t\le T_n}$ and $(L_s)_{0\le s\le S_n}$ are equivalent. In other words, they have positive density w.r.t. each other. The strong equivalence relation of radial and chordal SLE_6 is related to the so-called locality property: the SLE_6 hulls do not feel the boundary before hitting it.

The equivalence property ensures that for the same κ, radial SLE_κ and chordal SLE_κ behave similarly. For instance, if $\kappa \le 4$, and (K_t) is a radial or chordal SLE_κ in Ω, then a.s. there is a simple path $\beta : (0,\infty) \to \Omega$ such that for any $t \in [0,\infty)$, we have $K_t = \beta(0,t]$. If $\kappa > 4$ and $\partial\Omega$ is locally connected, then a.s. there is a non-simple path $\beta : (0,\infty) \to \overline{\Omega}$ such that for any $t \in [0,\infty)$, K_t is the hull generated by $\beta(0,t]$. This path β is called the SLE_κ trace.

Full plane SLE_κ: $(K_t, -\infty < t < \infty)$ grows in $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ from 0 to ∞. For any fixed $s \in \mathbb{R}$, the law of a certain conformal image of $(K_{s+t} \setminus K_s)$ is the same as the law of the standard radial SLE_κ, and is independent of $(K_r)_{-\infty<r\le s}$. Full plane SLE can be viewed as the limit of radial $\mathrm{SLE}_\kappa(\widehat{\mathbb{C}} \setminus \varepsilon\overline{\mathbb{D}}; \varepsilon \to \infty)$ as $\varepsilon \to 0^+$.

1.2. Definition of annulus SLE

For $p > 0$, we denote by $\mathbf{A}_p$ the standard annulus of modulus p:

$$\mathbf{A}_p = \{z \in \mathbb{C} : e^{-p} < |z| < 1\}.$$

Every doubly connected domain D with non-degenerate boundary is conformally equivalent to a unique $\mathbf{A}_p$, and $p = M(D)$ is the modulus of D. We may first define SLE on the standard annuli, and then extend the definition to arbitrary doubly connected domains via conformal maps.

Denote

$$\mathbf{S}_p(z) = \lim_{N\to\infty} \sum_{-N}^{N} \frac{e^{2kp} + z}{e^{2kp} - z}.$$

For $\chi \in \partial D$, let

$$\mathbf{S}_p(\chi, z) = \mathbf{S}_p(z/\chi).$$

The function $\mathbf{S}_p(\chi,\cdot)$ is a Schwarz kernel of $\mathbf{A}_p$ in the sense that if f is an analytic function in $\mathbf{A}_p$, continuous up to the boundary, and constant on the circle $\mathbf{C}_p := \{z \in \mathbb{C} : |z| = e^{-p}\}$, then for any $z \in \mathbf{A}_p$,

$$f(z) = \int_{\mathbf{C}_0} f(\chi)\mathbf{S}_p(\chi, z)d\mathbf{m} + iC,$$

where $\mathbf{m}$ is the uniform probability measure on $\mathbf{C}_0 = \partial\mathbb{D}$, and C is some real constant. Note that the Schwarz kernels are not unique. The choice of $\mathbf{S}_p(\chi, \cdot)$ here satisfies the rotation symmetry and reflection symmetry.

Let $\chi : [0, p) \to \mathbf{C}_0$ be a continuous function. Consider the following Loewner-type differential equation:

$$\partial_t \varphi_t(z) = \varphi_t(z)\mathbf{S}_{p-t}(\chi_t, \varphi_t(z)),\ 0 \le t < p,\ \ \varphi_0(z) = z. \tag{1.1}$$

For $0 \le t < p$, let K_t be the set of $z \in \mathbf{A}_p$ such that the solution $\varphi_s(z)$ blows up before or at time t. Let $D_t = \mathbf{A}_p \setminus K_t$, $0 \le t < p$. We call K_t (φ_t, resp.), $0 \le t < p$, the standard annulus LE hulls (maps, resp.) of modulus p driven by χ_t, $0 \le t < p$. We will see that for each $0 \le t < p$, φ_t maps D_t conformally onto $\mathbf{A}_{p-t}$, and maps $\mathbf{C}_p$ onto $\mathbf{C}_{p-t}$.

If we replace $\mathbf{S}_{p-t}(\chi_t, \varphi_t(z))$ in formula (1.1) by

$$\widehat{\mathbf{S}}_{p-t}(\chi_t, \varphi_t(z)) := \mathbf{S}_{p-t}(\chi_t, \varphi_t(z)) - \operatorname{Im}\mathbf{S}_{p-t}(\chi_t, e^{t-p}),$$

and let $\widehat{\varphi}_t(z)$ be the corresponding solutions. Then we have $\widehat{\varphi}_t(e^{-p}) = e^{t-p}$, $0 \le t < p$, since $\widehat{\mathbf{S}}_{p-t}(\chi_t, e^{t-p}) \equiv 1$. Actually $\widehat{\mathbf{S}}_p$ is the Schwarz kernel in [18]. We will use it in the proof of Proposition 2.1. We prefer $\mathbf{S}_p$ to $\widehat{\mathbf{S}}_p$ in the definition of SLE because if we use $\widehat{\mathbf{S}}_p$ then the driving function must contain a drift term besides a Brownian motion. See the definition of SLE_6 in [3].

We define standard annulus SLE_κ of modulus p to be the solution of (1.1) with $\chi_t = \exp(iB(\kappa t))$, $0 \le t < p$. The family of hulls grows from 1 to $\mathbf{C}_p$. Via a certain conformal map, we may extend the definition to $\mathrm{SLE}_\kappa(\Omega; a \to B)$ where Ω is a doubly connected domain with non-degenerate boundary, B is a boundary component, and a is a boundary point (prime end) on the other boundary component. Note that the conformal type of $\Omega \setminus K_t$ is always changing, so the annulus SLE_κ hulls cannot have identical increments in the sense of conformal equivalence. We may only require that for any fixed $s \in [0, p)$, the conformal image of $(K_{s+t} \setminus K_s)_{0 \le t < p-s}$ has the same law as the annulus SLE hulls of modulus $p - s$. This together with the symmetry property does not determine the driving process up to a single parameter. However, it turns out that $\exp(iB(\kappa t))$, a Brownian motion on $\mathbf{C}_0$ started from 1 with constant speed κ, is a reasonable choice for the driving process. The main goal of the paper is to justify this claim.

Two facts of doubly connected domains are used in the above definition of annulus SLE. First, the conformal type of a doubly connected domain can be described by a single number, which is the modulus. So we use the time parameter to describe the modulus. Second, given a boundary component B and a prime end P on the other boundary component of some doubly connected domain D, there is a self-conjugate-conformal map of $(D; B, P)$. This is clear when D is the standard annulus. We actually assume that the law of annulus $\mathrm{SLE}_\kappa(D; P \to B)$ is invariant under that map. Because of these, our definition of annulus SLE can be expressed by some nice differential equations. However, these two facts do not hold for n-connected domains when $n > 2$. Some other methods are needed to define the SLEs. The extensions of SLE to multiply connected domains and Riemann surfaces are now in preparation, and will appear elsewhere.

1.3. Main results

Suppose Ω is a simply connected domain, a is a prime end, and b is an interior point. Suppose $F \supsetneqq \{b\}$ is a contractible compact subset of Ω. Then $\Omega \setminus F$ is a doubly connected domain with two boundary components $\partial\Omega$ and ∂F. We call F a hull in Ω w.r.t. b. For a fixed $\kappa > 0$, let (K_t) be a radial $\text{SLE}_\kappa(\Omega; a \to b)$, and (L_s) an annulus $\text{SLE}_\kappa(\Omega \setminus F; a \to \partial F)$. Then we have

Theorem 1.1. *(i) If $\kappa = 6$, the law of $(K_t)_{0 \le t < T_F}$, is equal to that of $(L_s)_{0 \le s < p}$, up to a time-change.*
(ii) If $\kappa \ne 6$, there exist two sequences of stopping times $\{T_n\}$ and $\{S_n\}$ such that $T = \vee_n T_n$, $p = \vee_n S_n$, and for each $n \in \mathbb{N}$, the law of $(K_t)_{0 \le t \le T_n}$ is equivalent to that of $(L_s)_{0 \le s \le S_n}$, up to a time-change.

The second main result of the paper concerns the convergence of a loop-erased conditional random walk (LERW) with appropriate boundary conditions to an annulus SLE_2. For any plane domain Ω, and $\delta > 0$, let Ω^δ denote the graph defined as follows. The vertex set $V(\Omega^\delta)$ consists of the points in $\delta\mathbb{Z}^2 \cap \Omega$ and the intersection points of $\partial\Omega$ with edges of $\delta\mathbb{Z}^2$. The edge set $E(\Omega^\delta)$ consists of the unordered vertex pairs $\{u, v\}$ such that the line segment $(u, v) \subset \Omega$, and there is an edge of $\delta\mathbb{Z}^2$ that contains (u, v) as a subset.

Suppose D is a doubly connected domain with boundary components B_1 and B_2, $0 \in B_1$ and there is some $a > 0$ such that the line segment $(0, a]$ is contained in D. This line segment determines a prime end in D on B_1, denoted by 0_+. We may assume that δ is sufficiently small so that 0 and δ are adjacent vertices of D^δ, and there is a lattice path on D^δ connecting δ and $V(D^\delta) \cap B_2$.

Now let RW be a simple random walk on D^δ started from δ and stopped on hitting ∂D. Let CRW be RW conditioned on the event that RW hits B_2 before B_1. Let LERW be the loop-erasure of CRW, which is obtained by erasing the loops of CRW in the order that they appear. See [4] for details. Then LERW is a random simple lattice path on D^δ from δ to B_2. We may also view LERW as a random simple curve in D from δ to B_2. Taking with the segment $[0, \delta]$, we obtain a random simple curve in D from 0 to B_2. We parameterize this curve by $\beta^\delta[0, p]$ so that $\beta^\delta(0) = 0$, $\beta^\delta(p) \in B_2$, and $M(D \setminus \beta^\delta(0, t)) = p - t$, for $0 \le t < p$.

Now let $(K_t^0)_{0 \le t < p}$ be an annulus $\text{SLE}_2(D; 0_+ \to B_2)$. From Theorem 1.1 and the existence of radial SLE_κ traces, we know that a.s. there exists a random simple path $\beta^0(t)$, $0 < t < p$, such that $K_t^0 = \beta^0(0, t]$, for $0 \le t < p$.

Theorem 1.2. *For every $q \in (0, p)$ and $\varepsilon > 0$, there is a $\delta_0 > 0$ depending on q and ε such that for $\delta \in (0, \delta_0)$ there is a coupling of the processes β^δ and β^0 such that*

$$\mathbf{P}[\sup\{|\beta^\delta(t) - \beta^0(t)| : t \in [q, p)\} > \varepsilon] < \varepsilon.$$

Moreover, if the impression of the prime end 0_+ is a single point, then the theorem holds with $q = 0$.

Here a coupling of two random processes A and B is a probability space with two random processes A' and B', where A' and B' have the same law as A and B, respectively. In the above statement (as is customary) we don't distinguish between A and A' and between B and B'. The impression (see [13]) of a prime end is the intersection of the closure of all neighborhoods of that prime end.

For $\kappa = 2$, 8 and $8/3$, chordal SLE$_\kappa$ satisfies the reversibility property. That means the reversal of chordal SLE$_\kappa(D; a \to b)$ trace has the same law as chordal SLE$_\kappa(D; b \to a)$ trace, up to a time-change. For the annulus SLE trace, the starting point is a fixed prime end, but the end point (if it exists) is a random point on a boundary component. To get the reversibility property, we have to "average" the annulus SLE traces in the same domain started from different points of one boundary component. From Theorem 1.2 and the reversibility of LERW (see [4]), it then follows

Corollary 1.1. *The reversal of the annulus SLE*$_2(\mathbf{A}_p; \mathbf{x} \to \mathbf{C}_p)$ *trace has the same law as the annulus SLE*$_2(\mathbf{A}_p; \mathbf{y} \to \mathbf{C}_0)$ *trace, up to a time-change, where* $\mathbf{x}$ *and* $\mathbf{y}$ *are uniform random points on* $\mathbf{C}_0$ *and* $\mathbf{C}_p$*, respectively.*

The definition of annulus SLE enables us to define disc SLE$_\kappa$ that grows in a simply connected domain Ω from an interior point to the whole boundary. It can be viewed as the limit of annulus SLE$_\kappa$ as the modulus tends to infinity. The relation between disc SLE and annulus SLE is similar to that between full plane SLE and radial SLE.

From our methods, it follows that for any simply connected domain Ω that contains 0, the full plane SLE$_6$ before the hitting time of $\partial\Omega$ has the same law as the disc SLE$_6(\Omega; 0 \to \partial\Omega)$, up to a time-change. This gives an alternative proof of the following facts mentioned in [19][9]. The hitting point of full plane SLE$_6$ at $\partial\Omega$ has harmonic measure valued at 0, and therefore the full plane SLE$_6$ hull at the hitting time of $\partial\Omega$ has the same law as the hull generated by a plane Brownian motion started from 0 and stopped on exiting Ω.

We also show that the LERW on the grid approximation Ω^δ started from an interior vertex 0 to the boundary converges to the disc SLE$_2(\Omega; 0 \to \partial\Omega)$, as $\delta \to 0$. Together with the approximation result in [8], this implies that the reversal of the disc SLE$_2(\Omega; 0 \to \partial\Omega)$ has the same law as the radial SLE$_2(\Omega; \mathbf{z} \to 0)$, up to a time-change, where $\mathbf{z}$ is a random point on $\partial\Omega$ that has harmonic measure valued at 0.

1.4. Some comments about the proof

The discussion of the convergence of LERW to annulus SLE$_2$ basically follows the methods developed in [8]. In the same order as in [8], logically, we first find the observables for LERW; then prove they are martingales and converge to some continuous harmonic functions; these facts are used to show that the driving function of the LERW converges to the Brownian motion with speed 2; finally we use the nice behavior of LERW path to show that the path parameterized according to the modulus of the remaining domain converges to the annulus SLE$_2$ trace uniformly in probability.

However, some notations and proofs in [8] can not be transplanted to this paper immediately. For example, the observables in this paper has counterparts in simply connected domains, which are exactly the observables introduced in [8]. But the LERW studied there is from an interior vertex to the boundary, and the proof of Proposition 3.4 in [8] uses this construction. We have to prove the fact that they are martingales using a different method, which we believe shows some essence of this subject. Moreover, since the moduli change in time, some proofs here, e.g., that to Proposition 3.4, are much longer than their counterparts, e.g., part of the proof to Proposition 3.4 in [8].

The authors of [8] first use some subgraph of $\mathbb{Z}^2$ to approximate a simply connected plane domain, and they use the inner radius with respect to a fixed point (which is 0 there) to describe the extent that the graph approximates the domain. After some rescaling, the inner radius means the distance from 0 to the boundary of the domain divided by the length of the mesh. It seems not easy to find counterparts of the inner radius for doubly connected domains. So we proceed in another way by taking the limit of some sequence of domains. This results in a very long proof of Proposition 3.3. This method extends to the cases of multiply connected domains.

2. Equivalence of annulus and radial SLE

2.1. *Deterministic annulus LE hulls*

We recall some facts about the Schwarz function

$$\mathbf{S}_r(z) = \lim_{N\to\infty} \sum_{k=-N}^{N} \frac{e^{2kr}+z}{e^{2kr}-z}, \ r>0.$$

(i) $\mathbf{S}_r$ is analytic in $\mathbb{C}\setminus\{0\}\setminus\{e^{2kr}:k\in\mathbb{Z}\}$;
(ii) $\{e^{2kr}:k\in\mathbb{Z}\}$ are simple poles of S_r;
(iii) $\operatorname{Re}\mathbf{S}_r\equiv 1$ on $\mathbf{C}_r=\{z\in\mathbb{C}:|z|=e^{-r}\}$;
(iv) $\operatorname{Re}\mathbf{S}_r\equiv 0$ on $\mathbf{C}_0\setminus\{1\}$;
(v) $\operatorname{Re}\mathbf{S}_r>0$ in $\mathbf{A}_r$; and
(vi) $\operatorname{Im}\mathbf{S}_r\equiv 0$ on $\mathbb{R}\setminus\{0\}\setminus\{\text{poles}\}$.

Suppose f is an analytic function in $\mathbf{A}_r$, $\operatorname{Re} f$ is non-negative, and $\operatorname{Re} f(z)$ tends to a as $z\to\mathbf{C}_r$, then there is some positive measure $\mu=\mu(f)$ on $\mathbf{C}_0$ of total mass a such that

$$f(z)=\int_{\mathbf{C}_0}\mathbf{S}_r(z/\chi)d\mu(\chi)+iC, \tag{2.1}$$

for some real constant C. If $\operatorname{Re} f(z)$ tends to zero as z approaches the complement of an arc α of $\mathbf{C}_0$, then $\mu(f)$ is supported by $\overline{\alpha}$. Moreover, if f is bounded, then the radial limit of f on $\mathbf{C}_0$ exists a.e., and $d\mu(f)/d\mathbf{m}=f|_{C_0}$. The proof is similar to that of the Poisson integral formula.

Divide both sides of equation (1.1) by $\varphi_t(z)$and take the real part. We get

$$\partial_t \ln|\varphi_t(z)| = \operatorname{Re}\mathbf{S}_{p-t}(\varphi_t(z)/\chi_t).$$

From the values of $\operatorname{Re}\mathbf{S}_{p-t}$ on $\mathbf{C}_{p-t}$ and $\mathbf{C}_0$ we see that if $z\in\mathbf{C}_0\setminus\{1\}$, then $\varphi_t(z)\in\mathbf{C}_0\setminus\{1\}$ until it blows up; if $z\in\mathbf{C}_p$, then $\varphi_t(z)\in\mathbf{C}_{p-t}$ for $0\le t<p$.

Thus for $z \in \mathbf{A}_p$, $\varphi_t(z)$ stays between $\mathbf{C}_0$ and $\mathbf{C}_{p-t}$ until it blows up. So φ_t maps D_t into $\mathbf{A}_{p-t}$. The fact that $\mathbf{S}_{p-t}$ is analytic implies that for every $t \in [0, p)$, φ_t is a conformal map of D_t. By considering the backward flow, it is easy to see that φ_t maps D_t onto $\mathbf{A}_{p-t}$.

Definition 2.1. *Suppose D is a doubly connected domain with boundary components B and B'. We call $K \subset D$ a hull in D on B if $D \setminus K$ is a doubly connected domain that has B' as a boundary component. The capacity of K in D w.r.t. B', denoted by $C_{D,B'}(K)$, is the value of $M(D) - M(D \setminus K)$.*

Definition 2.2. *Suppose Ω is a simply connected domain. We call $K \subset \Omega$ a hull in Ω on $\partial\Omega$, if $\Omega \setminus K$ is a simply connected domain. If φ maps $\Omega \setminus K$ conformally onto Ω and for some $a \in \Omega \setminus K$, $\varphi(a) = a$ and $\varphi'(a) > 0$, then $\ln \varphi'(a) > 0$, and is called the capacity of K in Ω w.r.t. a, denoted by $C_{\Omega,a}(K)$.*

If K is a hull in $\mathbf{A}_p$ on $\mathbf{C}_0$, and ψ is any conformal map from $\mathbf{A}_p \setminus K$ onto $\mathbf{A}_{p-r}$ which takes $\mathbf{C}_p$ to $\mathbf{C}_{p-r}$, then the radial limit of ψ^{-1} on $\mathbf{C}_0$ exists a.e., and

$$C_{A_p,C_p}(K) = \int_{\mathbf{C}_0} -\ln|\psi^{-1}|d\mathbf{m}.$$

If K is a hull in $\mathbb{D}$ on $\mathbf{C}_0$ and φ maps $\mathbb{D} \setminus K$ onto $\mathbb{D}$ conformally so that $\varphi(0) = 0$, then the radial limit of φ^{-1} on $\mathbf{C}_0$ exists a.e., and

$$C_{\mathbb{D},0}(K) = \int_{\mathbf{C}_0} -\ln|\varphi^{-1}|d\mathbf{m}.$$

Similarly as Lemma 2.8 in [5], using the integral formulas for capacities of hulls in $\mathbb{D}$ and $\mathbf{A}_p$, it is not hard to derive the following Lemma:

Lemma 2.1. *Suppose $x, y \in \mathbf{C}_0$, and G is a conformal map from a neighborhood U of x onto a neighborhood V of y such that $G(U \cap \mathbb{D}) = V \cap \mathbb{D}$. Fix any $p > 0$. For every $\varepsilon > 0$, there is $r = r(\varepsilon) > 0$ such that if K is a non-empty hull in $\mathbb{D}$ on $\mathbf{C}_0$ and $K \subset \mathbf{B}(x; r)$, the open ball of radius r about x, then $K \subset U$, $G(K)$ is a hull in $\mathbf{A}_p$ on $\mathbf{C}_0$, and*

$$\left| \frac{C_{A_p,C_p}(G(K))}{C_{\mathbb{D},0}(K)} - |G'(x)|^2 \right| < \varepsilon.$$

Suppose D is a doubly connected domain with boundary components B_1 and B_2. We call $(K_s, a \le s < b)$ a Loewner chain in D on B_1 if every K_s is a hull in D on B_1, $K_{s_1} \subsetneqq K_{s_2}$ if $a \le s_1 < s_2 < b$, and for every $c \in (a, b)$, the extremal length (see [1]) of the family of curves in $D \setminus K_{s+u}$ that disconnect $K_{s+u} \setminus K_s$ from B_2 tends to 0 as $u \to 0^+$, uniformly in $s \in [a, c]$. If the area of D is finite, then the above condition holds iff the infimum length of all C^1 curves in $D \setminus K_{s+u}$ that disconnect B_2 from $K_{s+u} \setminus K_s$ tends to 0 as $u \to 0^+$, uniformly in $s \in [a, c]$.

Now we consider a Loewner chain in $\mathbf{A}_p$ on $\mathbf{C}_0$. The following proposition is similar to the theorems for chordal and radial LE in [5] and [11].

Proposition 2.1. *The following two statements are equivalent:*

1. *K_t, $0 \le t < p$, are the standard LE hulls of modulus p driven by some continuous function $\chi : [0, p) \to \mathbf{C}_0$;*
2. *$(K_t, 0 \le t < p)$ is a Loewner chain in $\mathbf{A}_p$ on $\mathbf{C}_0$, and $C_{A_p,C_p}(K_t) = M(\mathbf{A}_p) - M(\mathbf{A}_p \setminus K_t) = t$ for $0 \le t < p$.*

Moreover, $\{\chi_t\} = \cap_{u>0}\overline{\varphi_t(K_{t+u} \setminus K_t)}$, where φ_t is the standard annulus LE map. If $(L_s, a \le s < b)$ is any Loewner chain in $\mathbf{A}_p$ on $\mathbf{C}_0$, then $s \mapsto C_{A_p,C_p}(L_s)$ is a continuous (strictly) increasing function.

