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Exact L_2-small ball behavior of integrated Gaussian processes and spectral asymptotics of boundary value problems

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Abstract. We find the exact small deviation asymptotics for the L_2-norm of various m-times integrated Gaussian processes closely connected with the Wiener process and the Ornstein – Uhlenbeck process. Using a general approach from the spectral theory of linear differential operators we obtain the two-term spectral asymptotics of eigenvalues in corresponding boundary value problems. This enables us to improve the recent results from [15] on the small ball asymptotics for a class of m-times integrated Wiener processes. Moreover, the exact small ball asymptotics for the m-times integrated Brownian bridge, the m-times integrated Ornstein – Uhlenbeck process and similar processes appear as relatively simple examples illustrating the developed general theory.

1. Introduction

The problem of small ball behavior for norms of Gaussian processes has obtained much attention in recent years (see, for example, the reviews [1] and [2]). The easiest and most explored case is that of L_2-norm. Suppose we have a Gaussian process $X(t), 0 \le t \le 1$, with zero mean and covariance function $\sigma(s,t) = EX(t)X(s)$ for $s, t \in [0, 1]$. Let

$$||X||_2 = \left(\int_0^1 X^2(t)dt \right)^{1/2}$$

and consider

$$Q(X; \varepsilon) = P\{||X||_2 \le \varepsilon\}.$$

The problem is to define the behavior of $Q(X; \varepsilon)$ as $\varepsilon \to 0$. Theoretically the problem of small deviation asymptotics was solved in [3], but in an implicit way. Therefore, the efforts of many scientists starting from [4]–[8] were aimed at the

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simplification of the expression for $Q(X;\varepsilon)$ (see, e.g., the references in [2] and in [9]).

By the well-known Karhunen-Loève expansion, we have in distribution

$$||X||_2^2 = \int_0^1 X^2(t)dt = \sum_{n=1}^{\infty} \lambda_n \xi_n^2, \tag{1.1}$$

where ξ_n, $n \in \mathbb{N}$, are independent standard normal r.v.'s and $\lambda_n > 0$, $n \in \mathbb{N}$, $\sum\limits_n \lambda_n < \infty$, are the eigenvalues of the integral equation

$$\lambda f(t) = \int_0^1 \sigma(s,t) f(s) ds, \quad 0 \le t \le 1. \tag{1.2}$$

Observe that the substantial extension of (1.1) for general quadratic forms in Gaussian variables was obtained in [10].

Thus we are led to the equivalent problem of studying the asymptotic behavior as $\varepsilon \to 0$ of $P\left\{\sum_{n=1}^{\infty} \lambda_n \xi_n^2 \le \varepsilon^2\right\}$. This problem is considered as solved when the eigenvalues λ_n can be found explicitly. However, explicit formulas are known only for a limited number of examples (see [2], [9], [11] and [12]).

Let $W(t)$, $0 \le t \le 1$, be the Wiener process and denote by $W_m(t)$, $0 \le t \le 1$, $m \ge 1$ the result of m-times integration of this process, namely

$$W_m(t) = \int_0^t \int_0^{t_1} \dots \int_0^{t_{m-1}} W(s)\, ds dt_{m-1} \dots dt_1.$$

The small ball behavior of this process was studied in several recent publications. First step was made in [13], where it was proved for the one time integrated Wiener process W_1 that

$$\lim_{\varepsilon \to 0} \varepsilon^{2/3} \ln P\{||W_1||_2 \le \varepsilon\} = -3/8.$$

The m-times integrated Wiener process $W_m(t)$ for arbitrary integer m was considered later in [14]. It was shown there that, as $\varepsilon \to 0$, one has

$$\lim_{\varepsilon \to 0} \varepsilon^{2/(2m+1)} \ln P\{||W_m||_2 \le \varepsilon\} = -D_m,$$

where

$$D_m = \frac{1}{2}(2m+1)\left((2m+2)\sin\frac{\pi}{2m+2}\right)^{-\frac{2m+2}{2m+1}}. \tag{1.3}$$

Next contribution was made in [15]. It was proved that for a class of processes slightly generalizing the m-times integrated Wiener process one has as $\varepsilon \to 0$

$$P\{||W_m||_2 \le \varepsilon\} \sim C_m \varepsilon^{\frac{1}{2m+1}(1+k_0(2m+2))} \exp\left(-D_m \varepsilon^{-\frac{2}{2m+1}}\right), \tag{1.4}$$

where k_0 is some *indefinite* integer and C_m is some *unknown* constant. This result was obtained via refining the behavior of eigenvalues of the integral operator corresponding to the covariance of the m-times integrated process. It is worth noting that the authors of [15] prove again the eigenvalue asymptotics for boundary value problems (BVPs) which is well-known since the basic works of Birkhoff in 1908 – 1913, see [16] and the references therein.

In [17] the final result for the one-fold integrated Wiener process was obtained among a series of similar asymptotics. Namely, it was proved that

$$P\{||W_1||_2 \le \varepsilon\} \sim 1.414... \cdot (8/\sqrt{3\pi})\, \varepsilon^{1/3} \exp(-(3/8)\varepsilon^{-2/3}). \qquad (1.5)$$

Let underline that this formula refines the result in [15] for $m = 1$ as the constant is written out explicitly.

In the present paper we obtain the further refinement of the asymptotics (1.4) not only for the m-times integrated Wiener process, but for a larger class of integrated Gaussian processes including, for example, the m-times integrated *centered* Wiener process, the m-times integrated Brownian bridge and the m-times integrated Ornstein – Uhlenbeck process. Let underline that the power term is given explicitly and does not contain any indefinite number k_0. Part of our results was obtained independently in [18], [19].

The constant C_m in asymptotic expressions like (1.4) consists of two factors. The first factor arises from analytic arguments (see Section 6) while the second one which we call "distortion" constant can be evaluated numerically for any moderate value of m using the well-known comparison theorems in [11], see also [20]. When the eigenfunctions of (1.2) can be expressed in terms of elementary or special functions, e.g., for the process W_m and its conditional version $\mathbb{B}_m$ (see subsection 5.3), there exist explicit and rather unexpected *sharp* formulas for the distortion constants. In particular, it turns out that the constant 1.414... in (1.5) is in fact $\sqrt{2}$ while the analogous constant for the conditional process $\mathbb{B}_1$ equals to $\pi^2 2^{-5/2} 3^{-1/2}$. The derivation of these formulas is given in [12], independently some of equivalent results were obtained by a different method in [18], [19], where the constant C_m is calculated directly.

However, we doubt that an explicit formula for these constants exists in the general case.

The distinctive feature of this paper is that we develop a new approach demonstrating the explicit connection between the BVP corresponding to the initial Gaussian process X and the BVP for the m-times integrated process X_m, $m \ge 1$. Most crucial in our method is the application of a general point of view on the problem of small L_2–balls for a broad class of Gaussian processes based on spectral theory of BVPs. In this context the m-times integrated Wiener process and Brownian bridge, the m-times integrated Ornstein – Uhlenbeck process and other similar processes appear as interesting but relatively simple examples.

The problem of asymptotic behavior of eigenvalues and eigenfunctions of differential operators has been solved in rather general case. Recently the results on two-term spectral asymptotics in the multidimensional case were collected in [21]. However, we could not find in the literature corresponding results for the essentially more simple one-dimensional case. For example, in classical monographs

[16] and [22] the second term in the asymptotic expansion is given “up to constant” (see Section 3) that is not sufficient for our aims. Moreover, it seems that the proof of multidimensional results cannot be translated to the one-dimensional case. Therefore we adduce the proofs of necessary statements.

The paper is organized as follows. In Section 2 we establish the basic connection between the integrated kernels and the boundary value problems. Section 3 contains a preliminary result on spectral asymptotics. We underline that this result is well-known (see, e.g., [22], §4 and references therein, or [16], Ch.XIX, §4). For reader’s convenience we derive only the concrete formula for the special case of “separated” boundary conditions (we stress that any boundary condition in (3.3) contains the values of derivatives of u in only one endpoint). We note also that unknown constant n_0 in Theorem 3.1 corresponds to constant k_0 in (1.4). Then, in Sections 4-5 we eliminate this constant and derive the exact formulas of two-term spectral asymptotics for BVPs corresponding to some integral kernels of probability interest. In Section 6 we obtain the exact small ball asymptotics. As far as we get the precise asymptotics of spectrum, this result can be obtained after somewhat tiresome calculations from [9]. Further generalizations are outlined in Section 7.

Let us recall some notations. The function $G(s,t)$ is called Green function of BVP for the differential operator L if it satisfies the equation $LG = \delta(s-t)$ in the sense of distributions and satisfies the boundary conditions. The existence of Green function is equivalent to the invertibility of operator L with given boundary conditions, and $G(s,t)$ is the kernel of the integral operator L^{-1}.

The space $W_p^m(0,1)$ is the Banach space of functions u having continuous derivatives up to $(m-1)$-th order when $u^{(m-1)}$ is absolutely continuous on $[0,1]$ and $u^{(m)} \in L_p(0,1)$. If $p=2$ it is a Hilbert space.

We refer to [23], [24] and [25] for the properties of self-adjoint operators and their quadratic forms used later on.

2. The BVP corresponding to the integrated kernel

Consider the self-adjoint differential operator L of order $2\widehat{m}$ defined on the space $\mathcal{D}(L)$ of functions $u \in W_2^{2\widehat{m}}(0,1)$ satisfying $2\widehat{m}$ boundary conditions.

