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# The Brownian loop soup

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**Abstract.** We define a natural conformally invariant measure on unrooted Brownian loops in the plane and study some of its properties. We relate this measure to a measure on loops rooted at a boundary point of a domain and show how this relation gives a way to “chronologically add Brownian loops” to simple curves in the plane.

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## 1. Introduction

The recent study of conformally invariant scaling limits of two-dimensional lattice systems has shown that measures on paths that satisfy conformal invariance (or conformal covariance) and a certain restriction property are important. In particular, in [9], it is shown how to construct “restriction” measures by dynamically adding bubbles to Schramm-Loewner evolution (SLE) curves. As announced there, this construction has an equivalent formulation in terms of a Brownian soup of loops. The purpose of this paper is to describe these Brownian measures and to prove this equivalence.

This description will be given without reference to SLE and is interesting on its own, but since this is what initiated our own interest, let us now describe the link with SLE. In [11], Oded Schramm introduced the SLE processes. These are the only random non-self-crossing curves in a domain that combine conformal invariance and a certain Markovian-type property. The definition of SLE is based on these two facts and can be viewed as a dynamic construction: one constructs the law of the curve on the time-interval  $[t, t + dt]$  given  $\gamma[0, t]$  and then iterates this procedure. In [9], following ideas of [10] and partially motivated by the problem of the self-avoiding walks in the plane (see [8]), a different approach to SLE was described. Basically, one looks at how the law of the random curve (seen globally) is distorted by an infinitesimal perturbation of the domain it is defined in. It turns out that a one-dimensional family of random sets is in some sense invariant under such perturbation. These are called restriction measures in [9], where it is shown

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that all of these measures are closely related to Brownian excursions. The law of SLE, except for the special case of the SLE with parameter  $\kappa = 8/3$ , is not a restriction measure. However, one can measure precisely the “restriction defect” (i.e., the Radon-Nikodym derivative) with a term involving the Schwarzian derivatives of the corresponding conformal maps. One interpretation for  $\text{SLE}_\kappa$  with  $\kappa < 8/3$  goes as follows: if one adds a certain Poissonian cloud of Brownian bubbles to the SLE curve, then the resulting set is restriction invariant. This can be understood simply when  $\kappa = 2$ . In that case, the SLE curve is [7] the scaling limit of the loop-erased random walk. In the scaling limit, the corresponding random walk converges to the Brownian excursion (which is restriction invariant). Hence, it is not surprising that if one puts the erased Brownian bubbles back onto the  $\text{SLE}_2$  curve, one obtains a restriction measure. This Poissonian cloud of Brownian bubbles provides a simple geometric picture of the distortion of the law of SLE under perturbation of the boundary of a domain. This “variational” approach to SLE is closely related to some conformal field theory considerations of e.g. [1, 3], as pointed out in [5, 6]. The density of the Poissonian cloud in particular plays the role of the (negative of the) central charge of the corresponding model in the theoretical physics language.

We will describe various measures on Brownian paths with an emphasis on two measures, the Brownian loop measure and the Brownian bubble measure. The latter was already defined and used in [9] for the previously described reasons.

The Brownian loop measure is an infinite measure on unrooted Brownian loops in the plane. It is defined on the set of periodic continuous functions in the plane, where two functions are considered to be indistinguishable if one is obtained by a simple translation in time ( $t \mapsto t + c$ ), and we call these equivalence classes “unrooted loops”. The Brownian loop measure is scale invariant, and translation-invariant. Furthermore, it is conformally invariant in the following sense: If there exists a conformal map  $\phi$  from  $D$  onto  $D'$ , then the image under  $\phi$  of the Brownian loop measure restricted to those loops that stay in  $D$  is exactly the Brownian loop measure restricted to those loops that stay in  $D'$ . This property is in fact very closely related to the restriction property.

This measure can also be considered a measure on “hulls” (compact sets  $K$  such that  $\mathbb{C} \setminus K$  is connected) by “filling in” the bounded loops. It is possible to argue that the Brownian loop measure is the only measure on hulls that is conformally invariant in the previous sense.

The Brownian loop soup of intensity  $\lambda > 0$  is a realization of a Poisson point process of density  $\lambda$  times the Brownian loop measure. In other words, a sample of the Brownian loop soup is a countable family of Brownian unrooted loops. There is no non-intersection condition or other interaction between the loops. Each loop will intersect countably many other loops in the same realization of the loop soup. Although for some purposes it is sufficient to consider the hull generated by a loop, we will study the measure on loops with time parametrization in this paper. This is partially motivated by possible future applications.

A bubble in a domain  $D$  will be a continuous path  $\gamma[0, T]$  such that  $\gamma(0, T) \subset D$  and  $\gamma(0) = \gamma(T) \in \partial D$ . We say that the bubble is rooted at  $x$  if  $\gamma(0) = x$ . The Brownian bubble measure was introduced in [9] in order to construct the restriction measures via SLE. The Brownian bubble in  $D$ , rooted in  $x \in \partial D$  is a  $\sigma$ -finite

measure on Brownian loops that start and end at  $x$ , and otherwise stay in  $D$ . The description is simplest if the considered domain is the upper half-plane  $\mathbb{H}$ , and the root is the origin. We will see that it can be considered as a conditioned version of the Brownian loop measure. The relation between these two measures will lead to an equivalence that we now describe.

Loosely speaking, the relation is as follows. Imagine that a realization of the loop soup in  $\mathbb{H}$  has been chosen, but we cannot see a loop until we visit a point on that loop. Suppose that we travel along a simple curve  $\eta$  with  $\eta(0) = 0$  and  $\eta(0, \infty) \subset \mathbb{H}$ . Each time  $t$  at which one encounters a loop in the loop soup for the first time, we can see the whole loop. This prescribes the order in which one finds the loops that intersect the curve  $\eta[0, \infty)$ . These loops are a priori unrooted; however, we can make them into rooted loops by starting a loop found at time  $t$  at the point  $\eta(t)$ . If we use this point as a root, the loop becomes a bubble in the domain  $\mathbb{H} \setminus \eta[0, t]$ . The point is that this loop is “distributed” according to the bubble measure.

More precisely, let  $\eta$  be as before. We do not make smoothness assumptions on  $\eta$ ; in fact, the cases of most interest to us are SLE curves that have Hausdorff dimension greater than one. Assume that  $\eta$  is parametrized by its “half-plane capacity” (as is customary for Loewner chains in the upper half-plane), i.e., that for all  $t$ , there exists a (unique) conformal map  $\tilde{g}_t$  from  $\mathbb{H} \setminus \eta[0, t]$  onto  $\mathbb{H}$  such that  $\tilde{g}_t(z) = z + 2t/z + o(1/z)$  when  $z \rightarrow \infty$ . We let  $g_t(z) = \tilde{g}_t(z) - \tilde{g}_t(\eta_t)$ .

