Probab. Theory Relat. Fields 129, 261–289 (2004)
Digital Object Identifier (DOI) 10.1007/s00440-004-0338-y

C. Donati-Martin · A. Rouault · M. Yor · M. Zani

Large deviations for squares of Bessel and Ornstein-Uhlenbeck processes

Received: 12 February 2003 / Revised version: 11 January 2004 /
Published online: 25 March 2004 –

Abstract. Let $(X_t^{(\delta)}, t \geq 0)$ be the BESQ$^\delta$ process starting at δx. We are interested in large deviations as $\delta \to \infty$ for the family $\{\delta^{-1} X_t^{(\delta)}, t \leq T\}_\delta$, – or, more generally, for the family of squared radial OU$^\delta$ process. The main properties of this family allow us to develop three different approaches: an exponential martingale method, a Cramér–type theorem, thanks to a remarkable additivity property, and a Wentzell–Freidlin method, with the help of McKean results on the controlled equation. We also derive large deviations for Bessel bridges.

1. Introduction

Let $B^{(1)},\ B^{(2)}, \cdots B^{(n)},\ \cdots$ be a sequence of independent standard linear Brownian Motions, and consider

$$X_t^{(n)} = \sum_{k=1}^{n} (B_t^{(k)})^2 ,$$

a representation of the square of the "n–dimensional" Bessel process. Obviously, the sequence $X^{(n)}$ – considered as taking values in $\mathcal{C}(\mathbb{R}_+, \mathbb{R}_+)$ – lends itself to the application of the law of large numbers and of the central limit theorem, which, in this case, yields

$$\left(\sqrt{n}\Big(\frac{1}{n}X_t^{(n)} - t\Big),\ t \geq 0\right) \xrightarrow[n\to\infty]{(law)} \left(\sqrt{2}\beta_{t^2},\ t \geq 0\right) , \tag{1.1}$$

C. Donati-Martin: Laboratoire de Probabilités et Modèles Aléatoires, Université Paris 6, Site Chevaleret, 13 rue Clisson, F-75013 Paris, France. e-mail: donati@ccr.jussieu.fr

A. Rouault: LAMA, Bâtiment Fermat, Université de Versailles, F-78035 Versailles, France. e-mail: rouault@math.uvsq.fr

M. Yor: Laboratoire de Probabilités et Modèles Aléatoires, Université Paris 6, Site Chevaleret, 13 rue Clisson, F-75013 Paris, France.

M. Zani: Laboratoire d'Analyse et de Mathématiques Appliquées, CNRS UMR 8050, Université Paris 12-Val de Marne, 61, avenue du Général de Gaulle, F-94010 Créteil, France. e-mail: zani@univ-paris12.fr

Mathematics Subject Classification (2000): 60F10, 60J60

Key words or phrases: Bessel processes – Ornstein-Uhlenbeck processes – Additivity property – Large deviation principle

where $(\beta_u\,, u \ge 0)$ is a one dimensional Brownian Motion (see e.g. [23] Exercise 2.5.2), and the convergence in law corresponds to the weak convergence of probabilities on $C(\mathbb{R}_+, \mathbb{R}_+)$, endowed with the topology of uniform convergence on compact sets of $\mathbb{R}_+$. In fact, a (square of) Bessel process may be defined for every positive "dimension" δ (see e.g. [18], Chap XI), as solution of the stochastic equation

$$dX_t^{(\delta)} = \delta dt + 2\sqrt{X_t^{(\delta)}}\, dB_t, \quad X_0^{(\delta)} = x \ge 0\,,\; X_t^{(\delta)} \ge 0, \text{ for all } t \ge 0. \tag{1.2}$$

It is denoted by $BESQ_x^{(\delta)}$. The corresponding laws on $\mathcal{C}(\mathbb{R}_+, \mathbb{R}_+)$, denoted by $Q_x^{(\delta)}$, satisfy the additivity property

$$Q_x^{(\delta)} \oplus Q_y^{(\delta')} = Q_{x+y}^{(\delta+\delta')},\; x, y \ge 0,\; \delta, \delta' \ge 0 \tag{1.3}$$

(see e.g. [19], [17]). Now, the convergence (1.1) can be extended to

$$\left(\sqrt{\delta}\Big(\frac{1}{\delta}X_t^{(\delta)} - t\Big),\; t \ge 0\right) \xrightarrow[\delta\to\infty]{(law)} \left(\sqrt{2}\beta_{t^2},\; t \ge 0\right)\,. \tag{1.4}$$

Convergence results such as (1.1) and (1.4) have been motivated by, and are related to, the so-called Poincaré's lemma approximating the Gaussian distribution on $\mathbb{R}^d$ from uniform distributions on the spheres of radius $\sqrt{n}$ in $\mathbb{R}^n$ - see [23], [22] and references therein-.

The main purpose of this paper is to establish the corresponding large deviations result. For a precise definition of a Large Deviation Principle (LDP) and usual notions related to it, such as exponentiel tightness and weak LDP, we refer to [6].

Before stating our main results we indicate that this large deviations study may be understood in the framework of LDP for diffusions with a small parameter. Indeed we shall look for an LDP for

$$dX_t^\epsilon = b(X_t^\epsilon)dt + 2\epsilon\sqrt{|X_t^\epsilon|}\, dB_t\,,\quad X_0^\epsilon = a \ge 0\,, \tag{1.5}$$

where B is a one dimensional Brownian motion and b a Lipschitz function on $\mathbb{R}$ satisfying $b(0) \ge 0$. It is well known that this equation admits a unique pathwise solution (see [18] Chap. IX Th. 3.5). Nevertheless, the Freidlin-Wentzell theory on large deviations for diffusions does not apply since the diffusion coefficient $\sigma(x) = 2\sqrt{|x|}$ is not Lipschitz (see [10]). We note that this equation (1.5) leads to the above equation (1.2) when $b \equiv 1$ by denoting $\epsilon = \frac{1}{\sqrt{\delta}}$, $X_t^\epsilon = \frac{X_t^{(\delta)}}{\delta}$ and $a = \frac{x}{\delta}$. We shall freely go from one presentation to the other. We now present our results as well as our notations.

Let us fix $T > 0$, $\rho \ge 0$ and $a \ge 0$ and consider for $\delta > 0$ the distributions P_δ and $\hat{P}_\delta$ on $\mathcal{C}_a([0, T], \mathbb{R}_+)$ defined by

$$P_\delta(A) = Q_{\delta a}^{(\delta\rho)}\left(\{\delta^{-1}X_t, t \le T\} \in A\right)\,, \tag{1.6}$$

and

$$\widehat{P_\delta}(A) = Q_{\delta a}^{(\delta\rho)}\left(\{\sqrt{\delta^{-1}X_t}, t \le T\} \in A\right)\,. \tag{1.7}$$

We are interested in the large values of δ.

Proposition 1.1. *Let us assume $\rho > 0$.*

1) The family of distributions $\{P_\delta\}_\delta$ satisfies the LDP in $\mathcal{C}_a([0,T],\mathbb{R}_+)$ with speed δ^{-1} and good rate function defined by

$$J^\rho_{a,T}(\varphi) = \int_0^T \frac{(\dot\varphi_t - \rho)^2}{8\varphi_t}\, dt\,, \tag{1.8}$$

for φ non-negative, absolutely continuous on $[0,T]$, such that $t \mapsto \frac{\dot\varphi_t - \rho}{\sqrt{\varphi_t}} \in L^2([0,T])$, and by $J^\rho_{a,T}(\varphi) = \infty$ otherwise.

2) The family of distributions $\{\widehat{P}_\delta\}_\delta$ satisfies the LDP in $\mathcal{C}_{\sqrt{a}}([0,T],\mathbb{R}_+)$ with speed δ^{-1} and good rate function defined by

$$K^\rho_{a,T}(\varphi) = \frac{1}{2}\int_0^T \left(\dot\varphi_t - \frac{\rho}{2\varphi_t}\right)^2 dt\,, \tag{1.9}$$

for φ non-negative, absolutely continuous on $[0,T]$ such that $t \mapsto \dot\varphi_t - \frac{\rho}{2\varphi_t} \in L^2([0,T])$, and by $K^\rho_{a,T}(\varphi) = \infty$ otherwise.

Before we present our result with a general drift, we discuss the particular case of affine drifts i.e. we consider the family of squared radial Ornstein-Uhlenbeck (OU) processes, which are solution of

$$dY^{(\delta)}_t = \left(\delta + cY^{(\delta)}_t\right) dt + 2\sqrt{Y^{(\delta)}_t}\, dB_t\,, \quad Y^{(\delta)}_0 = x\,. \tag{1.10}$$

Their distributions denoted by ${}^cQ^{(\delta)}_x$ ([17]) satisfy the same additive property as (1.3):

$${}^cQ^{(\delta)}_x \oplus {}^cQ^{(\delta')}_y = {}^cQ^{(\delta+\delta')}_{x+y}\,,\ x, y \ge 0,\ \delta, \delta' \ge 0 \tag{1.11}$$

We consider the law ${}^cP_\delta$ defined on $\mathcal{C}_a([0,T],\mathbb{R}_+)$ by

$${}^cP_\delta(A) := {}^cQ^{(\delta\rho)}_{\delta a}\left(\{\delta^{-1}Y_t, t \le T\} \in A\right)\,.$$

Proposition 1.2. *Let us assume $\rho > 0$. The family of distributions $\{{}^cP_\delta\}_\delta$ satisfies the LDP with speed δ^{-1} and good rate function ${}^cJ^\rho_{a,T}$:*

$${}^cJ^\rho_{a,T}(\varphi) := \int_0^T \frac{[\dot\varphi_t - (c\varphi_t + \rho)]^2}{8\varphi_t}\, dt, \tag{1.12}$$

if $\left\{\dot\varphi_t - (c\varphi_t + \rho)\right\}/\sqrt{\varphi_t} \in L^2([0,T])$ and ${}^cJ^\rho_{x,T}(\varphi) = +\infty$ otherwise.

The preceding results may be generalized in the framework of (1.5).

Theorem 1.3. *Let us assume $\rho := b(0) \geq 0$ and $a \geq 0$, with $a > 0$ in the case $\rho = 0$. Then, the family of distributions of $\{(X_t^\epsilon), t \in [0,T]\}$ in $\mathcal{C}_a([0,T],\mathbb{R}_+)$ satisfies a LDP with speed ϵ^2 and good rate function*

$$I(\varphi) = \int_0^T \frac{[\dot{\varphi}_t - b(\varphi_t)]^2}{8\varphi_t}\, dt, \tag{1.13}$$

if $\{\dot{\varphi}_t - b(\varphi_t)\}/\sqrt{\varphi_t} \in L^2([0,T])$ and $I(\varphi) = +\infty$ otherwise.
If $a = \rho = 0$, then $X^\epsilon \equiv 0$ and $I(0) = 0$, $I(\varphi) = \infty$ for $\varphi \not\equiv 0$.

Remark. (1.13) has the following meaning:

- If $\rho > 0$, $I(\varphi) < \infty$ implies that $Leb\{t \in [0,T], \varphi(t) = 0\} = 0$.
- If $\rho = 0$, $I(\varphi) = \int_0^T \frac{[\dot{\varphi}_t - b(\varphi_t)]^2}{8\varphi_t} 1_{(\varphi_t > 0)}\, dt$.

Although Theorem 1.3 extends Proposition 1.1 and Proposition 1.2, we also give alternative proofs of these propositions, using specific properties of squares of Bessel and OU processes.

The first one is additivity, as said above. We mention that there are other additive families of Markov processes, e.g. the continuous state branching processes ([11]) for which similar arguments might lead to LDP results.

The second one comes from the particular form of the SDE satisfied by the Bessel process and is based on a study of the associated Ito map.