Theorem 2.1. *Let the kernel $G(x,y)$ be the Green function for the self-adjoint BVP*

$$Lu = \mu u \quad \text{on} \quad [0,1], \qquad u \in \mathcal{D}(L).$$

Then the m-times integrated kernel

$$\mathcal{G}_m(x,y) = \underbrace{\int_0^x \int_0^{x_1} \dots}_{m} \underbrace{\int_0^y \int_0^{y_1} \dots}_{m} \; G(s,t)\, dt \dots dy_1\, ds \dots dx_1 \tag{2.1}$$

is the Green function for the BVP

$$\mathcal{L}_m u \equiv (-1)^m \left(Lu^{(m)}\right)^{(m)} = \mu u \quad \text{on} \quad [0,1], \qquad u \in \mathcal{D}(\mathcal{L}_m), \tag{2.2}$$

where the space $\mathcal{D}(\mathcal{L}_m)$ *consists of functions* $u \in W_2^{2(m+\widehat{m})}(0,1)$ *satisfying following boundary conditions:*

$$\begin{aligned}
&u(0) = u'(0) = \dots = u^{(m-1)}(0) = 0;\\
&u^{(m)} \in \mathcal{D}(L);\\
&Lu^{(m)}(1) = \left(Lu^{(m)}\right)'(1) = \dots = \left(Lu^{(m)}\right)^{(m-1)}(1) = 0.
\end{aligned} \tag{2.3}$$

Proof. The kernel (2.1) can be rewritten as follows:

$$\mathcal{G}(x,y) \equiv \mathcal{G}_m(x,y) = \int_0^x \int_0^y \frac{(x-s)^{m-1}}{(m-1)!} \cdot \frac{(y-t)^{m-1}}{(m-1)!} \cdot G(s,t)\, dsdt. \tag{2.4}$$

Differentiating (2.4) with respect to x we derive for $k \le m-1$

$$\mathcal{G}_x^{(k)}(x,y) = \int_0^x \int_0^y \frac{(x-s)^{m-1-k}}{(m-1-k)!} \cdot \frac{(y-t)^{m-1}}{(m-1)!} \cdot G(s,t)\, dsdt,$$

and the first row in (2.3) follows. Next,

$$\mathcal{G}_x^{(m)}(x,y) = \int_0^y \frac{(y-t)^{m-1}}{(m-1)!} \cdot G(x,t)\, dt,$$

therefore $\mathcal{G}_x^{(m)} \in \mathcal{D}(L)$. Since $LG(x,y) = \delta(x-y)$ we have

$$L\mathcal{G}_x^{(m)}(x,y) = \frac{(y-x)_+^{m-1}}{(m-1)!},$$

and the last row in (2.3) follows. Finally,

$$\left(L\mathcal{G}_x^{(m)}\right)_x^{(m)}(x,y) = (-1)^m \delta(x-y),$$

and the statement is proved. □

Remark 1. The kernel (2.1) is the covariance function of the m-times integrated Gaussian process

$$X_m(x) = \underbrace{\int_0^x \int_0^{x_1} \dots}_{m} X(s)\, ds \dots dx_1, \tag{2.5}$$

where X is the Gaussian process with zero mean and covariance function $G(s,t)$.

If we replace, for example, the s-th integral in (2.5) by $\int\limits_{x_{s-1}}^{1}$, then the s-th integrals with respect to both variables in (2.1) are replaced similarly. The statement of Theorem 2.1 for this kernel remains true. We must only replace in (2.3) the corresponding pair of boundary conditions, namely

$$u^{(s-1)}(0) = \left(Lu^{(m)}\right)^{(m-s)}(1) = 0 \tag{2.6a}$$

by

$$u^{(s-1)}(1) = \left(Lu^{(m)}\right)^{(m-s)}(0) = 0. \tag{2.6b}$$

Later on we shall denote by $\xi_m^{[\beta_1,\dots,\beta_m]}(x)$ (here any β_j equals either zero or one) the m-times integrated Gaussian process with s-th integration from β_s. In particular, $\xi_m^{[0,0,\dots,0]}(x)$ is the usual m-times integrated process (2.5) and $\xi_m^{[1,0,1,\dots]}(x)$ is the so-called *Euler integrated* process (see [15]). By $\xi_m(x)$ we denote any m-times integrated process ξ.

3. Spectral asymptotics "up to constant" for a differential operator with "separated" boundary conditions

Let $\mathcal{A}$ be a differential operator of order 2ℓ

$$\mathcal{A}u \equiv (-1)^\ell u^{(2\ell)} + p_{2\ell-2}u^{(2\ell-2)} + \cdots + p_0 u \tag{3.1}$$

(here $p_k \in C[0,1]$, $k = 0,\dots,2\ell-2$). We consider the eigenvalue problem

$$\mathcal{A}u = \mu u \quad \text{on} \quad [0,1], \qquad u \in \mathcal{D}(\mathcal{A}), \tag{3.2}$$

assuming that the space $\mathcal{D}(\mathcal{A})$ consists of the functions $u \in W_2^{2\ell}(0,1)$ satisfying following boundary conditions:

$$\left.\begin{aligned} u^{(k_j)}(0) + \sum_{k<k_j} \alpha_{jk}^0 u^{(k)}(0) = 0,\\ u^{(k'_j)}(1) + \sum_{k<k'_j} \alpha_{jk}^1 u^{(k)}(1) = 0, \end{aligned}\right\} \quad j = 1,\dots,\ell \tag{3.3}$$

where α_{jk}^i are some constants (possibly complex) and

$$0 \le k_1 < \cdots < k_\ell \le 2\ell-1; \quad 0 \le k'_1 < \cdots < k'_\ell \le 2\ell-1. \tag{3.4}$$

We are interested in the asymptotics of eigenvalues of (3.2) for large $|\mu|$. Since we do not suppose the problem (3.2) to be self-adjoint, the eigenvalues μ are not necessary real.

Theorem 3.1. *Let μ_n, $n \in \mathbb{N}$, be the eigenvalues of (3.2), and $|\mu_1| \le |\mu_2| \le \dots$ (any eigenvalue is counted according to its multiplicity). Then there exists $n_0 \in \mathbb{Z}$ such that*

$$\mu_{n+n_0} = \Big(\pi n - \frac{\pi \varkappa}{2\ell} + O(n^{-1})\Big)^{2\ell}, \quad n \to \infty, \tag{3.5}$$

where $\varkappa = \sum\limits_{j=1}^{\ell} (k_j + k_j')$.

Proof. Using the asymptotics calculation algorithm given in [22], §4, Theorem 2, one obtains for boundary conditions (3.3)

$$\mu_{n+n_0} = \Big(\zeta_n + O(n^{-1})\Big)^{2\ell}, \quad n \to \infty. \tag{3.6}$$

Here $n_0 \in \mathbb{Z}$, while ζ_n are the zeros of the equation

$$\Phi(\zeta) \equiv \det(\mathfrak{A}_1)\det(\mathfrak{B}_1) - \det(\mathfrak{A}_2)\det(\mathfrak{B}_2) = 0,$$

where

$$\mathfrak{A}_1 = \begin{bmatrix} (iz)^{k_1} \dots (iz^{\ell-1})^{k_1} \; i^{k_1} \\ \dots\dots\dots\dots \\ (iz)^{k_\ell} \dots (iz^{\ell-1})^{k_\ell} \; i^{k_\ell} \end{bmatrix}; \quad \mathfrak{A}_2 = \begin{bmatrix} (iz)^{k_1} \dots (iz^{\ell-1})^{k_1} \; (-i)^{k_1} \\ \dots\dots\dots\dots \\ (iz)^{k_\ell} \dots (iz^{\ell-1})^{k_\ell} \; (-i)^{k_\ell} \end{bmatrix};$$

$$\mathfrak{B}_1 = \begin{bmatrix} (-i)^{k_1'} \exp(-i\zeta) \; (-iz)^{k_1'} \dots (-iz^{\ell-1})^{k_1'} \\ \dots\dots\dots\dots \\ (-i)^{k_\ell'} \exp(-i\zeta) \; (-iz)^{k_\ell'} \dots (-iz^{\ell-1})^{k_\ell'} \end{bmatrix};$$

$$\mathfrak{B}_2 = \begin{bmatrix} i^{k_1'} \exp(i\zeta) \; (-iz)^{k_1'} \dots (-iz^{\ell-1})^{k_1'} \\ \dots\dots\dots\dots \\ i^{k_\ell'} \exp(i\zeta) \; (-iz)^{k_\ell'} \dots (-iz^{\ell-1})^{k_\ell'} \end{bmatrix},$$

and $z = \exp(i\pi/\ell)$.

Direct calculation yields

$$\begin{aligned} &\det(\mathfrak{A}_1)\det(\mathfrak{B}_1) - \det(\mathfrak{A}_2)\det(\mathfrak{B}_2) \\ &= (-1)^{\ell} \cdot i^{\sum\limits_{j=1}^{\ell} k_j} \mathfrak{V}(z^{k_1}, \dots, z^{k_\ell}) \cdot (-i)^{\sum\limits_{j=1}^{\ell} k_j'} \mathfrak{V}(z^{k_1'}, \dots, z^{k_\ell'}) \cdot \\ &\quad \cdot (\exp(-i\zeta) - z^{\varkappa} \exp(i\zeta)), \end{aligned} \tag{3.7}$$

where $\mathfrak{V}(...)$ is the Vandermonde determinant.

The conditions (3.4) imply $z^{k_i} \ne z^{k_j}$ and $z^{k_i'} \ne z^{k_j'}$ for $i \ne j$. Hence both Vandermonde determinants are not equal to zero, and the formula (3.5) follows immediately from (3.6) and (3.7). □

Later on we apply the result of this theorem to the operators of the type considered in Theorem 2.1, with natural replacement $\ell = m + \widehat{m}$.