Proof. The method of the proof is a combination of extremal length comparison, the use of formula (2.1), and some estimation of Schwarz kernels. It is very similar to the proof of the counterparts in [5] and [11]. So we omit the most part of it. One thing we want to show here is how we derive φ_t from K_t in the proof of 2 implies 1. We first choose $\widehat{\varphi_t}$ that maps $\mathbf{A}_p \setminus K_t$ conformally onto $\mathbf{A}_{p-t}$ such that $\widehat{\varphi_t}(\mathbf{C}_p) = \mathbf{C}_{p-t}$ and $\widehat{\varphi_t}(e^{-p}) = e^{t-p}$. Then we prove that $\widehat{\varphi_t}$ satisfies the equation

$$\partial_t\widehat{\varphi_t}(z) = \widehat{\varphi_t}(z)(\mathbf{S}_{p-t}(\widehat{\varphi_t}(z)/\widehat{\chi_t}) - i\,\mathrm{Im}\,\mathbf{S}_{p-t}(e^{t-p}/\widehat{\chi_t})),$$

for some continuous $\widehat{\chi} : [0, p) \to \mathbf{C}_0$. And $\{\widehat{\chi_t}\} = \cap_{u>0}\overline{\widehat{\varphi_t}(K_{t+u} \setminus K_t)}$. Define

$$\theta(t) = \int_0^t \mathrm{Im}\,\mathbf{S}_{p-s}(e^{s-p}/\widehat{\chi_s})ds,$$

$\chi_t = e^{i\theta(t)}\widehat{\chi_t}$ and $\varphi_t(z) = e^{i\theta(t)}\widehat{\varphi_t}(z)$, for $t \in [0, p)$. Then $\varphi_0(z) = \widehat{\varphi_0}(z) = z$, φ_t maps $\mathbf{A}_p \setminus K_t$ conformally onto $\mathbf{A}_{p-t}$, $\{\chi_t\} = \cap_{u>0}\overline{\varphi_t(K_{t+u} \setminus K_t)}$, and

$$\partial_t \ln \varphi_t(z) = \partial_t \ln \widehat{\varphi_t}(z) + i\theta'(t) = \mathbf{S}_{p-t}(\widehat{\varphi_t}(z)/\widehat{\chi_t}) = \mathbf{S}_{p-t}(\varphi_t(z)/\chi_t).$$

Thus $\partial_t\varphi_t(z) = \varphi_t(z)\mathbf{S}_{p-t}(\varphi(z)/\chi_t)$. So K_t, $0 \le t < p$, are the standard annulus LE hulls of modulus p, driven by χ_t, $0 \le t < p$. □

2.2. *Proof of Theorem 1.1*

We may assume in Theorem 1.1 that $\Omega = \mathbb{D}$, $a = 1$ and $b = 0$. Then $(K_t, 0 \le t < \infty)$ is the standard radial SLE$_\kappa$. Suppose φ_t and χ_t, $0 \le t < \infty$, are the corresponding standard radial SLE$_\kappa$ maps and driving process, respectively. Then $\chi_t = e^{iB(\kappa t)}$, where $B(t)$ is a standard Brownian motion on $\mathbb{R}$ started from 0.

For $0 \le t < T_F$, $\mathbb{D} \setminus F \setminus K_t$ is a doubly connected domain. So K_t, $0 \le t < T_F$, are hulls in $\mathbb{D} \setminus F$ on $\mathbf{C}_0$. From [11] we know that $(K_t, 0 \le t < T_F)$ is a Loewner chain in $\mathbb{D}\setminus F$ on $\mathbf{C}_0$. Suppose W maps $\mathbb{D}\setminus F$ conformally onto $\mathbf{A}_p$ so that $W(1) = 1$. Then $(W(K_t), 0 \le t < T_F)$ is a Loewner chain in $\mathbf{A}_p$ on $\mathbf{C}_0$. From [15] we know that K_t approaches F as $t \nearrow T_F$, so $W(K_t)$ approaches $\mathbf{C}_p$ as $t \nearrow T_F$. This implies that $M(\mathbb{D} \setminus F \setminus K_t) \to 0$ as $t \nearrow T_F$. Let $u(t) = C_{D,\partial F}(K) = C_{A_p,C_p}(W(K))$. Then u is a continuous increasing function and maps $[0, T_F)$ onto $[0, p)$. Let v be the inverse of u. By Proposition 2.1, $W(K_{v(s)})$, $0 \le s < p$, are the standard annulus LE hulls of modulus p driven by some continuous $\nu : [0, p) \to \mathbf{C}_0$. Let ψ_s, $0 \le s < p$, be the corresponding standard annulus LE maps.

Now φ_t maps $\mathbb{D} \setminus F \setminus K_t$ conformally onto $\mathbb{D} \setminus \varphi_t(F)$. Let $f_t = \psi_{u(t)} \circ W \circ \varphi_t^{-1}$. Then f_t maps $\mathbb{D} \setminus \varphi_t(F)$ conformally onto $\mathbf{A}_{p-u(t)}$, and $f_t(\mathbf{C}_0) = \mathbf{C}_0$. By Schwarz reflection, we may extend f_t analytically to Σ_t, which is the union of $\mathbb{D}\setminus\varphi_t(F)$, $\mathbf{C}_0$, and the reflection of $\mathbb{D}\setminus\varphi_t(F)$ w.r.t. $\mathbf{C}_0$. And f_t is a conformal map on Σ_t. Note that f_t maps $\varphi_t(K_{t+a}\setminus K_t)$ to $\psi_{u(t)}(W(K_{t+a})\setminus W(K_t))$ for $a>0$. From Proposition 2.1, we see that $\{\nu_{u(t)}\}=\cap_{a>0}\overline{\psi_{u(t)}(W(K_{t+a})\setminus W(K_t))}$. And from the counterpart in [11] of Proposition 2.1, we know that $\{\chi_t\}=\cap_{a>0}\overline{\varphi_t(K_{t+a}\setminus K_t)}$. Thus $\nu_{u(t)}=f_t(\chi_t)$. Now $\varphi_t(K_{t+a}\setminus K_t)$ is a hull in $\mathbb{D}$, $\varphi_{t+a}\circ\varphi_t^{-1}$ maps $\mathbb{D}\setminus\varphi_t(K_{t+a}\setminus K_t)$ conformally onto $\mathbb{D}$, fixes 0, and $(\varphi_{t+a}\circ\varphi_t^{-1})'(0)=e^a$. So the capacity w.r.t. 0 of $\varphi_t(K_{t+a}\setminus K_t)$ is a. Similarly, $\psi_{u(t)}(W(K_{t+a}\setminus W(K_t))$ is a hull in $\mathbf{A}_{p-u(t)}$ on $\mathbf{C}_0$, and the capacity is $u(t+a)-u(t)$. From Lemma (2.1) we conclude that $u'_+(t)=|f_t'(\chi_t)|^2$.

Let $H=\{(t,z):0\le t<T_F, z\in\Sigma_t\}$ and $G(\chi)=\{(t,\chi_t):0\le t<T_F\}$. By the definition of f_t, we see that $(t,z)\mapsto f_t'(z)$ is continuous in $H\setminus G(\chi)$. Note that f_t' is analytic in Σ_t for each $t\in[0,T_F)$. The maximum principle implies that $(t,z)\mapsto f_t'(z)$ is continuous in H. In particular, $t\mapsto f_t'(\chi_t)$ is continuous. So we have

Lemma 2.2. *$u(t)$ is C^1 continuous, and $u'(t)=|f_t'(\chi_t)|^2$.*

The fact $W(\chi_0)=W(1)=1$ implies that $\nu_0=1$. We now lift f_t to the covering space. Write $\chi_t=e^{i\xi_t}$ and $\nu_s=e^{i\eta_s}$, where $\xi_t=B(\kappa t)$, $0\le t<\infty$, and η_s, $0\le s<p$, is a real continuous function with $\eta_0=0$. Let $\widetilde{\Sigma}_t=\{z\in\mathbb{C}:e^{iz}\in\Sigma_t\}$. Then there is a unique conformal map $\widetilde{f}_t$ on $\widetilde{\Sigma}_t$ such that $e^{i\widetilde{f}_t(z)}=f_t(e^{iz})$ and $\eta_{u(t)}=\widetilde{f}_t(\xi_t)$. And $\widetilde{f}_t$ takes real values on the real line. Moreover, $u'(t)=|f_t'(\chi_t)|^2=\widetilde{f}_t'(\xi_t)^2$.

Lemma 2.3. *$(t,x)\mapsto\widetilde{f}_t(x)$ is $C^{1,\infty}$ continuous on $[0,T_F)\times\mathbb{R}$. And for all $t\in[0,T_F)$, $\partial_t\widetilde{f}_t(\xi_t)=-3\widetilde{f}_t''(\xi_t)$.*

Proof. For any $t\in[0,T_F)$, and $z\in\mathbb{D}\setminus F\setminus K_t$, we have $f_t\circ\varphi_t(z)=\psi_{u(t)}\circ W(z)$. Taking the derivative w.r.t. t, we compute

$$\begin{aligned}&\partial_t f_t(\varphi_t(z))+f_t'(\varphi_t(z))\varphi_t(z)\frac{\chi_t+\varphi_t(z)}{\chi_t-\varphi_t(z)}\\&\quad=u'(t)\psi_{u(t)}(W(z))\mathbf{S}_{p-u(t)}(\psi_{u(t)}(W(z))/\eta_{u(t)}).\end{aligned}$$

By Lemma 2.2, $u'(t)=|f_t'(\chi_t)|^2$. Thus for any $t\in[0,T_F)$ and $z\in\mathbb{D}\setminus F\setminus K_t$,

$$\begin{aligned}\partial_t f_t(\varphi_t(z))&=|f_t'(\chi_t)|^2 f_t(\varphi_t(z))\mathbf{S}_{p-u(t)}(f_t(\varphi_t(z))/f_t(\chi_t))\\&\quad-f_t'(\varphi_t(z))\varphi_t(z)\frac{\chi_t+\varphi_t(z)}{\chi_t-\varphi_t(z)}.\end{aligned}$$

For any $t\in[0,T_F)$, and $w\in\mathbb{D}\setminus\varphi_t(F)$, we have $\varphi_t^{-1}(w)\in\mathbb{D}\setminus F\setminus K_t$. Thus

$$\partial_t f_t(w)=|f_t'(\chi_t)|^2 f_t(w)\mathbf{S}_{p-u(t)}(f_t(w)/f_t(\chi_t))-f_t'(w)w\frac{\chi_t+w}{\chi_t-w}.$$

Let $g_t(w)$ be the right-hand side of the above formula for $t\in[0,T_F)$ and $w\in\Sigma_t\setminus\{\chi_t\}$. Then for each $t\in[0,T_F)$, $g_t(w)$ is analytic in $\Sigma_t\setminus\{\chi_t\}$. And $(t,w)\mapsto g_t(w)$ is $C^{0,\infty}$ continuous on $H\setminus G(\chi)$.

Now fix $t_0 \in [0, T_F)$. Let us compute the limit of $g_{t_0}(w)$ when $w \to \chi_{t_0}$. Since

$$\mathbf{S}_{p-u(t_0)}(f_{t_0}(w)/f_{t_0}(\chi_{t_0})) - \frac{f_{t_0}(\chi_{t_0}) + f_{t_0}(w)}{f_{t_0}(\chi_{t_0}) - f_{t_0}(w)} \to 0, \text{ as } w \to \chi_{t_0},$$

so the limit of $g_{t_0}(w)$ is equal to the limit of the following function:

$$|f'_{t_0}(\chi_{t_0})|^2 f_{t_0}(w)\frac{f_{t_0}(\chi_{t_0}) + f_{t_0}(w)}{f_{t_0}(\chi_{t_0}) - f_{t_0}(w)} - f'_{t_0}(w)w\frac{\chi_{t_0} + w}{\chi_{t_0} - w}.$$

Let $w = e^{ix}$, we may express the above formula in term of x, ξ_{t_0} and $\widetilde{f}_{t_0}$, which is

$$\widetilde{f}'_{t_0}(\xi_{t_0})^2 e^{i\widetilde{f}_{t_0}(x)}\frac{e^{i\widetilde{f}_{t_0}(\xi_{t_0})} + e^{i\widetilde{f}_{t_0}(x)}}{e^{i\widetilde{f}_{t_0}(\xi_{t_0})} - e^{i\widetilde{f}_{t_0}(x)}} - \widetilde{f}'_{t_0}(x)e^{i\widetilde{f}_{t_0}(x)}\frac{e^{i\xi_{t_0}} + e^{ix}}{e^{i\xi_{t_0}} - e^{ix}}$$
$$= -ie^{i\widetilde{f}_{t_0}(x)}[\widetilde{f}'_{t_0}(\xi_{t_0})^2 \cot(\frac{\widetilde{f}_{t_0}(x) - \widetilde{f}_{t_0}(\xi_{t_0})}{2}) - \widetilde{f}'_{t_0}(x)\cot(\frac{x - \xi_{t_0}}{2})].$$

By expanding the Laurent series of $\cot(z)$ near 0, we see that the limit of the above formula is $3ie^{i\widetilde{f}_{t_0}(\xi_{t_0})}\widetilde{f}''_{t_0}(\xi_{t_0}) = 3if_{t_0}(\chi_{t_0})\widetilde{f}''_{t_0}(\xi_{t_0})$. Therefore g_t has an analytic extension to Σ_t for each $t \in [0, T_F)$. The maximum principle also implies that $g_t(w)$ is $C^{0,\infty}$ continuous in H, and $\partial_t f_t(w) = g_t(w)$ holds in the whole H. Thus $f_t(w)$ is $C^{1,\infty}$ continuous on $[0, T_F) \times \mathbf{C}_0$, and $\widetilde{f}_t(w)$ is $C^{1,\infty}$ continuous on $[0, T_F) \times \mathbb{R}$. Finally,

$$\partial_t \widetilde{f}_t(\xi_t) = \frac{i\partial_t f_t(\chi_t)}{f_t(\chi_t)} = \frac{ig_t(\chi_t)}{f_t(\chi_t)} = \frac{-3f_t(\chi_t)\widetilde{f}''_t(\xi_t)}{f_t(\chi_t)} = -3\widetilde{f}''_t(\xi_t). \qquad \square$$

Proof of Theorem 1.1. Note that $\eta_{u(t)} = \widetilde{f}_t(\xi_t)$, $\xi_t = B(\kappa t)$, and from Lemma 2.2, $\partial_t \widetilde{f}_t(\xi_t) = -3\widetilde{f}''_t(\xi_t)$. By Itô's formula, we have

$$d\eta_{u(t)} = \widetilde{f}'_t(\xi_t)d\xi_t + (\frac{\kappa}{2} - 3)\widetilde{f}''_t(\xi_t)dt.$$

Since $u'(t) = \widetilde{f}'_t(\xi_t)^2$, so

$$d\eta_s = d\widetilde{\xi}_s + (\frac{\kappa}{2} - 3)\widetilde{f}''_{v(s)}(\xi_t)/\widetilde{f}'_{v(s)}(\xi_t)^2 ds,$$

where $\widetilde{\xi}_s = \widetilde{B}(\kappa s), 0 \le s < p$, and $\widetilde{B}(s)$ is another standard Brownian motion on $\mathbb{R}$ started from 0. Note that $\eta_0 = 0$. If $\kappa = 6$, then $\eta_s = \widetilde{\xi}_s = \widetilde{B}(\kappa s), 0 \le s < p$. Thus $(W(K_{v(s)}))_{0\le s<p}$ has the same law as the standard annulus $\mathrm{SLE}_{\kappa=6}$ of modulus p. So $(K_{v(s)})_{0\le s<p}$ has the same law as $(L_s)_{0\le s<p}$.

If $\kappa \ne 6$, then $d\eta_s = d\widetilde{\xi}_s$ + drift term. The remaining part follows from Girsanov's Theorem ([14]). $\square$

Remark. This equivalence implies the a.s. existence of annulus SLE trace. Suppose (K_t) is an annulus $\mathrm{SLE}_\kappa(D; P \to B_2)$. If $\kappa \le 4$, the trace β is a simple curve in D such that every $K_t = \beta(0, t]$. If $\kappa > 4$ and B_1 is locally connected, then β is a non-simple curve in $D \cup B_1$ such that for every t, $D \setminus K_t$ is the connected component of $D \setminus \beta(0, t]$ that has B_2 as a boundary component.

3. Annulus SLE_2 and LERW

3.1. *Observables for SLE_2*

Suppose D is a doubly connected domain of modulus p with boundary components B_1 and B_2, P is a prime end on B_1. Let (K_t) be an annulus $\text{SLE}_2(D;\, P \to B_2)$ and β the corresponding trace. Let $D_t = D \setminus K_t, 0 \le t < p$. Then $\beta(t, t+\varepsilon)$ determines a prime end in D_t, denoted by $\beta(t_+)$. Now consider a positive harmonic function H_t in D_t, which has a harmonic conjugate and satisfies the following properties. As $z \in D_t$ and $z \to B_2$, we have $H_t(z) \to 1$; for any neighborhood V of $\beta(t_+)$, as $z \in D_t \setminus V$ and $z \to B_1 \cup K_t$, we have $H_t(z) \to 0$. The existence of the harmonic conjugate implies that for any smooth Jordan curve, say γ, that disconnects the two boundary components of D_t, we have $\int_\gamma \partial_{\mathbf{n}} H_t ds = 0$, where $\mathbf{n}$ are normal vectors on γ pointed towards B_1. Now we introduce another positive harmonic function P_t in D_t which satisfies that for any neighborhood V of $\beta(t_+)$, as $z \in D_t \setminus V$ and $z \to \partial D_t$, we have $P_t(z) \to 0$, and $\int_\gamma \partial_{\mathbf{n}} P_t ds = 2\pi$ for any smooth Jordan curve γ that disconnects the two boundary components of D_t.

Proposition 3.1. *For any fixed $z \in D$, $H_t(z)$ and $P_t(z)$, $0 \le t < p$, are local martingales.*

Proof. By conformal invariance, we may assume that $D = \mathbf{A}_p, B_1 = \mathbf{C}_0, B_2 = \mathbf{C}_p$, and $P = 1$. So $(K_t, 0 \le t < p)$ is the standard annulus SLE_2 of modulus p. Let χ_t and φ_t, $0 \le t < p$, be the corresponding driving function and conformal maps. Then $\chi_t = \exp(i\xi(t))$ and $\xi(t) = B(2t)$. Since φ_t maps D_t conformally onto $\mathbf{A}_{p-t}$ and by Proposition 2.1, $\varphi_t(\beta(t_+)) = \chi_t$, we have

$$H_t(z) = \operatorname{Re} \mathbf{S}_{p-t}(\varphi_t(z)/\chi_t), \text{ and } P_t(z) = \ln|\varphi_t(z)| + (p-t)H_t(z).$$

We want to use the Itô's formula. To simplify the computation, we lift the maps to the covering space. Let $\widetilde{D}_t, \widetilde{\mathbf{A}}_r$ and $\widetilde{\mathbf{C}}_r$ be the preimages of $D_t, \mathbf{A}_r$ and $\mathbf{C}_r$, respectively, under the map $z \mapsto e^{iz}$. We may lift φ_t to a conformal map $\widetilde{\varphi}_t$ from $\widetilde{D}_t$ onto $\widetilde{\mathbf{A}}_{p-t}$ so that $\exp(i\widetilde{\varphi}_t(z)) = \varphi_t(e^{iz})$, $\widetilde{\varphi}_0(z) = z$, and $\widetilde{\varphi}_t(z)$ is continuous in t. Let $\widetilde{\mathbf{S}}_r(z) = \frac{1}{i}\mathbf{S}_r(e^{iz})$. Then we have

$$\partial_t \widetilde{\varphi}_t(z) = \widetilde{\mathbf{S}}_{p-t}(\widetilde{\varphi}_t(z) - \xi(t)).$$

It is clear that $\widetilde{\mathbf{S}}_r$ has period 2π, is meromorphic in $\mathbb{C}$ with poles $\{2k\pi + i2mr : k, m \in \mathbb{Z}\}$, $\operatorname{Im} \widetilde{\mathbf{S}}_r \equiv 0$ on $\mathbb{R} \setminus \{\text{poles}\}$, and $\operatorname{Im} \widetilde{\mathbf{S}}_r \equiv -1$ on $\widetilde{\mathbf{C}}_r$. It is also easy to check that $\widetilde{\mathbf{S}}_r$ is an odd function, and the principal part of $\widetilde{\mathbf{S}}_r$ at 0 is $2/z$. So $\widetilde{\mathbf{S}}_r(z) = 2/z + az + O(z^3)$ near 0, for some $a \in \mathbb{R}$. It is possible to explicit this kernel using classical functions in [2]:

$$\widetilde{\mathbf{S}}_r(z) = 2\zeta(z) - \frac{2}{\pi}\zeta(\pi)z = \frac{1}{\pi}\frac{\partial_v \theta}{\theta}\left(\frac{z}{2\pi}, \frac{ir}{\pi}\right),$$

where ζ is the Weierstrass zeta function with basic periods $(2\pi, i2r)$, and $\theta = \theta(v, \tau)$ is Jacobi's theta function. The following lemma is a direct consequence of the heat-type differential equation satisfied by θ: $(\partial_v^2 - 4i\pi\partial_\tau)\theta = 0$.

But we prefer a proof using only basic complex analysis. The symbols $'$ and $''$ in the lemma denote the first and second derivatives w.r.t. z.

Lemma 3.1. $\partial_r\widetilde{\mathbf{S}}_r - \widetilde{\mathbf{S}}_r\widetilde{\mathbf{S}}_r' - \widetilde{\mathbf{S}}_r'' \equiv 0$.

Proof. Let $J = \partial_r\widetilde{\mathbf{S}}_r - \widetilde{\mathbf{S}}_r\widetilde{\mathbf{S}}_r' - \widetilde{\mathbf{S}}_r''$. Then J is odd, has period 2π, takes real values on $\mathbb{R}\setminus\{2k\pi : k\in\mathbb{Z}\}$, and is analytic on $\mathbb{C}\setminus\{2k\pi + i2mr : k,m\in\mathbb{Z}\}$. Since near 0, $\widetilde{\mathbf{S}}_r(z) = 2/z + az + O(z^3)$, so $\widetilde{\mathbf{S}}_r'(z) = -2/z^2 + a + O(z^2)$, and $\widetilde{\mathbf{S}}_r''(z) = 4/z^3 + O(z)$. Thus $\widetilde{\mathbf{S}}_r(z)\widetilde{\mathbf{S}}_r'(z) + \widetilde{\mathbf{S}}_r''(z) = O(z)$ near 0, i.e. 0 is a removable pole of $\widetilde{\mathbf{S}}_r\widetilde{\mathbf{S}}_r' + \widetilde{\mathbf{S}}_r''$. Since $\widetilde{\mathbf{S}}_r(z) - \frac{1}{i}\frac{1+e^{iz}}{1-e^{iz}}$ is analytic in a neighborhood of 0, and $\frac{1+e^{iz}}{1-e^{iz}}$ is constant in t, so 0 is also a removable pole of $\partial_r\widetilde{\mathbf{S}}_r$. Thus J extends analytically at 0. As J has period 2π, J extends analytically at $2k\pi$, for all $k\in\mathbb{Z}$. So J is analytic in $\{|\mathrm{Im}\, z| < 2r\}$. The fact that $\mathrm{Im}\,\widetilde{\mathbf{S}}_r \equiv 0$ on $\mathbb{R}\setminus\{\text{poles}\}$ implies $\mathrm{Im}\, J \equiv 0$ on $\mathbb{R}$.