We now describe the organisation of the paper. In Section 2, we prove some exponential tightness results which will be helpful in some of our proofs. In Section 3, we prove Theorem 1.3 by using the approach of exponential martingales. Our change of probability is slightly different from the classical one used for non-degenerate diffusions ([10] and [13]). In Section 4, we shall apply a slightly modified version of Cramer's theorem in Banach spaces to prove partially Proposition 1.1, in that we show the existence of a LDP but the computation of a rate function is presented only in a variational form[1]. In Section 5, we give another proof of Proposition 1.1, in which our discussion is given in terms of Bessel processes instead of their squares. Indeed, we would like to use the pathwise resolution of a one-dimensional SDE, due to Doss and Sussmann (see [18], Exercise IX.2.8), in order to apply a contraction principle. However, this method does not apply to the equation of square Bessel processes since the diffusion coefficient $\sigma(x) = 2\sqrt{|x|}$ is $1/2$ Hölder and the associated ODE $dy_t = \sigma(y_t)dt$ admits an infinite number of solutions. We are then tempted to prove directly the continuity of the Itô map for the equation of the Bessel processes, which has already been obtained by Mc Kean (see [14]). The case $\rho = 0$ is treated separately at the end of Section 5. In Section 6, we show how Proposition 1.1 leads to Theorem 1.3 with the help of the Girsanov transformation and Varadhan's lemma, and we prove Proposition 1.2. In Section

[1] The approach in Section 3 and 4 is somewhat similar to the usual one for large deviations of Brownian motion: Schilder's theorem (as seen in Chap. I of [7]) and additivity ([6] Exercise 6.1.19 or [7] Chap III).

7 we consider the distributions $\{Q^{(\delta)}_{x\to 0}\}$ of squared Bessel bridges r^2 which also enjoy the additivity property

$$Q^{(\delta)}_{x\to 0} \oplus Q^{(\delta')}_{y\to 0} = Q^{(\delta+\delta')}_{x+y\to 0},\ x, y \geq 0,\ \delta, \delta' \geq 0 \tag{1.14}$$

and we obtain the LDP for them, using the representation of r in terms of the corresponding Bessel process R:

$$r_t = (1-t) R_{\frac{t}{1-t}}$$

which requires to complete our previous discussion relative to the time interval $[0, T]$ to the positive half line.

In Section 8, we discuss the relation between the two expressions of the rate functions. Closely connected with this discussion is the well known fact that the Laplace transform of the $\{Q^{(\delta)}_x\}$ probabilities may be expressed explicitly in terms of solutions of the Sturm–Liouville equations (see [17] (Theorem 2.1), and [23]),

$$Q^{(\delta)}_x \left(\exp \int_0^T X_s \, d\mu(s) \right) = \phi_\mu(T)^{\delta/2} \ \exp(\frac{x}{2} \phi'_\mu(0)) \tag{1.15}$$

where ϕ_μ is the unique solution of

$$\frac{1}{2}\phi'' = -\mu\phi\,, \quad \phi(0) = 1\,, \quad \frac{1}{2}\phi'(T) = \mu(\{T\})\phi(T)\,. \tag{1.16}$$

In many instances, ϕ_μ is known explicitly, and the above mentioned variational formulae may be verified directly, as we have done in particular for μ a multiple of the Lebesgue measure (see Section 8.3). Another formula for the infinitely divisible laws $Q^{(\delta)}_x$ is provided in [17] (Theorem 4.1) by the following Lévy–Hinčin representation:

$$Q^{(\delta)}_x \left(\exp \int_0^T X_s \, d\mu(s) \right) = \exp \left\{ (xM + \delta N)(\exp(\int_0^T X_s \, d\mu(s)) - 1) \right\}, \tag{1.17}$$

where $M,\ N$ are σ–finite measures on $\mathcal{C}([0, T], \mathbb{R}_+)$, which are described explicitly in [17], [16] and [18].

We will use in some places the Cameron-Martin space

$$H^1_x[0, T] = \{h : [0, T] \to \mathbb{R};\ \ h_s = x + \int_0^s \dot{h}_u du\ ;\ \ \dot{h} \in L^2([0, T])\}\,. \tag{1.18}$$

2. Exponential tightness

We shall prove the exponential tightness of the law P^ϵ of the solution X^ϵ of (1.5). The exponential tightness of P_δ (resp. ${}^cP_\delta$) follows.
Let us fix $x \geq 0$, and let $\mathcal{C}_x^\alpha([0,T])$ be the space of α–Hölder continuous functions in $\mathcal{C}_x([0,T])$. Set, for any $f \in \mathcal{C}_x^\alpha([0,T])$

$$\|f\|_{\alpha,T} = \sup_{0 \leq t \neq s \leq T} \frac{|f(t)-f(s)|}{|t-s|^\alpha}.$$

Since the paths of X^ϵ are α-Hölderian for every $\alpha < 1/2$, we prove the exponential tightness in $\mathcal{C}_x^\alpha([0,T])$ by considering as compact sets, α'-Hölderian balls for $1/2 > \alpha' > \alpha$. (This is the method of Exercise 5.2.14 of ([6])).
For the sake of simplicity, we will assume in this subsection and in the following section that $T = 1$ and write $\|f\|_\alpha$ for $\|f\|_{\alpha,1}$. (See Remark after the proof of Proposition 2.2).

Let us first see why the condition $\rho := b(0) \geq 0$ ensures that the solution of (1.5) is a positive diffusion. Since b is Lipschitz there exists $\beta > 0$ such that:

$$-\beta x \leq b(x) \leq \beta x + \rho, \; x \geq 0.$$

Denote by Y^ϵ (resp. Z^ϵ), the solution of (1.5) for $b(x) = -\beta x$ (resp. $b(x) = \beta x + \rho$). Then $\epsilon^{-2}Y^\epsilon$, (resp. $\epsilon^{-2}Z^\epsilon$), is ${}^{-\beta}Q^0_{a\epsilon^{-2}}$ distributed, (resp. ${}^{\beta}Q^{(\rho\epsilon^{-2})}_{a\epsilon^{-2}}$ distributed). These two processes are positive a.s.. By a comparison theorem (see [18] Theorem IX.3.7), it holds that:

$$a.s., \; \forall t, \; Y_t^\epsilon \leq X_t^\epsilon \leq Z_t^\epsilon. \tag{2.1}$$

We now assume in the rest of the paper that b is Lipschitz with $\rho \geq 0$.

Proposition 2.1. *Let X^ϵ be the positive solution of (1.5). There exists $\lambda := \lambda(\beta,\rho,a) > 0$ such that:*

$$\forall \epsilon > 0, \;\; E\left[\exp\left(\lambda\epsilon^{-2}\sup_{t\in[0,1]} X_t^\epsilon\right)\right] \leq \exp([\epsilon^{-2}]+1). \tag{2.2}$$

Proof. Let $\widehat{X}^\epsilon := \sup_{t\in[0,1]} X_t^\epsilon$, $\widehat{Z}^\epsilon := \sup_{t\in[0,1]} Z_t^\epsilon$ and $\widehat{X} := \sup_{t\in[0,1]} X_t$. From the remark above we have

$$E\left[\exp\left(\lambda\epsilon^{-2}\widehat{X}^\epsilon\right)\right] \leq E\left[\exp\left(\lambda\epsilon^{-2}\widehat{Z}^\epsilon\right)\right] = {}^{\beta}Q^{(\rho\epsilon^{-2})}_{a\epsilon^{-2}}\left[\exp\lambda\widehat{X}\right]. \tag{2.3}$$

Taking $N := [\epsilon^{-2}]+1$, the comparison theorem gives

$${}^{\beta}Q^{(\rho\epsilon^{-2})}_{a\epsilon^{-2}}\left[\exp\lambda\widehat{X}\right] \leq {}^{\beta}Q^{(N\rho)}_{Na}\left[\exp\lambda\widehat{X}\right].$$

Using the infinite divisibility property (1.11) and the subadditivity of the mapping $X \to \sup_{t\in[0,1]} X_t$, we have:

$${}^{\beta}Q^{(\rho N)}_{aN}\left[\exp\lambda\widehat{X}\right] \leq \left({}^{\beta}Q^{(\rho)}_{a}\left[\exp\lambda\widehat{X}\right]\right)^N$$

for every positive integer N. Now, it is well known that, under ${}^{\beta}Q_a^{(\rho)}$, $\widehat{X}$ admits some exponential moments (for ρ integer, X is the square of a gaussian process, and it is a particular case of Fernique's Theorem ([9], see also [7] p.16)). Thus, there exists $\lambda := \lambda(\beta, a, \rho) > 0$ such that:

$$ {}^{\beta}Q_a^{(\rho)}\left[\exp \lambda \widehat{X}\right] \leq e. $$

Gathering all these inequalities we obtain (2.2). □

Proposition 2.2. *The family of distributions P_ϵ of X^ϵ is exponentially tight in $\mathcal{C}_a^\alpha([0,1])$, in scale ϵ^2.*

Proof. Let us fix $\alpha' \in (\alpha, 1/2)$ and $R > 0$. The Hölder ball $B_{\alpha'}(0, R)$ is a compact set of $C^\alpha([0,1])$.
From (1.5) we have $X_t^\epsilon = 2M_t^\epsilon + A_t^\epsilon$ where M^ϵ is the martingale $M_t^\epsilon = \epsilon \int_0^t \sqrt{X_u^\epsilon}\, dB_u$ and $A_t^\epsilon = \int_0^t b(X_u^\epsilon)\,du$, so that

$$ \|X^\epsilon\|_\alpha \leq \|A^\epsilon\|_\alpha + 2\|M^\epsilon\|_\alpha. $$

We shall bound the tails of $\|A^\epsilon\|_\alpha$ and $\|M^\epsilon\|_\alpha$.

a) Bounds for A^ϵ.

From

$$ |A_t^\epsilon - A_s^\epsilon| \leq \int_s^t |b(X_u^\epsilon)|du \leq |t-s|(\beta \widehat{X}^\epsilon + \rho)\,, $$

we get $\|A^\epsilon\|_\alpha \leq (\beta \widehat{X}^\epsilon + \rho)$, and then

$$ \begin{aligned} P(\|A^\epsilon\|_\alpha \geq R) &\leq P(\widehat{X}^\epsilon \geq R') \\ &\leq \exp(-\lambda R' \epsilon^{-2}) E[\exp(\lambda \epsilon^{-2} \widehat{X}^\epsilon)] \end{aligned} $$

with $R' = \frac{R-\rho}{\beta}$. Choosing $\lambda > 0$ as in Proposition 2.1 we get

$$ \limsup_{\epsilon \to 0} \epsilon^2 \ln P(\|A^\epsilon\|_\alpha \geq R) \leq -\lambda R' + 1 $$

and

$$ \lim_{R \to +\infty} \limsup_{\epsilon \to 0} \epsilon^2 \ln P(\|A^\epsilon\|_\alpha \geq R) = -\infty. $$

b) Bounds for M^ϵ. Fixing $0 < c < 1/2$ we use Garsia's lemma (see [20] p.47 or [2] p.203.) with $\Psi(x) = e^{c\epsilon^{-2}x} - 1$ and $p(x) = x^{1/2}$. So $\Psi^{-1}(y) = \frac{\epsilon^2}{c} \log(1+y)$.

Garsia's lemma asserts that if

$$ \int_0^1 \int_0^1 \Psi\Big(\frac{|M_t^\epsilon - M_s^\epsilon|}{p(|t-s|)}\Big) ds dt \leq K $$

then

$$ |M_t^\epsilon - M_s^\epsilon| \leq 8 \int_0^{|t-s|} \Psi^{-1}(4K/u^2) dp(u). $$

This yields easily that if

$$\int_0^1 \int_0^1 \exp\Big(c\epsilon^{-2}\frac{|M_t^\epsilon - M_s^\epsilon|}{|t-s|^{1/2}}\Big)\, dsdt \le K+1\,,$$

then

$$|M_t^\epsilon - M_s^\epsilon| \le \frac{8\epsilon^2}{c}(t-s)^{1/2}[K_1 + 2\log\frac{1}{t-s}]$$

with $K_1 = \log(4K+1)+4$, hence for any $\alpha < 1/2$

$$|M_t - M_s| \le (t-s)^\alpha R$$

where $R = \frac{8\epsilon^2}{c}(K_1+K_2)$ with $K_2 = 2\sup_{u\in[0,1]} u^{1/2-\alpha}\log\frac{1}{u}$.