Remark 2. We suppose that $p_{2\ell-1} \equiv 0$ in (3.1). The general case can be reduced to this one by transformation of the function u (see, e.g., [22], §4). If the main term of the operator $\mathcal{A}$ in (3.1) has the form $(-1)^\ell p_{2\ell}(x)u^{(2\ell)}$ with $p_{2\ell} \in C[0,1]$, $p_{2\ell}(x) > 0$, then one can reduce this problem to the case $p_{2\ell} \equiv 1$ by transformation of the variable x (see [22], §4). In this case one has to divide the expression in braces in (3.5) by $\int\limits_0^1 p_{2\ell}^{-1/(2\ell)}(x)\,dx$.

4. Two-term spectral asymptotics for $\mathcal{L}_m$

Let the self-adjoint operator L have "separated" boundary conditions. Integrating its Green function $G(x,y)$ m times from various endpoints we obtain kernels corresponding (in general) to 2^m various BVPs similar to (2.2) – (2.3). One can see from Remark 1 that the parameter $\varkappa = \sum\limits_{j=1}^{m+\widehat{m}}(k_j + k'_j)$ is the same for all these 2^m BVPs and hence by Theorem 3.1 these problems have the same spectral asymptotics "up to constant". However, due to self-adjointness we can refine this result.

Theorem 4.1. *Let the Green functions of operators $\mathcal{L}_m^{(1)}$ and $\mathcal{L}_m^{(2)}$ be obtained from the Green function of the self-adjoint operator L with "separated" boundary conditions by m-repeated integration (with various endpoints). Then the eigenvalues of operators $\mathcal{L}_m^{(1)}$ and $\mathcal{L}_m^{(2)}$ have the same two-term asymptotics, i.e. there exists $n_0 \in \mathbb{Z}$ such that*

$$\left.\begin{aligned} \mu_{n+n_0}^{(1)} &= \left(\pi n - \frac{\pi\varkappa}{2(m+\widehat{m})} + O(n^{-1})\right)^{2(m+\widehat{m})}, \\ \mu_{n+n_0}^{(2)} &= \left(\pi n - \frac{\pi\varkappa}{2(m+\widehat{m})} + O(n^{-1})\right)^{2(m+\widehat{m})}, \end{aligned}\right\} \quad n \to \infty.$$

Proof. Consider, for example, the operator $\mathcal{L}_m^{(1)}$ with boundary conditions (2.3) and the operator $\mathcal{L}_m^{(2)}$ with one replaced pair of boundary conditions (2.6). Other cases can be considered similarly.

The quadratic forms $Q_{\mathcal{L}_m^{(1)}}$ and $Q_{\mathcal{L}_m^{(2)}}$ of these operators have the same formal expression $\langle Lu^{(m)}, u^{(m)}\rangle$ (in the sense of distributions) but different domains:

$$\begin{aligned} \mathcal{D}(Q_{\mathcal{L}_m^{(1)}}) = \{u \in W_2^{m+\widehat{m}}(0,1) :\ & \\ u^{(i-1)}(0) = 0,\ \ 1 \le i \le m;\quad & u^{(m)} \in \mathcal{D}(Q_L)\}; \end{aligned}$$

$$\begin{aligned} \mathcal{D}(Q_{\mathcal{L}_m^{(2)}}) = \{u \in W_2^{m+\widehat{m}}(0,1) :\quad & u^{(s-1)}(1) = 0; \\ u^{(i-1)}(0) = 0,\ \ 1 \le i \le m, i \ne s;\quad & u^{(m)} \in \mathcal{D}(Q_L)\}. \end{aligned}$$

Now consider the auxiliary quadratic form $Q_{\mathcal{L}_m^{(3)}}$ with the same formal expression and the domain

$$\begin{aligned} &\mathcal{D}(Q_{\mathcal{L}_m^{(3)}}) = \mathcal{D}(Q_{\mathcal{L}_m^{(1)}}) \cap \mathcal{D}(Q_{\mathcal{L}_m^{(2)}}) = \{u \in W_2^{m+\widehat{m}}(0,1) :\quad u^{(s-1)}(1) = 0, \\ &u^{(i-1)}(0) = 0,\ \ 1 \le i \le m-1;\quad u^{(m)} \in \mathcal{D}(Q_L)\}. \end{aligned}$$

According to [25], Sec.10.2, the counting function of eigenvalues of self-adjoint operator $\mathcal{L}$ can be expressed in terms of its quadratic form as follows:

$$\begin{aligned}\mathcal{N}_{\mathcal{L}}(\mu) &\equiv \#(j:\ \mu_j(\mathcal{L}) \le \mu)\\ &= \sup\dim\{\mathcal{H}\subset\mathcal{D}(Q_{\mathcal{L}}):\ Q_{\mathcal{L}}(u,u)\le\mu\|u\|^2 \text{ on } \mathcal{H}\}.\end{aligned}\tag{4.1}$$

As $\dim\mathcal{D}(Q_{\mathcal{L}_m^{(1)}})\backslash\mathcal{D}(Q_{\mathcal{L}_m^{(3)}}) = \dim\mathcal{D}(Q_{\mathcal{L}_m^{(2)}})\backslash\mathcal{D}(Q_{\mathcal{L}_m^{(3)}}) = 1$, one can easily deduce from (4.1) that

$$\mu_n^{(1)} \le \mu_n^{(3)} \le \mu_{n+1}^{(1)} \quad\text{and}\quad \mu_n^{(2)} \le \mu_n^{(3)} \le \mu_{n+1}^{(2)}.\tag{4.2}$$

By Theorem 3.1 there exist such $n_0^{(1)}, n_0^{(2)}, n_0^{(3)} \in \mathbb{Z}$ that as $n\to\infty$

$$\mu^{(j)}_{n+n_0^{(j)}} = \left(\pi n - \frac{\pi\varkappa^{(j)}}{2(m+\widehat{m})} + O(n^{-1})\right)^{2(m+\widehat{m})}, \qquad j=1,2,3\tag{4.3}$$

(here $\varkappa^{(j)}$ stands for the parameter $\varkappa$ of operator $\mathcal{L}_m^{(j)}$).

But the operator $\mathcal{L}_m^{(3)}$ differs from $\mathcal{L}_m^{(1)}$ and $\mathcal{L}_m^{(2)}$ only by one of boundary conditions (2.6): $u^{(s-1)}(0) = u^{(s-1)}(1) = 0$. Hence

$$\varkappa^{(1)} = \varkappa^{(2)} = \varkappa^{(3)} + 2(m+\widehat{m}-s)+1.\tag{4.4}$$

Substituting (4.4) and (4.3) into (4.2) we obtain

$$0 \le n_0^{(3)} - n_0^{(1)} + \frac{2s-1}{2(m+\widehat{m})} + O(n^{-1}) \le 1,$$

$$0 \le n_0^{(3)} - n_0^{(2)} + \frac{2s-1}{2(m+\widehat{m})} + O(n^{-1}) \le 1.$$

Since $1\le s\le m$, we have $n_0^{(1)} = n_0^{(3)} = n_0^{(2)}$, and the statement follows. □

5. Examples

5.1. Integrated Wiener process

The covariance function of Wiener process $G_W(s,t) = s\wedge t$ is the Green function of the Sturm – Liouville problem

$$L_W u \equiv -u'' = \mu u \quad\text{on}\quad [0,1], \qquad u(0) = u'(1) = 0.$$

The BVPs corresponding to m-times integrated Wiener process were derived in [15]. In the same paper the spectral asymptotics "up to constant" was obtained for these problems and it was conjectured that the form of two-term asymptotics is independent of the integration endpoints. Now, we see that this conjecture is a corollary of Theorem 4.1.

Proposition 5.1. *The operator $\mathcal{L}_{W_m}$ corresponding to m-times integrated Wiener process has two-term spectral asymptotics*

$$\mu_n^{(W_m)} = \Big(\pi n - \frac{\pi}{2} + O(n^{-1})\Big)^{2(m+1)}, \quad n \to \infty. \tag{5.1}$$

Proof. By Theorem 4.1, all 2^m operators corresponding to m-times integrated Wiener processes (with various endpoints) have the same two-term spectral asymptotics. But one of these operators, namely, corresponding to Euler integrated Wiener process, coincides with $(L_W)^{m+1}$ (for m even) or with $\mathcal{R}(L_W)^{m+1}\mathcal{R}$ (for m odd), where $\mathcal{R}$ is the reflection of $[0, 1]$ with respect to the point $x = 1/2$. Hence the eigenvalues of this operator are exactly $(\pi n - \pi/2)^{2(m+1)}$ (in [26] these eigenvalues were calculated in a different way), and the statement follows. □

5.2. Integrated Brownian bridge

The covariance function of Brownian bridge $G_B(s, t) = s \wedge t - st$ is the Green function of the Sturm – Liouville problem

$$L_B u \equiv -u'' = \mu u \quad \text{on} \quad [0, 1], \qquad u(0) = u(1) = 0. \tag{5.2}$$

The Brownian bridge m-times integrated from zero was investigated in [27] and [28].