Suppose that the countable collection of loops  $\{\gamma_1, \gamma_2, \dots\}$  in  $\mathbb{H}$  is a realization of the Brownian loop soup in  $\mathbb{H}$  with intensity  $\lambda > 0$ . This is a random family of equivalence classes of curves  $\gamma_j : [0, t_j] \rightarrow \mathbb{H}$  with  $\gamma_j(0) = \gamma_j(t_j)$ , under the equivalence  $\gamma^1 \sim \gamma^2$  if the time-lengths  $t^1$  and  $t^2$  of  $\gamma^1$  and  $\gamma^2$  are identical, and if for some  $r$ ,  $\gamma^1(t) = \gamma^2(t + r)$  for all  $t$  (with addition modulo  $t^1$ ). For each  $j$ , let

$$r_j = \inf\{s : \eta(s) \in \gamma_j[0, t_j]\}$$

with  $r_j = \infty$  if  $\eta[0, \infty) \cap \gamma_j[0, t_j] = \emptyset$ . It is not difficult to see that with probability one for each  $j$  with  $r_j < \infty$  there is a unique  $t \in [0, t_j)$  such that  $\gamma_j(t) = \eta(r_j)$ . Then we can choose the representative  $\gamma_j$  so that  $\gamma_j(0) = \eta(r_j)$ . Note that  $\gamma_j$  is a bubble in  $\mathbb{H} \setminus \eta[0, r_j]$ . We define  $\tilde{\gamma}_{r_j}$  as the image of  $\gamma_j$  under the mapping  $g_{r_j}$  where the time-parametrization of  $\tilde{\gamma}_{r_j}$  is obtained from that of  $\gamma_j$  using the usual Brownian time-change under conformal maps. Note that each  $\tilde{\gamma}_{r_j}$  is a bubble rooted at the origin in  $\mathbb{H}$ . Here, the parametrization of  $\eta$  by its half-plane capacity is important since we index  $\tilde{\gamma}$  by the time  $r_j$ .

**Theorem 1.** *The process  $(\tilde{\gamma}_r, r \geq 0)$  is a Poisson point process with intensity  $\lambda$  times the Brownian bubble measure in  $\mathbb{H}$ .*

Of course, this statement depends on the precise definitions of these measures, but it shows that adding the Poisson cloud of bubbles (as in [9]) to the path  $\eta$  is exactly the same as adding to  $\eta$  the set of loops in a loop soup that it does intersect.

One direct application is that adding Brownian bubbles to  $\eta$  or to the time-reversal of  $\eta$  (that is, viewing  $\eta$  as a curve from  $\infty$  to the origin) is the same (from the point of view of the outside hulls). In the case where  $\eta$  is chordal  $SLE_2$ , it

corresponds to the fact that loop-erasing a random walk does not depend (in law) on the chosen time-orientation. This result is more generally closely related to the question of reversibility of the SLE’s.

One other application is for the “duality” conjecture of the SLEs, see [4]: Indeed, from the point of view of the outside hulls, adding the loops of the loop soup to a curve or to its outer boundary is the same. Hence, the same is true if one adds (dynamically) bubbles to a curve or to its outer boundary. This leads to an identity in law between the set obtained by adding the same loop soup to a process closely related to  $SLE_\kappa$  or to a process closely related to  $SLE_{16/\kappa}$ . See [4] for more details. Theorem 1 is also used in [13].

Another main point is just the definition of the Brownian loop measure. Despite its simplicity (and maybe its importance) and its nice properties, it does not seem (to our knowledge) to have been considered before.

The technical aspects of the present paper are not difficult. Once one has the correct definitions, the proofs are more or less standard exercises on Brownian motions, excursion theory and Green functions. In order to keep the pace of the paper flowing, we will at times be somewhat informal (we will not always describe precisely how to take the limit of one measure on paths, etc.), leaving the gaps to the interested reader. We will however not completely omit these problems (see, e.g., the next section).

The paper is organized as follows. In the next section, we mainly introduce some notation. In Section 3, we define some measures on Brownian paths, among which are the Brownian bubbles. These are not new, but it is convenient to summarize some of their features in order to simplify the relation with the Brownian loop measure. This measure is defined and studied in Section 4, the relation with the bubble measure is described in Section 5. The final section is devoted to the question of time-parametrization of the Brownian “loop-adding” procedure.

## 2. Notations

We will write  $\mathbb{D}$  for the unit disk,  $\mathbb{H} = \{x + iy : y > 0\}$  for the upper half-plane, and  $\mathbb{D}_+$  for  $\mathbb{D} \cap \mathbb{H} = \{z \in \mathbb{H} : |z| < 1\}$ .

Let  $\mathcal{K}$  be the set of all parametrized continuous planar curves  $\gamma$  defined on a finite time-interval  $[0, t_\gamma]$ . We consider  $\mathcal{K}$  as a metric space with the metric

$$d_{\mathcal{K}}(\gamma, \gamma^1) = \inf_{\theta} \left[ \sup_{0 \leq s \leq t_\gamma} |s - \theta(s)| + |\gamma(s) - \gamma^1(\theta(s))| \right], \tag{1}$$

where the infimum is over all increasing homeomorphisms  $\theta : [0, t_\gamma] \rightarrow [0, t_{\gamma^1}]$ . Note that  $\mathcal{K}$  under this metric does not identify curves that are the same modulo time-reparametrization.

If  $\mu$  is any measure on  $\mathcal{K}$ , we let  $|\mu| = \mu(\mathcal{K})$  denote the total mass. If  $0 < |\mu| < \infty$ , then we let  $\mu^\# = \mu/|\mu|$  be  $\mu$  normalized to be a probability measure.

Let  $\mathcal{M}$  denote the set of finite Borel measures on  $\mathcal{K}$ . This is a metric space under the Prohorov metric  $d$  (see [2, Appendix III], e.g., for details). When we say that a sequence of measures converges it will be with respect to this metric. Recall that one standard way to show that two probability measures  $\mu$  and  $\nu$  are close with

respect to this metric is via coupling: one finds a probability measure  $m$  on  $\mathcal{K} \times \mathcal{K}$  whose first marginal is  $\mu$ , whose second marginal is  $\nu$ , and such that

$$m[\{(\gamma^1, \gamma^2) : d_{\mathcal{K}}(\gamma^1, \gamma^2) > \epsilon\}] \leq \epsilon.$$

To show that a sequence of finite measures  $\mu_n$  converges to a finite measure  $\mu$ , it suffices to show that  $|\mu_n| \rightarrow |\mu|$  and  $\mu_n^\# \rightarrow \mu^\#$ .

If  $D$  is a domain, we say that  $\gamma$  is in  $D$  if  $\gamma(0, t_\gamma) \subset D$ ; note that we do not require the endpoints of  $\gamma$  to be in  $D$ . Let  $\mathcal{K}(D)$  be the set of  $\gamma \in \mathcal{K}$  that are in  $D$ . If  $z, w \in \mathbb{C}$ , let  $\mathcal{K}_z$  (resp.,  $\mathcal{K}^w$ ) be the set of  $\gamma \in \mathcal{K}$  with  $\gamma(0) = z$  (resp.,  $\gamma(t_\gamma) = w$ ). We let  $\mathcal{K}_z^w = \mathcal{K}_z \cap \mathcal{K}^w$  and we define  $\mathcal{K}_z(D), \mathcal{K}^w(D), \mathcal{K}_z^w(D)$  similarly.

If  $\gamma, \gamma^1 \in \mathcal{K}$  with  $\gamma(t_\gamma) = \gamma^1(0)$ , we define the concatenation  $\gamma \oplus \gamma^1$  by  $t_{\gamma \oplus \gamma^1} = t_\gamma + t_{\gamma^1}$  and

$$\gamma \oplus \gamma^1(t) = \begin{cases} \gamma(t), & 0 \leq t \leq t_\gamma \\ \gamma^1(t - t_\gamma), & t_\gamma \leq t \leq t_\gamma + t_{\gamma^1}. \end{cases}$$

For every  $w$ , the map  $(\gamma, \gamma^1) \mapsto \gamma \oplus \gamma^1$  is continuous from  $\mathcal{K}^w \times \mathcal{K}_w$  to  $\mathcal{K}$ .