Since $\mathrm{Im}\,\widetilde{\mathbf{S}}_r \equiv -1$ on $\widetilde{\mathbf{C}}_r = ir+\mathbb{R}$, we have $\mathrm{Im}\,\widetilde{\mathbf{S}}_r{}'' = \partial_x^2\mathrm{Im}\,\widetilde{\mathbf{S}}_r = \partial_x\mathrm{Im}\,\widetilde{\mathbf{S}}_r \equiv 0$, and $\partial_r\mathrm{Im}\,\widetilde{\mathbf{S}}_r = -\partial_y\mathrm{Im}\,\widetilde{\mathbf{S}}_r = -\partial_x\mathrm{Re}\,\widetilde{\mathbf{S}}_r$ on $\widetilde{\mathbf{C}}_r$. Therefore

$$\mathrm{Im}\,(\widetilde{\mathbf{S}}_r\widetilde{\mathbf{S}}_r') = \mathrm{Re}\,\widetilde{\mathbf{S}}_r\partial_x\mathrm{Im}\,\widetilde{\mathbf{S}}_r + \mathrm{Im}\,\widetilde{\mathbf{S}}_r\partial_x\mathrm{Re}\,\widetilde{\mathbf{S}}_r = -\partial_x\mathrm{Re}\,\widetilde{\mathbf{S}}_r$$

on $\widetilde{\mathbf{C}}_r$. Thus $\mathrm{Im}\, J = \mathrm{Im}\,\partial_r\widetilde{\mathbf{S}}_r - \mathrm{Im}\,(\widetilde{\mathbf{S}}_r\widetilde{\mathbf{S}}_r') - \mathrm{Im}\,\widetilde{\mathbf{S}}_r'' \equiv 0$ on $\widetilde{\mathbf{C}}_r$. Now $\mathrm{Im}\, J \equiv 0$ on both $\mathbb{R}$ and $ir+\mathbb{R}$, so it has to be zero everywhere. It then follows that $J\equiv C$ for some $C\in\mathbb{R}$. Since J is odd, $C = 0$ and $J\equiv 0$. □

Now we may express H_t and P_t by

$$H_t(e^{iz}) = \mathrm{Im}\,\widetilde{\mathbf{S}}_{p-t}(\widetilde{\varphi}_t(z) - \xi(t)), \text{ and } P_t(e^{iz}) = \mathrm{Im}\,\widetilde{\varphi}_t(z) + (p-t)H_t(z).$$

So it suffices to prove that for any $z\in\widetilde{\mathbf{A}}_p$,

$$M_1(t) = \widetilde{\mathbf{S}}_{p-t}(\widetilde{\varphi}_t(z) - \xi(t)), \text{ and } M_2(t) = \widetilde{\varphi}_t(z) + (p-t)M_1(t),$$

$0\le t < p$, are martingales. Using Itô's formula, we have

$$dM_1(t) = -\partial_r\widetilde{\mathbf{S}}_{p-t}dt + \widetilde{\mathbf{S}}_{p-t}'\cdot[d\widetilde{\varphi}_t(z) - d\xi(t)] + \widetilde{\mathbf{S}}_{p-t}''dt,$$

where $\partial_r\widetilde{\mathbf{S}}_{p-t}$, $\widetilde{\mathbf{S}}_{p-t}'$ and $\widetilde{\mathbf{S}}_{p-t}''$ are all valued at $\widetilde{\varphi}_t(z) - \xi(t)$. The last term is the drift term. Note that we use $\kappa = 2$ here. Since $d\widetilde{\varphi}_t(z) = \widetilde{\mathbf{S}}_{p-t}(\widetilde{\varphi}_t(z) - \xi(t))dt$, we have

$$dM_1(t) = (-\partial_r\widetilde{\mathbf{S}}_{p-t} + \widetilde{\mathbf{S}}_{p-t}'\widetilde{\mathbf{S}}_{p-t} + \widetilde{\mathbf{S}}_{p-t}'')dt - \widetilde{\mathbf{S}}_{p-t}'d\xi(t) = -\widetilde{\mathbf{S}}_{p-t}'d\xi(t)$$

by Lemma 3.1. Thus $(M_1(t), 0\le t < p)$ is a local martingale. Now

$$dM_2(t) = \widetilde{\mathbf{S}}_{p-t}(\widetilde{\varphi}_t(z) - \xi(t))dt + (p-t)dM_1(t) - M_1(t)dt = (p-t)dM_1(t).$$

Thus $(M_2(t), 0\le t < p)$ is also a local martingale. □

Remark. Similar observables also exist for radial and chordal SLE_2. For example, let K_t be radial SLE_2 in a simply connected domain Ω, let H_t be the positive harmonic function in $\Omega \setminus K_t$ which tends to 0 on $\partial(\Omega \setminus K_t)$ except at the "tip" point of K_t, and normalized so that the value of H_t at the target point is constant 1. Then for any fixed $z \in D$, $H_t(z)$, $0 \le t < \infty$, is a martingale. This observable was mentioned implicitly in the proof of Proposition 3.4 in [8]. As we want to define SLE for general domains, we conjecture that such kinds of observables always exist for SLE_2.

3.2. *Observables for LERW*

Let $G = (V, E)$ be a finite or infinite simple connected graph such that $\deg(v) < \infty$ for each $v \in V$. For a function f on V, and $v \in V$, let $\Delta_G f(v) = \sum_{w\sim v}(f(w) - f(v))$, where $w \sim v$ means that w and v are adjacent. A subset K of V is called reachable, if for any $v \in V \setminus K$, a symmetric random walk on G started from v will hit K in finite steps almost surely. For subsets S_1, S_2 and S_3 of V, let $\Gamma^{S_3}_{S_1,S_2}$ denote the set of all lattice paths $\gamma = (\gamma_0, \dots, \gamma_n)$ such that $\gamma_0 \in S_1$, $\gamma_n \in S_2$ and $\gamma_s \in S_3$ for $0 < s < n$. For a finite lattice path $\gamma = (\gamma_0, \dots, \gamma_n)$, write

$$P(\gamma) = 1/\prod_{j=0}^{n} \deg(\gamma_j), \quad P_0(\gamma) = 1/\prod_{j=0}^{n-1} \deg(\gamma_j), \quad \text{and} \quad P_1(\gamma) = 1/\prod_{j=1}^{n-1} \deg(\gamma_j).$$

Let $R(\gamma) = (\gamma_n, \dots, \gamma_0)$ be the reversal of γ, then $P(R(\gamma)) = P(\gamma)$ and $P_1(R(\gamma)) = P_1(\gamma)$. If S_1, S_2 and S_3 partition V, $v \in S_3$, then the probability that a random walk on G started from v hits S_2 before S_1 is equal to the summation of $P_0(\gamma)$, where γ runs over $\Gamma^{S_3}_{v,S_2}$.

Lemma 3.2. *Suppose A and B are disjoint subsets of V, and $A \cup B$ is reachable. Let $f(v)$ be the probability that the random walk on G started from v hits A before B. Then f is the unique bounded function on V that satisfies $f \equiv 1$ on A, $f \equiv 0$ on B, and $\Delta_G f \equiv 0$ on $C = V \setminus (A \cup B)$. Moreover $\sum_{v\in B} \Delta_G f(v) = -\sum_{v\in A} \Delta_G f(v) > 0$.*

Proof. The proof is elementary. For the last statement, note that $\sum_{v\in B} \Delta_G f(v) = \sum P_1(\gamma)$ where γ runs over the non-empty set $\Gamma^C_{B,A}$; and $-\sum_{v\in A} \Delta_G f(v) = \sum P_1(\gamma)$ where γ runs over $\Gamma^C_{A,B}$. The values of the two summations are equal because the reverse map R is a one-to-one correspondence between $\Gamma^C_{B,A}$ and $\Gamma^C_{A,B}$, and $P_1(\gamma) = P_1(R(\gamma))$. □

Let $L(A, B) = \sum_{v\in B} \Delta_G f(v)$ for the f in Lemma 3.2. Then $L(A, B) = L(B, A) > 0$. If any of A or B is a finite set, then we have $L(A, B) < \infty$.

Lemma 3.3. *Let A, B, C and f be as in Lemma 3.2. Fix $x \in C$. Let $h(v)$ be equal to the probability that a simple random walk on G started from v hits x before $A \cup B$. Then*

$$\sum_{v\in A} \Delta_G h(v) = f(x)(-\Delta_G h(x)).$$

Proof. From the proof of Lemma 3.2, we have

$$f(x) = \sum_{\alpha\in\Gamma^C_{x,A}} P_0(\alpha) = \sum_{\beta\in\Gamma^C_{x,x}} P(\beta) \sum_{\gamma\in\Gamma^{C\setminus\{x\}}_{x,A}} P_1(\gamma) = \sum_{\beta\in\Gamma^C_{x,x}} P(\beta) \sum_{v\in A} \Delta_G h(v),$$

and

$$1 = \sum_{\alpha\in\Gamma^C_{x,A\cup B}} P_0(\alpha) = \sum_{\beta\in\Gamma^C_{x,x}} P(\beta) \sum_{\gamma\in\Gamma^{C\setminus\{x\}}_{x,A\cup B}} P_1(\gamma) = \sum_{\beta\in\Gamma^C_{x,x}} P(\beta)(-\Delta_G h(x)).$$

So we proved this lemma. □

Lemma 3.4. *Let A, B, C and f be as in Lemma 3.2. Suppose $L(A, B) < \infty$. Fix $x \in C$ such that $f(x) > 0$. Then there is a unique bounded function g on V such that $g \equiv 1$ on A; $g \equiv 0$ on B; $\Delta_G g \equiv 0$ on $C \setminus \{x\}$; and $\sum_{v\in A} \Delta_G g(v) = 0$. Moreover, such g is non-negative and satisfies $\sum_{v\in B\cup\{x\}} \Delta_G g(v) = 0$ and $\Delta_G g(x) = -L(A, B)/f(x)$.*

Proof. Suppose g satisfies the first group of properties. Let $I = g - f$. Then I is bounded, $I \equiv 0$ on $A \cup B$ and $\Delta_G I \equiv 0$ on $C \setminus \{x\}$. Thus $I(v) = I(x)h(v)$, where h is as in Lemma 3.3. Then by Lemma 3.2 and 3.3,

$$0 = \sum_{v\in A} \Delta_G g(v) = \sum_{v\in A} \Delta_G(I + f)(v) = -I(x)f(x)\Delta_G h(x) - L(A, B).$$

Thus $I(x) = L(A, B)/(-f(x)\Delta_G h(x))$ is uniquely determined. Therefore g is unique.

On the other hand, if we define $g = f + hL(A, B)/(-f(x)\Delta_G h(x))$, then from the last paragraph, we see that g satisfies the first group of properties. Since f and h are non-negative, and $-\Delta_G h(x) = L(x, A \cup B) > 0$ by Lemma 3.2, so g is also non-negative. By Lemma 3.2 and 3.3,

$$\begin{aligned}\sum_{v\in B\cup\{x\}} \Delta_G g(v) &= L(A, B) + \Delta_G f(x) \\ &\quad + \sum_{v\in B\cup\{x\}} \Delta_G h(v)L(A, B)/(-f(x)\Delta_G h(x)) \\ &= L(A, B) - \sum_{v\in A} \Delta_G h(v)L(A, B)/(-f(x)\Delta_G h(x)) \\ &= L(A, B) - L(A, B) = 0.\end{aligned}$$

Finally, $\Delta_G g(x) = \Delta_G h(x) \cdot L(A, B)/(-f(x)\Delta_G h(x)) = -L(A, B)/f(x)$. □

From now on, let D be a doubly connected domain with boundary components B_1 and B_2, and satisfies $0 \in B_1$ and $(0, a] \subset D$ for some $a > 0$. We use the symbols D^δ and LERW defined in Section 1.3. Note that D^δ may not be connected. To apply the lemmas in above, we need to modify D^δ a little bit. Let $\mathcal{P}$ denote the set of all lattice paths on D^δ from δ to some boundary vertex whose vertices are inside D except the last vertex. Every path of $\mathcal{P}$ can be viewed as a subgraph of D^δ. Let $\widetilde{D^\delta}$ be the union of all paths in $\mathcal{P}$ as a subgraph of D^δ. Then $\widetilde{D^\delta}$ is a connected graph. And if we replace D^δ by $\widetilde{D^\delta}$ in the definition of LERW in Section 1.3, we will get the same LERW. So we can consider $\widetilde{D^\delta}$ instead of D^δ. For simplicity of notations, we write D^δ for $\widetilde{D^\delta}$.

By the definition, any two vertices of D^δ on ∂D are not adjacent, so the neighbors of boundary vertices of D^δ are those vertices lie in D, are in $\delta\mathbb{Z}^2$ and has exactly 4 neighbors. It follows that if any B_j is bounded, then there are finitely many vertices that lie on B_j. On the other hand, B_1 and B_2 can't be both unbounded. Now we denote

$$E^\delta_{-1} = V(D^\delta) \cap B_1,\ \ F^\delta = V(D^\delta) \cap B_2,\ \text{ and }\ N^\delta_{-1} = V(D^\delta) \cap D.$$

Then at least one of E^δ_{-1} and F^δ is a finite set. Write LERW as $y = (y_0, \dots, y_\upsilon)$, where $y_0 = \delta$ and $y_\upsilon \in B_2$. For $0 \le j < \upsilon$, let

$$E^\delta_j = E^\delta_{-1} \cup \{y_0, \dots, y_j\},\ \text{ and }\ N^\delta_j = N^\delta_{-1} \setminus \{y_0, \dots, y_j\}.$$

Then E^δ_j, N^δ_j and F^δ partition $V(D^\delta)$, for $-1 \le j < \upsilon$. The fact that the lattice $\mathbb{Z}^2$ is recurrent easily implies that $E^\delta_j \cup F^\delta$ is reachable in D^δ. Since one of E^δ_j and F^δ is a finite set, we have $L(E^\delta_j, F^\delta) < \infty$ for $-1 \le j < \upsilon$. For $-1 \le j < \upsilon$, let f_j be the f in Lemma 3.2 with $G = D^\delta$, $A = F^\delta$ and $B = E^\delta_j$. For $0 \le j < \upsilon$, since $(y_j, \dots, y_\upsilon)$ is a lattice path from y_j to F^δ not passing through E^δ_{j-1}, we have $f_{j-1}(y_j) > 0$. Let g_j be the g in Lemma 3.4 with $G = D^\delta$, $A = F^\delta$, $B = E^\delta_{j-1}$, and $x = y_j$, for $0 \le j < \upsilon$.

Lemma 3.5. *Conditioned on the event that $y_j = w_j$, $0 \le j \le k$, and $k < \upsilon$, the probability that $y_{k+1} = u$ is $f_k(u)/\sum_{v\sim w_k} f_k(v)$ if $u \sim w_k$; and is zero if $u \not\sim w_k$.*

Proof. This result is well known. See [4] for details. □

Proposition 3.2. *Let $\overline{F^\delta}$ be the union of F^δ and the set of vertices of D^δ that are adjacent to F^δ. Fix a vertex v_0 of D^δ. Conditioned on the event that $y_j = w_j$, $0 \le j \le k$, $w_k \notin \overline{F^\delta}$, and $f_k(v_0) > 0$, the expectation of $g_{k+1}(v_0)$ is equal to $g_k(v_0)$, which is determined by w_j, $0 \le j \le k$. Thus $g_k(v_0)$ is a discrete martingale up to the first time y_k hits $\overline{F^\delta}$, or $E^\delta_k = E^\delta_{-1} \cup \{y_0, \dots, y_k\}$ disconnects v_0 from F^δ in D^δ.*

Proof. Let S be the set of v such that $v \sim w_k$ and $f_k(v) > 0$. By lemma 3.5, the conditional probability that $y_{k+1} = u$ is $f_k(u)/\sum_{v\in S} f_k(v)$ for $u \in S$. For $v \in S$, let g^v_{k+1} be the g in Lemma 3.4 with $G = D^\delta$, $A = F^\delta$, $B = E^\delta_k$ and $x = v$. Then with probability $f_k(u)/\sum_{v\in S} f_k(v)$, $g_{k+1} = g^u_{k+1}$. Thus the conditional expectation of $g_{k+1}(v_0)$ is equal to $\widetilde{g}_k(v_0)$, where

$$\widetilde{g}_k(v) := \sum_{u\in S} f_k(u) g^u_{k+1}(v) / \sum_{u\in S} f_k(u).$$

Then $\widetilde{g}_k \equiv 0$ on E_k^δ, $\equiv 1$ on F^δ; $\Delta\widetilde{g}_k \equiv 0$ on $N_k^\delta \setminus S$, and $\sum_{v\in F^\delta} \Delta\widetilde{g}_k(v) = 0$. And

$$\Delta\widetilde{g}_k(v) = \frac{f_k(v)\Delta g_{k+1}^v(v)}{\sum_{u\in S} f_k(u)} = -\frac{L(E_k^\delta, F^\delta)}{\sum_{u\in S} f_k(u)}, \quad \forall v \in S,$$

by Lemma 3.4. Now define $\widehat{g}_k$ on $V(D^\delta)$ such that $\widehat{g}_k(w_k) = L(E_k^\delta, F^\delta)/\sum_{u\in S} f_k(u)$; for those $v \in N_k^\delta$ such that $f_k(v) = 0$, define $\widehat{g}_k(v)$ to be $\widehat{g}_k(w_k)$ times the probability that a simple random walk on D^δ started from v hits w_k before E_{k-1}^δ; and let $\widehat{g}_k(v) = \widetilde{g}_k(v)$ for other $v \in V(D^\delta)$. Then $\Delta\widehat{g}_k \equiv 0$ on N_k^δ, $\widehat{g}_k \equiv 0$ on $E_k^\delta \setminus \{w_k\}$, and $\widehat{g}_k \equiv 1$ on F^δ. Since $w_k \notin \overline{F^\delta}$, and for $v \in N_k^\delta$ such that $f_k(v) = 0$ we have $v \notin \overline{F^\delta}$, so $\sum_{v\in F^\delta} \Delta\widehat{g}_k(v) = \sum_{v\in F^\delta} \Delta\widetilde{g}_k(v) = 0$. Now $\widehat{g}_k$ satisfies all properties of g_k. The uniqueness of g_k implies that $\widehat{g}_k \equiv g_k$. Since $f_k(v_0) > 0$, we have $g_k(v_0) = \widehat{g}_k(v_0) = \widetilde{g}_k(v_0)$. □

Remark 1. The observable g_k corresponds to H_t in Proposition 3.1. We may define another kind of observables q_k to be the bounded function on the vertices of D^δ such that $q_k \equiv 0$ on $E_{k-1} \cup F$, $\Delta q_k \equiv 0$ on N_k, and $\sum_{v\in F} \Delta q_k(v) = 2\pi = -\sum_{v\in E_k} \Delta q_k(v)$. Then Proposition 3.2 still holds if g_k is replaced by q_k, and q_k corresponds to P_t in Proposition 3.1. The definition of q_k does not need the fact that $L(E_k, F) < \infty$. We may also use q_k to do the approximation.

Remark 2. Suppose α is a Jordan curve in D which disconnects E_k^δ from F and does not pass through any vertex of D^δ. Denote D_j the component of $D\setminus\alpha$ that has B_j as part of boundary, $j = 1, 2$. We also suppose that y_0 through y_k are in D_1. Let S be the set of vertex pair (v, w) such that $v \in D_1$, $w \in D_2$, and $v \sim w$. From the fact that $\Delta_{D^\delta} g_k \equiv 0$ on $V(D^\delta) \cap D$, we conclude $\sum_{(v,w)\in S}(g_k(v) - g_k(w)) = 0$. Similarly, $\sum_{(v,w)\in S}(q_k(v) - q_k(w)) = 2\pi$.

Now suppose α_1 and α_2 are two disjoint Jordan curves in D such that α_j disconnects α_{3-j} from B_j, $j = 1, 2$. For $j = 1, 2$, let U_j be the subdomain of D bounded by α_j and B_j, and $V_j^\delta = V(D^\delta) \cap U_j$. Let L^δ be the set of simple lattice paths of the form $w = (w_{-1}, w_0, \ldots, w_k)$, $k \geq 0$ such that $w_{-1} \in B_1$, $w_0, \ldots, w_k \in V_1^\delta$, and there is some lattice path from the last vertex $P(w) := w_k$ to B_2 without passing $w_0, \ldots, w_{k-1}$, and vertices on B_1. For $w \in L^\delta$, denote

$$E_w^\delta = E_{-1}^\delta \cup \{w_0, \ldots, w_k\}, \text{ and } N_w^\delta = N_{-1}^\delta \setminus \{w_0, \ldots, w_k\}.$$

Let g_w be the g in Lemma 3.4 with $G = D^\delta$, $A = F^\delta$, $B = E_w^\delta \setminus \{P(w)\}$, and $x = P(w)$. Now define $D_w = D \setminus \cup_{j=0}^k [w_{j-1}, w_j]$. Let u_w be the non-negative harmonic function in D_w whose harmonic conjugates exist, and whose continuation is constant 1 on B_2, and constant 0 on $\cup_{j=0}^k [w_{j-1}, w_j] \cup B_1$ except at $P(w)$. The existence of the harmonic conjugates implies that $\int_\alpha \partial_{\mathbf{n}} u_w ds = 0$ for any smooth Jordan curve α that disconnects B_2 from $\cup_{j=0}^k [w_{j-1}, w_j] \cup B_1$. It is intuitive to guess that g_w should be close to u_w. In fact, we have the following proposition. The proof is postponed to Section 5.

Proposition 3.3. *Given any $\varepsilon > 0$, there is $\delta(\varepsilon) > 0$ such that if $0 < \delta < \delta(\varepsilon)$ and $w \in L^\delta$, then $|g_w(v) - u_w(v)| < \varepsilon$, for any $v \in V_2^\delta$.*

3.3. Convergence of the driving process

Fix some small $\delta > 0$. We write LERW on D^δ by $y = (y_0, \dots, y_\upsilon)$ as in Section 3.2. Let $y_{-1} = 0$. Extend y to be a map from $[-1, \upsilon]$ into $\overline{D}$ such that y is linear on $[j-1, j]$ for each $0 \le j \le \upsilon$. It clear that $y(-1, s]$, $-1 \le s < \upsilon$, is a Loewner chain in D on B_1. And $y(-1, s]$ approaches B_2 as $s \nearrow \upsilon$. For $-1 \le s < \upsilon$, let $T(s) = C_{D,B_2}(y(-1, s])$, then T is a continuous increasing function, and maps $[-1, \upsilon)$ onto $[0, p)$, where $p = M(D)$. Let $S : [0, p) \to [-1, \upsilon)$ be the inverse of T. Let $\beta(t) = y(S(t))$, and $K_t = \beta(0, t]$, for $0 \le t < p$. Suppose W maps D conformally onto $\mathbf{A}_p$ so that $W(0_+) = 1$, i.e., $W(x) \to 1$ as $x \in \mathbb{R}^+$ and $x \to 0$. Then $(W(K_t), 0 \le t < p)$ is a Loewner chain in $\mathbf{A}_p$ on $\mathbf{C}_0$ such that $C_{A_p,C_p}(W(K_t)) = t$. By Proposition 2.1, $W(K_t)$, $0 \le t < p$, are the standard annulus LE hulls of modulus p driven by some continuous χ_t, $0 \le t < p$, on $\mathbf{C}_0$. Let $(\varphi_t, 0 \le t < p)$ be the corresponding standard annulus LE maps. Since $W(\beta(t)) \to 1$ as $t \to 0$, $\chi_0 = 1$. We may write $\chi_t = e^{i\xi_t}$, so that $\xi_0 = 0$, and ξ_t is continuous in t. We want to prove that the law of $(\xi_t)_{0 \le t < p}$, which depends on δ, converges to the law of $(B(2t))_{0 \le t < p}$.

For $a < b$, let $\mathbf{A}_{a,b}$ be the annulus bounded by $\mathbf{C}_a$ and $\mathbf{C}_b$. For any $0 < q < p$, there is a smallest $l(p, q) \in (0, p)$ such that if K is a hull in $\mathbf{A}_p$ on $\mathbf{C}_0$ with the capacity (w.r.t. $\mathbf{C}_p$) less than q, then K does not intersect $\mathbf{A}_{l(p,q),p}$. Using the fact that for any $0 < s \le r$, $\operatorname{Re} \mathbf{S}_r$ attains its unique maximum and minimum on $\overline{\mathbf{A}_{s,r}}$ at e^{-s} and $-e^{-s}$, respectively, it is not hard to derive the following Lemma.