Proposition 5.2. *The operator $\mathcal{L}_{B_m}$ corresponding to m-times integrated Brownian bridge has two-term spectral asymptotics*

$$\mu_n^{(B_m)} = \Big(\pi n - \frac{\pi m}{2(m+1)} + O(n^{-1})\Big)^{2(m+1)}, \quad n \to \infty. \tag{5.3}$$

Proof. By Theorem 4.1, all 2^m operators corresponding to m-times integrated Brownian bridges (with various endpoints) have the same two-term spectral asymptotics. Consider, for example, the process $B_m^{[0,0,\dots,0]}(x)$. By Theorem 2.1, the corresponding operator $\mathcal{L}_{B_m}$ can be written as follows:

$$\begin{aligned}
&\mathcal{L}_{B_m} u \equiv (-1)^{m+1} u^{(2m+2)} \quad \text{on} \quad [0, 1];\\
&\mathcal{D}(\mathcal{L}_{B_m}) = \{u \in W_2^{2(m+1)}(0, 1) : \quad u^{(m)}(0) = u^{(m)}(1) = 0;\\
&u^{(i)}(0) = u^{(2m+1-i)}(0) = 0,\ 0 \le i \le m-1\}.
\end{aligned}$$

Therefore, the parameter $\varkappa$ equals $(m + 1)(2m + 1) - 1$, and the formula (3.5) gives for an integer $\bar{n}_0$

$$\mu_{n+\bar{n}_0}^{(B_m)} = \Big(\pi n - \frac{\pi m}{2(m+1)} + O(n^{-1})\Big)^{2(m+1)}, \quad n \to \infty.$$

Next, the operator $\mathcal{L}_{B_m}$ differs from $\mathcal{L}_{W_m}$ only in one of boundary conditions. Comparing the quadratic form of $\mathcal{L}_{B_m}$ with that of $\mathcal{L}_{W_m}$ we get as in the proof of Theorem 4.1

$$\mu_n^{(W_m)} \le \mu_n^{(B_m)} \le \mu_{n+1}^{(W_m)},$$

and in view of Proposition 5.1 the statement follows. □

We note that due to symmetry of the problem (5.2) with respect to the point $x = 1/2$, any operator $\mathcal{L}_{B_m}$ coincides with $\mathcal{R}\widetilde{\mathcal{L}}_{B_m}\mathcal{R}$ where the operator $\widetilde{\mathcal{L}}_{B_m}$ corresponds to Brownian bridge integrated "symmetrically". So, the operators $\mathcal{L}_{B_m}$ and $\widetilde{\mathcal{L}}_{B_m}$ have the same eigenvalues.

5.3. "Bridged" integrated Wiener process

In an interesting paper [28] it was established that the conditional integrated Wiener process

$$\mathbb{B}_m(t) = (W_m(t)\Big| W_j(1) = 0, \quad 0 \le j \le m)$$

has the covariance function

$$G_{\mathbb{B}_m}(s,t) = \frac{(st)^{2n+1}}{(n!)^2} \int\limits_0^{1/(s\vee t)-1} \Big(\frac{1}{s} - 1 - z\Big)^n \Big(\frac{1}{t} - 1 - z\Big)^n dz$$

that is the Green function of the BVP

$$\begin{aligned}\mathcal{L}_{\mathbb{B}_m} u &\equiv (-1)^{m+1} u^{(2m+2)} = \mu u \quad \text{on} \quad [0,1];\\ \mathcal{D}(\mathcal{L}_{\mathbb{B}_m}) &= \{u \in W_2^{2(m+1)}(0,1) :\ u^{(i)}(0) = u^{(i)}(1) = 0,\ 0 \le i \le m\}.\end{aligned}$$

Proposition 5.3. *The operator $\mathcal{L}_{\mathbb{B}_m}$ corresponding to "bridged" m-times integrated Wiener process has two-term spectral asymptotics*

$$\mu_n^{(\mathbb{B}_m)} = \Big(\pi n + \pi\frac{m}{2} + O(n^{-1})\Big)^{2(m+1)}, \quad n \to \infty. \tag{5.4}$$

Proof. As in Proposition 5.2, we can "lower" step by step all the boundary conditions at the endpoint one that gives (5.4). □

5.4. Integrated centered Wiener process

The covariance function of (once) integrated centered Wiener process

$$\overline{W}_1(t) = \int_0^t (W(s) - \int_0^1 W(u)du)ds$$

(see, e.g., [29]) reads

$$G_{\overline{W}_1}(s,t) = \frac{1}{2}(s\wedge t)st - \frac{1}{6}(s\wedge t)^3 + \frac{st}{6}\Big(s^2 + t^2 - 3(s+t) + 2\Big)$$

and is the Green function of the BVP

$$L_{\overline{W}_1} u \equiv u^{IV} = \mu u \ \text{ on } \ [0,1], \qquad u(0) = u(1) = u''(0) = u''(1) = 0. \tag{5.5}$$

Proposition 5.4. *The operator $\mathcal{L}_{\overline{W}_m}$ corresponding to m-times integrated centered Wiener process has two-term spectral asymptotics*

$$\mu_n^{(\overline{W}_m)} = \left(\pi n - \frac{\pi(m-1)}{2(m+1)} + O(n^{-1})\right)^{2(m+1)}, \quad n \to \infty. \tag{5.6}$$

Proof. Due to symmetry of the problem (5.5) with respect to the point $x = 1/2$ the operator corresponding to $\overline{W}_m^{[0,\beta_2,\dots,\beta_m]}$ coincides with the operator corresponding to $\overline{W}_m^{[1,\beta_2,\dots,\beta_m]}$. By Theorem 4.1, all 2^{m-1} pairs of these operators have the same two-term spectral asymptotics. Consider, for example, the process $\overline{W}_m^{[0,0,\dots,0]}(x)$. By Theorem 2.1, the corresponding operator $\mathcal{L}_{\overline{W}_m}$ can be written as follows:

$$\mathcal{L}_{\overline{W}_m} u \equiv (-1)^{m+1} u^{(2m+2)} \quad \text{on} \quad [0,1];$$
$$\mathcal{D}(\mathcal{L}_{\overline{W}_m}) = \{u \in W_2^{2(m+1)}(0,1): \quad u^{(m-1)}(0) = u^{(m-1)}(1) = 0;$$
$$u^{(m+1)}(0) = u^{(m+1)}(1) = 0; \ \ u^{(i)}(0) = u^{(2m+1-i)}(1) = 0, \ 0 \le i \le m-2\}.$$

Therefore, the parameter $\varkappa$ equals $(m+1)(2m+1) - 2$, and formula (3.5) gives for an integer $\bar{n}_0$

$$\mu_{n+\bar{n}_0}^{(\overline{W}_m)} = \left(\pi n - \frac{\pi(m-1)}{2(m+1)} + O(n^{-1})\right)^{2(m+1)}, \quad n \to \infty.$$

Now, consider the auxiliary operator $\mathfrak{L}$:

$$\mathfrak{L} u \equiv (-1)^{m+1} u^{(2m+2)} \quad \text{on} \quad [0,1],$$
$$\mathcal{D}(\mathfrak{L}) = \{u \in W_2^{2(m+1)}(0,1): \quad u^{(m-1)}(0) = u^{(m-1)}(1) = 0;$$
$$u^{(m)}(0) = u^{(m+1)}(1) = 0; \quad u^{(i)}(0) = u^{(2m+1-i)}(1) = 0, \ 0 \le i \le m-2\}.$$

The parameter $\varkappa$ for $\mathfrak{L}$ equals $(m+1)(2m+1) - 3$, and the formula (3.5) gives for an integer $\widehat{n}_0$

$$\mu_{n+\widehat{n}_0}^{(\mathfrak{L})} = \left(\pi n - \frac{\pi(m-2)}{2(m+1)} + O(n^{-1})\right)^{2(m+1)}, \quad n \to \infty.$$

But the operator $\mathfrak{L}$ differs from $\mathcal{L}_{W_m}$ and $\mathcal{L}_{\overline{W}_m}$ only in one of boundary conditions. Comparing the quadratic form of $\mathfrak{L}$ with those of $\mathcal{L}_{W_m}$ and $\mathcal{L}_{\overline{W}_m}$ we get as in the proof of Theorem 4.1

$$\mu_n^{(W_m)} \le \mu_n^{(\mathfrak{L})} \le \mu_{n+1}^{(W_m)}, \qquad \mu_n^{(\overline{W}_m)} \le \mu_n^{(\mathfrak{L})} \le \mu_{n+1}^{(\overline{W}_m)},$$

whence $\widehat{n}_0 = \bar{n}_0 = 0$, and the statement follows. □

As in subsection 5.2, any operator $\mathcal{L}_{\overline{W}_m}$ coincides also with $\mathcal{R}\widetilde{\mathcal{L}}_{\overline{W}_m}\mathcal{R}$, where the operator $\widetilde{\mathcal{L}}_{\overline{W}_m}$ corresponds to centered Wiener process integrated "symmetrically". So, the operators $\mathcal{L}_{\overline{W}_m}$ and $\widetilde{\mathcal{L}}_{\overline{W}_m}$ have the same eigenvalues.

Remark 3. Since all considered operators have no lower order terms, one can easily deduce from the proof of Theorem 2, §4 in [22], see also [15], that the remainder term estimate in (5.1), (5.3), (5.4), (5.6) is in fact $O(\exp(-n\pi \sin \frac{\pi}{m+1}))$.