From the above assertion we have

$$P\Big(\|M^\epsilon\|_\alpha \ge R\Big) \le P\left(\int_0^1 \int_0^1 \exp\Big(c\epsilon^{-2}\frac{|M_t^\epsilon - M_s^\epsilon|}{|t-s|^{1/2}}\Big)\, dsdt \ge K+1\right) \quad (2.4)$$

with

$$K = \frac{1}{4}\Big(e^{\left(\frac{c\epsilon^{-2}R}{8}-K_2\right)-4} - 1\Big) \quad (2.5)$$

whenever $cR\epsilon^{-2} > 8K_2 + 32$. Now by Markov's inequality,

$$P\Big(\|M^\epsilon\|_\alpha \ge R\Big) \le \frac{1}{K+1}\int_0^1 \int_0^1 E\left[\exp\Big(c\epsilon^{-2}\frac{|M_t^\epsilon - M_s^\epsilon|}{|t-s|^{1/2}}\Big)\right] dsdt. \quad (2.6)$$

From the usual exponential inequality for continuous martingales $Ee^{\lambda Z_t} \le \big(Ee^{2\lambda^2 <Z>_t}\big)^{1/2}$, we deduce

$$\begin{aligned} E\left[\exp\left(c\epsilon^{-2}\frac{|M_t^\epsilon - M_s^\epsilon|}{|t-s|^{1/2}}\right)\right] &\le 2\left\{E\left[\exp\left(\frac{2c^2\epsilon^{-2}}{(t-s)}\int_s^t X_u^\epsilon du\right)\right]\right\}^{1/2} \\ &\le 2\Big\{\frac{1}{t-s}\int_s^t E\left[\exp\left(2c^2\epsilon^{-2}X_u^\epsilon\right)\right] du\Big\}^{1/2} \quad (2.7) \end{aligned}$$

(by Jensen's inequality). Thus, we obtain:

$$P(\|M^\epsilon\|_\alpha \ge R) \le \frac{2}{K+1}\left\{\sup_{u\in[0,1]} E\left[\exp(2c^2\epsilon^{-2}X_u^\epsilon)\right]\right\}^{1/2}, \quad (2.8)$$

where $K+1 = C\exp(cR\epsilon^{-2}/8)$ and C a constant. (See (2.5)).
Now, from (2.1),

$$E[\exp(2c^2\epsilon^{-2}X_u^\epsilon)] \le {}^{\beta}Q_{a\epsilon^{-2}}^{(\rho\epsilon^{-2})}[\exp(2c^2X_u)]\,.$$

From the representation of squared OU processes as deterministic time changes of squared Bessel processes (see [18]),

$$ {}^{\beta}Q_{a\epsilon^{-2}}^{(\rho\epsilon^{-2})}[\exp(2c^2 X_u)] = \left(1 - 4c^2 \frac{e^{2\beta u} - 1}{2\beta}\right)^{-\frac{\rho\epsilon^{-2}}{2}} \exp\left(\frac{2c^2 a}{\epsilon^2(1 - 4c^2 \frac{e^{2\beta u}-1}{2\beta})}\right). \tag{2.9} $$

Choosing $c > 0$ such that $1 - 4c^2 \frac{e^{2\beta}-1}{2\beta} > 0$, we obtain:

$$ P(\|M^\epsilon\|_\alpha \geq R) \leq C A^{\rho\epsilon^{-2}} B^{a\epsilon^{-2}} e^{-cR\epsilon^{-2}/8} $$

for positive constants A, B depending on β. Thus,

$$ \lim_{R\to+\infty} \limsup_{\epsilon\to 0} \epsilon^2 \ln P(\|M^\epsilon\|_\alpha \geq R) = -\infty. \qquad \square $$

Remark. For later purpose, we deduce from this proof a non asymptotic bound for the $BESQ^{(\delta)}$ process on $[0, T]$. Let us first note that by scaling

$$ \left(BESQ_x^{(\delta)}(t), 0 \leq t \leq T\right) \overset{\mathcal{D}}{=} \left(T\ BESQ_{x/T}^{(\delta)}(t/T), 0 \leq t \leq T\right) $$

and then

$$ Q_y^{(\delta)}(\|X\|_{\alpha,T} \geq R) = Q_{y/T}^{(\delta)}(\|X\|_{\alpha,1} \geq RT^{\alpha-1}) \tag{2.10} $$

Taking into account (2.8) and (2.9) we conclude that for any $\alpha \in (0, 1/2)$, there exist constants $\gamma, A, B, C, R_0 > 0$ such that

$$ Q_y^{(\delta)}(\|X\|_{\alpha,T} \geq R) \leq C A^\delta B^{y/T} e^{-\gamma R T^{\alpha-1}}. \tag{2.11} $$

for any $y, \delta, T > 0$ and $R > T^{1-\alpha}(\delta + R_0)$.

3. First method: Exponential martingale approach

We shall prove Theorem 1.3, i.e. the LDP in the space $\mathcal{C}_a^\alpha([0, 1], \mathbb{R}_+)$ of α-Hölder positive continuous functions, for $0 < \alpha < \frac{1}{2}$. Since we already have the exponential tightness, we need only the upper bound for compact sets. According to [6], we shall show:

i) Weak upper bound:

$$ \lim_{r\to 0} \limsup_{\epsilon\to 0} \epsilon^2 \ln P(X^\epsilon \in B_r(\varphi)) \leq -I(\varphi) \tag{3.1} $$

where $B_r(\varphi)$ denotes the open ball with center $\varphi \in \mathcal{C}_a^\alpha[0, 1]$ and radius r.

ii) Lower bound : for any open set $O \subset \mathcal{C}_a^\alpha[0, 1]$,

$$ \liminf_{\epsilon\to 0} \epsilon^2 \ln P(X^\epsilon \in O) \geq -\inf_{\varphi\in O} I(\varphi). \tag{3.2} $$

3.1. The upper bound

Set

$$H = \{h \in \mathcal{C}([0,1]) : \dot{h} \in L^2\} = \bigcup_{x \in \mathbb{R}} H^1_x([0,1])\,.$$

For $h \in H$ let

$$M_t^{\epsilon,h} = \exp\left(\frac{1}{\epsilon^2}\{\int_0^t h(s)(dX_s^\epsilon - b(X_s^\epsilon)ds)\ - \frac{1}{2}\int_0^t h^2(s)\sigma^2(X_s^\epsilon)ds\}\right).$$

$(M_t^{\epsilon,h})_t$ is a positive local martingale and thus a supermartingale, ensuring that $E(M_t^{\epsilon,h}) \le 1$. For the lower bound, we shall need a stronger result:

Lemma 3.1. *For $\epsilon > 0$, the process $\left(M_t^{\epsilon,h}\right)_t$ is a martingale. Thus, $E(M_t^{\epsilon,h}) = 1$.*

Proof. $M_t^{\epsilon,h} = \mathcal{E}(\frac{1}{\epsilon}\int_0^t \widehat{h}_s dB_s)$ where $\mathcal{E}(\mathcal{N})$ denotes the exponential martingale associated to the martingale $\mathcal{N}$ and

$$\widehat{h}_s = 2h(s)\sqrt{X_s^\epsilon}. \tag{3.3}$$

By Proposition 2.1, there exist positive constants β and γ such that $E(\exp(\beta\widehat{h}_s^2)) < \gamma$ for $s \le 1$. It follows by a Novikov's type criterion that $E(\mathcal{E}(\int_0^t \widehat{h}_s dB_s)) = 1$ (see [18, Exercice VIII.1.40]). □

By an integration by parts, we can write $M_1^{\epsilon,h} = \exp(\frac{1}{\epsilon^2}F(X^\epsilon; h))$ where

$$F(\varphi; h) = G(\varphi; h) - 2\int_0^1 h^2(s)\varphi_s ds \tag{3.4}$$

with

$$\begin{aligned} G(\varphi; h) = h(1)\left(\varphi_1 - \int_0^1 b(\varphi_u)du\right) - h(0)a \\ - \int_0^1 \left(\varphi_s - \int_0^s b(\varphi_u)du\right)\dot{h}(s)ds\,, \end{aligned} \tag{3.5}$$

or

$$G(\varphi; h) = h(1)\varphi_1 - h(0)a - \int_0^1 \varphi_s \dot{h}(s)ds - \int_0^1 b(\varphi_s)h(s)ds\,. \tag{3.6}$$

Of course, if φ is absolutely continuous (with $\varphi(0) = a$) then

$$G(\varphi; h) = \int_0^1 h(s)[\dot{\varphi}(s) - b(\varphi(s))]ds\,. \tag{3.7}$$

For $\varphi \in \mathcal{C}_a^\alpha[0,1]$ and $h \in H$, we have :

$$P\left(X^\epsilon \in B_r(\varphi)\right) = P\left(X^\epsilon \in B_r(\varphi);\ \frac{M_1^{\epsilon,h}}{M_1^{\epsilon,h}}\right)$$
$$\leq \exp\left(-\frac{1}{\epsilon^2}\inf_{\psi\in B_r(\varphi)} F(\psi;h)\right) E(M_1^{\epsilon,h})$$
$$\leq \exp\left(-\frac{1}{\epsilon^2}\inf_{\psi\in B_r(\varphi)} F(\psi;h)\right),$$

which yields :

$$\limsup_{\epsilon\to 0}\epsilon^2 \ln P\left(X^\epsilon \in B_r(\varphi)\right) \leq -\inf_{\psi\in B_r(\varphi)} F(\psi;h)\,.$$

For $h \in H$, the map $\varphi \longrightarrow F(\varphi;h)$ is continuous on $\mathcal{C}_a^\alpha([0,1])$, so that

$$\lim_{r\to 0}\limsup_{\epsilon\to 0}\epsilon^2 \ln P\left(X^\epsilon \in B_r(\varphi)\right) \leq -F(\varphi;h).$$

Minimizing in $h \in H$, we obtain:

$$\lim_{r\to 0}\limsup_{\epsilon\to 0}\epsilon^2 \ln P\left(X^\epsilon \in B_r(\varphi)\right) \leq -\sup_{h\in H} F(\varphi;h).$$

We shall now identify this supremum as I, defined in Theorem 1.3.

Proposition 3.2. *For $\varphi \in \mathcal{C}_a^\alpha([0,1],\mathbb{R}^+)$,*

$$\sup_{h\in H} F(\varphi;h) = I(\varphi)\,. \tag{3.8}$$

Proof. Coming back to (3.4) and (3.6), we first notice that if $h \in H$ and $\int_0^1 h^2(s)\varphi_s ds = 0$, then

$$F(\varphi;h) = G(\varphi;h) = -\rho\int_0^1 h(s)ds\,, \tag{3.9}$$

(since h and φ are continuous, we have $h \equiv 0$ and $\dot{h} \equiv 0$ on the open set $\{\varphi > 0\}$).

i) Let us first examine two degenerate cases.
If $\rho = 0$ and $\varphi \equiv 0$ (which occurs only when $a = 0$), $F(\varphi,h) = 0$ for all h and (3.8) holds, in view of the definition of I (see Remark after Theorem 1.3).
If $\rho > 0$ and $Leb\{s;\varphi_s = 0\} > 0$, we can find $h_0 \in H$ such that $\int_0^1 h_0^2(s)\varphi_s ds$ $= 0$ and $\int_0^1 h_0(s)ds > 0$. Then (3.9) yields $F(\varphi,h_0) = -\rho\int_0^1 h_0(s)ds$ which gives $\sup_H F(\varphi,h) \geq \sup_{\lambda<0} F(\phi,\lambda h_0) = +\infty$ and (3.8) holds again.

ii) In other cases, we define the finite positive measure

$$\mu(ds) = \varphi_s ds\,.$$

First, we will prove that

$$\sup_{h\in H} F(\varphi;h) = \sup\left\{F(\varphi;h);\ h\in H,\ \|h\|_{L^2(\mu)} > 0\right\} =: \mathcal{S}(\varphi)\,, \tag{3.10}$$

and then that

$$\mathcal{S}(\varphi) = I(\varphi)\,. \tag{3.11}$$

Replacing h by λh in (3.4), it is easy to see that

$$\mathcal{S}(\varphi) = \frac{1}{8} \sup \left\{ G(\varphi; h)^2;\ h \in H,\ \|h\|_{L^2(\mu)} = 1 \right\} \geq 0\,. \tag{3.12}$$

- If $\rho = 0$, the condition $\|h\|_{L^2(\mu)} = 0$ implies $F(\varphi, h) = 0$ (by (3.9)) and (3.10) holds.
- if $\rho \neq 0$ and $Leb\{s; \varphi_s = 0\} = 0$, the condition $\|h\|_{L^2(\mu)} = 0$ implies $h \equiv 0$ and again (3.10) holds.