Suppose  $f : D \rightarrow D'$  is a conformal transformation and  $\gamma \in \mathcal{K}(D)$ . Let

$$s_t = s_{t, \gamma} = \int_0^t |f'(\gamma(s))|^2 ds.$$

If  $s_t < \infty$  for all  $t < t_\gamma$ , we define  $f \circ \gamma$  by  $f \circ \gamma(s_t) = f(\gamma(t))$ . If  $s_{t_\gamma} < \infty$  and  $f$  extends continuously to the endpoints of  $\gamma$ , then  $f \circ \gamma \in \mathcal{K}(D')$  and  $t_{f \circ \gamma} = s(t_\gamma)$ . If  $\mu$  is a measure supported on the set of curves  $\gamma$  in  $\mathcal{K}(D)$  such that  $f \circ \gamma$  is well defined and in  $\mathcal{K}(D')$ , then  $f \circ \mu$  will denote the measure

$$f \circ \mu(V) = \mu[\{\gamma : f \circ \gamma \in V\}].$$

If  $\gamma \in \mathcal{K}$ , we let  $\gamma^R$  denote the time reversal of  $\gamma$ , i.e.,  $t_{\gamma^R} = t_\gamma$  and  $\gamma^R(s) = \gamma(t_\gamma - s), 0 \leq s \leq t_\gamma$ . Similarly if  $\mu$  is a measure on  $\mathcal{K}$ , we define the measure  $\mu^R$  in the obvious way.

Suppose  $\{\mu_D\}$  is a family of measures indexed by a family of domains  $D$  in  $\mathbb{C}$ . We say that  $\mu_D$  satisfies the *restriction property* if

- $\mu_D$  is supported on  $\mathcal{K}(D)$ ;
- if  $D' \subset D$ , then  $\mu_{D'}$  is  $\mu_D$  restricted to the curves in  $\mathcal{K}(D')$ .

Note that if  $\mu$  is any measure on  $\mathcal{K}$  and  $\mu_D$  is defined as  $\mu$  restricted to  $\mathcal{K}(D)$ , then the family  $\{\mu_D\}$  satisfies the restriction property. Conversely, suppose that

- $\{\mu_D\}$  satisfies the restriction property
- $D_n$  is an increasing sequence of domains whose union is  $\mathbb{C}$
- $\mu = \lim_{n \rightarrow \infty} \mu_{D_n}$ .

Then, for each  $D$ ,  $\mu_D$  is  $\mu$  restricted to  $\mathcal{K}(D)$ .

If  $A$  is any compact set, we define  $\text{rad}(A) = \sup\{|z| : z \in A\}$ . The *half-plane capacity* of a compact subset  $A$  of  $\overline{\mathbb{H}}$  is defined by

$$\text{hcap}(A) = \lim_{y \rightarrow \infty} y \mathbf{E}^{iy} [\text{Im}(B_{\rho_A})], \tag{2}$$

where  $\rho_A = \inf\{t : B_t \in A \cup \mathbb{R}\}$ . It is not difficult to see that the limit exists, satisfies the scaling rule  $\text{hcap}(rA) = r^2 \text{hcap}(A)$ , and is monotone in  $A$ . If  $A$  is such that  $\mathbb{H} \setminus A$  is simply connected, then we use  $\tilde{g}_A$  to denote the unique conformal transformation of  $\mathbb{H} \setminus A$  onto  $\mathbb{H}$  such that  $\tilde{g}_A(z) - z = o(1)$  as  $z \rightarrow \infty$ . Then  $\tilde{g}_A$  has an expansion at infinity

$$\tilde{g}_A(z) = z + \frac{\text{hcap}(A)}{z} + O(|z|^{-2}).$$

Since  $z \mapsto z + (1/z)$  maps  $\{z \in \mathbb{H} : |z| > 1\}$  conformally onto  $\mathbb{H}$ , we can see that  $\text{hcap}(\overline{\mathbb{D}}_+) = 1$ .

If  $\eta : [0, \infty) \rightarrow \mathbb{C}$  is a curve, we will sometimes write  $\tilde{g}_t$  for  $\tilde{g}_{\eta[0,t]}$  and define  $g_t(z) = \tilde{g}_t(z) - \tilde{g}_t(\eta_t)$  as in the introduction.

### 3. Brownian bridges, Brownian bubbles

We will start defining some bridge-type Brownian measures on curves that we will use. These are measures on Brownian paths with prescribed starting point and prescribed terminal point. Since we are interested in conformally invariant properties, the standard bridges with prescribed time duration are not well-suited.

In our notation,  $\mu_D(z, w)$  will always be a measure on Brownian paths that remain in the domain  $D$ , that start at  $z$  and end at  $w$ , but this notation will have different meanings depending on whether  $z, w$  are boundary or interior points of the domain  $D$ . We hope this will not cause confusion. Since the content of this section is rather standard, we will just review these definitions. The excursion measures have been defined in [10, 12, 9], the bubble measures in [9].

#### 3.1. First definitions

##### 3.1.1. Interior to interior

Let  $\mu(z, \cdot; t)$  denote the law of a standard complex Brownian motion  $(B_s, 0 \leq s \leq t)$ , with  $B_0 = z$ , viewed as an element of  $\mathcal{K}$ . We can write

$$\mu(z, \cdot; t) = \int_{\mathbb{C}} \mu(z, w; t) dA(w),$$

where  $A$  denotes area and  $\mu(z, w; t)$  is a measure supported on  $\gamma \in \mathcal{K}_z^w$  with  $t_\gamma = t$ . In other words,  $\mu(z, w; t)$  is  $|\mu(z, w; t)|$  times the law  $\mu^\#(z, w; t)$  of the Brownian bridge from  $z$  to  $w$  in time  $t$ , where  $|\mu(z, w; t)| = (2\pi t)^{-1} \exp\{-|z - w|^2/(2t)\}$ .

The measure  $\mu(z, w)$  is defined by

$$\mu(z, w) = \int_0^\infty \mu(z, w; t) dt.$$

This is a  $\sigma$ -finite measure (the integral explodes at infinity so that the total mass of large loops is infinite; when  $z = w$ , it also diverges at 0).

The measure  $\mu(z, z)$  is an infinite measure on Brownian loops that start and end at  $z$ . We can write

$$\mu(z, z) = \int_0^\infty \frac{1}{2\pi t} \mu^\#(z, z; t) dt.$$

where  $\mu^\#(z, z; t)$  is the usual probability measure of a Brownian bridge from  $z$  to  $z$ .

If  $D$  is a domain and  $z, w \in D$ , we define  $\mu_D(z, w)$  to be  $\mu(z, w)$  restricted to  $\mathcal{K}(D)$ . For fixed  $z, w$ , the family  $\{\mu_D(z, w), D \supset \{z, w\}\}$  clearly satisfies the restriction property.

If  $z \neq w$ , and if the domain  $D$  is such that a Brownian motion in  $D$  eventually exits  $D$ , then  $|\mu_D(z, w)| < \infty$ . In fact,

$$|\mu_D(z, w)| = \frac{G_D(z, w)}{\pi},$$

where  $G_D$  denotes the Green's function normalized so that  $G_{\mathbb{D}}(0, z) = -\log |z|$ . Note that  $\mu_D(z, z)$  is well defined and has infinite total mass. The reversibility of the Brownian bridge immediately implies that  $[\mu_D(z, w)]^R = \mu_D(w, z)$ .

### 3.1.2. Interior to boundary

Let  $D$  be a connected domain in  $\mathbb{C}$  whose boundary is a finite union of curves. We will call  $\partial D$  nice if it is piecewise analytic, i.e., if it is a finite union of analytic curves.. A nice boundary point will be any point at which the boundary is locally an analytic curve (and we say that infinity is a nice boundary point, if the origin is a nice boundary point of  $1/D$ ).