Lemma 3.6. *Fix $0 < q < p$, let $r \in (l(p, q), p)$. There are $\iota \in (0, 1/2)$ and $M > 0$ depending on p, q and r, which satisfy the following properties. Suppose φ_t, $0 \le t < p$, are some standard annulus LE maps of modulus p driven by χ_t, $0 \le t < p$. Then we have $|\partial_z \mathbf{S}_{p-t}(\varphi_t(z)/\chi_t)| \le M$, for all $t \in [0, q]$ and $z \in \mathbf{A}_{r,p}$. Moreover,*

$$\mathbf{A}_{\iota(p-t),p-t} \supset \varphi_t(\mathbf{A}_{r,p}) \supset \mathbf{A}_{(1-\iota)(p-t),p-t}, \ \ \forall t \in [0, q].$$

Now fix $q_0 \in (0, p)$. Let $q_1 = (q_0 + p)/2$. Choose $p_1 \in (l(p, q_1), p)$, and let $p_2 = (p_1 + p)/2$. Denote $\alpha_j = W^{-1}(\mathbf{C}_{p_j})$, $j = 1, 2$. Then α_1 and α_2 are disjoint Jordan curves in D such that α_j disconnects α_{3-j} from B_j, $j = 1, 2$. Let $n_\infty = \lceil S(q_0) \rceil$, where $\lceil x \rceil$ is the smallest integer that is not less than x. Then n_∞ is a stopping time w.r.t. $\{\mathcal{F}_k\}$, where $\mathcal{F}_k$ denotes the σ-algebra generated by y_0, y_1, $\dots$, $y_{k\wedge\upsilon}$. For $0 \le k \le n_\infty - 1$, $T(k) \le q_0 < q_1$, so from the choice of p_1, we see that $W(y_k)$ lies in the domain bounded by $\mathbf{C}_{p_1}$ and $\mathbf{C}_0$, so y_k lies in the domain bounded by B_1 and α_1. Note that $y_{-1} = 0 \in B_1$. So for $-1 \le k \le n_\infty - 1$, if δ is small, then $[y_k, y_{k+1}]$ can be disconnected from B_2 by an annulus centered at y_k with inner radius δ and outer radius $dist(\alpha_1, B_2)$. So as $\delta \to 0$, the conjugate extremal distance between B_2 and $[y_k, y_{k+1}]$ in $D_{y^k} = D \setminus \cup_{0 \le j \le k}[y_{j-1}, y_j]$ (the extremal length of the family of rectifiable curves in D_{y^k} that disconnect B_2 from $[y_k, y_{k+1}]$, see [1]) tends to 0, uniformly in $-1 \le k \le n_\infty - 1$. It then follows that $T(k+1) - T(k)$ and $\max\{|\xi_t - \xi_{T(k)}| : T(k) \le t \le T(k+1)\}$ tend to 0 as $\delta \to 0$, uniformly in $-1 \le k \le n_\infty - 1$. Since $T(n_\infty - 1) \le q_0$, we may choose δ small enough such that $T(n_\infty) < q_1$. We now use the symbols in the last part of Section 3.2 for Jordan curves α_1 and α_2 defined here. For $0 \le k \le n_\infty$, let $y^k = (y_{-1}, y_0, \dots, y_k) \in L^\delta$. Then $g_{y^k} = g_k$. By Proposition 3.2, for any fixed $v \in V_2^\delta$, $g_k(v)$, $0 \le k \le n_\infty$, is a discrete martingale w.r.t. $\{\mathcal{F}_k\}$.

Now fix $d > 0$. Define a non-decreasing sequence $(n_j)_{j\ge 0}$ inductively. Let $n_0 = 0$. Let n_{j+1} be the first integer $n \ge n_j$ such that $T(n) - T(n_j) \ge d^2$, or $|\xi_{T(n)} - \xi_{T(n_j)}| \ge d$, or $n \ge n_\infty$, whichever comes first. Then n_j's are stopping times w.r.t. $\{\mathcal{F}_k\}$, and they are bounded above by n_∞. If we let δ be smaller than some constant depending on d, then $T(n_{j+1}) - T(n_j) \le 2d^2$ and $|\xi_{T(s)} - \xi_{T(n_j)}| \le 2d$ for all $s \in [n_j, n_{j+1}]$ and $j \ge 0$. Let $\mathcal{F}'_j = \mathcal{F}_{n_j}$. Then for any $v \in V_2^\delta$, $\{g_{n_j}(v) : 0 \le j < \infty\}$ is a discrete martingale w.r.t. $\{\mathcal{F}'_j\}$. Since $\varphi_{T(k)} \circ W$ maps D_{y^k} conformally onto $\mathbf{A}_{p-T(k)}$ and takes $y_k = P(y^k)$ to $\chi_{T(k)}$, we have

$$u_{y^k}(z) = \operatorname{Re} \mathbf{S}_{p-T(k)}(\varphi_{T(k)} \circ W(z)/\chi_{T(k)}).$$

By Proposition 3.3, for any $z \in W(V_2^\delta)$ and $0 \le j \le k$,

$$\mathbf{E}\,[\operatorname{Re} \mathbf{S}_{p-T(n_k)}(\varphi_{T(n_k)}(z)/\chi_{T(n_k)})|\mathcal{F}'_j] = \operatorname{Re} \mathbf{S}_{p-T(n_j)}(\varphi_{T(n_j)}(z)/\chi_{T(n_j)}) + o_\delta(1).$$

As δ tends to 0, the set $W(V_2^\delta)$ tends to be dense in $\mathbf{A}_{p_2,p}$. So for any $z \in \mathbf{A}_{p_2,p}$, there is some $z_0 \in W(V_2^\delta)$ such that $|z - z_0| = o_\delta(1)$. Note that $T(n_j) \le T(n_k) \le T(n_\infty) \le q_1$ for $0 \le j \le k$. Using the boundedness of the derivative in Lemma 3.6 with $q = q_1$ and $r = p_2$, we then have that for all $z \in \mathbf{A}_{p_2,p}$,

$$\mathbf{E}\,[\operatorname{Re} \mathbf{S}_{p-T(n_k)}(\varphi_{T(n_k)}(z)/\chi_{T(n_k)})|\mathcal{F}'_j] = \operatorname{Re} \mathbf{S}_{p-T(n_j)}(\varphi_{T(n_j)}(z)/\chi_{T(n_j)}) + o_\delta(1).$$

Now consider the maps in the covering space. We use the notations in Section 3.1. And let $\widetilde{\mathbf{A}}_{a,b}$ be the preimage of $\mathbf{A}_{a,b}$ under the map $z \mapsto e^{iz}$. Then we have

$$\begin{aligned}&\mathbf{E}\,[\operatorname{Im} \widetilde{\mathbf{S}}_{p-T(n_k)}(\widetilde{\varphi}_{T(n_k)}(z) - \xi_{T(n_k)})|\mathcal{F}'_j] \\ &\quad = \operatorname{Im} \widetilde{\mathbf{S}}_{p-T(n_j)}(\widetilde{\varphi}_{T(n_j)}(z) - \xi_{T(n_j)}) + o_\delta(1).\end{aligned} \tag{3.1}$$

In Lemma 3.6, let $q = q_1$ and $r = p_2$, then we have some $\iota \in (0, 1/2)$ such that

$$\widetilde{\mathbf{A}}_{\iota(p-t),p-t} \supset \widetilde{\varphi}_t(\widetilde{\mathbf{A}}_{p_2,p}) \supset \widetilde{\mathbf{A}}_{(1-\iota)(p-t),p-t}, \tag{3.2}$$

for $0 \le t \le q_1$.

Proposition 3.4. *There are an absolute constant $C > 0$ and a constant $\delta(d) > 0$ such that if $\delta < \delta(d)$, then for all $j \ge 0$,*

$$\begin{aligned}&|\mathbf{E}\,[\xi_{T(n_{j+1})} - \xi_{T(n_j)}|\mathcal{F}'_j]| \le Cd^3, \text{ and} \\ &\quad |\mathbf{E}\,[(\xi_{T(n_{j+1})} - \xi_{T(n_j)})^2/2 - (T(n_{j+1}) - T(n_j))|\mathcal{F}'_j]| \le Cd^3.\end{aligned}$$

Proof. Fix some $j \ge 0$. Let $a = T(n_j)$ and $b = T(n_{j+1})$. Then $0 \le a \le b \le q_1$. And if δ is less than some $\delta_1(d)$, we have $|b-a| \le 2d^2$ and $|\xi_c - \xi_a| \le 2d$, for any $c \in [a,b]$. Now suppose $z \in \widetilde{\mathbf{A}}_{p_2,p}$, and consider

$$I := \widetilde{\mathbf{S}}_{p-b}(\widetilde{\varphi}_b(z) - \xi_b) - \widetilde{\mathbf{S}}_{p-a}(\widetilde{\varphi}_a(z) - \xi_a).$$

Then $I = I_1 + I_2$, where

$$I_1 := \widetilde{\mathbf{S}}_{p-b}(\widetilde{\varphi}_b(z) - \xi_b) - \widetilde{\mathbf{S}}_{p-b}(\widetilde{\varphi}_a(z) - \xi_a),$$

$$I_2 := \widetilde{\mathbf{S}}_{p-b}(\widetilde{\varphi}_a(z) - \xi_a) - \widetilde{\mathbf{S}}_{p-a}(\widetilde{\varphi}_a(z) - \xi_a).$$

Then for some $c_1 \in [a,b]$, $I_1 = I_3 + I_4 + I_5$, where

$$I_3 := \widetilde{\mathbf{S}}'_{p-b}(\widetilde{\varphi}_a(z) - \xi_a)[(\widetilde{\varphi}_b(z) - \widetilde{\varphi}_a(z)) - (\xi_b - \xi_a)],$$

$$I_4 := \widetilde{\mathbf{S}}''_{p-b}(\widetilde{\varphi}_a(z) - \xi_a)[(\widetilde{\varphi}_b(z) - \widetilde{\varphi}_a(z)) - (\xi_b - \xi_a)]^2/2,$$

$$I_5 := \widetilde{\mathbf{S}}'''_{p-b}(\widetilde{\varphi}_{c_1}(z) - \xi_{c_1})[(\widetilde{\varphi}_b(z) - \widetilde{\varphi}_a(z)) - (\xi_b - \xi_a)]^3/6.$$

And for some $c_2 \in [a,b]$, we have

$$I_2 = -\partial_r \widetilde{\mathbf{S}}_{p-b}(\widetilde{\varphi}_a(z) - \xi_a)(b-a) + \partial_r^2 \widetilde{\mathbf{S}}_{p-c_2}(\widetilde{\varphi}_a(z) - \xi_a)(b-a)^2/2. \quad (3.3)$$

Now for some $c_3 \in [a,b]$, we have

$$\widetilde{\varphi}_b(z) - \widetilde{\varphi}_a(z) = \partial_r \widetilde{\varphi}_{c_3}(z)(b-a) = \widetilde{\mathbf{S}}_{p-c_3}(\widetilde{\varphi}_{c_3}(z) - \xi_{c_3})(b-a). \quad (3.4)$$

For some $c_4 \in [c_3, b]$, we have

$$\widetilde{\mathbf{S}}_{p-c_3}(\widetilde{\varphi}_{c_3}(z) - \xi_{c_3}) = \widetilde{\mathbf{S}}_{p-b}(\widetilde{\varphi}_{c_3}(z) - \xi_{c_3}) + \partial_r \widetilde{\mathbf{S}}_{p-c_4}(\widetilde{\varphi}_{c_3}(z) - \xi_{c_3})(b-c_3). \quad (3.5)$$

For some $c_5 \in [a, c_3]$, we have

$$\widetilde{\mathbf{S}}_{p-b}(\widetilde{\varphi}_{c_3}(z) - \xi_{c_3}) = \widetilde{\mathbf{S}}_{p-b}(\widetilde{\varphi}_a(z) - \xi_a) + \widetilde{\mathbf{S}}'_{p-b}(\widetilde{\varphi}_{c_5}(z) - \xi_{c_5})[(\widetilde{\varphi}_{c_3}(z) - \widetilde{\varphi}_a(z)) - (\xi_{c_3} - \xi_a)]. \quad (3.6)$$

Once again, there is $c_6 \in [a, c_3]$ such that

$$\widetilde{\varphi}_{c_3}(z) - \widetilde{\varphi}_a(z) = \partial_r \widetilde{\varphi}_{c_6}(z)(c_3 - a) = \widetilde{\mathbf{S}}_{p-c_6}(\widetilde{\varphi}_{c_6}(z) - \xi_{c_6})(c_3 - a). \quad (3.7)$$

We have the freedom to choose d arbitrarily small. Now suppose $d < (1-\iota)(p-q_1)/2$. Then

$$p - a \le p - b + 2d \le (p-b) + (1-\iota)(p-q_1) \le (2-\iota)(p-b).$$

Thus for any $m \le M \in [a,b]$, $p - m \le (2-\iota)(p-M)$. By formula (3.2),

$$\widetilde{\varphi}_m(z) - \xi_m \in \widetilde{\mathbf{A}}_{\iota(p-m),p-m} \subset \widetilde{\mathbf{A}}_{\iota(p-M),(2-\iota)(p-M)}.$$

So the values of $\widetilde{\mathbf{S}}_{p-M}$, $\partial_r\widetilde{\mathbf{S}}_{p-M}$, $\partial_r^2\widetilde{\mathbf{S}}_{p-M}$, $\widetilde{\mathbf{S}}'_{p-M}$, $\widetilde{\mathbf{S}}''_{p-M}$ and $\widetilde{\mathbf{S}}'''_{p-M}$ at $\widetilde{\varphi}_m(z)-\xi_m$ are uniformly bounded. In formula (3.3), consider $m=a$ and $M=c_2$. Since $|b-a|\le 2d^2$, we have

$$I_2=-\partial_r\widetilde{\mathbf{S}}_{p-b}(\widetilde{\varphi}_a(z)-\xi_a)(b-a)+O(d^4).$$

Similarly, formula (3.7) implies

$$\widetilde{\varphi}_{c_3}(z)-\widetilde{\varphi}_a(z)=O(c_3-a)=O(d^2).$$

This together with formulae (3.5),(3.6) and $\xi_{c_3}-\xi_a=O(d)$ implies that

$$\widetilde{\mathbf{S}}_{p-c_3}(\widetilde{\varphi}_{c_3}(z)-\xi_{c_3})=\widetilde{\mathbf{S}}_{p-b}(\widetilde{\varphi}_a(z)-\xi_a)+O(d).$$

By formula (3.4), we have

$$\widetilde{\varphi}_b(z)-\widetilde{\varphi}_a(z)=\widetilde{\mathbf{S}}_{p-b}(\widetilde{\varphi}_a(z)-\xi_a)(b-a)+O(d^3)=O(d^2).$$

Thus $I_5=O(d^3)$,

$$I_4=\widetilde{\mathbf{S}}''_{p-b}(\widetilde{\varphi}_a(z)-\xi_a)(\xi_b-\xi_a)^2/2+O(d^3),\ \text{and}$$
$$I_3=\widetilde{\mathbf{S}}'_{p-b}(\widetilde{\varphi}_a(z)-\xi_a)[\widetilde{\mathbf{S}}_{p-b}(\widetilde{\varphi}_a(z)-\xi_a)(b-a)-(\xi_b-\xi_a)]+O(d^3).$$

Note that $I=I_2+I_3+I_4+I_5$. Using Lemma 3.1, we get

$$\begin{aligned}I=\;&\widetilde{\mathbf{S}}''_{p-b}(\widetilde{\varphi}_a(z)-\xi_a)[(\xi_b-\xi_a)^2/2-(b-a)]\\&-\widetilde{\mathbf{S}}'_{p-b}(\widetilde{\varphi}_a(z)-\xi_a)(\xi_b-\xi_a)+O(d^3).\end{aligned}$$

By formula (3.1), if δ is smaller than some $\delta_2(d)$, then the conditional expectation of

$$\operatorname{Im}\widetilde{\mathbf{S}}''_{p-b}(\widetilde{\varphi}_a(z)-\xi_a)[(\xi_b-\xi_a)^2/2-(b-a)]-\operatorname{Im}\widetilde{\mathbf{S}}'_{p-b}(\widetilde{\varphi}_a(z)-\xi_a)[\xi_b-\xi_a]$$

w.r.t. $\mathcal{F}'_j$ is bounded by C_1d^3.

By formula (3.2), for any $w\in\widetilde{\mathbf{A}}_{(1-\iota)(p-a),p-a}$, the conditional expectation of

$$\operatorname{Im}\widetilde{\mathbf{S}}''_{p-b}(w)[(\xi_b-\xi_a)^2/2-(b-a)]-\operatorname{Im}\widetilde{\mathbf{S}}''_{p-b}(w)[\xi_b-\xi_a] \tag{3.8}$$

w.r.t $\mathcal{F}'_j$ is bounded by C_1d^3, if δ is small enough (depending on d).

Now suppose $d<(p-q_1)\iota/(4-4\iota)$. Then

$$(1-\iota)(p-a)<(1-\iota/2)(p-b)<p-a.$$

Thus $i(1-\iota/2)(p-b)\in\widetilde{\mathbf{A}}_{(1-\iota)(p-a),p-a}$. We may check

$$\operatorname{Im}\widetilde{\mathbf{S}}''_{p-b}(i(1-\iota/2)(p-b))>0,\ \text{and}\ \operatorname{Im}\widetilde{\mathbf{S}}'_{p-b}(i(1-\iota/2)(p-b))=0.$$

So we can find $C_2>0$ such that for all $b\in[0,q_1]$, $\operatorname{Im}\widetilde{\mathbf{S}}''_{p-b}(i(1-\iota/2)(p-b))>C_2$. Let $w=i(1-\iota/2)(p-b)$ in formula (3.8), then we get

$$|\mathbf{E}\,[(\xi_b-\xi_a)^2/2-(b-a)|\mathcal{F}'_j]|\le C_3d^3.$$

Since $\mathrm{Im}\,\widetilde{\mathbf{S}}''_{p-b}(w)$ is uniformly bounded on $\widetilde{\mathbf{C}}_{(1-\iota/2)(p-b)}$, so for all $w \in \widetilde{\mathbf{C}}_{(1-\iota/2)(p-b)}$,

$$\mathrm{Im}\,\widetilde{\mathbf{S}}'_{p-b}(w)|\mathbf{E}\,[\xi_b - \xi_a|\mathcal{F}'_j]| \le C_4 d^3. \tag{3.9}$$

We may check that

$$x_b := \mathrm{Im}\,\widetilde{\mathbf{S}}_{p-b}(\pi + i(1-\iota/2)(p-b)) - \mathrm{Im}\,\widetilde{\mathbf{S}}_{p-b}(i(1-\iota/2)(p-b)) > 0.$$

So x_b is greater than some absolute constant $C_5 > 0$ for $b \in [0, q_1]$. Then there exists $w_b \in \widetilde{\mathbf{C}}_{(1-\iota/2)(p-b)}$ such that

$$|\mathrm{Im}\,\widetilde{\mathbf{S}}'_{p-b}(w_b)| = |\partial_x \mathrm{Im}\,\widetilde{\mathbf{S}}_{p-b}(w_b)| = x_b/\pi \ge C_5/\pi.$$

Plugging $w = w_b$ in formula (3.9), we then have $|\mathbf{E}\,[\xi_b - \xi_a|\mathcal{F}'_j]| \le C_6 d^3$. □

The following Theorem about the convergence of the driving process can be deduced from Proposition 3.4 by using the Skorokhod Embedding Theorem. It is very similar to Theorem 3.6 in [8]. So we omit the proof.

Theorem 3.1. *For every $q_0 \in (0, p)$ and $\varepsilon > 0$ there is a $\delta_0 > 0$ depending on q_0 and ε such that for $\delta < \delta_0$ there is a coupling of the processes ξ_t and $B(2t)$ such that*

$$\mathbf{P}[\sup\{|\xi_t - B(2t)\}| : t \in [0, q_0]\} > \varepsilon] < \varepsilon.$$

3.4. Convergence of the trace

In this subsection, we will prove Theorem 1.2. We use symbols y^δ, β^δ, K^δ and χ^δ to emphasize the fact that they depend on δ. Let $(K_t^0, 0 \le t < p)$ be the annulus SLE_2 in D from 0_+ to B_2. Let $\beta^0 : (0, p) \to D$ be the corresponding trace.

First, we need two well-known lemmas about simple random walks on $\delta\mathbb{Z}^2$. We use the superscript # to denote the spherical metric.

Lemma 3.7. *Suppose $v \in \delta\mathbb{Z}^2$ and K is a connected set on the plane that has Euclidean (spherical, resp.) diameter at least R. Then the probability that a simple random walk on $\delta\mathbb{Z}^2$ started from v will exit $\mathbf{B}(v; R)$ ($\mathbf{B}^\#(v; R)$, resp.) before using an edge of $\delta\mathbb{Z}^2$ that intersects K is at most $C_0((\delta + dist(v, K))/R)^{C_1}$ ($C_0((\delta + dist^\#(v, K))/R)^{C_1}$, resp.) for some absolute constants $C_0, C_1 > 0$.*

Lemma 3.8. *Suppose U is a plane domain, and has a compact subset K and a non-empty open subset V. Then there are positive constants δ_0 and C depending on U, V and K, such that when $\delta < \delta_0$, the probability that a simple random walk on $\delta\mathbb{Z}^2$ started from some $v \in \delta\mathbb{Z}^2 \cap K$ will hit V before exiting U is greater than C.*

The following lemma about simple random walks on D^δ is an easy consequence of the above two lemmas and the Markov property of random walks.

Lemma 3.9. *For every $d > 0$, there are $\delta_0, C > 0$ depending on d such that if $\delta < \delta_0$ and $v \in \delta\mathbb{Z}^2 \cap D$ is such that $dist^{\#}(v, B_1) > d$, then the probability that a simple random walk on D^δ started from v hits B_2 before B_1 is at least C.*

Lemma 3.10. *For every $q \in (0, p)$ and $\varepsilon > 0$, there are $d, \delta_0 > 0$ depending on q and ε such that for $\delta < \delta_0$, the probability that $dist^{\#}(\beta^\delta[q, p), B_1) \geq d$ is at least $1 - \varepsilon$.*

Proof. For $k = 1, 2, 3$, let $J_k = W^{-1}(\mathbf{C}_{q/k})$. Then J_1, J_2, J_3 are disjoint Jordan curves in D that separate B_2 from B_1. And J_2 lies in the domain, denoted by Λ, bounded by J_1 and J_3. Moreover, the modulus of the domain bounded by J_k and B_2 is $p - q/k$. Let τ^δ be the first n such that the edge $[y^\delta_{n-1}, y^\delta_n]$ intersects J_2. Then τ^δ is a stopping time. If δ is smaller than the distance between $J_1 \cup J_3$ and J_2, then $y^\delta_{\tau^\delta} \in \Lambda$ and $y^\delta[-1, \tau^\delta]$ does not intersect J_1. Thus $M(D \setminus y^\delta(-1, \tau^\delta]) \geq p - q$, and so $T(\tau^\delta) \leq q$. So it suffices to prove that when δ and d are small enough, the probability that y^δ will get within spherical distance d from B_1 after time τ^δ is less than ε. Let RW^δ denote a simple random walk on D^δ stopped on hitting ∂D, and CRW^δ denote that RW^δ conditioned to hit B_2 before B_1. Let RW^δ_v and CRW^δ_v denote that RW^δ and CRW^δ, respectively, started from v. Since y^δ is obtained by erasing loops of $\mathrm{CRW}^\delta_\delta$, it suffices to show that the probability that $\mathrm{CRW}^\delta_\delta$ will get within spherical distance d from B_1 after it hits Λ, tends to zero as $d, \delta \to 0$. Since CRW^δ is a Markov chain, it suffices to prove that the probability that CRW^δ_v will get within spherical distance d from B_1 tends to zero as $d, \delta \to 0$, uniformly in $v \in \delta\mathbb{Z}^2 \cap \Lambda$. By Lemma 3.9, there is $a > 0$ such that for δ small enough, the probability that RW^δ_v hits B_2 before B_1 is greater than a, for all $v \in \delta\mathbb{Z}^2 \cap \Lambda$. By Markov property, for every $v \in \delta\mathbb{Z}^2 \cap \Lambda$, the probability that CRW^δ_v will get within spherical distance d from B_1 is less than

$$\frac{1}{a} \cdot \sup\{\mathbf{P}\,[\mathrm{RW}^\delta_w \text{ hits } B_2 \text{ before } B_1] : w \in V(D^\delta) \cap D \text{ and } dist^{\#}(w, B_1) < d\},$$

which tends to 0 as $d, \delta \to 0$ by Lemma 3.7. So the proof is finished. □

Lemma 3.11. *For every $q \in (0, p)$ and $\varepsilon > 0$, there are $M, \delta_0 > 0$ depending on q and ε such that for $\delta < \delta_0$, the probability that $\beta^\delta[q, p) \subset \mathbf{B}(0; M)$ is at least $1 - \varepsilon$.*

Proof. We use the notations of the last lemma. It suffices to prove that the probability that $\mathrm{RW}^\delta_v \not\subset \mathbf{B}(0; M)$ tends to zero as $\delta \to 0$ and $M \to \infty$, uniformly in $v \in \delta\mathbb{Z}^2 \cap \Lambda$. Let $K = \mathbb{C} \setminus D$, then K is unbounded, and the distance between $v \in \Lambda$ and K is uniformly bounded from below by some $d > 0$. Let $r > 0$ be such that $\Lambda \subset \mathbf{B}(0; r)$. For $M > r$, let $R = M - r$, then for $v \in \delta\mathbb{Z}^2 \cap \Lambda$, RW^δ_v should exit $\mathbf{B}(v; R)$ before $\mathbf{B}(0; M)$. By Lemma 3.7, the probability that $\mathrm{RW}^\delta_v \not\subset \mathbf{B}(0; M)$ is less than $C_0((\delta + d)/(M - r))^{C_1}$, which tends to 0 as $\delta \to 0$ and $M \to \infty$, uniformly in $v \in \delta\mathbb{Z}^2 \cap \Lambda$. □