5.5. Integrated Ornstein – Uhlenbeck process

It is easy to check that the covariance function $G_U(s,t) = \exp(-|s-t|)$ of the Ornstein – Uhlenbeck process U is the Green function of the BVP

$$L_U u \equiv \frac{1}{2}(-u'' + u) = \mu u \quad \text{on} \quad [0,1], \qquad (u' - u)(0) = (u' + u)(1) = 0. \tag{5.7}$$

Proposition 5.5. *The operator $\mathcal{L}_{U_m}$ corresponding to m-times integrated Ornstein – Uhlenbeck process has the two-term spectral asymptotics*

$$\mu_n^{(U_m)} = \frac{1}{2}\left(\pi n - \frac{\pi(m+2)}{2(m+1)} + O(n^{-1})\right)^{2(m+1)}, \quad n \to \infty.$$

Proof. By Theorem 4.1, all 2^m operators corresponding to m-times integrated Ornstein – Uhlenbeck processes (with various endpoints) have the same two-term spectral asymptotics. Consider, for example, the Euler integrated process $U_m^{[1,0,1,\ldots]}(x)$. By Theorem 2.1, the corresponding operator $\mathcal{L}_{U_m}$ can be written as follows:

$$\begin{gathered}
2\mathcal{L}_{U_m} u \equiv (-1)^m(-u^{(2m+2)} + u^{(2m)}) \quad \text{on} \quad [0,1]; \\
\mathcal{D}(\mathcal{L}_{U_m}) = \{u \in W_2^{2(m+1)}(0,1) : \\
u^{(i-1)}(0) = u^{(i)}(1) = 0, \ \ i = m-1, m-3, \ldots; \\
(u^{(m+1)} - u^{(m)})(0) = (u^{(m+1)} + u^{(m)})(1) = 0; \\
(u^{(i+2)} - u^{(i)})(0) = (u^{(i+3)} - u^{(i+1)})(1) = 0, \ \ i = m, m+2, \ldots\}.
\end{gathered}$$

Therefore, the parameter $\varkappa$ equals $(m+1)(2m+1)+1$, and the formula (3.5) gives for an integer $\bar{n}_0$

$$2\mu_{n+\bar{n}_0}^{(U_m)} = \left(\pi n - \frac{\pi(m+2)}{2(m+1)} + O(n^{-1})\right)^{2(m+1)}, \quad n \to \infty. \tag{5.8}$$

Integrating by parts we obtain

$$2Q_{\mathcal{L}_{U_m}}(u,u) = \int_0^1 \left((u^{(m+1)})^2 + (u^{(m)})^2\right) dx \ + \ \left(u^{(m)}(0)\right)^2 + \left(u^{(m)}(1)\right)^2;$$

$$\begin{aligned}
\mathcal{D}(Q_{\mathcal{L}_{U_m}}) = \{u \in W_2^{m+1}(0,1) : \\
& u^{(i-1)}(0) = u^{(i)}(1) = 0, \ \ i = m-1, m-3, \ldots\}.
\end{aligned}$$

Now consider the auxiliary quadratic forms $Q_{\mathfrak{L}_j}$, $j = 2,3,4$, with the same formal expression

$$Q_{\mathfrak{L}_j}(u,u) = \int_0^1 \left((u^{(m+1)})^2 + (u^{(m)})^2\right) dx, \quad j = 2,3,4,$$

but different domains:

$$\mathcal{D}(Q_{\mathfrak{L}_2}) = \{u \in \mathcal{D}(Q_{\mathcal{L}_{U_m}}) : \quad u^{(m)}(0) = 0\};$$
$$\mathcal{D}(Q_{\mathfrak{L}_3}) = \mathcal{D}(Q_{\mathcal{L}_{U_m}});$$
$$\mathcal{D}(Q_{\mathfrak{L}_4}) = \{u \in \mathcal{D}(Q_{\mathcal{L}_{U_m}}) : \quad u^{(m)}(0) = u^{(m)}(1) = 0\}.$$

Integrating by parts we obtain that corresponding operators $\mathfrak{L}_j$, $j = 2, 3, 4$, have the same differential expression as $2\mathcal{L}_{U_m}$ but different domains:

$$\mathcal{D}(\mathfrak{L}_2) = \{u \in W_2^{2(m+1)}(0,1) : \quad u^{(i)}(0) = u^{(i+1)}(1) = 0, \ \ i - m \ \text{ even}\};$$

$$\begin{aligned}\mathcal{D}(\mathfrak{L}_3) = \{u \in W_2^{2(m+1)}(0,1) :& \\ & u^{(i-1)}(0) = u^{(i)}(1) = 0, \ \ i = m-1, m-3, \dots; \\ & u^{(i)}(1) = u^{(i+1)}(0) = 0, \ \ i = m+3, m+5, \dots; \\ & u^{(m+1)}(0) = u^{(m+1)}(1) = (u^{(m+2)} - u^{(m)})(0) = 0\};\end{aligned}$$

$$\begin{aligned}\mathcal{D}(\mathfrak{L}_4) = \{u \in W_2^{2(m+1)}(0,1) :& \\ & u^{(i-1)}(0) = u^{(i)}(1) = 0, \ \ i = m-1, m-3, \dots; \\ & u^{(i)}(0) = u^{(i+1)}(1) = 0, \ \ i = m+4, m+6, \dots; \\ & u^{(m)}(0) = u^{(m)}(1) = u^{(m+2)}(0) = (u^{(m+3)} - u^{(m+1)})(1) = 0\}.\end{aligned}$$

It is easy to see that the operator $(L_W)^{m+1} + (L_W)^m$ coincides with $\mathfrak{L}_2$ (for m even) or $\mathcal{R}\mathfrak{L}_2\mathcal{R}$ (for m odd). Hence

$$\mu_n^{(\mathfrak{L}_2)} = \left(\pi n - \frac{\pi}{2}\right)^{2(m+1)} + \left(\pi n - \frac{\pi}{2}\right)^{2m} = \left(\pi n - \frac{\pi}{2} + O(n^{-1})\right)^{2(m+1)}.$$

The operators $\mathfrak{L}_3$ and $\mathfrak{L}_4$ differ from $\mathfrak{L}_2$ only in one of boundary conditions. Comparing the quadratic form $Q_{\mathfrak{L}_2}$ with $Q_{\mathfrak{L}_3}$ and $Q_{\mathfrak{L}_4}$ we get as in the proof of Theorem 4.1

$$\mu_n^{(\mathfrak{L}_2)} \le \mu_n^{(\mathfrak{L}_4)} \le \mu_{n+1}^{(\mathfrak{L}_2)}, \qquad \mu_n^{(\mathfrak{L}_3)} \le \mu_n^{(\mathfrak{L}_2)} \le \mu_{n+1}^{(\mathfrak{L}_3)}. \tag{5.9}$$

Next, the parameter $\varkappa$ for $\mathfrak{L}_3$ and $\mathfrak{L}_4$ is equal to $(m+1)(2m+1)+1$ and to $(m+1)(2m+1)-1$ correspondingly, and by (3.5) and (5.9) we have as $n \to \infty$

$$\begin{aligned}\mu_n^{(\mathfrak{L}_3)} &= \left(\pi n - \frac{\pi(m+2)}{2(m+1)} + O(n^{-1})\right)^{2(m+1)} \\ \mu_n^{(\mathfrak{L}_4)} &= \left(\pi n - \frac{\pi m}{2(m+1)} + O(n^{-1})\right)^{2(m+1)},\end{aligned} \tag{5.10}$$

Now let compare the quadratic form of $2\mathcal{L}_{U_m}$ with those of $\mathfrak{L}_3$ and $\mathfrak{L}_4$. It is easy to see that $2Q_{\mathcal{L}_{U_m}}(u,u) \ge Q_{\mathfrak{L}_3}(u,u)$. On the other hand, $2Q_{\mathcal{L}_{U_m}}(u,u) = Q_{\mathfrak{L}_4}(u,u)$ for $u \in \mathcal{D}(Q_{\mathfrak{L}_4})$. By minimax principle (see [25], Sec.10.2) we have

$$\mu_n^{(\mathfrak{L}_3)} \le 2\mu_n^{(U_m)} \le \mu_n^{(\mathfrak{L}_4)},$$

and in view of (5.8) and (5.10) the statement follows. □

As in subsections 5.2 and 5.4, due to symmetry of the problem (5.7) with respect to the point $x = 1/2$, any operator $\mathcal{L}_{U_m}$ coincides with $\mathcal{R}\widetilde{\mathcal{L}}_{U_m}\mathcal{R}$ where the operator $\widetilde{\mathcal{L}}_{U_m}$ corresponds to Ornstein – Uhlenbeck process m-times integrated "symmetrically". Hence the operators $\mathcal{L}_{U_m}$ and $\widetilde{\mathcal{L}}_{U_m}$ have the same eigenvalues.

5.6. Stochastic integral with the time-varying upper limit

Let f be a function from $L_2(0, 1)$. Then the process

$$Z(t) = \int_0^t f(s)dW(s), \quad 0 \le t \le 1$$

is a Gaussian process with zero mean and covariance function

$$G_Z(s, t) = \int_0^{s\wedge t} f^2(u)du, \quad 0 \le s, t \le 1, \tag{5.11}$$

see, e.g. [30], p.143.

Denote by Z_m the m-times integrated process Z. The exact small ball asymptotics of this process in L_2-norm is an interesting unsolved problem. Our theory enables to obtain this asymptotics under some natural conditions on f. Suppose $f \neq 0$ and consider the BVP equivalent to the integral equation with the kernel (5.11). It reads

$$L_Z u \equiv -(u'/f^2)' = \mu u \quad \text{on} \quad [0, 1]; \qquad u(0) = (u'/f^2)(1) = 0.$$

Proposition 5.6. *Let non-vanishing function f belong to $W_\infty^{m+1}(0, 1)$. Then the operator $\mathcal{L}_{Z_m}$ corresponding to the m-times integrated process Z_m has the two-term spectral asymptotics*

$$\mu_n^{(Z_m)} = \left(\frac{\pi n - \pi/2}{\int_0^1 |f(x)|^{1/(m+1)}\, dx} + O(n^{-1})\right)^{2(m+1)}, \quad n \to \infty.$$

Proof. By Theorem 4.1, all 2^m operators corresponding to m-times integrated process Z (with various endpoints) have the same two-term spectral asymptotics. Consider the Euler integrated process $Z_m^{[1,0,1,\dots]}(x)$ as an example. By Theorem 2.1, the corresponding operator $\mathcal{L}_{Z_m}$ can be written as follows:

$$\begin{aligned}
&\mathcal{L}_{Z_m} u \equiv (-1)^{m+1}(u^{(m+1)}/f^2)^{(m+1)} \quad \text{on} \quad [0, 1];\\
&\mathcal{D}(\mathcal{L}_{Z_m}) = \{u \in W_2^{2(m+1)}(0, 1): \quad u^{(i)}(0) = u^{(i-1)}(1) = 0, \ \ i = m, m-2, \dots\\
&\qquad (u^{(m+1)}/f^2)^{(i)}(0) = (u^{(m+1)}/f^2)^{(i-1)}(1) = 0, \ \ i = 1, 3, \dots\}.
\end{aligned}$$

For simplicity we deal only with the case $m = 1$; the general case can be considered similarly. Without loss of generality we can suppose $f > 0$.