Let  $B$  be a Brownian motion starting at  $z \in D$  and stopped at its exit time of  $D$ , i.e., at

$$\tau_D = \inf\{t : B_t \notin D\}.$$

Define  $\mu_D(z, \partial D)$  to be the law of  $(B_t, 0 \leq t \leq \tau_D)$ . If  $D$  has a nice boundary we can write

$$\mu_D(z, \partial D) = \int_{\partial D} \mu_D(z, w) |dw|,$$

where  $\mu_D(z, w)$  for  $z \in D$  and  $w \in \partial D$  denotes a measure supported on  $\mathcal{K}_z^w(D)$  with total mass  $H_D(z, w)$ , where  $H_D(z, w)$  denotes the usual Poisson kernel. The normalized probability measure  $\mu_D^\#(z, w)$  is the law of Brownian motion conditioned to exit  $D$  "at  $w$ ".

## 3.2. First properties

### 3.2.1. Conformal invariance

It is well known that planar Brownian motion is conformally invariant. In our interior to interior notation, this can be phrased as follows. Suppose  $f : D \rightarrow D'$  is a

conformal transformation and  $z, w$  are two interior points in  $D$ . Then,

$$f \circ \mu_D(z, w) = \mu_{f(D)}(f(z), f(w)). \tag{3}$$

If  $z \neq w$  and the complement of  $D$  is not polar, this is a combination of the two classical results:  $G_{f(D)}(f(z), f(w)) = G_D(z, w)$  and  $[f \circ \mu_D]^\#(z, w) = \mu_{f(D)}^\#(f(z), f(w))$ . Otherwise (in which case the measures are infinite), one can prove this by taking a limit.

Similarly, in the interior to boundary case, if  $z \in D$  is an interior point and  $w$  a boundary point, and if both  $w$  and  $f(w)$  are nice, then

$$f \circ \mu_D(z, w) = |f'(w)| \mu_{f(D)}(f(z), f(w)). \tag{4}$$

This is a consequence of the two relations:  $H_D(z, w) = |f'(w)| H_{D'}(f(z), f(w))$  and  $f \circ \mu_D^\#(z, w) = \mu_{f(D)}^\#(f(z), f(w))$ . It implies that one can define the probability measure  $\mu_D^\#(z, w)$  for any simply connected  $D$  and any boundary point (i.e. prime end)  $w$  by conformal invariance. For instance, it suffices to put  $\mu_D^\#(z, w) = f \circ \mu_{\mathbb{D}}^\#(0, 1)$  where  $f : \mathbb{D} \rightarrow D$  is the conformal transformation with  $f(0) = z$  and  $f(1) = w$ .

### 3.2.2. Regularity

Note that the measures  $\mu_D(z, w)$  are continuous functions of  $z, w$  in the Prohorov metric. For instance, for any two interior points  $z_0 \neq w_0$  in the fixed domain  $D$ , the mapping  $(z, w) \mapsto \mu_D(z, w)$  is continuous at  $(z_0, w_0)$ . This can for instance be proved using a coupling argument.

Similarly, it is not difficult to show in the interior to boundary case that for a fixed boundary point  $w$ , the mapping  $z \mapsto \mu_D(z, w)$  is continuous. When one wishes to let  $w$  vary, one can for instance first note that  $w \mapsto \mu_{\mathbb{D}}(0, w)$  is clearly continuous on the unit circle. Furthermore, for a conformal map  $f$  from  $\mathbb{D}$  onto  $D$ , the derivative  $f'$  is uniformly bounded when restricted to any  $r\mathbb{D}$  for  $r < 1$ , so that one can control the variation of the time-parametrization. We will discuss this in more detail later in the (slightly more complicated) case of the excursion measures.

### 3.2.3. Relation between the two

If  $z, w$  are distinct points in  $D$ , then the normalized interior to interior measure  $\mu_D^\#(z, w)$  can be given as a limit of boundary measures. Let  $D_\epsilon = \{z' \in D : |z' - w| > \epsilon\}$ , and let  $\nu_\epsilon$  denote  $\mu(z, \partial D_\epsilon)$  restricted to curves whose terminal point is distance  $\epsilon$  from  $w$ . As  $\epsilon \rightarrow 0+$ ,  $|\nu_\epsilon| \sim G_D(z, w) [\log(1/\epsilon)]^{-1}$  and  $\nu_\epsilon^\# \rightarrow \mu_D^\#(z, w)$ .

The interior to boundary measure can also be viewed as the limit of an appropriately rescaled interior to interior measure: If  $w_n \in D$  and  $w_n \rightarrow w$  where  $w \in \partial D$ , then it is not hard to show that the corresponding probability measures converge  $\mu_D^\#(z, w_n) \rightarrow \mu_D^\#(z, w)$ , for instance using a coupling argument. Also, if  $w$  is a nice boundary point, and  $\mathbf{n}_w$  denotes the inward normal at  $w$ , then as  $\epsilon \rightarrow 0+$ ,

$$G_D(z, w - \epsilon \mathbf{n}_w) \sim 2\pi\epsilon H_D(z, w)$$

(the multiplicative constant can be worked out immediately using the case  $D = \mathbb{D}$ ,  $z = 0$ ,  $w = 1$ ). Hence,

$$\lim_{\epsilon \rightarrow 0^+} \frac{1}{2\epsilon} \mu_D(z, w - \epsilon \mathbf{n}_\epsilon) = \mu_D(z, w), \tag{5}$$

for any interior point  $z$  and any nice boundary point  $w$ .

### 3.3. Excursion measures

#### 3.3.1. Definition and conformal invariance

Suppose that  $D$  is a nice domain, and that  $z$  and  $w$  are different nice boundary points of  $D$ . We will define the Brownian measure on paths from  $z$  to  $w$  in  $D$ . This Brownian excursion measure  $\mu_D(z, w)$  can be defined by various means (see e.g. [10, 12, 9]). It can be viewed as limits of the previous measures:

$$\mu_D(z, w) = \lim_{\epsilon \rightarrow 0^+} \frac{1}{2\epsilon^2} \mu_D(z + \epsilon \mathbf{n}_z, w + \epsilon \mathbf{n}_w) = \lim_{\epsilon \rightarrow 0^+} \frac{1}{\epsilon} \mu_D(z + \epsilon \mathbf{n}_z, w). \tag{6}$$

Again we can write  $\mu_D(z, w) = H_D(z, w) \mu_D^\#(z, w)$  where

$$H_D(z, w) = \lim_{\epsilon \rightarrow 0^+} \epsilon^{-1} H_D(z + \epsilon \mathbf{n}_z, w).$$

Under this normalization  $H_{\mathbb{H}}(0, x) = 1/(\pi x^2)$ .

The probability measures  $\mu_D^\#(z, w)$  are conformally invariant, i.e.,

$$f \circ \mu_D^\#(z, w) = \mu_{f(D)}^\#(f(z), f(w))$$

for a conformal transformation such that the four boundary points  $z, w, f(z)$  and  $f(w)$  are nice. This shows that one can define  $\mu_D^\#(z, w)$  by conformal invariance even if  $z, w$  are not nice boundary points.

It is sometimes easier to consider  $\mu_{\mathbb{H}}^\#(0, \infty)$  where  $\mathbb{H}$  denotes the upper half-plane. This is the distribution of  $\mathbb{H}$ -excursions, which are Brownian motions in the first component and independent three-dimensional Bessel processes in the second component, see [12, 9]. One could choose this as the definition of  $\mu_{\mathbb{H}}^\#(0, \infty)$ , define the measures  $\mu_D^\#(z, w)$  by conformal invariance, and define the measures  $\mu_D(z, w)$  by multiplying by the total mass, and finally verify (6). (The measure  $\mu_{\mathbb{H}}^\#(0, \infty)$  is not supported on  $\mathcal{K}$  since curves under this measure have infinite time duration; however, this does not present a problem. In particular, the image of  $\mu_{\mathbb{H}}^\#(0, \infty)$  under a conformal transformation onto a bounded domain is supported on paths of finite time duration.)