Lemma 3.12. *For every $\varepsilon > 0$, there are $q \in (0, p)$ and $\delta_0 > 0$ depending on ε such that when $\delta < \delta_0$, with probability greater than $1 - \varepsilon$, the diameter of $\beta^\delta[q, p)$ is less than ε.*

Proof. The idea is as follows. Note that as $q \to p$, the modulus of $D \setminus \beta^\delta(0, q]$ tends to zero. So for any fixed $a \in (0, p)$, the spherical distance between $\beta^\delta[a, q]$ and B_2 tends to zero as $q \to p$. By Lemma 3.11, if M is big and δ is small, the fact that $\beta^\delta[a, q]$ does not lie in $\mathbf{B}(0; M)$ is an event of small probability. Thus on the complement of this event, the Euclidean distance between $\beta^\delta[a, q]$ and B_2 tends to zero, which means that β^δ gets to some point near B_2 in the Euclidean metric before time q. By Lemma 3.7, RW^δ_v does not go far before hitting ∂D if v is near B_2. The same is true for CRW^δ_v because by Lemma 3.9, RW^δ_v hits B_2 before B_1 with a probability bigger than some positive constant when v is near B_2. Since y^δ is the loop-erasure of CRW^δ, y^δ does not go far after it gets near B_2, nor does β^δ. So the diameter of $\beta^\delta[q, p)$ is small. □

Definition 3.1. *Let $z \in \mathbb{C}$, $r, \varepsilon > 0$. A (z, r, ε)-quasi-loop in a path ω is a pair $a, b \in \omega$ such that $a, b \in \mathbf{B}(z; r)$, $|a - b| \le \varepsilon$, and the subarc of ω with endpoints a and b is not contained in $\mathbf{B}(z; 2r)$. Let $\mathcal{L}^\delta(z, r, \varepsilon)$ denote the event that $\beta^\delta[0, p)$ has a (z, r, ε)-quasi-loop.*

Lemma 3.13. *If $\overline{\mathbf{B}(z; 2r)} \cap B_1 = \emptyset$, then $\lim_{\varepsilon \to 0} \mathbf{P}[\mathcal{L}^\delta(z, r, \varepsilon)] = 0$, uniformly in δ.*

Proof. This lemma is very similar to Lemma 3.4 in [16]. There are two points of difference between them. First, here we are dealing with the loop-erased *conditional* random walk. With Lemma 3.9, the hypothesis $\overline{\mathbf{B}(z; 2r)} \cap B_1 = \emptyset$ guarantees that for some v near $\partial\mathbf{B}(z; 2r)$, the probability that RW^δ_v hits B_2 before B_1 is bounded away from zero uniformly. Second, our LERW is stopped when it hits B_2, while in Lemma 3.4 in [16], the LERW is stopped when it hits some single point. It turns out that the current setting is easier to deal with. See [16] for more details. □

Proposition 3.5. *For every $q \in (0, p)$ and $\varepsilon > 0$, there are $\delta_0, a_0 > 0$ depending on q and ε such that for $\delta < \delta_0$, with probability at least $1 - \varepsilon$, β^δ satisfies the following property. If $q \le t_1 < t_2 < p$, and $|\beta^\delta(t_1) - \beta^\delta(t_2)| < a_0$, then the diameter of $\beta^\delta[t_1, t_2]$ is less than ε.*

Proof. For $d, M > 0$, let $\Lambda_{d,M}$ denote the set of $z \in \mathbf{B}(0; M)$ such that $dist^\#(z, B_1) \ge d$, and $\mathcal{A}^\delta_{d,M}$ denote the event that $\beta^\delta[q, p) \subset \Lambda_{d,M}$. By Lemma 3.10 and 3.11, there are $d_0, M_0, \delta_0 > 0$ such that for $\delta < \delta_0$, $\mathbf{P}[\mathcal{A}^\delta_{d_0,M_0}] > 1 - \varepsilon/2$. Note that the Euclidean distance between Λ_{d_0,M_0} and B_1 is greater than $d_0/2$. Choose $0 < r < \min\{\varepsilon/4, d_0/4\}$. There are finitely many points $z_1, \dots, z_n \in \Lambda_{d_0,M_0}$ such that $\Lambda_{d_0,M_0} \subset \cup_1^n \mathbf{B}(z_j; r/2)$. For $a > 0$, $1 \le j \le n$, let $\mathcal{B}^\delta_{j,a}$ denote the event that $\beta^\delta[0, p)$ does not have a (z_j, r, a)-quasi-loop. Since $r < d_0/4$, we have $\overline{\mathbf{B}(z_j; 2r)} \cap B_1 = \emptyset$. By Lemma 3.13, there is $a_0 \in (0, r/2)$ such that $\mathbf{P}[\mathcal{B}^\delta_{j,a_0}] \ge 1 - \varepsilon/(2n)$ for $1 \le j \le n$. Let $\mathcal{C}^\delta = \cap_1^n \mathcal{B}^\delta_{j,a_0} \cap \mathcal{A}^\delta_{d_0,M_0}$. Then $\mathbf{P}[\mathcal{C}^\delta] > 1 - \varepsilon$ if $\delta < \delta_0$. And on the event $\mathcal{C}^\delta$, if there are $t_1 < t_2 \in [q, p)$ satisfying $|\beta^\delta(t_1) - \beta^\delta(t_2)| < a_0$, then $\beta^\delta(t_1)$ lies in some ball $\mathbf{B}(z_j; r/2)$, so $\beta^\delta(t_2) \in \mathbf{B}(z_j; r)$ as $a_0 < r/2$. Since β^δ does not have a (z_j, r, a_0)-quasi-loop, $\beta^\delta[t_1, t_2] \subset \mathbf{B}(z_j; 2r)$. This then implies that the diameter of $\beta^\delta[t_1, t_2]$ is not bigger than $4r$, which is less than ε. □

Before the proof of Theorem 1.2, we need the notation of convergence of plane domain sequences. We say that a sequence of plane domains $\{\Omega_n\}$ converges to a plane domain Ω, or $\Omega_n \to \Omega$, if

(i) every compact subset of Ω lies in Ω_n, for n large enough;
(ii) for every $z \in \partial\Omega$ there exists $z_n \in \partial\Omega_n$ for each n such that $z_n \to z$.

Note that a sequence of domains may have more than one limits. The following lemma is similar to Theorem 1.8, the Carathéodory kernel theorem, in [13].

Lemma 3.14. *Suppose $\Omega_n \to \Omega$, f_n maps Ω_n conformally onto G_n, and f_n converges to some function f on Ω uniformly on each compact subset of Ω. Then either f is constant on Ω, or f maps Ω conformally onto some domain G. And in the latter case, $G_n \to G$ and f_n^{-1} converges to f^{-1} uniformly on each compact subset of G.*

Proof of Theorem 1.2. Suppose $(\chi^0_t, 0 \le t < p)$ is the driving function of $(W(K^0_t), 0 \le t < p)$. By Theorem 3.1, we may assume that all χ^δ and χ^0 are in the same probability space, so that for every $q \in (0, p)$ and $\varepsilon > 0$ there is an $\delta_0 > 0$ depending on q and ε such that for $\delta < \delta_0$,

$$\mathbf{P}\,[\sup\{|\chi^\delta_t - \chi^0_t| : t \in [0, q]\} > \varepsilon] < \varepsilon.$$

Since β^δ and β^0 are determined by χ^δ and χ^0, respectively, all β^δ and β^0 are also in the same probability space. For the first part of this theorem, it suffices to prove that for every $q \in (0, p)$ and $\varepsilon > 0$ there is $\delta_0 = \delta_0(q, \varepsilon) > 0$ such that for $\delta < \delta_0$,

$$\mathbf{P}\,[\sup\{|\beta^\delta(t) - \beta^0(t)| : t \in [q, p)\} > \varepsilon] < \varepsilon. \tag{3.10}$$

Now choose any sequence $\delta_n \to 0$. Then it contains a subsequence δ_{n_k} such that for each $q \in (0, p)$, $\chi^{\delta_{n_k}}$ converges to χ^0 uniformly on $[0, q]$ almost surely. Here we use the fact that a sequence converging in probability contains an a.s. converging subsequence. For simplicity, we write δ_n instead of δ_{n_k}. Let $\varphi^{\delta_n}_t$ (φ^0_t, resp.), $0 \le t < p$, be the standard annulus LE maps of modulus p driven by $\chi^{\delta_n}_t$ (χ^0_t, resp.), $0 \le t < p$. Let $\Omega^{\delta_n}_t := \mathbf{A}_p \setminus W(\beta^{\delta_n}(0, t])$, and $\Omega^0_t := \mathbf{A}_p \setminus W(\beta^0[0, t])$. Fix $q \in (0, p)$. Suppose K is a compact subset of Ω^0_q. Then for every $z \in K$, $\varphi^0_t(z)$ does not blow up on $[0, q]$. Since the driving function χ^{δ_n} converges to χ^0 uniformly on $[0, q]$, so if n is big enough, then for every $z \in K$, $\varphi^{\delta_n}_t(z)$ does not blow up on $[0, q]$, which means that $K \subset \Omega^{\delta_n}_q$. Moreover, $\varphi^{\delta_n}_q$ converges to φ^0_q uniformly on K. It follows that $\Omega^{\delta_n}_q \cap \Omega^0_q \to \Omega^0_q$ as $n \to \infty$. By Lemma 3.14, $(\varphi^{\delta_n}_q)^{-1}$ converges to $(\varphi^0_q)^{-1}$ uniformly on each compact subset of $\mathbf{A}_{p-q}$, and so $\Omega^{\delta_n}_q = (\varphi^{\delta_n}_q)^{-1}(\mathbf{A}_{p-q}) \to (\varphi^0_q)^{-1}(\mathbf{A}_{p-q}) = \Omega^0_q$. Now we denote $D^{\delta_n}_t := D \setminus \beta^{\delta_n}(0, t] = W^{-1}(\Omega^{\delta_n}_t)$, and $D^0_t := D \setminus \beta^0(0, t] = W^{-1}(\Omega^0_t)$. Then we have $D^{\delta_n}_q \to D^0_q$ for every $q \in (0, p)$.

Fix $\varepsilon > 0$ and $q_1 < q_2 \in (0, p)$. Let $q_0 = q_1/2$ and $q_3 = (q_2 + p)/2$. By Proposition 3.5, there are $n_1 \in \mathbb{N}$ and $a \in (0, \varepsilon/2)$ such that for $n \ge n_1$, with probability at least $1 - \varepsilon/3$, β^{δ_n} satisfies: if $q_0 \le t_1 < t_2 < p$, and $|\beta^{\delta_n}(t_1) - \beta^{\delta_n}(t_2)| < a$, then the diameter of $\beta^{\delta_n}[t_1, t_2]$ is less than $\varepsilon/3$. Let $\mathcal{A}_n$ denote the corresponding event. Since β^0 is continuous, there is $b > 0$ such that with probability $1-\varepsilon/3$, we have $|\beta^0(t_1)-\beta^0(t_2)| < a/2$ if $t_1, t_2 \in [q_0, q_3]$ and $|t_1 - t_2| \le b$. Let $\mathcal{B}$ denote the corresponding event. We may choose $q_0 < t_0 < t_1 = q_1 < \cdots < t_{m-1} = q_2 < t_m < q_3$ such that $t_j - t_{j-1} < b$ for $1 \le j \le m$. Since $\beta^0(t_j) \notin \beta^0(0, t_{j-1}]$ for $1 \le j \le m$, there is $r \in (0, a/4)$ such that with probability at least $1-\varepsilon/3$, $\overline{\mathbf{B}(\beta^0(t_j); r)} \subset D^0_{t_{j-1}}$ for all $0 \le j \le m$. We now use the convergence of $D^{\delta_n}_t$ to D^0_t for $t = t_0, \ldots, t_m$. There exists $n_2 \in \mathbb{N}$ such that for $n \ge n_2$, with probability at least $1-\varepsilon/3$, $\overline{\mathbf{B}(\beta^0(t_j); r)} \subset D^{\delta_n}_{t_{j-1}}$, and there is some $z^n_j \in \partial D^{\delta_n}_{t_j} \cap \mathbf{B}(\beta^0(t_j); r)$, for all $1 \le j \le m$. Let $\mathcal{C}_n$ denote the corresponding event. Then on the event $\mathcal{C}_n$, $z^n_j \in \partial D^{\delta_n}_j \backslash \partial D^{\delta_n}_{j-1}$, so $z^n_j = \beta^{\delta_n}(s^n_j)$ for some $s^n_j \in (t_{j-1}, t_j]$. Let $\mathcal{D}_n = \mathcal{A}_n \cap \mathcal{B} \cap \mathcal{C}_n$. Then $\mathbf{P}[\mathcal{D}_n] \ge 1-\varepsilon$, for $n \ge n_1 + n_2$. And on the event $\mathcal{D}_n$,

$$|z^n_j - z^n_{j+1}| \le 2r + |\beta^0(t_j) - \beta^0(t_{j+1})| \le 2r + a/2 < a, \ \forall 1 \le j \le m-1,$$

as $|t_j - t_{j+1}| \le b$. Thus the diameter of $\beta^{\delta_n}[s^n_j, s^n_{j+1}]$ is less than $\varepsilon/3$. It follows that for any $t \in [s^n_j, s^n_{j+1}] \subset [t_{j-1}, t_{j+1}]$,

$$\begin{aligned}|\beta^0(t) - \beta^{\delta_n}(t)| &\le |\beta^0(t) - \beta^0(t_j)| + |\beta^0(t_j) - z^n_j| + |z^n_j - \beta^{\delta_n}(t)| \\ &\le a/2 + r + \varepsilon/3 < \varepsilon.\end{aligned}$$

Since $[q_1, q_2] = [t_1, t_{m-1}] \subset \cup_{j=1}^{m-1}[s^n_j, s^n_{j+1}]$, we have now proved that for n big enough, with probability at least $1-\varepsilon$, $|\beta^{\delta_n}(t) - \beta^0(t)| < \varepsilon$ for all $t \in [q_1, q_2]$. By Lemma 3.12, for any $\varepsilon > 0$, there is $q(\varepsilon) \in (0, p)$ such that if n is big enough, with probability at least $1-\varepsilon$, the diameter of $\beta^{\delta_n}[q(\varepsilon), p)$ is less than ε. For any $S \in [q(\varepsilon), p)$, by the uniform convergence of β^{δ_n} to β^0 on the interval $[q(\varepsilon), S]$, it follows that with probability at least $1-\varepsilon$, the diameter of $\beta^0[q(\varepsilon), S)$ is no more than ε, nor is the diameter of $\beta^0[q(\varepsilon), p)$. Now for fixed $q \in (0, p)$ and $\varepsilon > 0$, choose $q_1 \in (q, p) \cap (q(\varepsilon/3), p)$. Then with probability at least $1-\varepsilon/3$, the diameter of $\beta^0[q_1, p)$ is less than $\varepsilon/3$. And if n is big enough, then with probability at least $1-\varepsilon/3$, the diameter of $\beta^{\delta_n}[q_1, p)$ is less than $\varepsilon/3$. Moreover, if n is big enough, we may require that with probability at least $1-\varepsilon/3$, $|\beta^{\delta_n}(t) - \beta^0(t)| \le \varepsilon/3$ for all $t \in [q, q_1]$. Thus $|\beta^{\delta_n}(t) - \beta^0(t)| \le \varepsilon$ for all $t \in [q, p)$ with probability at least $1-\varepsilon$, if n is big enough. Since $\{\delta_n\}$ is chosen arbitrarily, we proved formula (3.10).

Now suppose that the impression of 0_+ is the a single point, which must be 0. From [13], we see that $W^{-1}(z) \to 0$ as $z \in \mathbf{A}_p$ and $z \to 1$. From above, it suffices to prove that for any $\varepsilon > 0$, we can choose $q \in (0, p)$ and $\delta_0 > 0$ such that for $\delta < \delta_0$, with probability at least $1-\varepsilon$, the diameters of $\beta^\delta(0, q]$ and $\beta^0(0, q]$ are less than ε. Since W^{-1} is continuous at 1, we need only to prove the same is true for the diameters of $W(\beta^\delta(0, q])$ and $W(\beta^0(0, q])$. Note that they are the standard annulus LE hulls of modulus p at time q, driven by χ^δ_t and χ^0_t, respectively. By Theorem 3.1, if δ and q are small, then the diameters of $\chi^\delta[0, q]$ and $\chi^0[0, q]$ are uniformly small with probability near 1, so are the diameters of $W(\beta^\delta(0, q])$ and $W(\beta^0(0, q])$. □

Corollary 3.1. *Almost surely* $\lim_{t\to p} \beta^0(t)$ *exists on* B_2 *. And the law is the same as the hitting point of a Brownian excursion in* D *started from* 0_+ *conditioned to hit* B_2.

A Brownian excursion in D started from 0_+ conditioned to hit B_2 is a random closed subset of D whose law is the weak limit as $\varepsilon \to 0$ of the laws of Brownian motions in D started from $\varepsilon > 0$ stopped on hitting ∂D and conditioned to hit B_2.

Proof of Corollary 1.1. Now we consider the Riemann surface $R_p = (\mathbb{R}/(2\pi\mathbb{Z})) \times (0, p)$. Let $X_0 = (\mathbb{R}/(2\pi\mathbb{Z})) \times \{0\}$ and $X_p = (\mathbb{R}/(2\pi\mathbb{Z})) \times \{p\}$ be the two boundary components of R_p. Then $(x, y) \mapsto e^{-y+ix}$ is a conformal map from R_p onto $\mathbf{A}_p$, and it maps X_0 and X_p onto $\mathbf{C}_0$ and $\mathbf{C}_p$, respectively. So it suffices to prove this corollary with $\mathbf{A}_p$, $\mathbf{C}_p$ and $\mathbf{C}_0$ replaced by R_p, X_p and X_0, respectively.

For $n \in \mathbb{N}$, let G_n be a graph that approximates R_p. The vertex set $V(G_n)$ is

$$\{(2k\pi/n, 2m\pi/n) : 1 \le k \le n, 0 \le m \le \lfloor pn/(2\pi)\rfloor\} \cup \{(2k\pi/n, p) : 1 \le k \le n\},$$

where $\lfloor x \rfloor$ is the maximal integer that is not bigger than x. And two vertices are connected by an edge iff the distance between them is not bigger than $2\pi/n$. If $n > 2\pi/p$, then for every vertex v on X_0 or X_p, there is a unique $u \in V(G_n) \cap R_p$ that is adjacent to v. We write $u = N(v)$. For $v \in V(G_n) \cap X_0$, let RW be a simple random walk on G_n started from $N(v)$ and stopped on hitting $X_0 \cup X_p$. Let CRW be that RW conditioned to hit X_p before X_0. Take the loop-erasure of CRW, and then add the vertex v at the beginning of the loop-erasure. Then we get a simple lattice path from v to X_p. We call this lattice path the LERW from v to X_p. Similarly, for each $v \in X_p$, we may define the LERW from v to X_0. Suppose $v \in V(D^\delta) \cap X_0$ and $u \in V(D^\delta) \cap X_p$. Let $P_{v,u}$ be the LERW from v to X_p, conditioned to hit u, and $P_{u,v}$ be the LERW from u to X_0, conditioned to hit v. By Lemma 7.2.1 in [4], the reversal of $P_{v,u}$ has the same law as $P_{u,v}$. Now we define the LERW from X_0 to X_p to be the LERW from a uniformly distributed random vertex on X_0 to X_p. Similarly, we may define the LERW from X_p to X_0. It is clear that the hitting point at X_p of the LERW from X_0 to X_p is uniformly distributed. So the reversal of the LERW from X_0 to X_p has the same law as the LERW from X_p to X_0. Using the method in the proof of Theorem 1.2, we can show that the law of LERW from X_0 to X_p converges to that of annulus SLE_2 in R_p from a uniform random point on X_0 towards X_p. The same is true if we exchange X_0 with X_p. This ends the proof. □

4. Disc SLE

In this section, we will define another version of SLE: disc SLE, which describes a random process of growing compact subsets of a simply connected domain. Suppose Ω is a simply connected domain and $x \in \Omega$. Recall that a hull, say F, in Ω w.r.t. x, is a contractible compact subset of Ω that properly contains x. Then $\Omega \setminus F$ is a doubly connected domain with boundary components $\partial\Omega$ and ∂F. We say that $(F_t, a < t < b)$ is a Loewner chain in Ω w.r.t. x, if (i) each F_t is a hull in Ω w.r.t. x; (ii) $F_s \subsetneq F_t$ when $a < s < t < b$; and (iii) for any fixed $t_0 \in (a, b)$, $(F_t \setminus F_{t_0}, t_0 \le t < b)$ is a Loewner chain in $\Omega \setminus F_{t_0}$ on ∂F_{t_0}.

Proposition 4.1. *Suppose $\chi : (-\infty, 0) \to \mathbf{C}_0$ is continuous. Then there is a Loewner chain $(F_t, -\infty < t < 0)$, in $\mathbb{D}$ w.r.t. 0, and a family of maps g_t, $-\infty < t < 0$, such that each g_t maps $\mathbb{D} \setminus F_t$ conformally onto $\mathbf{A}_{|t|}$ with $g_t(\mathbf{C}_0) = \mathbf{C}_{|t|}$, and*

$$\begin{cases} \partial_t g_t(z) = g_t(z)\mathbf{S}_{|t|}(g_t(z)/\chi_t), & -\infty < t < 0; \\ \lim_{t\to-\infty} e^t/g_t(z) = z, & \forall z \in \mathbb{D}\setminus\{0\}. \end{cases} \tag{4.1}$$

Moreover, such F_t and g_t are uniquely determined by χ_t. We call F_t and g_t, $-\infty < t < 0$, the standard disc LE hulls and maps, respectively, driven by χ_t, $-\infty < t < 0$.