Let $\varphi \in \mathcal{C}_a^\alpha([0,1])$ such that $\mathcal{S}(\varphi) < \infty$. The linear form $G_\varphi : h \mapsto G(\varphi; h)$ (defined in (3.5) for $h \in H$) can be extended to $L^2(\mu)$ and by Riesz theorem, there exists $k \in L^2(\mu)$ such that, (keeping the notation G_φ for the extended linear form)

$$G_\varphi(h) = \langle k, h\rangle_{L^2(\mu)} = \langle k\varphi, h\rangle_{L^2} \tag{3.13}$$

Comparing with (3.5) we see that the function $\varphi - \int_0^\cdot b(\varphi(s))ds$ is absolutely continuous and from(3.7) we deduce

$$\dot{\varphi}_s = b(\varphi_s) + k(s)\varphi_s\,. \tag{3.14}$$

Applying the Cauchy-Schwarz inequality to (3.13) we get

$$G(\varphi; h)^2 \leq \|k\|^2_{L^2(\mu)} \|h\|^2_{L^2(\mu)} = 8I(\varphi)\|h\|^2_{L^2(\mu)} \tag{3.15}$$

with equality for h proportional to k. We conclude that $\mathcal{S}(\varphi) \leq I(\varphi)$ and then the equality (3.11) holds since H is dense in $L^2(\mu)$.

If $I(\varphi) < \infty$, φ is absolutely continuous and we use (3.14) to define $k \in L^2(\mu)$ and (3.13) holds. Hence, by (3.15) $\mathcal{S}(\varphi) \leq I(\varphi) < \infty$, which yields $\mathcal{S}(\varphi) = I(\varphi)$ in all cases. This ends the proof of the above proposition, hence the proof of the weak upper bound. □

3.2. The lower bound

In order to establish the lower bound, following the classical way, it is enough to find a subclass $\mathcal{H} \subset \mathcal{C}_a^\alpha([0,1])$ such that for all $\varphi \in \mathcal{H}$ and all $r > 0$,

$$\liminf_{\epsilon \to 0} \epsilon^2 \log P(X^\epsilon \in B_r(\varphi)) \geq -I(\varphi)\,, \tag{3.16}$$

and to prove that this class is rich enough, that is for any φ satisfying $I(\varphi) < \infty$, there exists a sequence φ_n of elements of $\mathcal{H}$ such that $\varphi_n \to \varphi$ in $\mathcal{C}_a^\alpha([0,1])$ and $I(\varphi_n) \to I(\varphi)$.

Let $\mathcal{H}$ be the set of elements $\varphi \in \mathcal{C}_a^\alpha([0,1])$ satisfying $I(\varphi) < \infty$ and such that h given by

$$h(s) := \frac{\dot{\varphi}(s) - b(\varphi(s))}{4\varphi(s)} \tag{3.17}$$

belongs to H. As before, for h chosen as above, we introduce the martingale $\left(M_t^{\epsilon,h}\right)_t$ and

$$P^{\epsilon,h} = M_1^{\epsilon,h} \cdot P \tag{3.18}$$

where P denotes the Wiener measure on $\mathcal{C}_a^\alpha([0,1])$. By Girsanov's theorem, X^ϵ is solution to the following SDE:

$$X_t^\epsilon = a + 2\epsilon \int_0^t \sqrt{|X_s^\epsilon|}\ d\beta_s + \int_0^t (4h(s)X_s^\epsilon + b(X_s^\epsilon))ds \tag{3.19}$$

where β is a $P^{\epsilon,h}$-Brownian motion. Since h is continuous and b is Lipschitz, the ODE:

$$\begin{cases} \dot{y}_t = 4h(t)y_t + b(y_t) \\ y_0 = a \end{cases} \tag{3.20}$$

admits φ as unique solution. From (3.19) we see that $\lim_{\epsilon\to 0} P^{\epsilon,h}(X^\epsilon \in B_r(\varphi)) = 1$ for every r. Now,

$$\begin{aligned} P\left(X^\epsilon \in B_r(\varphi)\right) &= P^{\epsilon,h}\left(X^\epsilon \in B_r(\varphi); \frac{1}{M_1^{\epsilon,h}}\right) \\ &\geq \exp\left(-\frac{1}{\epsilon^2} \sup_{\psi\in B_r(\varphi)} F(\psi;h)\right) P^{\epsilon,h}(X^\epsilon \in B_r(\varphi)) \end{aligned}$$

Thus,

$$\epsilon^2 \log P\left(X^\epsilon \in B_r(\varphi)\right) \geq - \sup_{\psi\in B_r(\varphi)} F(\psi;h) + \epsilon^2 \log P^{\epsilon,h}\left(X^\epsilon \in B_r(\varphi)\right)$$

and

$$\liminf_{\epsilon\to 0} \epsilon^2 \log P\left(X^\epsilon \in B_r(\varphi)\right) \geq - \sup_{\psi\in B_r(\varphi)} F(\psi;h)\,,$$

and by continuity of $F(.;h)$

$$\lim_{r\to 0} \liminf_{\epsilon\to 0} \epsilon^2 \ln P\left(X^\epsilon \in B_\delta(\varphi)\right) \geq -F(\varphi;h).$$

Since $F(\varphi;h) = I(\varphi)$, we have proved (3.16). It remains to prove:

Proposition 3.3. *For any φ satisfying $I(\varphi) < \infty$, there exists a sequence φ_n of elements of $\mathcal{H}$ such that $\varphi_n \to \varphi$ in $\mathcal{C}_a^\alpha([0,1])$ and $I(\varphi_n) \to I(\varphi)$.*

Proof. Step 1. Since the problem lies when φ hits 0 we will reduce it to the case $\varphi_t > 0$ for $t > 0$, setting for $\gamma > 0$:

$$\varphi_t^{(\gamma)} = \varphi_t + \gamma t^2\,.$$

Indeed we get immediately $\lim_{\gamma\to 0}\varphi^{(\gamma)} = \varphi$ in $\mathcal{C}_a^\alpha([0,1])$. Moreover,

$$\frac{\dot\varphi_t^{(\gamma)} - b(\varphi_t^{(\gamma)})}{\sqrt{\varphi_t^{(\gamma)}}} = \frac{\dot\varphi_t + 2\gamma t - b(\varphi_t + \gamma t^2)}{\sqrt{\varphi_t + \gamma t^2}} \underset{\gamma\to 0}{\longrightarrow} \frac{\dot\varphi_t - b(\varphi_t)}{\sqrt{\varphi_t}} \quad \text{a.e.}$$

and

$$\frac{|\dot\varphi_t^{(\gamma)} - b(\varphi_t^{(\gamma)})|}{\sqrt{\varphi_t^{(\gamma)}}} \le \frac{|\dot\varphi_t - b(\varphi_t)|}{\sqrt{\varphi_t}} + 2\sqrt{\gamma} + \beta\sqrt{\gamma}t \in L^2([0,1])\,,$$

so that, by dominated convergence, $\lim\limits_{\gamma\to 0} I(\varphi^{(\gamma)}) = I(\varphi)$.

Step 2. We take φ as in the first step.

The case $a > 0$: We have $\inf_{t\in[0,1]}\varphi_t = m > 0$. Then, $\dot\varphi \in L^2$, so it can be approximated by a smooth function $\dot\varphi^{(n)}$ and set $\varphi_t^{(n)} = a + \int_0^t \dot\varphi_s^{(n)} ds$. Since

$$\|\varphi - \varphi^{(n)}\|_\alpha \le \|\dot\varphi - \dot\varphi^{(n)}\|_{L^2},$$

then $\lim_{n\to\infty}\varphi^{(n)} = \varphi$ in $\mathcal{C}_a^\alpha([0,1])$ and for n large enough $\inf_{t\in[0,1]}\varphi_t^{(n)} \ge m/2 > 0$. Now, it is easy to see that $\varphi^{(n)} \in \mathcal{H}$ (for n large enough) and $\lim_{n\to\infty} I(\varphi^{(n)}) = I(\varphi)$. This ends the proof in that case.

The case $a = 0$. We assume $\rho = b(0) > 0$ since the case $\rho = 0$ is trivial. In i) we will reduce the problem to the case where $h := (\dot\varphi - \rho)/4\varphi \in L^2([0,1])$ and in ii) we will find an approximating sequence in $\mathcal{H}$.

i) Let us first remark that the condition $I(\varphi) < \infty$ implies $\lim\limits_{t\to 0}\varphi_t/t = \rho$ (see Feng [8]). For $r \in (0,1]$, let us define an absolutely continuous function $\psi^{(r)}$ such that $\dot\psi^{(r)} = \rho$ on $[0, r/2)$, $\psi^{(r)} > 0$ on $[r/2, 1]$ and $\psi^{(r)} = \varphi$ on $[r, 1]$. More precisely let $a_r := \frac{2\varphi_r}{r} - \rho$ and
   - $\psi_t^{(r)} = \rho t, \qquad t \in [0, \frac{r}{2}]$,
   - $\psi_t^{(r)} = \rho\frac{r}{2} + a_r(t - \frac{r}{2}),\ t \in [\frac{r}{2}, r]$,
   - $\psi_t^{(r)} = \varphi_t, \qquad t \in [r, 1]$.

   We are now sure that $(\dot\psi^{(r)} - \rho)/\psi^{(r)} \in L^2$ and $\lim_{r\to 0}\dot\psi^{(r)} = \dot\varphi$ in $L^2([0,1])$. To prove that $\lim_{r\to 0} I(\psi^{(r)}) = I(\varphi)$, it remains only to check that the contribution of $[0, r]$ to $I(\psi^{(r)})$ tends to 0, since $\psi^{(r)} = \varphi$ on $[r, 1]$. We have

$$\int_0^r \frac{(\dot\psi_t^{(r)} - b(\psi_t^{(r)}))^2}{4\psi_t^{(r)}}\,dt \le \int_0^r \frac{(\dot\psi_t^{(r)} - \rho)^2}{2\psi_t^{(r)}}\,dt + \int_0^r \frac{(b(\psi_t^{(r)}) - \rho)^2}{2\psi_t^{(r)}}\,dt$$

   We can choose r_0 such that $\varphi_r/r \in [\rho/2, 2\rho]$ for $r \le r_0$, so the first term of the right hand side is bounded by $2(a_r - \rho)^2/\rho$ and the second one by $\beta^2 r\varphi_r$. Since $\lim\limits_{r\to 0} a_r = \rho$, we conclude that $\lim_{r\to 0} I(\psi^{(r)}) = I(\varphi)$.

ii) Take $h^{(n)}$ a sequence of smooth functions converging to h in $L^2([0,1])$ and define $\varphi^{(n)} \in \mathcal{H}$ as the solution of the differential equation

$$\begin{cases} \dot{\varphi}_t^{(n)} = 4h_t^{(n)}\varphi_t^{(n)} + \rho \\ \varphi_0^{(n)} = 0 \end{cases}$$