If  $f : D \rightarrow D'$  is a conformal transformation, and  $z, w, f(z), f(w)$  are nice boundary points, then [12, 9]

$$f \circ \mu_D(z, w) = |f'(z)| |f'(w)| \mu_{f(D)}(f(z), f(w)) \tag{7}$$

The “integrated measure”

$$\mu_{\partial D} := \int_{\partial D} \int_{\partial D} \mu_D(z, w) |dz| |dw|,$$

is therefore conformally invariant:

$$f \circ \mu_{\partial D} = \mu_{\partial f(D)} \tag{8}$$

as was pointed out in [10].

### 3.3.2. Regularity

We now study the regularity of the excursion measures with respect to the domain  $D$ . For this we will need some simple lemmas.

**Lemma 2.** *For any simply connected domain  $D$  and any two distinct points  $w$  and  $w'$  on the boundary of  $D$ , the expected time spent in an open subset  $U$  of  $D$  by an excursion defined under the probability measure  $\mu_D^\#(w, w')$  is bounded from above by  $2 \text{area}(U)/\pi$ .*

*Proof.* If  $z \in D$ , let  $G_D^\#(w, w'; z)$  denote the Green’s function for  $\mu_D^\#(w, w')$ . This can be obtained as the limit of the interior to boundary Green’s functions  $G_D^\#(w_n, w'; z)$  where  $w_n$  is a sequence of points in  $D$  converging to  $w$ . If  $f : D \rightarrow D'$  is a conformal transformation, then

$$G_D^\#(w, w'; z) = G_{f(D)}^\#(f(w), f(w'); f(z)).$$

Also

$$G_{\mathbb{H}}^\#(0, \infty; z) = \lim_{\epsilon \rightarrow 0^+} \frac{\text{Im}(z)}{2\epsilon} \log \frac{\text{Re}(z)^2 + (\epsilon + \text{Im}(z))^2}{\text{Re}(z)^2 + (\epsilon - \text{Im}(z))^2} = 2 \frac{\text{Im}(z)^2}{|z|^2} \leq 2. \tag{9}$$

By conformal invariance, we get  $G_D^\#(w, w'; z) \leq 2$  for all simply connected  $D$  and all  $w, w', z$ . This readily implies the lemma.  $\square$

**Lemma 3.** *There is a constant  $c < \infty$  such that the following holds. Suppose  $D, D'$  are simply connected domains and  $f : D \rightarrow D'$  is a conformal transformation with  $|f(z_1) - z_1| \leq \delta \leq 1$  for all  $z_1 \in D$ . Then for any path  $\gamma$  in  $D$ ,*

$$\begin{aligned} d_{\mathcal{K}}(\gamma, f \circ \gamma) &\leq c \left[ \delta + \delta^{1/2} t_\gamma + \int_0^{t_\gamma} 1_{\{\text{dist}(\gamma(s), \partial D) \leq \delta^{1/2}\}} ds \right. \\ &\quad \left. + \int_0^{t_{f \circ \gamma}} 1_{\{\text{dist}(f \circ \gamma(u), \partial D') \leq c\delta^{1/2}\}} du \right]. \end{aligned}$$

*Proof.* For any  $\gamma$  let

$$\theta_\gamma(s) = \int_0^s |f'(\gamma(r))|^2 dr.$$

Then,

$$d_{\mathcal{K}}(\gamma, f \circ \gamma) \leq \sup_{0 \leq s \leq t_\gamma} |s - \theta_\gamma(s)| + \sup_{0 \leq s \leq t_\gamma} |\gamma(s) - f(\gamma(s))|.$$

The second term on the right is bounded above by  $\delta$  and the first term is bounded above by  $\int_0^{t_\gamma} Y_s ds$ , where  $Y_s = ||f'(\gamma(s))|^2 - 1|$ . Let

$$\tilde{D} = \tilde{D}_\delta = \{z \in D : \text{dist}(z, \partial D) \leq \delta^{1/2}\}.$$

For  $z \in D \setminus \tilde{D}$ , a standard estimate gives  $|f'(z) - 1| \leq c \delta / \text{dist}(z, \partial D) \leq c \delta^{1/2}$ . Hence,

$$\int_0^{t_\gamma} Y_s 1_{\gamma(s) \notin \tilde{D}} ds \leq c \delta^{1/2} t_\gamma.$$

For the other part, write

$$\int_0^{t_\gamma} Y_s 1_{\gamma(s) \in \tilde{D}} ds \leq \int_0^{t_\gamma} 1_{\gamma(s) \in \tilde{D}} ds + \int_0^{t_\gamma} |f'(\gamma(s))|^2 1_{\gamma(s) \in \tilde{D}} ds.$$

The first term on the right hand side is the amount of time that  $\gamma$  spends within distance  $\delta^{1/2}$  of the boundary. Since  $|f(z) - z| \leq \delta$ , the second term on the right is less than the amount of time that  $f \circ \gamma$  spends within distance  $3\delta^{1/2}$  of  $\partial D'$ . Combining these estimates gives the lemma.  $\square$

**Lemma 4.** *Suppose that  $D \subset \mathbb{D}_+$  is a simply connected domain with  $\mathbb{D}_+ \setminus D \subset \delta \mathbb{D}_+$  for some  $\delta > 0$ . Let  $z, z', w$  on  $\partial D$  with  $|z| = |z'| = 1, |w| \leq \delta$  and  $|z - z'| \leq \delta$ . Then, the distance between  $\mu_D^\#(z, w)$  and  $\mu_{\mathbb{D}_+}^\#(z', 0)$  goes to zero with  $\delta$ , uniformly with respect to the choice of  $w, z, z'$  and  $D$ .*

*Proof.* Let  $f$  denote the conformal mapping from  $D$  onto  $\mathbb{D}_+$  such that  $f(w) = 0, f(z) = z'$  and  $|f'(i)| = 1$ . It is standard that independently of  $\delta \leq 1$  and  $D$  as above, there exists a constant  $c$  such that  $|f(x) - x| \leq c\delta$  for all  $x \in D$ . The total area in  $D$  or  $D'$  of the set of points that is at distance less than  $\delta^{1/2}$  from the boundary is no larger than  $c'\delta^{1/2}$ . Hence, a combination of the two previous lemmas shows that

$$\mathbf{E}[d_{\mathcal{K}}(\gamma, f \circ \gamma)] \leq c \delta^{1/2},$$

where the expectation is with respect to  $\mu_D^\#(z, w)$ . Since the law of  $f \circ \gamma$  is  $\mu_{\mathbb{D}_+}^\#(z', 0)$ , the lemma follows.  $\square$

### 3.4. Brownian Bubbles

#### 3.4.1. Definition

The Brownian bubble measure in  $\mathbb{H}$  at the origin is the  $\sigma$ -finite measure

$$\mu_{\mathbb{H}}^{\text{bub}}(0) = \lim_{z \rightarrow 0} \frac{\pi}{\text{Im}(z)} \mu_{\mathbb{H}}(z, 0) \quad (z \in \mathbb{H}) \tag{10}$$

or equivalently (see (5)),

$$\mu_{\mathbb{H}}^{\text{bub}}(0) = \lim_{z,w \rightarrow 0} \frac{\pi}{2 \operatorname{Im}(z) \operatorname{Im}(w)} \mu_{\mathbb{H}}(z, w) \quad (z, w \in \mathbb{H}). \tag{11}$$