Proof. For fixed $r \in (-\infty, 0)$, let φ_t^r, $r \le t < 0$, be the solution of

$$\partial_t \varphi_t^r(z) = \varphi_t^r(z)\mathbf{S}_{|t|}(\varphi_t^r(z)/\chi_t),\ \ \varphi_r^r(z) = z. \tag{4.2}$$

For $r \le t < 0$, let K_t^r be the set of $z \in \mathbf{A}_{|r|}$ such that $\varphi_s^r(z)$ blows up at some time $s \in [r, t]$. Then $(K_t^r, r \le t < 0)$ is a Loewner chain in $\mathbf{A}_{|r|}$ on $\mathbf{C}_0$, and φ_t^r maps $\mathbf{A}_{|r|} \setminus K_t^r$ conformally onto $\mathbf{A}_{|t|}$ with $\varphi_t^r(\mathbf{C}_{|r|}) = \mathbf{C}_{|t|}$. By the uniqueness of the solution of ODE, if $t_1 \le t_2 \le t_3 < 0$, then $\varphi_{t_3}^{t_2} \circ \varphi_{t_2}^{t_1}(z) = \varphi_{t_3}^{t_1}(z)$, for $z \in \mathbf{A}_{|t_1|} \setminus K_{t_3}^{t_1}$. For $t < 0$, define $R_t(z) = e^t/z$. Then R_t maps $\mathbf{A}_{|t|}$ conformally onto itself, and exchanges the two boundary components. Define $\widehat{\varphi}_t^r = R_t \circ \varphi_t^r \circ R_r$, and $\widehat{K}_t^r = R_r(K_t^r)$. Then $\widehat{K}_t^r$ is a hull in $\mathbf{A}_{|r|}$ on $\mathbf{C}_{|r|}$, and $\widehat{\varphi}_t^r$ maps $\mathbf{A}_{|r|} \setminus \widehat{K}_t^r$ conformally onto $\mathbf{A}_{|t|}$ with $\widehat{\varphi}_t^r(\mathbf{C}_0) = \mathbf{C}_0$. We also have $\widehat{\varphi}_{t_3}^{t_2} \circ \widehat{\varphi}_{t_2}^{t_1}(z) = \widehat{\varphi}_{t_3}^{t_1}(z)$, for $z \in \mathbf{A}_{|t_1|} \setminus \widehat{K}_{t_3}^{t_1}$, if $t_1 \le t_2 \le t_3 < 0$. And $\widehat{\varphi}_t^r$ satisfies

$$\partial_t \widehat{\varphi}_t^r(z) = \widehat{\varphi}_t^r(z)\widehat{\mathbf{S}}_{|t|}(\widehat{\varphi}_t^r(z)/\overline{\chi_t}),\ \ \widehat{\varphi}_r^r(z) = z,$$

where $\widehat{\mathbf{S}}_p(z) = 1 - \mathbf{S}_p(e^{-p}/z)$ for $p > 0$. A simple computation gives:

$$|\widehat{\mathbf{S}}_p(z)| \le 8e^{-p}/|z|,\ \text{ if } 4e^{-p} \le |z| \le 1.$$

We then have

$$|\widehat{\varphi}_t^r(z) - z| \le 8e^t,\ \text{ if }\ r \le t < 0,\ \text{ and }\ 12e^t \le |z| \le 1. \tag{4.3}$$

Now let $\widehat{\psi}_t^r$ be the inverse of $\widehat{\varphi}_t^r$. If $t_1 \le t_2 \le t_3$, then $\widehat{\psi}_{t_2}^{t_1} \circ \widehat{\psi}_{t_3}^{t_2}(z) = \widehat{\psi}_{t_3}^{t_1}(z)$, for any $z \in \mathbf{A}_{|t_3|}$. For fixed $t \in (-\infty, 0)$, $\{\widehat{\psi}_t^r : r \in (-\infty, t]\}$ is a family of uniformly bounded conformal maps on $\mathbf{A}_{|t|}$, so is a normal family. This implies that we can find a sequence $r_n \to -\infty$ such that for any $m \in \mathbb{N}$, $\{\widehat{\psi}_{-m}^{r_n}\}$ converges to some $\widehat{\psi}_{-m}$, uniformly on each compact subset of $\mathbf{A}_m$. Let $\beta_n = \widehat{\psi}_{-m}^{r_n}(\mathbf{C}_{m/2})$. Then β_n is a Jordan curve in $\mathbf{A}_{|r_n|} \setminus \widehat{K}_{-m}^{r_n}$ that separates the two boundary components. So 0 is contained in the Jordan domain determined by β_n. Note that $\{\widehat{\psi}_{-m}^{r_n}\}$ maps $\mathbf{A}_{m/2}$ onto the domain bounded by β_n and $\mathbf{C}_0$, whose modulus has to be $m/2$. So β_n is not contained in $\mathbf{B}(0; e^{-m/2})$. This implies that the diameter of β_n is not less than $e^{-m/2}$. So $\widehat{\psi}_{-m}$ can't be a constant. By Lemma 3.14, $\widehat{\psi}_{-m}$ maps $\mathbf{A}_m$ conformally onto some domain D_{-m}, and $\widehat{\psi}_{-m}^{r_n}(\mathbf{A}_m) \to D_{-m}$. Since $\widehat{\psi}_{-m}^{r_n}(\mathbf{A}_m) = \mathbf{A}_{|r_n|} \setminus \widehat{K}_{-m}^{r_n} \subset \mathbb{D}\setminus\{0\}$, $D_{-m} \subset \mathbb{D}\setminus\{0\}$. Since $M(\mathbf{A}_{|r_n|} \setminus \widehat{K}_{-m}^{r_n}) = m$, there is some $a_m \in (0, 1)$ such that $\overline{\mathbf{B}(0; e^{r_n})} \cup \widehat{K}_{-m}^{r_n} \subset \mathbf{B}(0; e^{-a_m})$ for all r_n. So $\mathbf{A}_{a_m}$ contains no boundary points of $\mathbf{A}_{|r_n|} \setminus \widehat{K}_{-m}^{r_n} = \widehat{\psi}_{-m}^{r_n}(\mathbf{A}_m)$. Since these domains converge to D_{-m} as $n \to \infty$, so $\mathbf{A}_{a_m}$ contains no boundary points of D_{-m}, which means that either $\mathbf{A}_{a_m} \subset D_{-m}$ or $\mathbf{A}_{a_m} \cap D_{-m} = \emptyset$. Now let $\gamma_n = \widehat{\psi}_{-m}^{r_n}(\mathbf{C}_{a_m/2})$. For the same reason as β_n, we have $\gamma_n \not\subset \mathbf{B}(0; e^{-a_m/2})$. So there is $z_n \in \mathbf{C}_{a_m/2}$ such that $|\widehat{\psi}_{-m}^{r_n}(z_n)| \ge e^{-a_m/2}$. Let z_0 be any subsequential limit of $\{z_n\}$, then $z_0 \in \mathbf{C}_{a_m/2} \subset \mathbf{A}_m$ and $|\widehat{\psi}_{-m}(z_0)| \ge e^{-a_m/2}$, so $\widehat{\psi}_{-m}(z_0) \in \mathbf{A}_{a_m}$. Thus $D_{-m} \cap \mathbf{A}_{a_m} \neq \emptyset$, and so $\mathbf{A}_{a_m} \subset D_{-m}$. Hence D_{-m} has one boundary component $\mathbf{C}_0$. Using similar arguments, we have $\widehat{\psi}_t(\mathbf{C}_0) = \mathbf{C}_0$.

If $r_n < -m_1 < -m_2$, then $\widehat{\psi}^{r_n}_{-m_1} \circ \widehat{\psi}^{-m_1}_{-m_2} = \widehat{\psi}^{r_n}_{-m_2}$, which implies $\widehat{\psi}_{-m_1} \circ \widehat{\psi}^{-m_1}_{-m_2} = \widehat{\psi}_{-m_2}$. For $t \in (-\infty, 0)$, choose $m \in \mathbb{N}$ with $-m \le t$, define $\widehat{\psi}_t = \widehat{\psi}_{-m} \circ \widehat{\psi}^{-m}_t$ and $D_t = \widehat{\psi}_t(\mathbf{A}_{|t|})$. It is easy to check that the definition of $\widehat{\psi}_t$ is independent of the choice of m, and the following properties hold. For all $t \in (-\infty, 0)$, D_t is a doubly connected subdomain of $\mathbb{D} \setminus \{0\}$ that has one boundary component $\mathbf{C}_0$, and $\widehat{\psi}_t(\mathbf{C}_0) = \mathbf{C}_0$; $\widehat{\psi}^{r_n}_t$ converges to $\widehat{\psi}_t$, uniformly on each compact subset of $\mathbf{A}_{|t|}$. If $r < t < 0$, then $\widehat{\psi}_t = \widehat{\psi}_r \circ \widehat{\psi}^r_t$; $D_t \subsetneqq D_r$, and $D_r \setminus D_t = \widehat{\psi}_r(\widehat{K}^r_t)$.

Let $\widehat{\varphi}_t$ on D_t be the inverse of $\widehat{\psi}_t$. By Lemma 3.14, $\widehat{\varphi}^{r_n}_t$ converges to $\widehat{\varphi}_t$ as $n \to \infty$, uniformly on each compact subset of D_t. Thus from formula (4.3), we have $|\widehat{\varphi}_t(z) - z| \le 8e^t$, if $12e^t \le |z| < 1$. It follows that $\lim_{t\to-\infty} \widehat{\varphi}_t(z) = z$, for any $z \in \mathbb{D} \setminus \{0\}$. We also have $\widehat{\varphi}_t(z) = \widehat{\varphi}^{-m}_t \circ \widehat{\varphi}_{-m}(z)$, if $-m \le t < 0$ and $z \in D_t$. Let $g_t = R_t \circ \widehat{\varphi}_t$ on D_t. Then g_t maps D_t conformally onto $\mathbf{A}_{|t|}$, takes $\mathbf{C}_0$ to $\mathbf{C}_{|t|}$, and

$$\lim_{t\to-\infty} e^t/g_t(z) = \lim_{t\to-\infty} \widehat{\varphi}_t(z) = z, \text{ for any } z \in \mathbb{D} \setminus \{0\}.$$

If $-m \le t$, then $g_t(z) = \varphi^{-m}_t \circ R_{-m} \circ \widehat{\varphi}_{-m}(z)$, $\forall z \in D_t$. By formula (4.2), we have

$$\partial_t g_t(z) = g_t(z)\mathbf{S}_{|t|}(g_t(z)/\chi_t), \quad -m \le t < 0.$$

Since we may choose $m \in \mathbb{N}$ arbitrarily, formula (4.1) holds.

Let $F_t = \mathbb{D} \setminus D_t$. Since D_t is a doubly connected subdomain of $\mathbb{D} \setminus \{0\}$ with a boundary component $\mathbf{C}_0$, F_t is a hull in $\mathbb{D}$ w.r.t. 0. If $t_1 < t_2 < 0$, then $F_{t_1} \subsetneqq F_{t_2}$, as $D_{t_1} \supsetneqq D_{t_2}$. Fix any $r \in (-\infty, 0)$. For $t \in [r, 0)$, $F_t \setminus F_r = D_r \setminus D_t = \widehat{\psi}_r(\widehat{K}^r_t)$. From Proposition 2.1 and the conformal invariance, $(\widehat{\psi}_r(\widehat{K}^r_t), r \le t < 0)$ is a Loewner chain in D_r on ∂F_r. Thus $(F_t, -\infty < t < 0)$ is a Loewner chain in $\mathbb{D}$ w.r.t. 0.

Suppose F^*_t, $-\infty < t < 0$, is a family of hulls in $\mathbb{D}$ on 0, and g^*_t, $-\infty < t < 0$, is a family of maps such that for each t, g^*_t maps $\mathbb{D} \setminus F^*_t$ conformally onto $\mathbf{A}_{|t|}$ and formula (4.1) holds with g_t replaced by g^*_t. By the uniqueness of the solution of ODE, we have $g^*_t = \varphi^r_t \circ g^*_r$, if $r \le t < 0$. So $R_t \circ g^*_t = \widehat{\varphi}^r_t \circ R_r \circ g^*_r$. Now choose $r = r_n$ and let $n \to \infty$. Since $R_{r_n} \circ g^*_{r_n} \to \mathrm{id}$ by formula (4.1) and $\widehat{\varphi}^{r_n}_t \to \widehat{\varphi}_t$, so $R_t \circ g^*_t = \widehat{\varphi}_t$, from which follows that $g^*_t = R_t \circ \widehat{\varphi}_t = g_t$ and $F^*_t = F_t$. □

Proposition 4.2. *Suppose* $(F_t, -\infty < t < 0)$ *is a Loewner chain in* $\mathbb{D}$ *w.r.t.* 0 *such that* $M(\mathbb{D} \setminus F_t) = |t|$ *for each* t. *Then there is a continuous* $\chi : (-\infty, 0) \to \mathbf{C}_0$ *such that* F_t, $-\infty < t < 0$, *are the standard disc LE hulls driven by* χ_t, $-\infty < t < 0$.

Proof. For each $t < 0$, choose φ^*_t which maps $\mathbb{D} \setminus F_t$ conformally onto $\mathbf{A}_{|t|}$ so that $\varphi^*_t(1) = 1$. Let $g^*_t = R_t \circ \varphi^*_t$, where $R_t(z) = e^t/z$. Then g^*_t maps $\mathbb{D} \setminus F_t$ conformally onto $\mathbf{A}_{|t|}$ with $g^*_t(\mathbf{C}_0) = \mathbf{C}_{|t|}$ and $g^*_t(1) = e^t$. For any $r \le t < 0$, let $K^*_{r,t} = g^*_r(F_t \setminus F_r)$. Then for fixed $r < 0$, $(K^*_{r,t}, r \le t < 0)$ is a Loewner chain in $\mathbf{A}_{|r|}$ on $\mathbf{C}_0$. Now $g^*_t \circ (g^*_r)^{-1}$ maps $\mathbf{A}_{|r|} \setminus K^*_{r,t}$ conformally onto $\mathbf{A}_{|t|}$, and satisfies $g_t^* \circ (g_r^*)^{-1}(e^r) = e^t$. From the proof of Proposition 2.1, there exists some continuous $\chi_{r,\cdot}^* : [r, 0) \to \mathbf{C}_0$ such that for $r \le t < 0$,

$$\partial_t g_t^* \circ (g_r^*)^{-1}(w) = g_t^* \circ (g_r^*)^{-1}(w)[\mathbf{S}_{|t|}(g_t^* \circ (g_r^*)^{-1}(w)/\chi_{r,t}^*) - i\mathrm{Im}\,\mathbf{S}_{|t|}(e^t/\chi_{r,t}^*)].$$

It then follows that

$$\partial_t g_t^*(z) = g_t^*(z)[\mathbf{S}_{|t|}(g_t^*(z)/\chi_{r,t}^*) - i\mathrm{Im}\,\mathbf{S}_{|t|}(e^t/\chi_{r,t}^*)],\ r \le t < 0.$$

So $\chi_{r_1,t}^* = \chi_{r_2,t}^*$ if $r_1, r_2 \le t$. We then have a continuous $\chi^* : (-\infty, 0) \to \mathbf{C}_0$, such that

$$\partial_t g_t^*(z) = g_t^*(z)[\mathbf{S}_{|t|}(g_t^*(z)/\chi_t^*) - i\mathrm{Im}\,\mathbf{S}_{|t|}(e^t/\chi_t^*)],\ -\infty \le t < 0.$$

Consequently,

$$\partial_t \varphi_t^*(z) = \varphi_t^*(z)[\widehat{\mathbf{S}}_{|t|}(\varphi_t^*(z)/\overline{\chi_t^*}) - i\mathrm{Im}\,\widehat{\mathbf{S}}_{|t|}(\chi_t^*)],\ -\infty \le t < 0.$$

Since $|\widehat{\mathbf{S}}_{|t|}(z)| \le 8e^t$ when $4e^t \le |z| \le 1$, $|\mathrm{Im}\,\widehat{\mathbf{S}}_{|t|}(\chi_t^*)|$ decays exponentially as $t \to -\infty$. Let $\theta(t) = \int_{-\infty}^t \mathrm{Im}\,\widehat{\mathbf{S}}_{|s|}(\chi_s^*)ds$, $\varphi_t(z) = e^{i\theta(t)}\varphi_t^*(z)$, and $\chi_t = e^{-i\theta(t)}\chi_t^*$. Then φ_t maps $\mathbb{D} \setminus F_t$ conformally onto $\mathbf{A}_{|t|}$ with $\varphi_t(\mathbf{C}_0) = \mathbf{C}_0$, and

$$\partial_t \ln \varphi_t(z) = \partial_t \ln \varphi_t^*(z) + i\theta'(t) = \widehat{\mathbf{S}}_{|t|}(\varphi_t^*/\overline{\chi_t^*}) = \widehat{\mathbf{S}}_{|t|}(\varphi_t/\overline{\chi_t}).$$

Thus $\partial_t \varphi_t(z) = \varphi_t(z)\widehat{\mathbf{S}}_{|t|}(\varphi_t(z)/\overline{\chi_t})$. From the estimation of $\widehat{\mathbf{S}}_{|t|}$, we have

$$|\varphi_t(z) - \varphi_r(z)| \le 8e^t,\ \text{if}\ 12e^t \le |\varphi_r(z)| \le 1,\ \text{and}\ r \le t < 0.$$

Since F_t contains 0 and $M(\mathbb{D} \setminus F_t) = |t|$, the diameter of F_t tends to zero as $t \to -\infty$. Let $D_t = \mathbb{D} \setminus F_t$. Then for any sequence $t_n \to -\infty$, we have $D_{t_n} \to \mathbb{D} \setminus \{0\}$. Since φ_{t_n} is uniformly bounded, there is a subsequence that converges to some function φ on $\mathbb{D} \setminus \{0\}$ uniformly on each compact subset of $\mathbb{D} \setminus \{0\}$. By checking the image of $\mathbf{C}_1$ under φ_{t_n} similarly as in the proof of Proposition 4.1, we see that φ cannot be constant. So by Lemma 3.14, φ maps $\mathbb{D} \setminus \{0\}$ conformally onto some domain D_0 which is a subsequential limit of $\mathbf{A}_{|t_n|} = \varphi_{t_n}(D_{t_n})$. Since $t_n \to -\infty$, D_0 has to be $\mathbb{D}\setminus\{0\}$ and so $\varphi(z) = \chi z$ for some $\chi \in \mathbf{C}_0$. Now this χ may depend on the subsequence of $\{t_n\}$. But we always have $\lim_{t\to-\infty} |\varphi_t(z)| = |z|$ for any $z \in \mathbb{D}\setminus\{0\}$. Now fix $z \in \mathbb{D}\setminus\{0\}$, there is $s(z) < 0$ such that when $r \le t < s(z)$, we have $12e^t \le |\varphi_r(z)| \le 1$. Therefore $|\varphi_t(z) - \varphi_r(z)| \le 8e^t$ for $r \le t < s(z)$. Thus $\lim_{t\to-\infty} \varphi_t(z)$ exists for every $z \in \mathbb{D} \setminus \{0\}$. Since we have a sequence $t_n \to -\infty$ such that $\{\varphi_{t_n}\}$ converges pointwise to $z \mapsto \chi^* z$ on $\mathbb{D} \setminus \{0\}$ for some $\chi^* \in \mathbf{C}_0$, so $\lim_{t\to-\infty} \varphi_t(z) = \chi^* z$, for all $z \in \mathbb{D}\setminus\{0\}$. Finally, let $g_t(z) = R_t \circ \varphi_t(z/\chi^*)$. Then g_t maps $\mathbb{D} \setminus F_t$ conformally onto $\mathbf{A}_{|t|}$, takes $\mathbf{C}_0$ to $\mathbf{C}_{|t|}$, and satisfies (4.1). □

We still use $B(t)$ to denote a standard Brownian motion on $\mathbb{R}$ started from 0. Let $\mathbf{x}$ be some uniform random point on $\mathbf{C}_0$, independent of $B(t)$. For $\kappa > 0$ and $-\infty < t < 0$, write $\chi_t^\kappa = \mathbf{x}e^{iB(\kappa|t|)}$. The process (χ^κ) is determined by the following properties: for any fixed $r < 0$, $(\chi_t^\kappa/\chi_r^\kappa, r \le t < 0)$ has the same law as $(e^{iB(\kappa(t-r))}, r \le t < 0)$ and is independent from χ_r^κ. If F_t and g_t, $-\infty < t < 0$, are the standard disc LE hulls and maps, respectively, driven by χ_t^κ, $-\infty < t < 0$, then we call them the standard disc SLE_κ hulls and maps, respectively. From the properties of χ_t^κ, we see that for any fixed $r < 0$, $g_r(F_{r+t} \setminus F_r)$, $0 \le t < |r|$, is an annulus $\mathrm{SLE}_\kappa(\mathbf{A}_{|r|}; \chi_r^\kappa \to \mathbf{C}_{|r|})$. The existence of standard annulus SLE_κ trace then implies the a.s. existence of standard disc SLE_κ trace, which is a curve $\gamma : [-\infty, 0) \to \mathbb{D}$ such that $\gamma(-\infty) = 0$, and for each $t \in (-\infty, 0)$, F_t is the hull generated by $\gamma[-\infty, t]$, i.e., the complement of the unbounded component of $\mathbb{C} \setminus \gamma[-\infty, t]$. If $\kappa \le 4$, the trace is a simple curve; otherwise, it is not simple. Suppose D is a simply connected domain and $a \in D$. Let f map $\mathbb{D}$ conformally onto D so that $f(0) = a$ and $f'(0) > 0$. Then we define $f(F_t)$ and $f(\gamma(t))$, $-\infty \le t < 0$, to be the disc $\mathrm{SLE}_\kappa(D; a \to \partial D)$ hulls and trace.

The next theorem is about the equivalence of disc SLE_6 and full plane SLE_6. First, let's review the definition of full plane SLE. It was proved in [12] that for any continuous $\chi : (-\infty, +\infty) \to \mathbf{C}_0$, there is a Loewner chain $(F_t, -\infty < t < +\infty)$, in $\mathbb{C}$ w.r.t. 0, and a family of maps g_t, $-\infty < t < +\infty$, such that for each t, g_t maps $\widehat{\mathbb{C}} \setminus F_t$ conformally onto $\mathbb{D}$ with $g_t(\infty) = 0$, and

$$\begin{cases} \partial_t g_t(z) = g_t(z)\frac{1+g_t(z)/\chi_t}{1-g_t(z)/\chi_t}, & -\infty < t < +\infty; \\ \lim_{t\to-\infty} e^t/g_t(z) = z, & \forall z \in \mathbb{C} \setminus \{0\}. \end{cases}$$

Such F_t and g_t, $-\infty < t < +\infty$, are unique, and are called the full plane LE hulls and maps, respectively, driven by χ_t, $-\infty < t < +\infty$. The diameter of F_t tends to 0 as $t \to -\infty$; and tends to ∞ as $t \to +\infty$.

The driving process of full plane SLE_κ is an extension of χ_t^κ to $\mathbb{R}$ defined as follows. Choose another standard Brownian motion $B'(t)$ on $\mathbb{R}$ started from 0, which is independent of $B(t)$ and $\mathbf{x}$. For $t \ge 0$, let $\chi_t^\kappa = \mathbf{x}e^{iB'(t)}$. Then for any fixed $r \in \mathbb{R}$, $\chi_t^\kappa/\chi_r^\kappa$, $r \le t < +\infty$, have the same distribution as $e^{iB(\kappa(t-r))}$, $r \le t < +\infty$. This implies that for full plane SLE_κ hulls F_t, $t \in \mathbb{R}$, and any fixed $r \in \mathbb{R}$, $(g_r(F_{r+t} \setminus F_r))$ has the same law as radial $\mathrm{SLE}_\kappa(\mathbb{D}; \chi_r^\kappa \to 0)$.

Suppose Ω is a simply connected plane domain that contains 0. Let τ be the first t such that full plane SLE_κ hull $F_t \not\subset \Omega$. Then as $t \nearrow \tau$, F_t approaches $\partial\Omega$, and $(F_t, -\infty < t < \tau)$ is a Loewner chain in Ω w.r.t. 0. Let $u(t) = -M(\Omega \setminus F_t)$, for $-\infty < t < \tau$. Then u is a continuous increasing function, and maps $(-\infty, \tau)$ onto $(-\infty, 0)$. Let v be the inverse of u, and choose f that maps $\mathbb{D}$ onto Ω with $f(0) = 0$ and $f'(0) > 0$. Then $f^{-1}(F_{v(s)})$, $-\infty < s < 0$, are the standard disc LE hulls driven by some function. Using the same method in the proof of Theorem 1.1, we can prove that this driving function has the same law as $(\chi_t^6)_{-\infty<t<0}$. So we have

Theorem 4.1. *Suppose Ω is a simply connected domain that contains 0. Let $(K_t, -\infty < t < +\infty)$ be full plane SLE_6 hulls, and $(L_s, -\infty < s < 0)$ be the disc $SLE_6(\Omega; 0 \to \partial\Omega)$. Let τ be the first t that $K_t \not\subset \Omega$. Then up to a time-change, $(K_t, -\infty < t < \tau)$ has the same law as L_s, $-\infty < s < 0$.*

Corollary 4.1. *The distribution of the hitting point of full plane SLE_6 trace at $\partial\Omega$ is the harmonic measure valued at 0.*

An immediate consequence of this corollary is that the plane SLE_6 hull stopped at the hitting time of $\partial\Omega$ has the same law as the hull generated by a plane Brownian motion started from 0 and stopped on exiting Ω. See [19] and [9] for details.