Using an explicit expression or a Gronwall type argument we see that $\varphi^{(n)}$ converges uniformly to φ as $n \to \infty$. Moreover $\dot{\varphi} = \lim \dot{\varphi}^{(n)}$ in L^2 so that $\varphi = \lim_{n\to\infty} \varphi^{(n)}$ in $\mathcal{C}_a^\alpha([0,1])$. Now,

$$\begin{aligned} \frac{\dot{\varphi}_t^{(n)} - b(\varphi_t^{(n)})}{2\sqrt{\varphi_t^{(n)}}} &= 2h_t^{(n)}\sqrt{\varphi_t^{(n)}} + \frac{\rho - b(\varphi_t^{(n)})}{\sqrt{\varphi_t^{(n)}}} \\ &\xrightarrow[n\to\infty]{L^2} 2h_t\sqrt{\varphi_t} + \frac{\rho - b(\varphi_t)}{\sqrt{\varphi_t}} = \frac{\dot{\varphi}_t - b(\varphi_t)}{2\sqrt{\varphi_t}} \end{aligned}$$

Thus, $\lim\limits_{n\to\infty} I(\varphi^{(n)}) = I(\varphi)$ and the proof is finished. □

Remark. Our diffusion satisfies the SDE

$$dX_t^\epsilon = b(X_t^\epsilon)dt + \epsilon\sigma(X_t^\epsilon)dB_t$$

with $\sigma(x) = 2\sqrt{|x|}$ and we used in the above subsections (see (3.18), (3.17) and (3.3)) the change of probability whose Radon-Nikodym derivative is $\mathcal{E}(\frac{1}{\epsilon}\int_0^t \widehat{h}_s dB_s)$ with

$$\widehat{h}_s = \frac{\dot{\varphi}_s - b(\varphi_s)}{\sigma^2(\varphi_s)}\sigma(X_s^\epsilon)$$

For a non-degenerate diffusion, Lipster-Pukhalskii [13] used an exponential martingale with

$$\widehat{h}_s = \frac{\dot{\varphi}_s - b(X_s^\epsilon)}{\sigma(X_s^\epsilon)},$$

(this change of probability was introduced by Wentzell-Freidlin in [10]).

4. Second method: additivity

For $\rho > 0, a \geq 0$ or $\rho = 0, a > 0$ the additivity property (1.3) gives

$$Q_{\delta a}^{(\delta\rho)} \oplus Q_{\delta' a}^{(\delta'\rho)} = Q_{(\delta+\delta')a}^{((\delta+\delta')\rho)} \tag{4.1}$$

for $\delta, \delta' > 0$. We are in the situation of a Cramer's theorem for the family (P_δ) (defined by (1.6)) in the Banach space $\mathcal{C}_a([0,T])$ whose dual is $\mathcal{M}[0,T]$, the space of bounded signed measures on $[0,T]$.

4.1. LDP (Cramer's theorem)

On $\mathcal{M}[0,T]$, let us define the logarithmic moment generating function

$$\Lambda^{\delta}_{a,T}(\mu) = \log Q^{(\delta)}_a\big[\exp \int_0^T X_s \, d\mu(s)\big]\,. \tag{4.2}$$

Theorem 4.1. *For $\rho > 0$, $a \geq 0$ or $\rho = 0$ and $a > 0$, the family $\{P_\delta\}_\delta$ of distributions on $\mathcal{C}_a([0,T])$ (equipped with the topology of uniform convergence) satisfies the LDP with speed δ^{-1} and good rate function Λ^*, (the Fenchel-Legendre dual of Λ) given by*

$$\Lambda^*(\varphi) = \sup_{\mu \in \mathcal{M}[0,T]} \{\int_0^T \varphi(t)\, d\mu(t) - \Lambda^{\rho}_{a,T}(\mu)\} \tag{4.3}$$

for $\varphi \in \mathcal{C}_a([0,T])$. More generally the LDP holds in $\mathcal{C}^{\alpha}_a([0,T])$ for any $\alpha \in (0,1/2)$.

Proof. We will use additivity in δ as in the standard proofs of Cramer's theorem (see for instance [7] chap. III, [6] chap. 6, [1]). This yields a weak LDP

First step. For A a convex measurable subset of $\mathcal{C}([0,T])$ the function $f_A(\delta) := -\log P_\delta(A)$ is subadditive. Let us prove that if $P_\delta(A) > 0$ for some positive δ, then there exists η_0 such that

$$\inf_{\eta \geq \eta_0} P_\eta(A) > 0\,.$$

We follow the path of [7] p.60. One can find a convex compact $K \subset A$ such that $P_\delta(K) > 0$. Let $\epsilon < d(K, A^c)/2$ where $d(.,.)$ is the uniform norm distance, and define $G := \{f : \|f - K\| < \epsilon\}$. For $\eta > \delta$, define $q \in [0,\infty)$ and $r \in [0,\delta)$ so that $\eta = q\delta + r$. From the additivity (4.1) we have

$$P_\eta(A) = Q^{(\eta\rho)}_{a\eta}(\eta A) \geq Q^{(q\delta)}_{q\delta a}(\eta G)\; Q^{(r)}_{ra}(\|X\| < \eta\epsilon)\,. \tag{4.4}$$

From the tightness study in the above paragraph, we deduce

$$\lim_{R\to\infty} \sup_{r\leq\delta} Q^{(r)}_{ra}(\|X\| < R) = 1 \tag{4.5}$$

so that there exists η_1 with $\sup_{r\leq\delta} Q^{(r)}_{ra}(\|X\| < \eta\epsilon) > 1/2$ for $\eta > \eta_1$. Besides,

$$Q^{(q\delta\rho)}_{q\delta a}(\eta G) = Q^{(q\delta)}_{q\delta a}(\inf_{k\in K} \|X - \eta k\| \leq \eta\epsilon)\,. \tag{4.6}$$

Now, if $X \in q\delta K$ then $\inf_{k\in K} \|X - \eta k\| \leq \delta M$ where $M := \sup\{\|q\|,\ q \in K\}$ and then, if $\eta > \eta_2 := \delta M/\epsilon$ we get

$$Q^{(q\delta d)}_{q\delta a}(\eta G) \geq Q^{(q\delta d)}_{q\delta a}(q\delta K)$$

From the subadditivity and (4.4) we conclude

$$\inf_{\eta > \max\{\eta_1,\eta_2\}} P_\eta(A) \geq \frac{1}{2}\left[Q^{(\delta\rho)}_{\delta a}(K)\right]^q = \frac{1}{2}\,[P_\delta(K)]^q > 0\,.$$

Second step. Fix an open convex subset $A \subset \mathcal{C}$. Either $P_\delta(A) = 0$ for all δ, in which case $\mathcal{L}_A := -\lim_{\delta\to+\infty} \frac{1}{\delta} \log P_\delta(A) = +\infty$, or else the limit $\mathcal{L}_A$ exists by Lemma 4.2.5 of [7]. Then we apply Theorem 4.1.11 and Lemma 4.1.21 of [6]. This yields a weak LDP with a convex rate function I. Then, following Lemma 6.1.8 of [6] we have, for every open, convex subset $A \subset \mathcal{C}$

$$\lim_{\delta\to\infty} \frac{1}{\delta} \log P_\delta(A) = -\inf_A I\,.$$

Last step. Identification of the rate function. We follow the proof of Theorem 6.1.3. of [6], the only new point is that we need a Cramer's theorem for the additive family of real r.v. $\int_0^T X_t^{(\delta)} \mu(dt)$ and an equivalent of Corollary 2.2.19. of [6]. We don't know of any explicit reference for this fact, but the proof is straightforward. $\square$

5. Third method: Wentzell-Freidlin approach

Here we consider the normalized process $\xi_t^{(\delta)} = \delta^{-1} X_t^{(\delta\rho)}$ with $X_0^{(\delta)} = \delta a$, which satisfies

$$d\xi_t^{(\delta)} = \rho\, dt + \frac{2}{\sqrt{\delta}}\sqrt{\xi_t^{(\delta)}}\, dB_t\,,\ \ \xi_0^{(\delta)} = a\,. \tag{5.1}$$

When $\delta \to \infty$, we are in the domain of the Wentzell-Freidlin asymptotics of LDP for diffusions. There is a broad literature on these problems, [1] [3] [13] [21]... The difficulty here lies in the singularity of the diffusion coefficient.

Let us assume $\rho > 0$. Since we are interested in the large values of δ, we may assume $\delta \geq 2/\rho$, in which case, $\xi_t^{(\delta)} > 0$ for $t > 0$. In the Bessel (not squared Bessel) notations, the process $\eta_t^{(\delta)} = \sqrt{\xi_t^{(\delta)}}$ satisfies

$$d\eta_t^{(\delta)} = \left(\rho - \frac{1}{\delta}\right)\frac{dt}{2\eta_t^{(\delta)}} + \frac{1}{\sqrt{\delta}}\, dB_t\,,\ \ \eta_0^{(\delta)} = \sqrt{a}\,. \tag{5.2}$$

When the diffusion coefficient is constant (as here) and the drift coefficient is smooth, Azencott's method ([1]) uses the relation linking the process and its driving (rescaled) Brownian motion, to apply the contraction principle. Here we adapt this scheme to the process $\eta^{(\delta)}$ (with a non smooth drift) and begin with a study of the controlled equation.

5.1. The controlled equation

The controlled equation corresponding to (5.2) is the integral equation

$$u(t) = \ell(t) + \frac{\rho}{2}\int_0^t \frac{ds}{u(s)}\,,\ \ u \geq 0 \tag{5.3}$$

with ℓ continuous and $\ell(0) \geq 0$.

We may use the results of H.P McKean ([14] and [15] p.80) obtained for $\rho = 2$.

Proposition 5.1 (Mc Kean). *For $\ell \in \mathcal{C}([0,T])$, $\ell(0) \geq 0$, there exists a unique non negative solution of (5.3) when $\rho = 2$. Let us denote it by $S_2(\ell)$. Given ℓ_1, ℓ_2 we have*

$$\|S_2(\ell_1) - S_2(\ell_2)\|_\infty \leq 2\|\ell_1 - \ell_2\|_\infty$$

where $\|.\|_\infty$ is the uniform norm.

Now, let us remark that if $\gamma > 0$, the function $v = \gamma S_2(\ell)$ is solution of

$$v(t) = \gamma\ell(t) + \gamma^2 \int_0^t \frac{ds}{v(s)},$$

and the converse holds. So, (5.3) has a unique solution

$$S_\rho(\ell) := \sqrt{\frac{\rho}{2}} S_2\left(\sqrt{\frac{2}{\rho}}\ell\right), \tag{5.4}$$

and we have

$$\|S_\rho(\ell_1) - S_\rho(\ell_2)\|_\infty \leq 2\|\ell_1 - \ell_2\|_\infty. \tag{5.5}$$

Proposition 5.2. *If $\ell \in H_x^1([0,T])$, and $x \geq 0$ then, $S_\rho(\ell)(t) > 0$ for $t \in (0,T]$.*

Proof. Let us first assume $x > 0$. For $\ell \in \mathcal{C}_x([0,T])$, McKean [14, 15] constructed a positive solution on $[0, t_1)$ where $t_1 = \inf\{s : \ell(s) = 0\}$ and extended it to a non-negative solution on $[0,T]$. We claim that if $\ell \in H_x^1([0,T])$ this (unique) solution is actually strictly positive.

First, from (5.3), we have (as long as u is positive)

$$\dot{u} = \dot{\ell} + \frac{\rho}{2u} \tag{5.6}$$

hence

$$2u\dot{u} = 2u\dot{\ell} + \rho \leq u^2 + \dot{\ell}^2 + \rho \tag{5.7}$$

and by Gronwall's inequality:

$$u^2(t) \leq (x^2 + \rho T + \|\dot{\ell}\|_2^2)e^T =: M. \tag{5.8}$$

Now, (5.3) and (5.6) entail

$$(\dot{\log u}) = \frac{\dot{u}}{u} = \frac{\dot{\ell}}{u} + \frac{\rho}{2u^2} \tag{5.9}$$

$$(\log u(t))^- + \int_0^t \frac{\rho\, ds}{2u^2(s)} = (\log u(t))^+ - \log x - \int_0^t \frac{\dot{\ell}(s)ds}{u(s)} \tag{5.10}$$

$$\leq \frac{1}{2}(\log M)^+ - \log x + \int_0^t \frac{\rho\, ds}{4u^2(s)} + \rho^{-1}\|\dot{\ell}\|_2^2 \tag{5.11}$$

for $t \leq T$ hence

$$(\log u(t))^- + \int_0^t \frac{\rho ds}{4u^2(s)} \leq \frac{1}{2}(\log M)^+ - \log x + \rho^{-1}\|\dot{\ell}\|_2^2 =: \gamma\,. \quad (5.12)$$

This yields $u(t) \geq e^{-\gamma}$.