When we speak of the limit, we mean that for every  $r > 0$ , if we restrict the measures on the right to loops that intersect the circle of radius  $r$  (so that this is a finite measure), then the limit exists and equals  $\mu_{\mathbb{H}}^{\text{bub}}(0; r)$  which is  $\mu_{\mathbb{H}}^{\text{bub}}(0)$  restricted to loops that intersect  $\{|z| = r\}$ . It is not hard to show the limit exists and the normalization is chosen so that  $|\mu_{\mathbb{H}}^{\text{bub}}(0; r)| = 1/r^2$ . If  $r > 0$  and  $f_r(z) = rz$ , then  $\mu_{\mathbb{H}}^{\text{bub}}(0)$  satisfies the scaling rule

$$f_r \circ \mu_{\mathbb{H}}^{\text{bub}}(0) = r^2 \mu_{\mathbb{H}}^{\text{bub}}(0).$$

We can also define  $\mu_D^{\text{bub}}(z)$  for other domains, at least if  $\partial D$  is smooth near  $z$ , using conformal covariance,

$$f \circ \mu_D^{\text{bub}}(z) = |f'(z)|^2 \mu_{f(D)}^{\text{bub}}(f(z)). \tag{12}$$

These measures satisfy the restriction property: if  $D' \subset D$ , then  $\mu_{D'}^{\text{bub}}(0)$  is  $\mu_D^{\text{bub}}(0)$  restricted to loops that are in  $D'$ .

Suppose  $D \subset \mathbb{H}$  is a simply connected domain containing  $r\mathbb{D}_+$  for some  $r > 0$ , and let  $A$  be the image of  $\mathbb{H} \setminus D$  under the map  $z \mapsto -1/z$ . Then [9, (7.2)] tells us that the  $\mu_{\mathbb{H}}^{\text{bub}}(0)$  measure of the set of loops that do not stay in  $D$  is  $\text{heap}(A)$ . The reader can check that both the definition in the present paper and the definition in [9] give measure 1 to the set of loops that intersect the unit circle, and hence the two definitions use the same normalization. In particular, this shows immediately that

$$\mu_{\mathbb{H}}^{\text{bub}}(0)[\{\gamma : \gamma(0, t_\gamma) \not\subset D\}] = \frac{-S_\Phi(0)}{6}, \tag{13}$$

where  $\Phi$  is a conformal map from  $D$  onto  $\mathbb{H}$  that keeps the origin fixed, say, and  $S_\Phi$  denotes the Schwarzian derivative

$$S_\Phi(z) = \frac{\Phi'''(z)}{\Phi'(z)} - \frac{3\Phi''(z)^2}{2\Phi'(z)^2}.$$

### 3.4.2. Path decomposition

The next proposition relates  $\mu_{\mathbb{H}}^{\text{bub}}(0)$  to excursion measures. This expression for  $\mu_{\mathbb{H}}^{\text{bub}}(0)$  splits the bubble at the point  $se^{i\theta}$  at which its distance to the origin is maximal.

**Proposition 5.** *One has*

$$\mu_{\mathbb{H}}^{\text{bub}}(0) = \pi \int_0^\infty \int_0^\pi [\mu_{r\mathbb{D}_+}(0, re^{i\theta}) \oplus \mu_{r\mathbb{D}_+}(re^{i\theta}, 0)] r \, d\theta \, dr \tag{14}$$

$$= \int_0^\infty \frac{4}{\pi r^3} \int_0^\pi [\mu_{r\mathbb{D}_+}^\#(0, re^{i\theta}) \oplus \mu_{r\mathbb{D}_+}^\#(re^{i\theta}, 0)] \sin^2 \theta \, d\theta \, dr. \tag{15}$$

*Proof.* Let  $r > 0, \delta > 0$ . By the strong Markov property,

$$\begin{aligned} & \mu_{\mathbb{H}}^{\text{bub}}(0; r) - \mu_{\mathbb{H}}^{\text{bub}}(0; r + \delta) \\ &= \lim_{\epsilon \rightarrow 0^+} \frac{\pi}{\epsilon} \int_0^\pi [\mu_{r\mathbb{D}_+}(\epsilon i, r e^{i\theta}) \oplus \mu_{(r+\delta)\mathbb{D}_+}(r e^{i\theta}, 0)] r \, d\theta. \\ &= \pi \int_0^\pi [\mu_{r\mathbb{D}_+}(0, r e^{i\theta}) \oplus \mu_{(r+\delta)\mathbb{D}_+}(r e^{i\theta}, 0)] r \, d\theta. \end{aligned}$$

But as

$$\lim_{\delta \rightarrow 0^+} \delta^{-1} \mu_{(r+\delta)\mathbb{D}_+}(r e^{i\theta}, 0) = \lim_{\delta \rightarrow 0^+} \delta^{-1} \mu_{r\mathbb{D}_+}(r^2 e^{i\theta} / (r + \delta), 0) = \mu_{r\mathbb{D}_+}(r e^{i\theta}, 0),$$

we get that

$$\frac{d}{dr} \mu_{\mathbb{H}}^{\text{bub}}(0; r) = -\pi \int_0^\pi [\mu_{r\mathbb{D}_+}(0, r e^{i\theta}) \oplus \mu_{r\mathbb{D}_+}(r e^{i\theta}, 0)] r \, d\theta,$$

which gives (14). Identity (15) follows from the fact (see Lemma 9) that

$$|\mu_{r\mathbb{D}_+}(0, r e^{i\theta})| = r^{-2} |\mu_{\mathbb{D}_+}(0, e^{i\theta})| = \frac{2 \sin \theta}{\pi r^2}. \quad \square$$

Note that

$$\frac{d}{dr} |\mu_{\mathbb{H}}^{\text{bub}}(0; r)| = -4/(\pi r^3) \int_0^\pi \sin^2 \theta \, d\theta = -2/r^3,$$

which is consistent with  $|\mu_{\mathbb{H}}^{\text{bub}}(0; r)| = r^{-2}$ .

Similarly, one can decompose the Brownian bubble measure in  $\mathbb{H}$  at the point where the imaginary part is maximal. This gives a joint description of the real and imaginary parts using one-dimensional excursions and Brownian bridges (as briefly mentioned in [9]).

#### 4. (Unrooted) loop measure

##### 4.1. Definition, restriction and conformal invariance

We will now define the most important object for this paper, the Brownian loop measure  $\mu^{\text{loop}}$ . Let  $\tilde{\mathcal{K}}$  be the set of loops, i.e., the set of  $\gamma \in \mathcal{K}$  with  $\gamma(0) = \gamma(t_\gamma)$ . Such a  $\gamma$  can also be considered as a function with domain  $(-\infty, \infty)$  satisfying  $\gamma(s) = \gamma(s + t_\gamma)$ .

Define  $\theta_r : \tilde{\mathcal{K}} \rightarrow \tilde{\mathcal{K}}$  by  $t_{\theta_r \gamma} = t_\gamma$  and  $\theta_r \gamma(s) = \gamma(s + r)$ . We say that two loops  $\gamma$  and  $\gamma'$  are equivalent if for some  $r, \gamma' = \theta_r \gamma$ . We write  $[\gamma]$  for the equivalence class of  $\gamma$ . Let  $\tilde{\mathcal{K}}_U$  be the set of *unrooted loops*, i.e., the equivalence classes in  $\tilde{\mathcal{K}}$ . Note that  $\tilde{\mathcal{K}}_U$  is a metric space under the metric

$$d_{p,U}(\gamma, \gamma') = \inf_{r \in [0, t_\gamma]} d_{\mathcal{K}}(\theta_r \gamma, \gamma').$$

Any measure supported on  $\tilde{\mathcal{K}}$  gives a measure on  $\tilde{\mathcal{K}}_U$  by “forgetting the root”, i.e., by considering the map  $\gamma \mapsto [\gamma]$ . If  $D$  is a domain, we define  $\tilde{\mathcal{K}}(D), \tilde{\mathcal{K}}_U(D)$  to be the set of loops that lie entirely in  $D$ , i.e.,  $\gamma[0, t_\gamma] \subset D$ .