Disc SLE_2 is also interesting. Suppose Ω is a simply connected domain that contains 0. Let RW be a simple random walk on Ω^δ started from 0, and stopped on hitting $\partial\Omega$. Let LERW be the loop-erasure of RW. Then LERW is a simple lattice path from 0 to $\partial\Omega$. Write LERW as $y = (y_0, \dots, y_\upsilon)$ with $y_0 = 0$ and $y_\upsilon \in \partial\Omega$. We may extend y to be defined on $[0, \upsilon]$ so that it is linear on each $[j-1, j]$ for $1 \le j \le \upsilon$. Then it is clear that $(y(0, s], 0 \le s < \upsilon)$ is a Loewner chain in Ω w.r.t. 0. Let $T(s) = -M(\Omega \setminus y(0, s])$, for $0 < s < \upsilon$. Then T is a continuous increasing function, and maps $(0, \upsilon)$ onto $(-\infty, 0)$. Let S be the inverse of T. Define $\beta^\delta(t) = y(S(t))$, for $-\infty < t < 0$, and $\beta^\delta(-\infty) = 0$. Let $\beta^0 : [-\infty, 0) \to \Omega$ be the trace of disc $\mathrm{SLE}_2(\Omega; 0 \to \partial\Omega)$.

Theorem 4.2. *For any $\varepsilon > 0$, there is $\delta_0 > 0$ such that for $\delta < \delta_0$, we may couple β^δ with β^0 so that*

$$\mathbf{P}\,[\sup\{|\beta^\delta(t) - \beta^0(t)| : -\infty \le t < 0\} \ge \varepsilon] < \varepsilon.$$

Proof. Note that Ω^δ may not be connected, we replace it by its connected component that contains 0. Let g_0 be constant 1 on $V(\Omega^\delta)$. For $0 < j < \upsilon_\delta$, let g_j be the g in Lemma 3.4 with $A = V(\Omega^\delta) \cap \partial\Omega$, $B = \{y_0, \dots, y_{j-1}\}$, and $x = y_j$. Similarly as Proposition 3.2 and 3.3, g_j's are observables for the LERW here, and they approximate the observables for disc SLE_2. We may follow the process in proving Theorem 1.2. □

Corollary 4.2. *Suppose Ω is a simply connected plane domain, and $a \in \Omega$. Let $\beta(s)$, $-\infty < s < 0$, be the disc $SLE_2(\Omega; a \to \partial\Omega)$ trace. Let $\gamma(t)$, $0 < t < \infty$, be the radial $SLE_2(\Omega; \mathbf{x} \to 0)$ trace, where $\mathbf{x}$ is a random point on $\partial\Omega$ with harmonic measure at a. Then the reversal of β has the same law as γ, up to a time-change.*

Proof. This follows immediately from Theorem 4.2, the approximation of LERW to radial SLE_2 in [8], and the reversibility property of LERW in [4]. □

5. Convergence of the observables

This is the last section of this paper. The goal is to prove Proposition 3.3. The proof is sort of long. The main difficulty is that we need the approximation to be uniform in the domains. The tool we can use is Lemma 3.14. However, the limit of a domain sequence in general does not have good boundary conditions, even if every domain in the sequence has. Prime ends and crosscuts are used to describe the boundary correspondence under conformal maps. Some ideas of the proof come from [8].

We will often deal with a function defined on a subset of $\delta\mathbb{Z}^2$. Suppose f is such a function. For $v \in \delta\mathbb{Z}^2$ and $z \in \mathbb{Z}^2$, if $f(v)$ and $f(v + \delta z)$ are defined, then define

$$\nabla_z^\delta f(v) = (f(v + \delta z) - f(v))/\delta,$$

We say that f is δ-harmonic in $\Omega \subset \mathbb{C}$ if f is defined on $\delta\mathbb{Z}^2 \cap \Omega$ and all $v \in \delta\mathbb{Z}^2$ that are adjacent to vertices of $\delta\mathbb{Z}^2 \cap \Omega$ so that for all $v \in \delta\mathbb{Z}^2 \cap \Omega$,

$$f(v+\delta) + f(v-\delta) + f(v+i\delta) + f(v-i\delta) = 4f(v).$$

The following lemma is well known.

Lemma 5.1. *Suppose Ω is a plane domain that has a compact subset K. For $l \in \mathbb{N}$, let $z_1, \dots, z_l \in \mathbb{Z}^2$. Then there are positive constants δ_0 and C depending on Ω, K, and $z_1, \dots, z_l$, such that for $\delta < \delta_0$, if f is non-negative and δ-harmonic in Ω, then for all $v_1, v_2 \in \delta\mathbb{Z}^2 \cap K$,*

$$\nabla^\delta_{z_1} \cdots \nabla^\delta_{z_l} f(v_1) \le Cf(v_2).$$

This is also true for $l = 0$, which means that $f(v_1) \le Cf(v_2)$.

For $a, b \in \delta\mathbb{Z}$, denote

$$S^\delta_{a,b} := \{(x, y) : a \le x \le a+\delta, b \le y \le b+\delta\}.$$

Suppose A is a subset of $\delta\mathbb{Z}^2$, let S^δ_A be the union of all $S^\delta_{a,b}$ whose four vertices are in A. If f is defined on A, we may define a continuous function $\mathrm{CE}^\delta f$ on S^δ_A, as follows. For $(x, y) \in S^\delta_{a,b} \subset S^\delta_A$, define

$$\begin{aligned}\mathrm{CE}^\delta f(x, y) = (1-s)(1-t)f(a, b) + (1-s)tf(a, b+\delta)\\ +s(1-t)f(a+\delta, b) + stf(a+\delta, b+\delta),\end{aligned}$$

where $s = (x-a)/\delta$ and $t = (y-b)/\delta$. Then $\mathrm{CE}^\delta f$ is well defined on S^δ_A, and agrees with f on $S^\delta_A \cap A$. Moreover, on $S^\delta_{a,b}$, $\mathrm{CE}^\delta f$ has a Lipschitz constant not bigger than two times the maximum of $|\nabla^\delta_{(1,0)} f(a, b)|$, $|\nabla^\delta_{(0,1)} f(a, b)|$, $|\nabla^\delta_{(1,0)} f(a, b+\delta)|$, $|\nabla^\delta_{(0,1)} f(a+\delta, b)|$. And for any $u \in \mathbb{Z}^2$,

$$\mathrm{CE}^\delta \nabla^\delta_u f(z) = (\mathrm{CE}^\delta f(z+\delta u) - \mathrm{CE}^\delta f(z))/\delta,$$

when both sides are defined.

Proof of Proposition 3.3. Suppose the proposition is not true. Then we can find $\varepsilon_0 > 0$, a sequence of lattice paths $w_n \in L^{\delta_n}$ with $\delta_n \to 0$, and a sequence of points $v_n \in V_2^{\delta_n}$, such that $|g_{w_n}(v_n) - u_{w_n}(v_n)| > \varepsilon_0$ for all $n \in \mathbb{N}$. For simplicity of notations, we write g_n for g_{w_n}, u_n for u_{w_n}, and D_n for D_{w_n}. Let p_n be the modulus of D_n. The remaining of the proof is composed of four steps.

5.1. The limits of domains and functions

By comparison principle of extremal length, we have $p \ge p_n \ge M(U_2) > 0$. By passing to a subsequence, we may assume that $p_n \to p_0 \in (0, p]$. Then

$\mathbf{A}_{p_n} \to \mathbf{A}_{p_0}$. Let Q_n map D_n conformally onto $\mathbf{A}_{p_n}$ so that $Q_n(z) \to 1$ as $z \in D_n$ and $z \to P(w_n)$. Then $u_n = \mathrm{Re}\,\mathbf{S}_{p_n} \circ Q_n$. Now Q_n^{-1} maps $\mathbf{A}_{p_n}$ conformally onto $D_n \subset D$. Thus $\{Q_n^{-1}\}$ is a normal family. By passing to a subsequence, we may assume that Q_n^{-1} converges to some function J uniformly on each compact subset of $\mathbf{A}_{p_0}$. Using some argument similar to that in the proof of Theorem 1.2, we conclude that J maps $\mathbf{A}_{p_0}$ conformally onto some domain D_0, and $D_n \to D_0$. Let $Q_0 = J^{-1}$ and $u_0 = \mathrm{Re}\,\mathbf{S}_{p_0} \circ Q_0$. Then Q_n and u_n converge to Q_0 and u_0, respectively, uniformly on each compact subset of D_0. Moreover, we have $U_2 \cup \alpha_2 \subset D_0 \subset D$. Thus B_2 is one boundary component of D_0. Let B_1^n and B_1^0 denote the boundary component of D_n and D_0, respectively, other than B_2.

Let $\{K_m\}$ be a sequence of compact subsets of D_0 such that $D_0 = \cup_m K_m$, and for each m, K_m disconnects B_1^0 from B_2 and $K_m \subset \mathrm{int}\, K_{m+1}$. Let $K_m^n = K_m \cap \delta_n \mathbb{Z}^2$. Now fix m. If n is big enough depending on m, we can have the following properties. First, $K_m \subset D_n$ and $K_m^n \subset V(D^{\delta_n})$, so g_n is δ_n-harmonic on K_m. Second, K_m^n disconnects all lattice paths on D^{δ_n} from B_2 to B_1^n. Now let RW_v^n be a simple random walk on D^{δ_n} started from $v \in V(D^{\delta_n})$, and τ_m^n the hitting time of RW_v^n at $B_2 \cup K_m^n$. By the properties of g_n, if v is in D and between K_m and B_2, then $(g_n(\mathrm{RW}_v^n(j)), 0 \le j \le \tau_n^m)$ is a martingale, so $g_n(v) = \mathbf{E}\,[g_n(\mathrm{RW}_n^v(\tau_n^m))]$. Now suppose $g_n(v) > 1$ for all $v \in K_n^m$. Choose $v_0 \in V(D^{\delta_n}) \cap D$ that is adjacent to some vertex of $F^{\delta_n} = V(D^{\delta_n}) \cap B_2$. Then $g_n(v_0) = \mathbf{E}\,[g_n(\mathrm{RW}_n^{v_0}(\tau_n^m))] \ge 1$. The equality holds iff there is no lattice path on D^{δ_n} from v_0 to K_m^n. By the definition of D^{δ_n}, we know that the equality can not always hold. It follows that $\sum_{u \in F^{\delta_n}} \Delta_{D^{\delta_n}} g_n(u) > 0$, which contradicts the definition of g_n. Thus there is $v \in K_m^n$ such that $g_n(v) \le 1$. Note that g_n is non-negative. By Lemma 5.1, if n is big enough depending on m, then g_n on K_m^n is uniformly bounded in n. Similarly for any $z_1, \dots, z_l \in \mathbb{Z}^2$, $\nabla_{z_1}^{\delta_n} \cdots \nabla_{z_l}^{\delta_n} g_n$ on K_m^n is uniformly bounded in n, if n is big enough depending on m, and $z_1, \dots, z_l \in \mathbb{Z}^2$.

We just proved that for a fixed m, if n is big enough depending on m, then g_n on K_{m+1}^n is δ_n-harmonic and uniformly bounded in n. We may also choose n big such that every lattice square of $\delta_n \mathbb{Z}^2$ that intersects K_m is contained in K_{m+1}, and so $\mathrm{CE}^{\delta_n} g_n$ on K_m is well defined, and is uniformly bounded in n. Using the boundedness of $\nabla_u^{\delta_n} g_n$ on K_{m+1}^n for $u \in \{1, i\}$, we conclude that $\{\mathrm{CE}^{\delta_n} g_n\}$ on K_m is uniformly continuous. By Arzela-Ascoli Theorem, there is a subsequence of $\{\mathrm{CE}^{\delta_n} g_n\}$, which converges uniformly on K_m. By passing to a subsequence, we may assume that $\mathrm{CE}^{\delta_n} g_n$ converges uniformly on each K_m. Let g_0 on D_0 be the limit function. Similarly, for any $z_1, \dots, z_l \in \mathbb{Z}^2$, there is a subsequence of $\{\mathrm{CE}^{\delta_n} \nabla_{z_1}^{\delta_n} \cdots \nabla_{z_l}^{\delta_n} g_n\}$ which converges uniformly on each K_m. By passing to a subsequence again, we may assume that for any $z_1, \dots, z_l \in \mathbb{Z}^2$, $\mathrm{CE}^{\delta_n} \nabla_{z_1}^{\delta_n} \cdots \nabla_{z_l}^{\delta_n} g_n$ converges to $g_0^{z_1,\dots,z_l}$ on D_0, uniformly on each K_m. It is easy to check that

$$g_0^{z_1,\dots,z_l} = (a_1 \partial_x + b_1 \partial_y) \cdots (a_l \partial_x + b_l \partial_y) g_0,$$

if $z_j = (a_j, b_j)$, $1 \le j \le l$. Since g_n is δ_n-harmonic on K_m for n big enough, we have $(\nabla_1^{\delta_n} \nabla_{-1}^{\delta_n} + \nabla_i^{\delta_n} \nabla_{-i}^{\delta_n}) g_n \equiv 0$ on K_m^n. Thus $(\partial_x^2 + \partial_y^2) g_0 = 0$, which means that g_0 is harmonic.

Now suppose $x_n \in V(D^{\delta_n}) \cap D \to B_2$ in the spherical metric. Since the spherical distance between K_1 and B_2 is positive, the probability that a simple random walk on D^{δ_n} started from x_n hits K_1 before B_2 tends to zero by Lemma 3.7. If n is big enough, K_1 is a subset of D_n and disconnects B_2 from B_1^n. We have proved that g_n is uniformly bounded on $\delta_n\mathbb{Z}^2 \cap K_1$, if n is big enough. And by definition $g_n \equiv 1$ on $V(D^{\delta_n}) \cap B_2$. By Markov property, we have $g_n(x_n) \to 1$. Since g_0 is the limit of $\mathrm{CE}^{\delta_n} g_n$, this implies that $g_0(z) \to 1$ as $z \in D_0$ and $z \to B_2$ in the spherical metric. Thus $g_0 \circ J(z) \to 1$ as $z \in \mathbf{A}_{p_0}$ and $z \to \mathbf{C}_{p_0}$.

Now let us consider the behavior of u_n and u_0 near B_2. If $z \in D_n$ and $z \to B_2$ in the spherical metric, then $Q_n(z) \to \mathbf{C}_{p_n}$, and so $u_n(z) = \mathrm{Re}\,\mathbf{S}_{p_n} \circ Q_n(z) \to 1$. Using a plane Brownian motion instead of a simple random walk in the above argument, we conclude that $u_n(z) \to 1$ as $z \in D_n$ and $z \to B_2$ in the spherical metric, uniformly in n.

Suppose $\{v_n\}$, chosen at the beginning of this proof, has a subsequence that tends to B_2 in the spherical metric. By passing to a subsequence, we may assume that $v_n \to B_2$ in the spherical metric. From the result of the last two paragraphs, we see that $g_n(v_n) \to 1$ and $u_n(v_n) \to 1$. This contradicts the hypothesis that $|g_n(v_n) - u_n(v_n)| \geq \varepsilon_0$. Thus $\{v_n\}$ has a positive spherical distance from B_2. Since the domain bounded by α_1 and α_2 disconnects U_2 from B_1^0, and $\{v_n\} \subset U_2$, so $\{v_n\}$ has a positive spherical distance from B_1 too. Thus $\{v_n\}$ has a subsequence that converges to some $z_0 \in D_0$. Again we may assume that $v_n \to z_0$. Then $u_0(z_0) = \lim u_n(v_n)$ and $g_0(z_0) = \lim g_n(v_n)$, and so $|u_0(z_0) - g_0(z_0)| \geq \varepsilon_0$. We will get a contradiction by proving that $g_0 \equiv u_0$ in D_0.

Note that g_0 is non-negative, since each g_n is non-negative. We can find a Jordan curve β in D_0 which satisfies the following properties. It disconnects B_2 from B_1^0; it is the union of finite line segments which are parallel to either x or y axis; and it does not intersect $\cup_n \delta_n\mathbb{Z}^2$. By Remark 2 in Section 3 and the uniform convergence of $\nabla_1^{\delta_n} g_n$ to $\partial_x g_0$, and $\nabla_i^{\delta_n} g_n$ to $\partial_y g_0$ on some neighborhood of β, we have $\int_\beta \partial_{\mathbf{n}} g_0 ds = 0$, where $\mathbf{n}$ is the unit norm vector on β pointed towards B_1. Thus g_0 has a harmonic conjugate, and so does $g_0 \circ J$. We will prove $g_0 \circ J = \mathrm{Re}\,\mathbf{S}_{p_0}$, from which follows that $g_0 = u_0$. We have proved that $g_0 \circ J(z) \to 1$ as $\mathbf{A}_{p_0} \ni z \to \mathbf{C}_{p_0}$. It suffices to show that $g_0 \circ J(z) \to 0$ as $\mathbf{A}_{p_0} \setminus U \ni z \to \mathbf{C}_0$ for any neighborhood U of 1.

5.2. The existence of some sequences of crosscuts

For a doubly connected domain Ω and one of its boundary component X, we say that γ is a crosscut in Ω on X if γ is an open simple curve in D whose two ends approach two points (need not be distinct) of X in Euclidean distance. For such γ, $\Omega \setminus \gamma$ has two connected components, one is a simply connected domain, and the other is a doubly connected domain. Let $U(\gamma)$ denote the simply connected component of $D \setminus \gamma$. Then $\partial U(\gamma)$ is the union of γ and a subset of X.

Now Q_0 maps D_0 conformally onto $\mathbf{A}_{p_0}$, and $Q_0(B_1^0) = \mathbf{C}_0$. Similarly as Theorem 2.15 in [13], we can find a sequence of crosscuts $\{\gamma^k\}$ in D_0 on B_1^0 which satisfies

(i) for each k, $\overline{\gamma^{k+1}} \cap \overline{\gamma^k} = \emptyset$ and $U(\gamma^{k+1}) \subset U(\gamma^k)$;
(ii) $Q_0(\gamma^k)$, $k \in \mathbb{N}$, are mutually disjoint crosscuts in $\mathbf{A}_{p_0}$ on $\mathbf{C}_0$; and
(iii) $U(Q_0(\gamma^k))$, $k \in \mathbb{N}$, forms a neighborhood basis of 1 in $\mathbf{A}_{p_0}$.

Note that $U(Q_0(\gamma^k)) = Q_0(U(\gamma^k))$, so $U(Q_0(\gamma^{k+1})) \subset U(Q_0(\gamma^k))$, for all $k \in \mathbb{N}$. We will prove that there is some crosscut γ_n^k in each D_n on B_1^n such that γ_n^k and $Q_n(\gamma_n^k)$ converge to γ^k and $Q_0(\gamma^k)$, respectively, in the sense that we will specify.

Now fix $k \in \mathbb{N}$ and $\varepsilon > 0$. Parameterize $\overline{\gamma^k}$ and $\overline{Q_0(\gamma^k)}$ as the image of the function $a : [0, 1] \to D \cup B_1^0$ and $b : [0, 1] \to \mathbf{A}_{p_0} \cup \mathbf{C}_0$, respectively, so that $b(t) = Q_0(a(t))$, for $t \in (0, 1)$. We may choose $s_1 \in (0, 1/2)$ such that the diameters of $a[0, s_1]$ and $a[1 - s_1, 1]$ are both less than $\varepsilon/3$. There is $r_1 \in (0, \varepsilon) \cap (0, (1 - e^{-p_0})/2)$ such that the curve $b[s_1, 1 - s_1]$ and the balls $\overline{\mathbf{B}(b(0); r_1)}$ and $\overline{\mathbf{B}(b(1); r_1)}$ are mutually disjoint. Suppose γ^k is contained in $\mathbf{B}(0; M)$ for some $M > \varepsilon$. There is $C_M > 0$ such that the spherical distance between any $z_1, z_2 \in \mathbf{B}(0; 2M)$ is at least $C_M|z_1 - z_2|$. So for every smooth curve γ in $\mathbf{B}(0; 2M)$, we have $L^\#(\gamma) \geq C_M L(\gamma)$, where L and $L^\#$ denote the Euclidean length and spherical length, respectively. Let $r_2 = r_1 \exp(-72\pi^2/(C_M^2\varepsilon^2))$. Then we may choose $s_2 \in (0, s_1)$ such that $b[0, s_2] \subset \mathbf{B}(b(0); r_2)$ and $b[1 - s_2, 1] \subset \mathbf{B}(b(1); r_2)$.

For $j = 0, 1$, let Γ_j be the set of crosscuts γ in $\mathbf{A}_{p_0}$ on $\mathbf{C}_0$ such that

$$\mathbf{B}(b(j); r_2) \cap \mathbb{D} \subset U(\gamma) \subset \mathbf{B}(b(j); r_1).$$

Then the extremal length of Γ_j is less than

$$2\pi/(\ln r_1 - \ln r_2) = C_M^2\varepsilon^2/(36\pi).$$

If n is big enough, then $\mathbf{B}(b(j); r_1) \cap \mathbb{D} \subset \mathbf{A}_{p_n}$, so all $\gamma \in \Gamma_j$ are in $\mathbf{A}_{p_n}$. Then the extremal length of $Q_n^{-1}(\Gamma_j)$ is also less than $C_M^2\varepsilon^2/(36\pi)$. Since the spherical area of $Q_n^{-1}(\mathbf{A}_{p_n})$ is not bigger than that of $\mathbb{C}$, which is 4π, there is some $\beta_{n,j}$ in $Q_n^{-1}(\Gamma_j)$ of spherical length less than $C_M\varepsilon/3$. Since

$$J(b[s_2, 1 - s_2]) = a[s_2, 1 - s_2] \subset \gamma^k \subset \mathbf{B}(0; M),$$

and Q_n^{-1} converges to J uniformly on $b[s_2, 1 - s_2]$, so if n is big enough, then $Q_n^{-1}(b[s_2, 1 - s_2]) \subset \mathbf{B}(0; 1.5M)$. Every curve in Γ_j intersects $b[s_2, 1 - s_2]$, so $\beta_{n,j} \in Q_n^{-1}(\Gamma_j)$ intersects $Q_n^{-1}(b[s_2, 1 - s_2]) \subset \mathbf{B}(0; 1.5M)$. If $\beta_{n,j} \not\subset \mathbf{B}(0; 2M)$, then there is a subarc γ of $\beta_{n,j}$ that is contained in $\mathbf{B}(0; 2M)$ and connects $\partial\mathbf{B}(0; 1.5M)$ with $\partial\mathbf{B}(0; 2M)$. So $L^\#(\gamma) \geq C_M L(\gamma) \geq C_M M/2$. This is impossible since $L^\#(\gamma) \leq L^\#(\beta_{n,j}) \leq C_M\varepsilon/3 < C_M M/2$. Thus $\beta_{n,j} \subset \mathbf{B}(0; 2M)$, and so $L(\beta_{n,j}) \leq L^\#(\beta_{n,j})/C_M < \varepsilon/3$. Since $\beta_{n,j}$ has finite length, it is a crosscut in D_n on B_1^n. Let $s_{n,0}$ be the biggest s such that $Q_n^{-1}(b(s)) \in \beta_{n,0}$, and $s_{n,1}$ the biggest s such that $Q_n^{-1}(b(1 - s)) \in \beta_{n,1}$. Then $s_{n,0}, s_{n,1} \in [s_2, s_1]$. Let $\beta'_{n,0}$ and $\beta'_{n,1}$ denote any one component of $\beta_{n,0} \setminus \{Q_n^{-1}(b(s_{n,0}))\}$ and $\beta_{n,1} \setminus \{Q_n^{-1}(b(1 - s_{n,1}))\}$, respectively. Let

$$\gamma_n^k := Q_n^{-1}(b[s_{n,0}, 1 - s_{n,1}]) \cup \beta'_{n,0} \cup \beta'_{n,1}.$$

Then γ_n^k is a crosscut in D_n on B_1^n. As $r_1 < \varepsilon$, the symmetric difference between $Q_n(\gamma_n^k)$ and $Q_0(\gamma^k)$ is contained in $\mathbf{B}(b(0); \varepsilon) \cup \mathbf{B}(b(1); \varepsilon)$. Since $b[s_{n,0}, 1-s_{n,1}]$ is contained in $b[s_2, 1-s_2]$, which is a compact subset of D_0, so if n is big enough, then the Hausdorff distance between $Q_n^{-1}(b[s_{n,0}, 1-s_{n,1}])$ and $a[s_{n,0}, 1-s_{n,1}]$ is less than $\varepsilon/3$. Now the Hausdorff distance between $Q_n^{-1}(b[s_{n,0}, 1-s_{n,1}])$ and γ_n^k is not bigger than the bigger diameter of $\beta'_{n,0}$ and $\beta'_{n,1}$, which is less than $\varepsilon/3$. And the Hausdorff distance between $a[s_{n,0}, 1-s_{n,1}]$ and γ^k is not bigger than the bigger diameter of $a[0, s_{n,0}]$ and $a[1-s_{n,1}, 1]$, which is also less than $\varepsilon/3$. So the Hausdorff distance between γ_n^k and γ^k is less than ε. Now we proved that we can choose crosscuts γ_n^k in D_n on B_1^n such that γ_n^k converges to γ^k, and the symmetric difference of $Q_n(\gamma_n^k)$ and $Q_0(\gamma^k)$ converges to the two end points of $Q_0(\gamma^k)$, respectively, both in the Hausdorff distance, as n tends to infinity.