Remark 2 gives for an integer $\bar n_0$

$$\mu^{(Z_1)}_{n+\bar n_0} = \left(\frac{\pi n - \pi/2}{\int_0^1 f(x)^{1/2}\,dx} + O(n^{-1}) \right)^4, \quad n \to \infty.$$

Now consider the auxiliary operator $\mathfrak{L}_5$:

$$\mathfrak{L}_5 u \equiv -(u'/f)' \quad \text{on} \quad [0,1]; \qquad u(1) = (u'/f)(0) = 0.$$

It is well-known (see, e.g., [31], Vol.IV, Ch.IV, §188) that

$$\mu^{(\mathfrak{L}_5)}_{n} = \left(\frac{\pi n - \pi/2}{\int_0^1 f(x)^{1/2}\,dx} + O(n^{-1}) \right)^2.$$

Integrating by parts we have

$$\begin{aligned} Q_{\mathfrak{L}_6}(u,u) &\equiv Q_{\mathcal{L}_{Z_1}}(u,u) + \frac{u'^2(1)}{f(1)} \left[\frac{1}{f}\right]'(1) \\ &= Q_{\mathfrak{L}_5^2}(u,u) + \int\limits_0^1 \frac{u'^2(t)}{f(t)} \left[\frac{1}{f(t)}\right]'' dt. \end{aligned} \tag{5.12}$$

Denote $c_1 = \sup\limits_{t\in[0,1]} |(1/f(t))''|$ and consider a pair of auxiliary operators $\mathfrak{L}_7^{\pm} = \mathfrak{L}_5^2 \pm c_1 \mathfrak{L}_5$. Then (5.12) yields

$$Q_{\mathfrak{L}_7^-}(u,u) \le Q_{\mathfrak{L}_6}(u,u) \le Q_{\mathfrak{L}_7^+}(u,u).$$

Therefore, as $n \to \infty$,

$$\mu^{(\mathfrak{L}_6)}_{n} = \left(\mu^{(\mathfrak{L}_5)}_{n}\right)^2 + O(n^2) = \left(\frac{\pi n - \pi/2}{\int_0^1 f(x)^{1/2}\,dx} + O(n^{-1}) \right)^4.$$

Finally, consider the auxiliary quadratic form $Q_{\mathfrak{L}_8}$ with the same formal expression as $Q_{\mathcal{L}_{Z_1}}$ and the domain

$$\mathcal{D}(Q_{\mathfrak{L}_8}) = \{u \in W_2^2(0,1) :\ u(1) = u'(0) = u'(1) = 0\}.$$

Obviously, one has $Q_{\mathcal{L}_{Z_1}}(u,u) = Q_{\mathfrak{L}_6}(u,u) = Q_{\mathfrak{L}_8}(u,u)$ for $u \in \mathcal{D}(Q_{\mathfrak{L}_8})$. As $\dim \mathcal{D}(Q_{\mathcal{L}_{Z_1}}) \setminus \mathcal{D}(Q_{\mathfrak{L}_8}) = 1$, we have as in the proof of Theorem 4.1

$$\mu^{(Z_1)}_{n} \le \mu^{(\mathfrak{L}_8)}_{n} \le \mu^{(Z_1)}_{n+1} \quad \text{and} \quad \mu^{(\mathfrak{L}_6)}_{n} \le \mu^{(\mathfrak{L}_8)}_{n} \le \mu^{(\mathfrak{L}_6)}_{n+1},$$

therefore $\bar n_0 = 0$, and the statement follows. □

6. Exact small ball asymptotics

The results of previous sections give rather precise information on the asymptotics of eigenvalues in our boundary problems. Therefore we are able to get the exact small ball behavior of processes under consideration.

For any zero mean Gaussian process $X(t), 0 \le t \le 1$, with covariance $G_X(s,t)$ denote by $\lambda_n^{(X)}, n \ge 1$ the eigenvalues of the integral equation (1.2) with the kernel $G_X(s,t)$. The results obtained above give for these eigenvalues in all considered cases the asymptotics of the type

$$\lambda_n = \lambda_n^{(X)} = \Big(\vartheta(n + \delta + O(n^{-1}))\Big)^{-d}, \quad n \to \infty, \tag{6.1}$$

where $\vartheta, d > 0$ and $\delta > -1$ are some constants, possibly depending on m.

Denote by $\Lambda_n = \Lambda_n^{(X)}, n \ge 1$ the approximate eigenvalues of integral equation (1.1) given by (6.1) without the remainder $O(n^{-1})$. By the comparison principle of Li [11] the exact small deviation asymptotics of two infinite random forms $\sum_{j=1}^{\infty} \lambda_j \xi_j^2$ and $\sum_{j=1}^{\infty} \Lambda_j \xi_j^2$ differs only in a distortion constant

$$C_{\rm dist}(X) \equiv \prod_{n=1}^{\infty} \left(\frac{\Lambda_n}{\lambda_n}\right)^{1/2}. \tag{6.2}$$

As the relative error of our approximaton is $O(n^{-2})$ this infinite product converges. For any fixed moderate value of m we can calculate some first values of $\lambda_n^{(X)}$ and find numerically the value of distortion constant. Due to this we shall consider in this section only the form $\sum_{j=1}^{\infty} \Lambda_j \xi_j^2$ with relatively simple "approximate" eigenvalues.

Next step is using the results of [9] to get the desired exact small ball asymptotics. This technique was already used in [17] for various one time integrated processes. Temporarily let assume that in (6.1) $\vartheta = 1$, later we will incorporate the influence of ϑ in the final formula.

The following result is a concretization of Corollary 3.2 from [9] within the framework of our conditions. Note that with our choice of ϕ and f all regularity conditions assumed in [9] are fulfilled.

Lemma 6.1. *Denote, for* $t, u \ge 0$, $d > 1$ *and* $\delta > -1$

$$\phi(t) = (t+\delta)^{-d}, \quad f(t) = (1+2t)^{-1/2},$$

$$I_0(u) = \int_1^{\infty} \ln f(u\phi(t))dt, \quad I_1(u) = \int_1^{\infty} u\phi(t)(\ln f)'(u\phi(t))dt,$$

$$I_2(u) = \int_1^{\infty} (u\phi(t))^2(\ln f)''(u\phi(t))dt, \ C_\phi = \frac{1}{2}\sum_{j=1}^{\infty}\int_0^1 \ln \frac{\phi(j)\phi(j+1)}{\phi^2(t+j)}\, dt.$$

Then as $r \to 0$,

$$P\left\{\sum_{j=1}^{\infty}\phi(j)\xi_j^2 \le r\right\} \sim \sqrt{\frac{f(u\phi(1))}{2\pi I_2(u)}}\exp(I_0(u)+ur-C_\phi/2), \tag{6.3}$$

where $u = u(r)$ is any function satisfying

$$\lim_{r\to 0}\frac{I_1(u)+ur}{\sqrt{I_2(u)}} = 0. \tag{6.4}$$

We begin by the asymptotic analysis of $I_s(u)$, $s = 0, 1, 2,$ as $u \to \infty$. Using [32], formulas 3.241.2 and 3.241.5 we have

$$I_0(u) \sim \frac{(1+\delta)}{2}\ln\frac{2u}{(1+\delta)^d} - \frac{\pi(2u)^{1/d}}{2\sin\frac{\pi}{d}} + d(1+\delta)/2.$$

$$I_1(u) = -u\int_{1+\delta}^{\infty}\frac{dt}{2u+t^d} \sim -\frac{\pi(2u)^{1/d}}{2d\sin\frac{\pi}{d}},$$

$$I_2(u) = 2u^2\int_{1+\delta}^{\infty}\frac{dt}{(2u+t^d)^2} \sim \frac{(d-1)\pi(2u)^{1/d}}{2d^2\sin\frac{\pi}{d}}.$$

Now, if we choose u in such a way that

$$u = \frac{1}{2}\left(\frac{\pi r^{-1}}{d\sin\frac{\pi}{d}}\right)^{\frac{d}{d-1}},$$

then u satisfies condition (6.4). It remains to calculate the constant C_ϕ. Consider the integral (see [32], formula 3.427.4)

$$I(q) = \int_0^{\infty}\left(\frac{1}{e^t-1}+\frac{1}{2}-\frac{1}{t}\right)\frac{e^{-qt}}{t}\,dt = \ln\Gamma(q) - (q-\frac{1}{2})\ln q + q - \frac{1}{2}\ln(2\pi)$$

defined for any complex q with $Re\, q > 0$. We have after some simplifications

$$\begin{aligned} C_\phi &= d\sum_{j=1}^{\infty}\left[(j+1/2+\delta)\ln\frac{j+1+\delta}{j+\delta} - 1\right] \\ &= d\sum_{j=1}^{\infty}[I(j+\delta) - I(j+1+\delta)] = d\cdot I(1+\delta) \\ &= d\cdot[\ln\Gamma(1+\delta) - (1/2+\delta)\ln(1+\delta) + 1 + \delta - \frac{1}{2}\ln(2\pi)]. \end{aligned}$$

Now we can obtain the exact asymptotics for $P\left\{\sum_{j=1}^{\infty}\phi(j)\xi_j^2 \le \varepsilon^2\right\}$. The passage to the probability $P\left\{\sum_{j=1}^{\infty}\Lambda_j\xi_j^2 \le \varepsilon^2\right\}$ is simple rescaling because $\Lambda_j = \vartheta^{-d}\phi(j),\ j \ge 1$. Finally we arrive to the following result.