We define the *Brownian loop measure*  $\mu^{\text{loop}}$  on  $\tilde{\mathcal{K}}_{\mathbb{U}}$  by

$$\mu^{\text{loop}} = \int_{\mathbb{C}} \frac{1}{t_{\gamma}} \mu(z, z) dA(z) = \int_{\mathbb{C}} \int_0^{\infty} \frac{1}{2\pi t^2} \mu^{\#}(z, z; t) dt dA(z), \tag{16}$$

where  $dA$  denote the Lebesgue measure on  $\mathbb{C}$ . We insist on the fact that the measure  $\mu^{\text{loop}}$  is a measure on *unrooted* loops (so that in fact  $\mu^{\text{loop}}$  is the image of the measure on the right-hand side of the definition under the projection  $\gamma \mapsto [\gamma]$ ).

We will call a Borel measurable function  $T : \tilde{\mathcal{K}} \rightarrow [0, \infty)$  a *unit weight* if for every  $\gamma \in \tilde{\mathcal{K}}$ ,

$$\int_0^{t_{\gamma}} T(\theta_r \gamma) dr = 1.$$

One example of a unit weight is  $T(\gamma) = 1/t_{\gamma}$ . Note that  $\mu^{\text{loop}}$  satisfies

$$\mu^{\text{loop}} = \int_{\mathbb{C}} T \mu(z, z) dA(z) \tag{17}$$

(considered as a measure on  $\tilde{\mathcal{K}}_{\mathbb{U}}$ ) for any unit weight  $T$ .

If  $D$  is a domain, we define  $\mu_D^{\text{loop}}$  to be  $\mu^{\text{loop}}$  restricted to the curves in  $\tilde{\mathcal{K}}_{\mathbb{U}}(D)$ ; this is the same as the right-hand side of (17) with  $D$  replacing  $\mathbb{C}$  and  $\mu_D(z, z)$  replacing  $\mu(z, z)$ . By construction, the family  $\{\mu_D^{\text{loop}}\}$  satisfies the restriction property. Not as obviously, these measures are also conformally invariant:

**Proposition 6.** *If  $f : D \rightarrow D'$  is a conformal transformation, then  $f \circ \mu_D^{\text{loop}} = \mu_{f(D)}^{\text{loop}}$ .*

*Proof.* Showing this requires two observations. One, which we have already noted, is the conformal invariance of interior to interior measures,

$f \circ \mu_D(z, z) = \mu_{f(D)}(f(z), f(z))$ . The other is the fact that we can define a unit weight  $T_f$  by  $T_f(\gamma) = 1/t_{\gamma}$  if  $\gamma \notin \tilde{\mathcal{K}}(D)$ , and if  $\gamma \in \tilde{\mathcal{K}}(D)$ ,  $T_f(\gamma) = |f'(\gamma(0))|^2/t_{f \circ \gamma}$ . To check that this is a unit weight, note that

$$\int_0^{t_{\gamma}} T_f(\theta_r \gamma) dr = (1/t_{f \circ \gamma}) \int_0^{t_{\gamma}} |f'(\gamma(r))|^2 dr = 1.$$

Therefore,

$$\begin{aligned} f \circ \mu_D^{\text{loop}} &= f \circ \int_D T_f \mu_D(z, z) dA(z) \\ &= \int_D (1/t_{f \circ \gamma}) |f'(z)|^2 f \circ \mu_D(z, z) dA(z) \\ &= \int_D (1/t_{f \circ \gamma}) \mu_{D'}(f(z), f(z)) [|f'(z)|^2 dA(z)] \\ &= \int_{D'} T \mu_{D'}(w, w) dA(w) = \mu_{D'}^{\text{loop}}. \end{aligned}$$

Here  $T$  denotes the simple unit weight  $T(\gamma) = 1/t_{\gamma}$ . □

Note that the same argument shows that  $\mu^{\text{loop}}$  is invariant under the inversions  $z \mapsto 1/(z - z_0)$  for all fixed  $z_0$ .

4.2. Decompositions

The definition of  $\mu^{\text{loop}}$  makes it conformally invariant and hence independent of the choice of coordinate axes. It will be however convenient to have expressions for  $\mu^{\text{loop}}$  that do depend on the axes. We will write the measure on unrooted loops  $[\gamma]$  as a measure on rooted loops by choosing the representative  $\gamma$  whose initial point is the (unique) point on the loop of minimal imaginary part (the same works of course also for the maximal imaginary part). Note that this choice of “root” of the loop is not conformally invariant.

**Proposition 7.**

$$\mu^{\text{loop}} = \frac{1}{\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \mu_{\mathbb{H}+iy}^{\text{bub}}(x + iy) \, dx \, dy$$

*Proof.* There are various simple ways to prove this. The main point is to get the multiplicative constants right. We therefore opt for a self-contained elementary proof that does not rely on other multiplicative conventions (i.e., excursions). We start by recalling some facts about one dimensional Brownian motion. Suppose  $Y_t$  is a one-dimensional Brownian motion started at the origin. Let  $t^*$  be the time in  $[0, 1]$  at which  $Y_t$  is minimal, let  $M = Y_{t^*}$ , and let  $\Psi = (Y_0 - M)(Y_1 - M)$ . It is easy to see that the law of  $t^*$  is the arcsine law with density  $1/(\pi\sqrt{t(1-t)})$  on  $[0, 1]$ . Given  $t^*$ ,  $Y_0 - M$  and  $Y_1 - M$  are independent random variables with the distribution of Brownian motion “conditioned to stay positive”. It is not difficult to show that  $\mathbf{E}[Y_0 - M \mid t^* = t] = \sqrt{\pi t/2}$  and hence that  $\mathbf{E}[\Psi] = 1/2$ .

We now define a unit weight  $T_\epsilon$  on  $\gamma \in \tilde{\mathcal{K}}$  that will approximate the Dirac mass at the time of the minimal imaginary part of  $\gamma$ . If  $t_\gamma < \epsilon$ , then  $T_\epsilon(\gamma) = 1/t_\gamma$ . Suppose now that  $t_\gamma \geq \epsilon$  and that there is a unique  $r_0 \in [0, t_\gamma]$  such that  $\text{Im}[\gamma(r_0)] < \text{Im}[\gamma(t)]$  for  $t \in [0, t_\gamma] \setminus \{r_0\}$ . Then, we define

$$T_\epsilon(\gamma) = \epsilon^{-1} 1_{r_0 \leq \epsilon}.$$

If no such unique  $r_0$  exists, set  $T_\epsilon(\gamma) = 1/t_\gamma$ . This last choice is in fact irrelevant here since the measures  $\mu(z, z)$  are supported on loops for which a unique  $r_0$  exists.