Given this uniform lower bound, it is classical that the solution may be extended.

Let us now study the case $x = 0$. Since, by definition of u in (5.3) u^{-1} is in L^1, for every $t_0 > 0$ there exists $0 < t' < t_0$ such that $u(t') > 0$. Then we have

$$u(.+t') = S(\ell(.+t') - \ell(t') + u(t'))$$

(which is positive from the above results) and u is positive on the full interval $[0, T]$. □

A useful property of the family $S_u(\ell)$, $u > 0$ is given by the following lemma.

Lemma 5.3. *If* $u, v > 0$ *and* $\ell \in \mathcal{C}_0([0, T])$ *then*

$$\|S_u(\ell) - S_v(\ell)\| \leq |\sqrt{u} - \sqrt{v}|\left[\sqrt{T} + \frac{4\|\ell\|_\infty}{\sqrt{u}}\right] \quad (5.13)$$

Proof of Lemma 5.3. Since $S_u(0)(t) = \sqrt{ut}$, a first application of (5.5) entails

$$\|S_u(\ell)\|_\infty \leq \sqrt{uT} + 2\|\ell\|_\infty \quad (5.14)$$

Moreover from (5.4)

$$\begin{aligned} S_u(\ell) - S_v(\ell) &= S_u(\ell) - \sqrt{vu^{-1}}S_u\left(\sqrt{uv^{-1}}\ell\right) \\ &= \left(1 - \sqrt{vu^{-1}}\right)S_u(\ell) + \sqrt{vu^{-1}}\left[S_u(\ell) - S_u\left(\sqrt{uv^{-1}}\ell\right)\right] \end{aligned}$$

It is enough to apply (5.14) and (5.5) again. □

5.2. Another proof of Proposition 1.1

Proof. We prove only part 2 of that theorem , since the mapping $\eta \mapsto \eta^2$ allows to carry it to part 1 by the contraction principle. Moreover we will assume $a = 0$ to simplify.

It is not possible to apply the classical Wentzell-Freidlin results since the drift is neither bounded nor Lipschitz. Since the diffusion coefficient in (5.2) is constant, we may use a version of the contraction principle.

We have $\eta^{(\delta)} = S_{\rho-\delta^{-1}}(B^\delta)$ where $B^\delta := \frac{1}{\sqrt{\delta}}B$. From Proposition 5.1 the mapping S_ρ is continuous. Schilder's theorem ([6] Th. 5.2.3) yields a LDP for B^δ with good rate function $\ell \mapsto \frac{1}{2}\|\dot{\ell}\|_2^2$. Corollary 4.2.21 of [6] on approximate contractions will allow to use $S_{\rho-\delta^{-1}}$ instead of S_ρ. It says that if

$$\limsup_{\delta\to\infty} \frac{1}{\delta}\log P\left(\|S_\rho(B^\delta) - S_{\rho-\delta^{-1}}(B^\delta)\|_\infty \geq R\right) = -\infty \quad (5.15)$$

for every $R > 0$, then $\eta^{(\delta)} = S_{\rho-\delta^{-1}}(B^\delta)$ satisfies the LDP with good rate function

$$I'(\varphi) := \inf\left\{\frac{1}{2}\|\dot{\ell}\|_2^2\,;\ \ell \in H_0^1([0,T])\,;\ \varphi = S_\rho(\ell)\right\}. \tag{5.16}$$

From Proposition 5.1 the mapping S_ρ is clearly injective. From Prop. 5.2, if $I'(\varphi) < \infty$ then $\varphi(t) > 0$ for $t > 0$ and from (5.6) $I'(\varphi) = K^\rho_{0,T}(\varphi)$. Conversely, if $\varphi(0) = 0$ and $K^\rho_{0,T}(\varphi) < \infty$, then $\dot{\varphi} - \frac{\rho}{2\varphi} \in L^2$ and $\varphi - \int_0^\cdot \frac{\rho}{2\varphi} \in H_0^1$. Hence Proposition 1.1 is proved modulo (5.15). Now we apply lemma 5.3 with $u = \rho$ and $v = \rho - \delta^{-1}$.

$$\|S_\rho(\ell) - S_{\rho-\delta^{-1}}(\ell)\|_\infty \le \delta^{-1}\rho^{-1/2}(\sqrt{T} + 4\rho^{-1/2}\|\ell\|_\infty) \tag{5.17}$$

It remains to apply the classical exponential inequality $P(\|B\| \ge a) \le 4e^{-\frac{a^2}{2T}}$ to obtain (5.15). □

5.3. The case $\rho = 0$

When $\rho = 0$ and $a > 0$ (BESQ of 0-dimension starting at δa and rescaled by δ), our Theorem 4.1 says that the family $\{P_\delta\}$ satisfies the LDP with good rate function the dual of $\Lambda^\rho_{a,T}$. Nevertheless, the case $\rho = 0$ is excluded from the previous analysis of this section. Feng ([8]) gives a partial answer to the problem. He showed that if φ is a path starting from a and reaching 0 at $\tau(\varphi)$ then

$$\begin{aligned}\limsup_{\gamma\to 0}&\left(\limsup_{\delta\to\infty}\frac{1}{\delta}\log P\left(\sup_{0\le t\le\tau(\varphi)} \mid \xi_t^{(\delta)} - \varphi(t) \mid \le \gamma\right)\right)\\ &\le -J^0_{a,\tau(\varphi)}(\varphi) = -\frac{1}{8}\int_0^{\tau(\varphi)}\frac{\dot{\varphi}_s^2}{\varphi_s}\,ds\,.\end{aligned} \tag{5.18}$$

He claims that “In this case a LDP can also be established by an approximation argument”.

Actually we can explain how to get the LDP on $[0, T]$ by means of a Girsanov transformation on BESQ processes. In the sequel of this subsection, we will denote P^0_δ the distribution P_δ given by formula (1.6) not to forget that we are handling the case $\rho = 0$. According to [22] formula (2c), if $\tau = \tau(X) = \inf\{u > 0;\, X_u = 0\}$ and $\mathcal{F}_t = \sigma(X_s, s \le t)$, then

$$Q^{(0)}_{a|\mathcal{F}_t\cap(t<\tau)} = \left(\frac{a}{X_t}\right)\cdot Q^{(4)}_{a|\mathcal{F}_t} \tag{5.19}$$

Let φ belongs to $\mathcal{C}_a([0,T];[0,\infty))$ and $J^0_{a,T}(\varphi) < \infty$. Let $B = \{f \in \mathcal{C}_a([0,T];[0,\infty)) : \sup_{t\in[0,T]}|f(t) - \varphi(t)| \le \gamma\}$. If $\varphi > 0$ on $[0, T]$ and γ small enough, then $\tau(f) > T$ for every $f \in B$. Moreover there exist two positive constants A_1 and A_2 such that $A_1 \le |f(t)| \le A_2$ for all $t \in [0, T]$ and $f \in B$. Scaling by δ we get

$$P^0_\delta(B) = P^0_\delta(B \cap (T < \tau)) = Q^{(4)}_{\delta a}\left(\frac{\delta a}{X_T}1_B\left(\frac{X_\cdot}{\delta}\right)\right) \le \left(\frac{a}{A_1}\right)Q^{(4)}_{\delta a}(\delta B)\,, \tag{5.20}$$

and

$$P^0_\delta(\overset{\circ}{B}) = P^0_\delta(\overset{\circ}{B} \cap (T < \tau)) = Q^{(4)}_{\delta a}\left(\frac{\delta a}{X_T} 1_{\overset{\circ}{B}}\right) \geq \left(\frac{a}{A_2}\right) Q^{(4)}_{\delta a}(\delta \overset{\circ}{B}), \tag{5.21}$$

so that

$$\liminf_{\delta\to\infty} P^0_\delta(\overset{\circ}{B}) = \liminf_{\delta\to\infty} Q^{(4)}_{\delta a}(\delta \overset{\circ}{B}) \leq \limsup_{\delta\to\infty} Q^{(4)}_{a\delta}(\delta B) = \limsup_{\delta\to\infty} P^0_\delta(B)\,. \tag{5.22}$$

We have now to study large deviations for the rescaled BESQ^4 process $Q^{(4)}_{\delta a}(\delta\cdot)$. Since, under $Q^{(4)}_a$,

$$X_t = \left(\sqrt{a} + B^{(1)}_t\right)^2 + \left(B^{(2)}_t\right)^2 + \left(B^{(3)}_t\right)^2 + \left(B^{(4)}_t\right)^2, \tag{5.23}$$

we have by rescaling, under $Q^{(4)}_{\delta a}$

$$\frac{X_t}{\delta} = \left(\sqrt{a} + \frac{B^{(1)}_t}{\sqrt{\delta}}\right)^2 + \left(\frac{B^{(2)}_t}{\sqrt{\delta}}\right)^2 + \left(\frac{B^{(3)}_t}{\sqrt{\delta}}\right)^2 + \left(\frac{B^{(4)}_t}{\sqrt{\delta}}\right)^2, \tag{5.24}$$

We can now use the Schilder theorem and the contraction principle ([6]) to get the following action functional for $(X_t/\delta,\ t \in [0, T])$

$$\mathcal{I}(\varphi) = \inf\left\{\frac{1}{2}\int_0^T \left[\sum_{i=1}^4 \dot{\psi}_1(t)^2\right] dt;\ \left(\sqrt{a} + \psi_1(t)\right)^2 + \sum_{i=2}^4 \psi_i(t)^2 = \varphi(t),\ \forall t \in [0, T]\right\}$$

It is easy to see that the above infimum is reached at $\psi_1 = \sqrt{\varphi} - \sqrt{a}$, $\psi_2 = 0$, $\psi_3 = 0$, $\psi_4 = 0$ and is then equal to

$$J^0_{a,T}(\varphi) = \frac{1}{8}\int_0^T \frac{\dot{\varphi}_s^2}{\varphi_s}\, ds \quad \text{if } \varphi_0 = a\,.$$

Afterwards, a detailed study of paths touching 0 as in [8] would allow to get the LDP.

Remark. A heuristic explanation involving another relation between BESQ^0 and BESQ^4 can also be given. According to classical results ([18] Ex. 1.23 p. 451), the law of $\{X_t, t \leq \tau\}$ under $Q^{(0)}_b$ is the same as the law of $\{X_{L_b - t}, t \leq L_b\}$ under $Q^{(4)}_0$, where $L_b = \sup\{t : X_t = b\}$.

Scaling again by δ we are lead to the large deviations for $Q^{(4)}_0(\delta\cdot)$ in reversed time until its last visit to a. We may imagine that it corresponds to reverse the time in the integral of (5.18).

6. Non-constant drift

6.1. The family of squared radial OU^δ processes

For Y defined by (1.10), let us notice that the family $y^\epsilon := \frac{\epsilon^2}{2} Y^{(\rho/2\epsilon^2)}$ satisfies the SDE

$$dy_t^\epsilon = \left(cy_t^\epsilon + \rho\right) dt + \epsilon \sqrt{y_t^\epsilon}\, dB_t\,, \tag{6.1}$$

and we are in the situation of Feng ([8]). We recover his result (see p.117 and remark on top of p. 122), but our method is different.