Theorem 6.2. *Consider the form* $\sum_{j=1}^{\infty} \Lambda_j \xi_j^2$ *with*

$$\Lambda_j = (\vartheta(j+\delta))^{-d},$$

where $\vartheta > 0, \delta > -1$ *and* $d > 1$ *are some constants. Then as* $\varepsilon \to 0$ *it holds*

$$P\left\{\sum_{j=1}^{\infty} \Lambda_j \xi_j^2 \leq \varepsilon^2\right\}$$
$$\sim \mathcal{C}(\vartheta, d, \delta) \cdot \varepsilon^{\gamma} \exp\left(-\frac{d-1}{2}\left(\frac{\pi/d}{\vartheta \sin(\pi/d)}\right)^{\frac{d}{d-1}} \varepsilon^{-\frac{2}{d-1}}\right), \tag{6.5}$$

where

$$\gamma = \frac{2-d-2\delta d}{2(d-1)},$$

while the constant $\mathcal{C}(\vartheta, d, \delta)$ *is given by the following expression:*

$$\mathcal{C}(\vartheta, d, \delta) = \frac{(2\pi)^{d/4} \vartheta^{d\gamma/2} (\sin(\pi/d))^{\frac{1+\gamma}{2}}}{(d-1)^{1/2} (\pi/d)^{1+\frac{\gamma}{2}} \Gamma^{d/2}(1+\delta)}. \tag{6.6}$$

Theorem 6.2 generalizes the Corollary 4.2 in [9], where the case $\vartheta = 1$ and $\delta = 0$ was considered, as well as the formula (3.4) in [11] covering the case $\delta > -1$, $d = 2$. This Theorem opens the possibility to write out the exact small ball asymptotics in Examples 5.1 – 5.6.

To get the final result (6.3) we should multiply (6.5) by the distortion constant (6.2) which can be calculated analytically or numerically. We underline that the “particularity” of a process consists in the constants ϑ, δ and d appearing in the formula (6.5) and the distortion constant C_{dist}.

Now we apply the “abstract” Theorem 6.2 to the examples 5.1 – 5.6. We recall that in the considered examples m is the number of successive integrations.

Proposition 6.3. *The exact small ball asymptotics for the processes considered in Section 5 are as follows:*

$$P\{||X||_2 \leq \varepsilon\} \sim C_{\text{dist}}(X) \cdot \mathcal{C}(\vartheta, d, \delta) \cdot \varepsilon^{\gamma} \exp(-D_X \varepsilon^{-\frac{2}{2m+1}}), \tag{6.7}$$

where $d = 2m+2$, *while the constants* $C_{\text{dist}}(X)$ *and* $\mathcal{C}(\vartheta, d, \delta)$ *are defined in (6.2) and (6.6) correspondingly. Other parameters in (6.7) are collected in Table 1 (here* $I_m(f) = \int_0^1 |f(x)|^{1/(m+1)}\, dx$, *and* D_m *is the constant defined in (1.3)).*

Let us examine briefly the power of ε. In the case of m-times integrated Wiener process this power is equal to $\frac{1}{2m+1}$ for any m. This refines the result of [15]. It is curious to note that this power is equal to 0 for any m-times integrated Brownian bridge while in other cases this power depends on m.

The exact small ball asymptotics for the integrated Ornstein – Uhlenbeck process is obtained for the first time. This result is new even for the rough asymptotics and the simplest case $m = 1$ when the constant in the exponent changes from 3/8

Table 1. Parameters for the exact small ball asymptotics

X	ϑ	δ	γ	D_X
W_m	π	$-\frac{1}{2}$	$\frac{1}{2m+1}$	D_m
B_m	π	$-\frac{m}{2m+2}$	0	D_m
$\mathbb{B}_m$	π	$\frac{m}{2}$	$-\frac{m(m+2)}{2m+1}$	D_m
$\overline{W}_m$	π	$-\frac{m-1}{2m+2}$	$-\frac{1}{2m+1}$	D_m
U_m	$\pi 2^{-\frac{1}{2m+2}}$	$-\frac{m+2}{2m+2}$	$\frac{2}{2m+1}$	$D_m 2^{\frac{1}{2m+1}}$
Z_m	$\pi/I_m(f)$	$-\frac{1}{2}$	$\frac{1}{2m+1}$	$D_m(I_m(f))^{\frac{2m+2}{2m+1}}$

(which was typical for different forms of integrated Wiener process and Brownian bridge) to $3/2^{8/3}$.

The exact small ball asymptotics of the process Z_m from subsection 5.6 is also obtained for the first time. Previously only the logarithmic small deviation asymptotics for the process Z was known which follows from more general results in [11], namely

$$\lim_{\varepsilon\to 0} \varepsilon^2 \ln P\{||Z||_2 \le \varepsilon\} = -\frac{1}{8}\left(\int_0^1 |f(s)|ds\right)^2 .$$

Note that our formulas in Proposition 6.3 hold true also in the non-integrated case $m = 0$. The matter is that the ordinary (non-integrated) centered Wiener process $\overline{W}$ is exceptional (corresponding BVP is not of the form described in subsection 5.4). Happily it was shown in [33] that $||\overline{W}||_2 = ||B||_2$ in distribution, another proof can be found in [17]. Therefore we can use the well-known small ball asymptotics for the Brownian bridge, namely as $\varepsilon \to 0$

$$P\{||\overline{W}||_2 \le \varepsilon\} = P\{||B||_2 \le \varepsilon\} \sim \frac{2\sqrt{2}}{\sqrt{\pi}} \exp\left(-\frac{1}{8\varepsilon^2}\right). \tag{6.8}$$

The non-integrated Ornstein – Uhlenbeck process is distinguished for another reason. In fact, according to Proposition 5.5 this process corresponds to $\delta = -1$, and the general formula (6.5) is not applicable. This case is degenerate from the point of view of our approach ($\Lambda_1 = \infty$). However we can surmount this obstacle using some appropriate (though delicate) passage to the limit as $\delta \to -1$.

An alternative direct variant of proof is as follows. Take as an approximate system of eigenvalues the set $\Lambda_1 = 1$, $\Lambda_n = 2(\pi(n-1))^{-2}$, $n \ge 2$. Then the comparison principle of Li [11] and Theorem 6.2 imply

$$P\left\{\sum_{j=1}^{\infty} \lambda_j^{(U)} \xi_j^2 \le \varepsilon^2\right\} \sim \left(1/\lambda_1^{(U)}\right)^{1/2} \prod_{n=2}^{\infty} \left(\frac{2(\pi(n-1))^{-2}}{\lambda_n^{(U)}}\right)^{1/2} \cdot$$
$$\cdot P\left\{\xi_0^2 + 2\sum_{j=1}^{\infty} (\pi j)^{-2} \xi_j^2 \le \varepsilon^2\right\},$$

where ξ_0 is a standard Gaussian variable independent from $\{\xi_n\}$, $n \in \mathbb{N}$. The probability in the right-hand side is a convolution of two distribution functions with known behavior near zero. In fact as $r \to 0$ one has

$$\frac{d}{dr} P\{\xi_0^2 < r\} = \frac{d}{dr}[2\Phi(\sqrt{r}) - 1] \sim \frac{1}{\sqrt{2\pi r}},$$

and using (6.8) we get

$$P\left\{\xi_0^2 + 2\sum_{j=1}^{\infty} \lambda_j \xi_j^2 \le \varepsilon^2\right\} \sim \frac{2}{\pi} \int_0^{\varepsilon^2} \frac{1}{\sqrt{x}} \exp\left(-\frac{1}{4(\varepsilon^2 - x)}\right) dx.$$

The integrand is close to zero and its point of maximum is of order $O(\varepsilon^4)$, hence we can linearize the function in the exponent near zero, replacing it by $-1/(4\varepsilon^2) - x/(4\varepsilon^4)$. But

$$\int_0^{\varepsilon^2} \frac{1}{\sqrt{x}} \exp\left(-\frac{x}{4\varepsilon^4}\right) dx \sim 2\Gamma(1/2)\,\varepsilon^2, \quad \varepsilon \to 0,$$

so that we arrive to

Proposition 6.4. *The exact small ball asymptotics for the ordinary Ornstein – Uhlenbeck process is*

$$P\{||U||_2 \le \varepsilon\} \sim \widetilde{C}_{\mathrm{dist}}(U) \cdot 4\pi^{-1/2} \varepsilon^2 \exp\left(-\frac{1}{4}\varepsilon^{-2}\right),$$

where

$$\widetilde{C}_{\mathrm{dist}}(U) = \left(\frac{1}{\lambda_1^{(U)}}\right)^{1/2} \prod_{n=2}^{\infty} \frac{1}{\pi(n-1)} \left(\frac{2}{\lambda_n^{(U)}}\right)^{1/2}.$$

This proposition refines Theorem 5 from [11], see also [34], where it was shown that

$$\lim_{\varepsilon \to 0} \varepsilon^2 \ln P\{||U||_2 \le \varepsilon\} = -\frac{1}{4}.$$

7. Further generalizations

We present now the general formula of two-term spectral asymptotics for self-adjoint operator with "separated" boundary conditions.