5.3. Constructing hooks that hold the boundary

Now fix $k \geq 2$. We still parameterize $\overline{\gamma^k}$ and $\overline{Q_0(\gamma^k)}$ as the image of the function $a : [0,1] \to D \cup B_1^0$ and $b : [0,1] \to \mathbf{A}_{p_0} \cup \mathbf{C}_0$, respectively, such that $b(t) = Q_0(a(t))$, for $t \in (0,1)$. Let Ω^k denote the domain bounded by $Q_0(\gamma^{k-1})$ and $Q_0(\gamma^{k+1})$ in $\mathbf{A}_{p_0}$. Then $\partial\Omega^k$ is composed of $Q_0(\gamma^{k-1})$, $Q_0(\gamma^{k+1})$, and two arcs on $\mathbf{C}_0$. Let ρ_0^k and ρ_1^k denote these two arcs such that $b(j) \in \rho_j^k$, $j = 0, 1$. If n is big enough, from the convergence of $Q_n(\gamma_n^{k\pm1})$ to $Q_0(\gamma^{k\pm1})$, we have $\overline{Q_n(\gamma_n^{k-1})} \cap \overline{Q_n(\gamma_n^{k+1})} = \emptyset$, and $U(Q_n(\gamma_n^{k+1})) \subset U(Q_n(\gamma_n^{k-1}))$. Let Ω_n^k denote the domain bounded by $Q_n(\gamma_n^{k-1})$ and $Q_n(\gamma_n^{k+1})$ in $\mathbf{A}_{p_n}$. Then the boundary of Ω_n^k is composed of $Q_n(\gamma_n^{k-1})$, $Q_0(\gamma_n^{k+1})$, and two disjoint arcs on $\mathbf{C}_0$. If n is big enough, then each of these two arcs contains one of $b(0)$ and $b(1)$. Let $\rho_{n,0}^k$ and $\rho_{n,1}^k$ denote these two arcs so that $b(j) \in \rho_{n,j}^k$, $j = 0, 1$. Now suppose $c : (-1, +1) \to \Omega^k$ is a crosscut in Ω^k with $c(\pm 1) \in Q_0(\gamma^{k\pm1})$. Then $c(-1,+1)$ divides Ω^k into two parts: Ω_0^k and Ω_1^k, so that $\rho_j^k \subset \partial\Omega_j^k$, $j = 0, 1$. If n is big enough, then $c(\pm1) \in Q_n(\gamma_n^{k\pm1})$, and $c(-1,+1) \subset \Omega_n^k$. Thus $c(-1,+1)$ also divides Ω_n^k into two parts: $\Omega_{n,0}^k$ and $\Omega_{n,1}^k$, so that $\rho_{n,j}^k \subset \partial\Omega_{n,j}^k$. Let λ_j ($\lambda_{n,j}$, resp.) be the extremal distance between $Q_0(\gamma^{k-1})$ ($Q_n(\gamma_n^{k-1})$, resp.) and $Q_0(\gamma^{k+1})$ ($Q_n(\gamma_n^{k+1})$, resp.) in Ω_j^k ($\Omega_{n,j}^k$, resp.), $j = 0, 1$. It is clear that $\lambda_{n,j} \to \lambda_j$ as $n \to \infty$, and $\lambda_j < \infty$. Thus $\{\lambda_{n,j}\}$ is bounded by some $I_k > 0$.

Since $\overline{\gamma^k} \cap \overline{\gamma^{k\pm1}} = \emptyset$ and $\gamma_n^{k\pm1}$ converges to $\gamma^{k\pm1}$ in the Hausdorff distance, there is $d_k > 0$ such that the distance between γ^k and $\gamma_n^{k\pm1}$ is greater than d_k, if n is big enough. For $x \in D_0$ and $r > 0$, let $\widetilde{\mathbf{B}}_0(x; r)$ and $\widetilde{\mathbf{B}}_n(x; r)$ denote the connected component of $\mathbf{B}(x; r) \cap D_0$ and $\mathbf{B}(x; r) \cap D_n$, respectively, that contains x. Since $D_n \to D_0$, it is easy to prove that $\widetilde{\mathbf{B}}_n(x; r) \to \widetilde{\mathbf{B}}_0(x; r)$. Let $e_k = d_k \exp(-2\pi I_k)$. Suppose $s_0 \in (0,1)$ is such that the diameter of $a(0, s_0)$ is less than e_k. By the construction of γ_n^k, we have $\Omega_n^k \to \Omega^k$, so $Q_n^{-1}(\Omega_n^k) \to Q_0^{-1}(\Omega^k)$. Now $a(s_0) \in \gamma^k \subset Q_0^{-1}(\Omega^k)$. Hence $a(s_0) \in Q_n^{-1}(\Omega_n^k)$ if n is big enough. Since the distance from $a(s_0)$ to $\gamma_n^{k\pm1}$ is bigger than $d_k > e_k$, $\widetilde{\mathbf{B}}_n(a(s_0); e_k)$ is contained in $Q_n^{-1}(\Omega_n^k)$. We claim that $\widetilde{\mathbf{B}}_n(a(s_0); e_k) \subset Q_n^{-1}(\Omega_{n,0}^k)$, if n is big enough.

Since $a(0) \in \partial Q_0^{-1}(\Omega^k)$, $|a(0) - a(s_0)| < e_k$, and $Q_n^{-1}(\Omega_n^k) \to Q_0^{-1}(\Omega^k)$, so the distance from $a(s_0)$ to $\partial Q_n^{-1}(\Omega_n^k))$ is less than e_k, if n is big enough. Now choose $z_n \in \partial Q_n^{-1}(\Omega_n^k)$ that is the nearest to $a(s_0)$. Then the line segment $[a(s_0), z_n) \subset \widetilde{\mathbf{B}}_n(a(s_0); e_k)$. Hence $Q_n[a(s_0), z_n)$ is a simple curve in Ω_n^k such that $Q_n(z)$ tends to some $z_n' \in \partial\Omega_n^k$, as $z \in [a(s_0), z_n)$ and $z \to z_n$. Since $z_n \notin \gamma_n^{k\pm1}$, $z_n' \notin Q_n(\gamma_n^{k\pm1})$. Thus z_n' is on $\rho_{n,j}^k$ for some $j \in \{0, 1\}$. Since $Q_n(\widetilde{\mathbf{B}}_n(a(s_0); e_k)) \to Q_0(\widetilde{\mathbf{B}}_0(a(s_0); e_k)) \ni b(s_0)$, and $b(s_0) \in \Omega_{n,0}^k$, so if n is big enough, $Q_n(\widetilde{\mathbf{B}}_n(a(s_0); e_k))$ intersects $\Omega_{n,0}^k$. For such n, if $z_n' \in \rho_{n,1}^k$, then all curves in $Q_n^{-1}(\Omega_{n,0}^k)$ that go from γ_n^{k-1} to γ_n^{k-1} will pass $\widetilde{\mathbf{B}}_n(a(s_0); e_k)$. And so they all cross some annulus centered at $a(s_0)$ with inner radius e_k and outer radius greater than d_k. So the extremal distance between γ_n^{k-1} and γ_n^{k+1} in $Q_n^{-1}(\Omega_{n,j}^k)$ is greater than $(\ln d_k - \ln e_k)/(2\pi) = I_k$. However, by conformal invariance, this extremal distance is equal to $\lambda_{n,j}$, which is not bigger than I_k if n is big enough. Thus $z_n' \in \rho_{n,0}^k$ for n big enough. Similarly, $z_n' \in \rho_{n,0}^k$ and $Q_n(\widetilde{\mathbf{B}}_n(a(s_0); e_k)) \cap \overline{\Omega_{n,1}^k} \neq \emptyset$ can not happen at the same time when n is big enough. So if n is big enough, $Q_n(\widetilde{\mathbf{B}}_n(a(s_0); e_k))$ is contained in $\Omega_{n,0}^k$. Similarly, we let $s_1 \in (s_0, 1)$ be such that the diameter of $a(s_1, 1)$ is less than e_k, then $Q_n(\widetilde{\mathbf{B}}_n(a(s_1); e_k)) \subset \Omega_{n,1}^k$, if n is big enough.

For $j = 0, 1$, $a(s_j)$ and $a(j)$ determine a square of side length $l_j = |a(j) - a(s_j)|$ with vertices $v_{0,j} := a(s_j)$, $v_{2,j}$, $v_{1,j}$, and $v_{3,j}$, in the clockwise order, so that $a(j)$ is on one middle line $[(v_{0,j} + v_{3,j})/2, (v_{1,j} + v_{2,j})/2]$. This square is contained in $\overline{\mathbf{B}(a(s_j); \sqrt{2}l_j)} \subset \mathbf{B}(a(s_j); 0.8e_k)$, since $l_j < e_k/2$. And the union of line segments $[v_{0,j}, v_{1,j}]$, $[v_{1,j}, v_{2,j}]$ and $[v_{2,j}, v_{3,j}]$ surrounds $\mathbf{B}(a(j); l_j/8)$.

For $j = 0, 1$, let N_j be the $l_j/20$-neighborhood of $[v_{0,j}, v_{1,j}] \cup [v_{1,j}, v_{2,j}] \cup [v_{2,j}, v_{3,j}]$. Then $N_j \subset \mathbf{B}(a(s_j); e_k)$. Choose $q_j \in (0, l_j/30)$ such that $\overline{\mathbf{B}(a(s_j); q_j)} \subset Q_0^{-1}(\Omega^k)$. For $m = 0, 1, 2, 3$, let $W_{m,j} = \overline{\mathbf{B}(v_{m,j}; q_j)}$. When n is big enough, $W_{0,j} \subset Q_n^{-1}(\Omega_n^k)$, and $\mathbf{B}(a(j); l_j/30)$ intersects $\partial Q_n^{-1}(\Omega_n^k)$. Suppose β_j is a curve in N_j which starts from $W_{0,j}$, and reaches $W_{1,j}$, $W_{2,j}$ and $W_{3,j}$ in the order. Then β_j disconnects a subset of $\partial Q_n^{-1}(\Omega_n^k)$ from ∞, if n is big enough. Since $Q_n^{-1}(\Omega_n^k)$ is a simply connected domain, β_j hits $\partial Q_n^{-1}(\Omega_n^k)$. Let β_j^n be the part of β_j before hitting $\partial Q_n^{-1}(\Omega_n^k)$. Then $\beta_j^n \subset \widetilde{\mathbf{B}}_n(a(s_j); e_k) \subset Q_n^{-1}(\Omega_{n,j}^k)$, if n is big enough. So $Q_n(\beta_j^n)$ is a curve in $\Omega_{n,j}^k$ that tends to some point of $\partial\Omega_{n,j}^k$ at one end. This point is not on $Q_n(\gamma_n^{k\pm1})$, because the distance between γ^k and γ_n^{k+1} is greater than e_k. Hence $\overline{Q_n(\beta_j^n)}$ intersects $\rho_{n,j}^k$.

Suppose I is a closed ball in $Q_0^{-1}(\Omega^k)$. For $j = 0, 1$, let Π_j be a subdomain of $Q_0^{-1}(\Omega^k)$ that contains $I \cup W_{0,j}$ such that $\overline{\Pi_j}$ is a compact subset of $Q_0^{-1}(\Omega^k)$. Then Π_j is contained in $Q_n^{-1}(\Omega_n^k)$ for n big enough. For $x \in \delta_n\mathbb{Z}^2 \cap I$, let $\mathcal{A}_{n,j}^x$ be the set of lattice paths of $\delta_n\mathbb{Z}^2$ that start from x, and hit $W_{0,j}$, $W_{1,j}$, $W_{2,j}$ and $W_{3,j}$ in the order before exiting $\Pi_j \cup N_j$. We may view $\beta \in \mathcal{A}_{n,j}^x$ as a continuous curve. Let β^{D_n} denote the part of $\beta \in \mathcal{A}_{n,j}^x$ before exiting $Q_n^{-1}(\Omega_n^k)$. Then β^{D_n} can be viewed as a lattice path on D^{δ_n}. We proved in the last paragraph that if n is big enough, $\overline{Q_n(\beta^{D_n})}$ intersects $\rho_{n,j}^k$, for any $\beta \in \mathcal{A}_{n,j}^x$, $x \in \delta_n\mathbb{Z}^2 \cap I$, $j = 0, 1$.

Thus for any $\beta_0 \in \mathcal{A}^x_{n,0}$ and $\beta_1 \in \mathcal{A}^x_{n,1}$, $\beta_0^{D_n} \cup \beta_1^{D_n}$ disconnects γ_n^{k-1} from γ_n^{k+1} in $Q_n^{-1}(\Omega_n^k)$.

5.4. The behaviors of $g_0 \circ J$ *outside any neighborhood of* 1

Let $P^x_{n,j}$ be the probability that a simple random walk on $\delta_n\mathbb{Z}^2$ started from x belongs to $\mathcal{A}^x_{n,j}$. By Lemma 3.8, if n is big enough, then $P^x_{n,j}$ is greater than some $a_k > 0$ for all $x \in \delta_n\mathbb{Z}^2 \cap I$, $j = 0, 1$. We may also choose n big enough such that $V(D^{\delta_n}) \cap I$ is non-empty, and $g_n(x)$ is less than some $b_k \in (0, \infty)$ for all $x \in \delta_n\mathbb{Z}^2 \cap I$. We claim that if n is big enough, then $g_n(x) \leq \max\{b_k/a_k, 1\}$ for every $x \in \delta_n\mathbb{Z}^2 \cap (D_n \setminus U(\gamma_n^{k-1}))$. Suppose for infinitely many n, there are $x_n \in \delta_n\mathbb{Z}^2 \cap D_n \setminus U(\gamma_n^{k-1})$ such that $g_n(x_n) \geq M > \max\{b_k/a_k, 1\}$. Since g_n is discrete harmonic on $\delta_n\mathbb{Z}^2 \cap D_n$, and $g_n \leq 1$ on the boundary vertices of D_n except at $P(w_n)$, the tip point of w_n, so there is a lattice path β_n in D_n that goes from x_n to $P(w_n)$ such that the value of g_n at each vertex of β_n is not less than M. By the construction of γ_n^{k+1}, if n is big enough, then $U(Q_n(\gamma_n^{k+1}))$ is some neighborhood of 1 in $\mathbf{A}_{p_n}$, and so $U(\gamma_n^{k+1})$ is some neighborhood of $P(w_n)$ in D_n. Thus β_n intersects both γ_n^{k-1} and γ_n^{k+1}. Choose $v_0 \in \delta_n\mathbb{Z}^2 \cap I$. For every $\rho_{n,0} \in \mathcal{A}^{v_0}_{n,0}$ and $\rho_{n,1} \in \mathcal{A}^{v_0}_{n,1}$, the path $\rho_{n,0}^{D_n} \cup \rho_{n,1}^{D_n}$ disconnects γ_n^{k-1} from γ_n^{k+1}. Therefore $\rho_{n,0}^{D_n} \cup \rho_{n,1}^{D_n}$ intersects β_n. This implies that for some $j_n \in \{0, 1\}$, for every $\rho \in \mathcal{A}^{v_0}_{n,j}$, we have ρ^{D_n} intersects β_n. Thus the probability that a simple random walk on $\delta_n\mathbb{Z}^2$ started from v_0 hits β_n before ∂D_n is greater than a_k. Let τ_n be the first time this random walk hits $\beta_n \cup \partial D_n$. Since g_n is non-negative, bounded, and discrete harmonic on $\delta_n\mathbb{Z}^2 \cap D_n$, so $g_n(v_0) = \mathbf{E}[g_n(\mathrm{RW}^x_{v_0}(\tau_n))] \geq a_k M > b_k$, which is a contradiction. So the claim is proved.

By passing to a subsequence depending on k, we can now assume the following. $U(\gamma_n^{k+1})$ is some neighborhood of $P(w_n)$ in D_n; the value of g_n on $\delta_n\mathbb{Z}^2 \cap D_n \setminus U(\gamma_n^{k+1})$ is bounded by some $M_k \geq 1$; $U(\gamma_n^{k+1}) \subset U(\gamma_n^k) \subset U(\gamma_n^{k-1})$; the spherical distance between γ_n^k and γ_n^{k-1} is greater than some $R_k > 0$; and the (Euclidean) distance between γ_n^k and γ_n^{k+1} is greater than δ_n. Since the end points of γ_n^k and γ_n^{k-1} are on B_1^n, the spherical diameter of B_1^n is at least R_k. Let R be the spherical distance between B_2 and α_2. Then the spherical distance between B_2 and B_1^n is at least R, as α_2 disconnects B_2 from B_1^n. Suppose $v \in V(D^{\delta_n}) \cap D_n \setminus U(\gamma_n^{k-1})$, and $dist^\#(v, B_1^n) = d < R/2$. Then $dist^\#(v, B_2) > R/2$. Let RW_v^n be a simple random walk on $\delta_n\mathbb{Z}^2$ started from v, and τ_n^k be the first time that RW_v^n leaves $D_n \setminus U(\gamma_n^k)$. Then $\mathrm{RW}_v^n(\tau_n^k)$ is either on B_2, or on B_1^n, or in $U(\gamma_n^k)$. In the first case, $g_n(\mathrm{RW}_v^n(\tau_n^k)) = 1$, and v should first exit $\mathbf{B}^\#(v, R/2)$ before hitting B_2. In the second and third cases, since $\mathrm{RW}_v^n(\tau_n^k - 1) \in D_n \setminus U(\gamma_n^k)$, and the Euclidean distance between γ_n^k and γ_n^{k+1} is greater than δ by construction, so $[\mathrm{RW}_v^n(\tau_n^k - 1), \mathrm{RW}_v^n(\tau_n^k)]$ does not intersect γ_n^{k+1}. Thus in the second case, $\mathrm{RW}_v^n(\tau_n^k) \neq P(w_n)$, and so $g_n(\mathrm{RW}_v^n(\tau_n^k)) = 0$. In the third case, $\mathrm{RW}_v^n(\tau_n^k) \in D_n \setminus U(\gamma_n^{k+1})$, so $g_n(\mathrm{RW}_v^n(\tau_n^k)) \leq M_k$; and RW_v^n first uses some edge that intersects γ_n^{k-1}, then uses some edge that intersects γ_n^k at time τ_n^k. So the spherical diameter of $\mathrm{RW}_v^n[0, \tau_n^k]$ is at least R_k. This implies that RW_v^n should first exit $\mathbf{B}^\#(v; R_k/2)$ before hitting $U(\gamma_n^k)$. Let

$R'_k = \min\{R/2, R_k/2\}$, then by Lemma 3.7,

$$\mathbf{P}[\mathrm{RW}^n_v(\tau^k_n) \notin B^n_1] \le C_0((\delta_n + d)/R'_k)^{C_1},$$

for some absolute constants $C_0, C_1 > 0$. So we have $g_n(v) \le M_k C_0((\delta_n + d)/R'_k)^{C_1}$.

Suppose $z \in D_0 \setminus U(\gamma^{k-1}) \setminus \gamma^{k-1}$, and $dist^{\#}(z, B^0_1) = d < R/4$. Choose $r \in (0, d/2)$ such that $\mathbf{B}^{\#}(z, r)$ is bounded and $\overline{\mathbf{B}^{\#}(z; r)} \subset D_0 \setminus U(\gamma^{k-1}) \setminus \gamma^{k-1}$. If n is big enough, then $\overline{\mathbf{B}^{\#}(z; r)} \subset D_n \setminus U(\gamma^{k-1}_n)$, and the spherical distance from every $v \in \mathbf{B}^{\#}(z; r)$ to B^n_1 is less than $2d < R/2$. Thus

$$g_n(v) \le M_k C_0((\delta_n + 2d)/R'_k)^{C_1}, \qquad \forall v \in \delta_n\mathbb{Z}^2 \cap \mathbf{B}^{\#}(z; r).$$

Since g_0 is the limit of g_n, $g_0(z) \le M_k C_0(2d/R)^{C_1}$. Thus for every $k \ge 2$, $g_0(z) \to 0$, as $z \in D_0 \setminus U(\gamma^{k-1}) \setminus \gamma^{k-1}$, and $z \to B_1$ in the spherical metric, and so $g_0 \circ J(z) \to 0$ as $z \in \mathbf{A}_{p_0} \setminus U(Q_0(\gamma^{k-1}))$, and $z \to \mathbf{C}_0$. Since $U(Q_0(\gamma^k))$, $k \in \mathbb{N}$, forms a neighborhood basis of 1 in $\mathbf{A}_{p_0}$, so for any $r > 0$, $g_0 \circ J(z) \to 0$ if $z \in \mathbf{A}_{p_0} \setminus \mathbf{B}(1, r)$ and $z \to \mathbf{C}_0$. This is what we need at the end of 5.1. □

Acknowledgements. This work was proceeded under the instruction of Professor Nikolai Makarov, who let the author be interested in this subject, and gave many valuable comments on this paper.

References

[1] Ahlfors, L.V.: Conformal invariants: topics in geometric function theory. New York: McGraw-Hill Book Co., 1973

[2] Chandrasekharan, K.: Elliptic functions. Springer-Verlag Berlin Heidelberg, 1985

[3] Dubédat, J.: Critical percolation in annuli and SLE_6, arXiv:math.PR/0306056

[4] Lawler, G.F.: Intersection of random walks. Boston: Birkhäuser, 1991

[5] Lawler, G.F., Schramm, O., Werner, W.: Values of Brownian intersection exponents I: Half-plane exponents. Acta Mathematica **187**, 237–273 (2001)

[6] Lawler, G.F., Schramm, O., Werner, W.: Values of Brownian intersection exponents II: Plane exponents. Acta Mathematica **187**, 275–308 (2001)

[7] Lawler, G.F., Schramm, O., Werner, W.: Values of Brownian intersection exponents III: two-sided exponents. Ann. Int. Henri Poincaré **38**, 109–123, (2002)

[8] Lawler, G.F., Schramm, O., Werner, W.: Conformal invariance of planar loop-erased random walks and uniform spanning trees, arXiv: math.PR/012234

[9] Lawler, G.F., Schramm, O., Werner, W.: Conformal restriction: the chordal case. J. Amer. Math. Soc. **16** (4), 917–955 (2003)

[10] Oksendal, B.: Stochastic Differential Equations: an introduction with applications. Berlin-Heidelberg-New York: Springer-Verlag, 1995

[11] Pommerenke, C.: On the Löwner differential equation. Michigan Math. J. **13**, 435–443 (1968)

[12] Pommerenke, C.: Univalent functions. Göttingen: Vandenhoeck & Ruprecht, 1975

[13] Pommerenke, C.: Boundary behaviour of conformal maps. Berlin-Heidelberg-New York: Springer-Verlag, 1991

[14] Revuz, D., Yor, M.: Continuous Martingales and Brownian Motion. Springer-Verlag, 1991

[15] Rohde, S., Schramm, O.: Basic properties of SLE, arXiv:math.PR/ 0106036
[16] Schramm, O.: Scaling limits of loop-erased random walks and uniform spanning trees. Israel J. Math. **118**, 221–228 (2000)
[17] Smirnov, S.: Critical percolation in the plane: conformal invariance, Cardy's formula, scaling limits. C. R. Acad. Sci. Paris Sér. I Math. **333** (3), 239–244 (2001)
[18] Villat, H.: Le problème de Dirichlet dans une aire annulaire. Rend. circ. mat. Palermo 134–175 (1912)
[19] Werner, W.: Critical exponents, conformal invariance and planar Brownian motion. Proceedings of the 3rd Europ. Congress Math., Prog. Math. **202**, 87–103 (2001)