We use a time change. Let $e_c(t) = c^{-1}\left(1 - e^{-ct}\right)$ and

$$\begin{aligned} G:\ \mathcal{C}\left([0,T];(0,\infty)\right) &\to \mathcal{C}\left([0,e_c(T)];(0,\infty)\right) \\ x &\mapsto \left(t \mapsto e^{ct} x(e_c(t))\right) . \end{aligned}$$

The mapping G is continuous for the uniform topologies. Since $Y \overset{\mathcal{D}}{=} G(X)$, we may apply Proposition 1.1 and the contraction principle to see that the action functional for the LDP of ${}^cP_\delta$ is $J^\rho_{a,(e_c)^{-1}(T)}\left(G^{-1}(\varphi)\right)$ which after an obvious change of variable is equal to ${}^cJ^\rho_{a,T}$.

6.2. Case of the general drift

We show how Proposition 1.1 leads to Theorem 1.3 using Varadhan's lemma, under some smoothness assumptions on the drift coefficient b. We assume that $\hat{b}(x) := \frac{b(x)-\rho}{4x}$ is well defined on $\mathbb{R}_+$ and differentiable, with $\hat{b}'$ Lipschitz.
From Girsanov's theorem, denoting by Q_ϵ, resp. P_ϵ, the law of Y^ϵ solution of (1.5) associated to $b(x) = \rho$, resp. the law of X^ϵ solution of (1.5) associated to b, we have the following absolute continuity relation:

$$\frac{dP_\epsilon}{dQ_\epsilon} = \exp\left(\frac{1}{\epsilon^2}\left\{\int_0^T \hat{b}(X_s)(dX_s - \rho ds)\ - 2\int_0^T \hat{b}^2(X_s)\, X_s\, ds\right\}\right).$$

Now, from Itô's formula, for $f(t) := \int_0^t \hat{b}(s)ds$,

$$f(Y_T^\epsilon) - f(a) = \int_0^T \hat{b}(Y_s^\epsilon)dY_s^\epsilon + 2\epsilon^2\int_0^T \hat{b}'(Y_s^\epsilon)Y_s^\epsilon ds$$

Thus, $\frac{dP_\epsilon}{dQ_\epsilon} = \exp\left(\frac{1}{\epsilon^2}F_\epsilon(X)\right)$ with

$$\begin{aligned} F_\epsilon(X) =&\ f(X_T) - f(a) - \int_0^T (\rho\hat{b}(X_s) + 2X_s\hat{b}^2(X_s))\, ds \\ &-\epsilon^2\int_0^T 2X_s\hat{b}'(X_s)\, ds \\ =:&\ F(X) + \epsilon^2 G(X) \end{aligned}$$

and F, G continuous.

We are almost [2] in the situation of Varadhan's Lemma ([6] Theorem 4.3.1 and Exercise 4.3.11). The validity of Eq. (4.3.2) of [6] Theorem 4.3.1 follows from the exponential tighness result. We conclude that (X^ϵ) satisfies the LDP with good rate function $J^{b(0)}_{a,T}(\varphi) - F(\varphi)$. It is easy to see that we recover the expression given in (1.13).

7. The family of squared Bessel bridges

We use the fact ([18] ex. 3.6) that under Q^δ_x, the process

$$X^{br}(u) := (1-u)^2 X_{u/(1-u)}, \quad 0 \le u < 1\,,$$

(and $X^{br}(1) = 0$) has the law $Q^{(\delta)}_{x\to 0}$. Set

$$\bar{\mathcal{C}}_a = \left\{\varphi \in \mathcal{C}([0,\infty);\mathbb{R}) : \varphi(0) = a \text{ and } \lim_{t\to\infty} \frac{\varphi(t)}{t^2} = 0\right\} \tag{7.1}$$

equipped with the norm

$$\| \varphi \| = \sup_{t\ge 0} \frac{|\varphi(t)|}{(1+t)^2}$$

which makes it a separable Banach space. The mapping $\varphi \mapsto \varphi^{br}$ is continuous (Lipschitzian) from $\bar{\mathcal{C}}_a$ to the space $\mathcal{C}^{0,1}_{a,0} := \{\varphi \in \mathcal{C}([0,1];\mathbb{R}) : \varphi(0) = a \text{ and } \varphi(1) = 0\}$ equipped with the uniform norm.

7.1. Extension of the previous results to $\mathcal{C}([0,\infty);[0,\infty))$

The extension of the results of Proposition 1.1 1) to $\bar{\mathcal{C}}_a$ follows from a theorem of Dawson and Gärtner [5] on LDP for projective limits. From this theorem we see that the LDP holds for the topology of uniform convergence on $[0,T]$ for every T. To strenghten it, it is enough to show exponential tightness. We will follow the scheme of [7] Lemma 1.3.25. Set

$$V(\varphi) = \sum_{n=1}^{\infty} \frac{1}{2^n} \sup_{0\le s<t\le n} \frac{|\varphi(s)-\varphi(t)|}{|t-s|^{1/4}} + \sup_{t\ge 1} \frac{|\varphi(t)|}{t^{3/2}}\,.$$

For any $L > 0$ the set $\{\varphi : V(\varphi) \le L\}$ is compact ([7] p.17) so we have to prove

$$\lim_{L\to\infty} \limsup_{\delta\to\infty} \delta^{-1} \log Q^{\delta\rho}_{\delta a}(V(X) > \delta L) = -\infty\,. \tag{7.2}$$

[2] Here, the continuous functional F_ϵ depends on ϵ with $F_\epsilon \to F$. We can show that the proof of Varadhan's Lemma remains true in this case.

First, for any $\theta \in (1/2, 1)$

$$Q_{\delta a}^{\delta\rho}\left(\sum_{n=1}^{\infty}\frac{1}{2^n}\sup_{0\le s<t\le n}\frac{|X_s - X_t|}{|t-s|^{1/4}} > \delta L\right) \le \sum_{n=1}^{\infty} Q_{\delta a}^{\delta\rho}\left(\| X \|_{1/4,n} > \frac{\delta L}{\theta(1-\theta)}(2\theta)^n\right)$$
$$\le C A^{\delta\rho}\sum_n B^{\delta a/n}\exp\left[-\gamma\frac{\delta L(2\theta)^n}{\theta(1-\theta)n^{3/4}}\right] \quad (7.3)$$

(from (2.11). There exists $\lambda > 0$ such that $(2\theta)^n \ge n^{7/4}\lambda$ for every $n \ge 1$ so,

$$\limsup_{\delta\to\infty}\delta^{-1}\log Q_{\delta a}^{\delta\rho}\left(\sum_{n=1}^{\infty}\frac{1}{2^n}\sup_{0\le s<t\le n}\frac{|X_s - X_t|}{|t-s|^{1/4}} > \delta L\right)$$
$$\le \rho\log A + x\log B - \frac{\gamma L\lambda}{\theta(1-\theta)} \quad (7.4)$$

and letting $L \to \infty$ gives half the result. It remains to prove that

$$\lim_{L\to\infty}\limsup_{\delta\to\infty}\delta^{-1}\log Q_{\delta a}^{\delta\rho}(X^* > \delta L) = -\infty \quad (7.5)$$

where $X^* := \sup_{t\ge 1}\frac{X_t}{t^{3/2}}$. Without loss of generality we will assume that $\rho = 1$. Let n such that $n-1 \le \delta < n$. By additivity we may assume that the processes $X^{(n-1)}$, $X^{(\delta)}$, $X^{(n)}$ are defined on the same probability space and satisfy for every t $X_t^{(n-1)} \le X_t^{(\delta)} \le X_t^{(n)}$; then (by the comparison theorem and the exponential inequality)

$$I := Q_{\delta a}^{\delta}\left(X^* > \delta L\right) \le Q_{na}^{n}(X^* > \delta L) \le e^{-\lambda\delta L} Q_{na}^{n}\left(\exp\lambda X^*\right). \quad (7.6)$$

Now,

$$Q_{na}^{(n)}\left(\exp\lambda X^*\right) \le \left[Q_a^{(1)}\left(\exp\lambda X^*\right)\right]^n$$

and by Fernique's Theorem ([9] and [7] p. 17-18) there exists $\lambda > 0$ such that $U := Q_a^{(1)}(\exp\lambda X^*) < \infty$, which gives,

$$I \le U^{\delta+1}e^{-\lambda\delta L}. \quad (7.7)$$

Letting δ and L go to infinity gives the expected conclusion. □

7.2. Main result

Theorem 7.1. *The family of distributions* $Q_{\delta a\to 0}^{(\delta\rho)}\left(\delta^{-1}X_t,\ 0 \le t \le 1 \in .\right)$ *on* $\mathcal{C}_{a,0}^{0,1}$ *satisfies a LDP with good rate function*

$$J^{br}(\varphi) := \int_0^1 \frac{\left(\dot\varphi(u) + \frac{2\varphi(u)}{1-u} - \rho\right)^2}{8\varphi(u)}\,du. \quad (7.8)$$

Proof of Theorem 7.1. Apply the contraction principle and the above result. □

Remark. It is worthwile to notice that the rate function J^{br} can be obtained informally by applying the Wentzell-Freidlin formula for small parameter diffusions to the squared Bessel bridge which is actually solution of

$$X_t = 2\int_0^t \sqrt{X_s} dB_s + \int_0^t \left(\delta - \frac{2X_s}{1-s}\right) ds\,, \tag{7.9}$$

(see [18] Exercise 3.11).

8. Rate functions and variational formulae

8.1. Laplace transform

We now present two known expressions of the log-Laplace transform $\Lambda^{\rho}_{x,T}$ defined in (4.2). The first one comes from (1.15):

$$\Lambda^{\rho}_{x,T}(\mu) = \frac{\rho}{2}\log\phi_\mu(T) + \frac{x}{2}\phi'_\mu(0)\,. \tag{8.1}$$

where ϕ_μ is solution of the Sturm–Liouville equation,

$$\frac{1}{2}\phi'' = -\mu\phi\,, \quad \phi(0) = 1\,, \quad \frac{1}{2}\phi'(T) = \mu(\{T\})\phi(T)\,. \tag{8.2}$$

This result is usually found in the literature when μ is a negative measure. It may be extended to measures μ such that ϕ_μ is strictly positive on $[0, T]$.

Noting as above, that under $Q_x^{(1)}$, X is B^2, the square of Brownian motion, a classical Gaussian calculus is possible. In the most simple case $x = 0$ it gives

$$Q_0^{(1)}\big[\exp\int_0^T X_s\, d\mu(s)\big] = \det\,[I - 2A_\mu]^{-1/2} \tag{8.3}$$

where A_μ is the symmetric trace-class operator on the Cameron-Martin space $H_0^1([0, T])$ (see (1.18)), given by

$$< A_\mu h, h > = \int_0^T h_s^2\, d\mu(s)\,. \tag{8.4}$$

This holds as soon as the greatest eigenvalue $\lambda_1(\mu)$ of A_μ, satisfies $2\lambda_1(\mu) < 1$.

In the general case $x \neq 0$ it can be proved that

$$Q_x^{(1)}\big[\exp\int_0^T X_s\, d\mu(s)\big] = \det[I - 2A_\mu]^{-1/2} e^{xk(\mu)} \tag{8.5}$$

with

$$k(\mu) = \Big[\int_0^T d\mu(s) + 2\sum_0^\infty \frac{(\int_0^T f_n d\mu)^2}{1-2\lambda_n}\Big]$$

where (f_n) is an orthonormal basis of eigenvectors of A_μ. It is of the same form as in (1.15). Comparing with (1.15) , we get

$$\phi_\mu(T) = \det[I - 2A_\mu]^{-1}\,, \quad \frac{1}{2}\phi'_\mu(0) = k(\mu)\,.$$

8.2. Variational formulae

We will restrict ourselves to the BESQ case to simplify the discussion. Using three methods, we obtained three variational formulae.