Theorem 7.1. *Let $\mathcal{A}$ be a self-adjoint operator of order 2ℓ*

$$\mathcal{A}u \equiv (-1)^{\ell} \left(p_{2\ell} u^{(\ell)}\right)^{(\ell)} + \left(p_{2\ell-2} u^{(\ell-1)}\right)^{(\ell-1)} + \cdots + p_0 u \tag{7.1}$$

with "separated" boundary conditions. Let $p_{2k} \in L_1(0,1),\ k = 0, \dots, \ell-2;$ $p_{2\ell-2} \in L_\infty(0,1);\ p_{2\ell} \in W_\infty^\ell(0,1);\ p_{2\ell}(x) > 0$. *Then the eigenvalues of operator* $\mathcal{A}$ *have two-term asymptotics*

$$\mu_n = \left(\frac{\pi \cdot \left[n + \ell - 1 - \frac{\varkappa}{2\ell} \right]}{\int\limits_0^1 p_{2\ell}^{-1/(2\ell)}(x)\,dx} + O(n^{-1}) \right)^{2\ell}, \quad n \to \infty, \tag{7.2}$$

where parameter $\varkappa$ *is defined in Section 3*

Proof. The way to derive (7.2) is similar to ideas considered in Section 5. So we give only the sketch of proof.

The quadratic form $Q_{\mathcal{A}}$ can be written as follows:

$$Q_{\mathcal{A}}(u,u) = \int\limits_0^1 \sum_{k=0}^{\ell} p_{2k} \left(u^{(k)}\right)^2 dx \ + \ Q_0(u,u) \ + \ Q_1(u,u), \tag{7.3}$$

where $Q_0(u,u)$ and $Q_1(u,u)$ contain boundary terms at the endpoints zero and one, correspondingly.

First of all, using the minimax principle we eliminate the boundary terms from $Q_{\mathcal{A}}$ in the same way as in Proposition 5.5. Then, "raising" and "lowering" boundary conditions as in Propositions 5.2 - 5.4 we reduce the problem to the case of simplest boundary conditions

$$u(0) = u'(1) = \cdots = u^{(2\ell-2)}(0) = u^{(2\ell-1)}(1) = 0.$$

In this case the result can be obtained as in Proposition 5.6. □

Proposition 7.2. *Let the covariance* $G_X(s,t)$ *of zero mean Gaussian process* $X(t)$, $0 \le t \le 1$, *be the Green function for the self-adjoint operator (7.1) satisfying the conditions of Theorem 7.1. Let* $\varkappa < 2\ell^2$. *Then*

$$P\{||X||_2 \le \varepsilon\} \sim C_{\text{dist}}(X) \cdot \mathcal{C}\left(\frac{\pi}{J_{2\ell}},\ 2\ell,\ \ell - 1 - \frac{\varkappa}{2\ell}\right) \cdot$$
$$\cdot \varepsilon^{-\ell + \frac{\varkappa+1}{2\ell-1}} \exp\left(-\frac{2\ell-1}{2} \left(\frac{J_{2\ell}}{2\ell \sin\frac{\pi}{2\ell}} \right)^{\frac{2\ell}{2\ell-1}} \varepsilon^{-\frac{2}{2\ell-1}} \right), \tag{7.4}$$

where $J_{2\ell} = \int\limits_0^1 p_{2\ell}^{-1/(2\ell)}(x)\,dx$, *the constant* $\mathcal{C}(\vartheta, d, \delta)$ *is defined in (6.11) while* $C_{\text{dist}}(X)$ *is the distortion constant (6.2).*

Proof. This statement is direct corollary of Theorem 7.1 and Theorem 6.2. □

Remark 4. In the case $\varkappa \ge 2\ell^2$ one must modify formula (7.4) in the way shown for the Ornstein – Uhlenbeck process at the end of Section 6.

Now we can obtain the rough (logarithmic) asymptotics of L_2-norm of Gaussian process in highly general case.

Theorem 7.3. *Let the covariance $G_X(s,t)$ of zero mean Gaussian process $X(t)$, $0 \le t \le 1$, be the Green function for the self-adjoint operator (7.1) (here we do not suppose that the boundary conditions are separated). Let $p_{2k} \in L_1(0,1)$, $k = 0, \dots, \ell$ and $p_{2\ell}(x) \ge c > 0$. Then*

$$\lim_{\varepsilon\to 0} \varepsilon^{2/(2\ell-1)} \ln P\{||X||_2 \le \varepsilon\} = -\frac{2\ell-1}{2}\left(\frac{J_{2\ell}}{2\ell \sin\frac{\pi}{2\ell}}\right)^{\frac{2\ell}{2\ell-1}}.$$

Proof. Consider two auxiliary quadratic forms $Q_{\mathfrak{L}_9}$ and $Q_{\mathfrak{L}_9^{(D)}}$ with the same formal expression

$$Q_{\mathfrak{L}_9}(u,u) = \int\limits_0^1 p_{2\ell}\left(u^{(\ell)}\right)^2\,dx$$

and different domains: $\mathcal{D}(Q_{\mathfrak{L}_9}) = \mathcal{D}(Q_{\mathcal{A}})$ while $\mathcal{D}(Q_{\mathfrak{L}_9^{(D)}})$ is the closure of the set of smooth functions with compact support in $]0,1[$ with respect to the norm generated by $Q_{\mathfrak{L}_9}(u,u)$.

It follows from [35] that as $n \to \infty$,

$$\mu_n^{(\mathfrak{L}_9^{(D)})} = \left(\frac{\pi n}{J_{2\ell}} + o(n)\right)^{2\ell}.$$

Next, the difference of the operators $\mathfrak{L}_9$ and $\mathfrak{L}_9^{(D)}$ is a finite-dimensional operator, and therefore

$$\mu_n^{(\mathfrak{L}_9)} = \left(\frac{\pi n}{J_{2\ell}} + o(n)\right)^{2\ell}.$$

Now we estimate the lower order terms in (7.3). For example, integrating by parts we obtain

$$\int\limits_0^1 p_{2\ell-2}\left(u^{(\ell-1)}\right)^2\,dx = \left(u^{(\ell-1)}(1)\right)^2\int\limits_0^1 p_{2\ell-2}\,dx - \int\limits_0^1 2u^{(\ell-1)}(x)u^{(\ell)}(x)\int\limits_0^x p_{2\ell-2}(t)\,dtdx.$$

Due to the well-known inequality

$$\max_{x\in[0,1]} |f(x)|^2 \le \int\limits_0^1 2|f(x)f'(x)|\,dx + \int\limits_0^1 (f(x))^2\,dx, \qquad f \in W_2^1(0,1)$$

we have

$$\left|\int_0^1 p_{2\ell-2}\left(u^{(\ell-1)}\right)^2\,dx\right| \le c\left[\int_0^1 \left|u^{(\ell-1)}u^{(\ell)}\right|\,dx + \int_0^1 \left(u^{(\ell-1)}\right)^2 dx\right]. \qquad (7.5)$$

Other lower order terms in (7.3) can be estimated similarly.

The estimate (7.5) implies that $\mathcal{A} = \mathfrak{L}_9 \cdot (I + \mathcal{T})$, where $\mathcal{T}$ is compact operator. By Weyl Theorem ([36]; see also [35]) the lower order terms do not influence upon the one-term spectral asymptotics, and hence

$$\mu_n^{(\mathcal{A})} = \left(\frac{\pi n}{J_{2\ell}} + o(n)\right)^{2\ell}. \qquad (7.6)$$

It follows from (7.6) that for any $\tau > 0$ there exists $c_2(\tau)$ such that

$$\left(\frac{\pi n}{J_{2\ell}+\tau} - c_2(\tau)\right)^{2\ell} \le \mu_n^{(\mathcal{A})} \le \left(\frac{\pi n}{J_{2\ell}-\tau} + c_2(\tau)\right)^{2\ell}.$$

Using Theorem 6.2 we have

$$\liminf_{\varepsilon\to 0} \varepsilon^{2/(2\ell-1)} \ln P\{||X||_2 \le \varepsilon\} \ge -\,\frac{2\ell-1}{2}\left(\frac{J_{2\ell}+\tau}{2\ell\sin\frac{\pi}{2\ell}}\right)^{\frac{2\ell}{2\ell-1}};$$

$$\limsup_{\varepsilon\to 0} \varepsilon^{2/(2\ell-1)} \ln P\{||X||_2 \le \varepsilon\} \le -\,\frac{2\ell-1}{2}\left(\frac{J_{2\ell}-\tau}{2\ell\sin\frac{\pi}{2\ell}}\right)^{\frac{2\ell}{2\ell-1}}.$$

Passage to the limit as $\tau \to 0$ completes the proof. □

Remark 5. If in (7.1) $p_{2k} \equiv 0, k = 0, \ldots, \ell-1$, then the condition $p_{2\ell}(x) \ge c > 0$ can be weakened to $p_{2\ell}(x) > 0$ a.e., $p_{2\ell}^{-1} \in L_1(0, 1)$. In particular, rough asymptotics for the process Z_m (see subsection 5.6) holds true under the conditions $f^2 \in L_1(0, 1)$, $f^{-2} \in L_1(0, 1)$.

Though not all Gaussian processes satisfy the conditions of Theorem 7.3, many processes of probability interest do in case they are obtained by various transformations of Wiener process.

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