- The "exponential martingale method" yields

$$J^{\rho}_{x,T}(\varphi) = \sup_h \left(G(\varphi, h) - 2\int_0^1 h^2(s)\varphi_s ds \right) \tag{8.6}$$

- The "Cramer method" yields

$$J^{\rho}_{x,T}(\varphi) = \sup_\mu \left(\int_0^T \varphi_s d\mu(s) - \Lambda^{\rho}_{x,T}(\mu) \right) \tag{8.7}$$

- The "Wentzell-Freidlin method" yields

$$J^{\rho}_{x,T}(\varphi) = \inf \{ \frac{1}{2} \|\dot{\ell}\|^2; \ell \in H^1_{\sqrt{x}},\ S_\rho(\ell) = \sqrt{\varphi} \}\,. \tag{8.8}$$

Our aim is to explain the relations between the different quantities involved in these formulae.

1) For $\varphi \in \mathcal{H}$ we saw in section 3.2 (see (3.20)) that the optimal h in (8.6) is given by

$$h = \frac{\dot{\varphi} - \rho}{4\varphi}\,. \tag{8.9}$$

2) Let us make the correspondence precise between φ and the optimal μ in (8.7) (if any). If μ is a negative measure, then ϕ_μ is positive on $[0, T]$ and $\Lambda(\mu) < \infty$. From [18] p.550, it is known that $F_\mu := \frac{\phi'_\mu}{\phi_\mu}$ is solution of the Ricatti equation (in the Schwartz distribution sense)

$$\dot{F} + F^2 = -2\mu \text{ on } (0, T), \quad F(T) = 2\mu_T\,. \tag{8.10}$$

which entails that

$$\Lambda^{\rho}_{x,T}(\mu) = \frac{\rho}{2}\int_0^T F_\mu(s) ds + \frac{a}{2} F_\mu(0)\,. \tag{8.11}$$

Now, if φ is absolutely continuous, an integration par parts using (8.10) gives

$$\int_0^T \varphi_t d\mu(t) - \Lambda^{\rho}_{x,T}(\mu) = \frac{1}{2}\int_0^T \left[(\dot{\varphi}_t - \rho) F_\mu(t) - \varphi_t F_\mu(t)^2) \right] dt\,. \tag{8.12}$$

Since φ_t is nonnegative the integrand is bounded above by $\frac{(\dot{\varphi}_t - \rho)^2}{4\varphi_t}$ and then the right hand side is bounded above by $J^{\rho}_{x,T}(\varphi)$. It takes exactly this value if

$$F_\mu = \frac{\dot{\varphi} - \rho}{2\varphi} \tag{8.13}$$

(which is close to (8.9)). That means

$$\frac{\phi'_\mu}{\phi_\mu} = \frac{\dot\varphi - \rho}{2\varphi}\,. \tag{8.14}$$

Giving φ we can find μ by (8.10) where F is taken as in (8.13).

Conversely, it is interesting to look for φ when μ is given, solving the dual problem. Indeed, by the duality lemma (Lemma 4.5.8 in Dembo-Zeitouni ([6]))

$$\sup_{\varphi:\varphi(0)=x} \left(\int_0^T \varphi_s \, d\mu(s) - J^\rho_{x,T}(\varphi) \right) = \Lambda^\rho_{x,T}(\mu)\,, \tag{8.15}$$

If we denote

$$g(y,z) = \frac{(z-\rho)^2}{8y}$$

then the optimal path φ solves the Euler-Lagrange equation (in the Schwartz distribution sense)

$$\mu + \frac{d}{dt}\Big(\frac{\partial g}{\partial z}(\varphi,\dot\varphi)\Big) = \Big(\frac{\partial g}{\partial y}\Big)(\varphi,\dot\varphi) \quad \text{on } (0,T) \tag{8.16}$$

with the initial condition $\varphi(0) = x$ and the final condition (free end point)

$$(\frac{\partial g}{\partial z}(\varphi,\dot\varphi)\Big)_{t=T} = \mu_T\,. \tag{8.17}$$

It is then easy to see that the auxiliary function $\frac{\dot\varphi-\rho}{2\varphi}$ is solution of the Riccati equation (8.10), so by unicity, equation (8.14) is satisfied. Solving it (this is a linear equation in φ) we get

$$\varphi(t) = \rho\phi_\mu(t)^2 \int_0^t \frac{ds}{\phi_\mu(s)^2} + x\phi_\mu(t)^2 \tag{8.18}$$

We may obtain another expression of the path φ, introducing as in ([18]) Ex. 1.34, the function

$$\psi_\mu(t) = \phi_\mu(t) \int_0^t \frac{ds}{\phi^2_\mu(s)}\,. \tag{8.19}$$

which solves also the Sturm–Liouville equation with the boundary conditions

$$\psi_\mu(0) = 0, \quad \psi'_\mu(0) = 1. \tag{8.20}$$

(and it satisfies the Wronskian relation $\phi_\mu\psi'_\mu - \phi'_\mu\psi_\mu \equiv 1$). The optimal path φ becomes

$$\varphi(t) = \rho\phi_\mu(t)\psi_\mu(t) + x\phi_\mu(t)^2. \tag{8.21}$$

3) Going back to Mc Kean's construction, we see that $S_\rho(\ell) = \sqrt{\varphi}$ gives

$$\ell = \frac{\dot\varphi - \rho}{2\sqrt{\varphi}} = 2h\sqrt{\varphi}$$

which we saw already in the exponential martingale (3.3).

8.3. Examples

In the above subsection we considered only negative measures, in order to make sure that the Sturm-Liouville machinery works. If we take the particular case $\mu(ds) := -\beta\, \epsilon_1(ds)$ for $s \in [0, 1]$ (Dirac mass in 1), then we get easily $\phi_\mu(t) = \frac{1+\beta(2-t)}{1+2\beta}$, $\Psi_\mu(t) = t$, $F_\mu(t) = \frac{-\beta}{1+\beta(2-t)}$ and

$$\varphi(t) = \rho\, \frac{t\,(1+\beta(2-t))}{1+2\beta} + x\, \frac{(1+\beta(2-t))^2}{(1+2\beta)^2}$$

(the equation of a parabola). Actually, everything holds true for $\beta > -1/2$.

Let us give a detailed analysis of a more interesting case (well studied, as far as computation of distributions is concerned). Assume that $d\mu(s) = -\frac{\beta^2}{2} ds$ for $s \in [0, 1]$. From [17] and [18], Corollary 1 p.445, it is known that

$$\phi_\mu(t) = \cosh \beta t - \tanh \beta\ \sinh \beta t = \frac{\cosh \beta(1-t)}{\cosh \beta} \tag{8.22}$$

which gives

$$\Lambda^\rho_{x,1}(\mu) = -\frac{\rho}{2} \log\cosh \beta - \frac{x}{2} \beta \tanh \beta\,. \tag{8.23}$$

The operator approach (8.4) gives here :

$$Ah(t) = -\frac{\beta^2}{2} \int_0^1 (s \wedge t) h(s) ds, \quad t \in [0, 1]. \tag{8.24}$$

Its eigenvalues and associated eigenvectors are ($n \geq 0$)

$$\lambda_n = \frac{-2\beta^2}{\pi^2(2n+1)^2}\,, \quad f_n(t) = \frac{2^{3/2}}{(2n+1)\pi} \sin \frac{(2n+1)\pi t}{2}. \tag{8.25}$$

This entails

$$\det(I - 2A) = \prod_{n=0}^{\infty} \Big[1 + \frac{4\beta^2}{\pi^2(2n+1)^2} \Big] = \cosh \beta, \tag{8.26}$$

and

$$\begin{aligned} k(\mu) &= -\frac{\beta^2}{2} + \frac{16\beta^4}{\pi^2} \sum_0^\infty \frac{1}{(2n+1)^2[(2n+1)^2\pi^2 + 4\beta^2]} \\ &= -\frac{\beta}{2} \tanh \beta\,. \end{aligned} \tag{8.27}$$

For a study of the second parts of equalities (8.26) and (8.27), see [12] and [4].

To identify the optimal path, let us remark that $F_\mu(t) = -\beta \tanh \beta(1-t)$. From (8.20) we deduce that $\psi_\mu(t) = \beta^{-1} \sinh \beta t$, and then

$$\varphi(t) = \rho\, \frac{\sinh \beta t \cosh \beta(1-t)}{\beta \sinh \beta} + x\, \frac{\cosh^2 \beta(1-t)}{\cosh^2 \beta}\,. \tag{8.28}$$

References

[1] Azencott, R.: Grandes déviations et applications. In: Ecole d'été de Probabilités de Saint-Flour VIII, Lecture Notes in Mathematics, **774**, Springer-Verlag, 1978, pp. 1–176

[2] Barlow, M.T., Yor, M.: Semi-martingale inequalities via the Garsia-Rodemich-Rumsey lemma and applications to local time. J. Funct. Anal. **49**, 198–299 (1982)

[3] BenArous, G., Ledoux, M.: Grandes Déviations de Freidlin-Wentzell en Norme Höldérienne. In: Séminaire de Probabilités XXVIII, Lecture Notes in Mathematics, **1583**, Springer-Verlag, 1994, pp. 293–299

[4] Biane, P., Yor, M.: Variations sur une formule de Paul Lévy. Ann. Inst. H. Poincaré Probab. Statist. **23**, 359–377 (1987)

[5] Dawson, D.A., Gartner, J.: Large deviations for Mc-Kean Vlasov limit of weakly interacting diffusions. Stochastics **20**, 247–308 (1987)

[6] Dembo, A., Zeitouni, O.: Large deviations techniques and applications. Second edition, Springer, 1998

[7] Deuschel, J.D., Stroock, D.W.: Large deviations. Academic Press, 1989

[8] Feng, S.: The behaviour near the boundary of some degenerate diffusions under random perturbation. Stochastic models (Ottawa ON 1998), Providence RI, Amer. Math. Soc., 2000, pp. 115–123

[9] Fernique, X.: Régularité des trajectoires des fonctions aléatoires gaussiennes. In: Ecole d'été de Probabilités de Saint-Flour IVLecture Notes in Mathematics, **480** Springer-Verlag, 1974, pp. 1–97

[10] Freidlin, M.I., Wentzell, A.D.: Random perturbations of dynamical systems. Springer-Verlag, 1984

[11] Lamperti, J.: Continuous state branching processes. Bull. Am. Math. Soc. **73**, 382–386 (1967)

[12] Lévy, P.: Wiener's random function and other Laplacian random functions. In: Second Berkeley Symposium, Berkeley and Los Angeles, University of California Press, 1951, pp. 171–187

[13] Liptser, R.S., Pukhalskii, A.A.: Limit Theorems on Large Deviations for Semimartingales. Stochastics Stochastics Rep. **38**, 201–249 (1992)

[14] McKean, H.P.: The Bessel motion and a singular integral equation. Mem. Coll. Sci. Univ. Kyoto. Ser. A Math. **33**, 317–322 (1960/1961)

[15] McKean H.P.: Stochastic integrals. Academic Press, New York, 1969

[16] Pitman, J.: Cyclically stationary Brownian local time processes. Probab. Theory Related. Fields **106**, 299–329 (1996)

[17] Pitman, J., Yor, M.: A decomposition of Bessel Bridges. Z. Wahrscheinlichkeitstheorie. Verw. Geb. **59**, 425–457 (1982)

[18] Revuz, D., Yor, M.: Continuous martingales and Brownian motion. 3rd ed., Springer, Berlin, 1999

[19] Shiga, T., Watanabe, S.: Bessel diffusions as a one–parameter family of diffusion. processes Z. Wahrscheinlichkeitstheorie. Verw. Geb. **27**, 37–46 (1973)

[20] Stroock, D.W., Varadhan, S.R.S.: Multidimensional diffusion processes. Springer-Verlag, Berlin, 1979

[21] Wentzell, A., Freidlin, M.: On small random perturbations of dynamical systems. Russian Math. Surveys **25**, 1–55 (1970)

[22] Yor, M.: On some exponential functionals of Brownian motion. Adv. Appl. Probab. **24**, 509–531 (1992)

[23] Yor, M.: Some aspects of Brownian motion Part I: Some special functionals. In: Lectures in Mathematics, ETH Zürich, Birkhäuser, Basel, 1992