It is easy to see that  $T_\epsilon$  is a unit weight, and that for every  $\epsilon$ ,

$$\mu^{\text{loop}} = \int_{\mathbb{C}} T_\epsilon \mu(z, z) \, dA(z) = \lim_{\epsilon \rightarrow 0+} \int_{\mathbb{C}} \epsilon^{-1} \mu(z, z; \geq \epsilon) \, dA(z),$$

where  $\mu(z, z; \geq \epsilon)$  denotes  $\mu(z, z)$  restricted to curves  $\gamma$  with  $t_\gamma \geq \epsilon$  and

$$\inf\{\text{Im}(\gamma(t)); 0 \leq t \leq \epsilon\} = \inf\{\text{Im}(\gamma(t)) : 0 \leq t < t_\gamma\}.$$

For fixed  $\epsilon$ ,  $\int_{\mathbb{C}} \epsilon^{-1} \mu(z, z; \geq \epsilon) \, dA(z)$  is the same as  $\mu^{\text{loop}}$  restricted to curves with  $t_\gamma \geq \epsilon$ . Let us consider the measure  $\epsilon^{-1} \mu(z, z; \geq \epsilon)$ . For ease let  $z = 0$ . Start a Brownian motion  $B_t$  at 0 and let it run until time  $\epsilon$ ; let us write  $w = B_\epsilon/\sqrt{\epsilon}$  and

$$-b\sqrt{\epsilon} = \min\{\text{Im}(B_t) : 0 \leq t \leq \epsilon\}.$$

Then given  $(B_t, 0 \leq t \leq \epsilon)$ , the remainder of the curve is obtained from the measure  $\epsilon^{-1} \mu_{\mathbb{H}-ib\sqrt{\epsilon}}(0, w\sqrt{\epsilon})$ . As  $\epsilon \rightarrow 0+$ , this is equivalent to  $\epsilon^{-1} \mu_{\mathbb{H}}(ib\sqrt{\epsilon}, (ib+w)\sqrt{\epsilon})$  (this is just an infinitesimal shift of the previous one), which in turn has the same limit as  $\epsilon^{-1} b[b + \text{Im}(w)] \mu_{\mathbb{H}}(i\sqrt{\epsilon}, i\sqrt{\epsilon})$ . Hence (see (11)),

$$\begin{aligned} \lim_{\epsilon \rightarrow 0+} \epsilon^{-1} \mu(0, 0; \epsilon) &= \lim_{\epsilon \rightarrow 0+} \mathbf{E}[b(b + \text{Im}(w))] \epsilon^{-1} \mu_{\mathbb{H}}(i\sqrt{\epsilon}, i\sqrt{\epsilon}) \\ &= \frac{1}{\pi} \mu_{\mathbb{H}}^{\text{bub}}(0). \end{aligned} \quad \square$$

The next proposition is similar. It gives an expression for  $\mu_{\mathbb{H}}^{\text{loop}}$  by associating to an unrooted loop the rooted loop whose root has maximal absolute value.

**Proposition 8.**

$$\mu_{\mathbb{H}}^{\text{loop}} = \frac{1}{\pi} \int_0^\infty \int_0^\pi \mu_{r\mathbb{D}_+}^{\text{bub}}(re^{i\theta}) d\theta r dr.$$

*Proof.* Let

$$\begin{aligned} \mathcal{R} &= \{x + iy : -\infty < x < \infty, 0 < y < \pi\} \\ \mathcal{R}_b &= \{z \in \mathcal{R} : \text{Re}(z) < b\} \end{aligned}$$

and let  $\phi(z) = e^z$ . Conformal invariance tells us that  $\phi \circ \mu_{\mathcal{R}}^{\text{loop}} = \mu_{\mathbb{H}}^{\text{loop}}$ . But Proposition 7 (rotated ninety degrees) and restriction tell us that

$$\mu_{\mathcal{R}}^{\text{loop}} = \frac{1}{\pi} \int_{-\infty}^\infty \int_0^\pi \mu_{\mathcal{R}_x}^{\text{bub}}(x + iy) dy dx.$$

Conformal covariance (12) of  $\mu^{\text{bub}}$  gives  $\phi \circ \mu_{\mathcal{R}_x}^{\text{bub}}(x + iy) = e^{2x} \mu_{e^x\mathbb{D}_+}^{\text{bub}}(e^{x+iy})$ . Therefore

$$\begin{aligned} \mu_{\mathbb{H}}^{\text{loop}} = \phi \circ \mu_{\mathcal{R}}^{\text{loop}} &= \frac{1}{\pi} \int_{-\infty}^\infty \int_0^\pi e^{2x} \mu_{e^x\mathbb{D}_+}^{\text{bub}}(e^{x+iy}) dy dx \\ &= \frac{1}{\pi} \int_0^\infty \int_0^\pi \mu_{r\mathbb{D}_+}^{\text{bub}}(re^{iy}) dy r dr. \end{aligned} \quad \square$$

*Remark.* If we combine this description with the invariance of the unrooted loop measure under the inversion  $z \mapsto -1/z$ , we get that, when  $r \rightarrow 0$ , the measure  $\mu_{\mathbb{H}}^{\text{loop}}$  restricted to those loops that intersect  $r\mathbb{D}_+$  is close to a multiple of the Brownian bubble measure in  $\mathbb{H}$  at the origin.

**5. Bubbles and loops**

The goal of this section is to derive the relation between the Poissonian cloud of loops that intersect a given curve and the Poisson point process of bubbles that we briefly described in the introduction. In order to prove this, we need a clean generalization of the previous remark to shapes other than disks, and to show that the convergence holds uniformly over all shapes.

5.1. Some estimates

We will need some standard estimates about the Poisson kernel on rectangles and half-infinite rectangles, or, more precisely, on the images of these domains under the exponential map.

**Lemma 9.** *There exists a constant  $c$  such that if  $r \in (0, 1/2)$  and  $\theta, \varphi \in (0, \pi)$ ,*

$$|H_{\mathbb{D}_+}(re^{i\theta}, e^{i\varphi}) - \frac{2}{\pi} r \sin \theta \sin \varphi| \leq cr^2 \sin \theta \sin \varphi, \tag{18}$$

$$|H_{\mathbb{H} \setminus \overline{\mathbb{D}_+}}(r^{-1}e^{i\theta}, e^{i\varphi}) - \frac{2}{\pi} r \sin \theta \sin \varphi| \leq cr^2 \sin \theta \sin \varphi. \tag{19}$$

*Proof.* The map  $f(z) = -z - (1/z)$  maps  $\mathbb{D}_+$  onto  $\mathbb{H}$ . Hence

$$\begin{aligned} H_{\mathbb{D}_+}(re^{i\theta}, e^{i\varphi}) &= |f'(e^{i\varphi})| H_{\mathbb{H}}(f(re^{i\theta}), f(e^{i\varphi})) \\ &= 2 \sin \varphi H_{\mathbb{H}}(f(re^{i\theta}), f(e^{i\varphi})). \end{aligned}$$

But if  $|z| \geq 3/2, |x'| \leq 2,$

$$H_{\mathbb{H}}(z, x') = \frac{\text{Im}(z)}{\pi [(\text{Re}(z) - x')^2 + \text{Im}(z)^2]} = \frac{\text{Im}(z)}{\pi |z|^2} [1 + O(\frac{1}{|z|})],$$

and

$$f(re^{i\theta}) = \frac{1}{r} e^{i(\pi-\theta)} + O(r),$$

$$\text{Im}[f(re^{i\theta})] = \frac{\sin \theta}{r} + \sin \theta O(r).$$

This gives the first expression, and the second is obtained from the first using the map  $z \mapsto -1/z$ . □

**Lemma 10.** *There exists a constant  $c$  such that if  $e^{-s} \in (3/4, 1), r \in (0, 1/2),$  and  $\theta, \varphi \in (0, \pi),$  then*

$$|H_{\mathbb{D}_{+,r}}(e^{-s+i\theta}, re^{i\varphi}) - \frac{4}{\pi} \sinh s \sin \theta \sin \varphi| \leq crs \sin \theta \sin \varphi, \tag{20}$$

where  $\mathbb{D}_{+,r} = \{z \in \mathbb{D}_+ : |z| > r\}.$

*Proof.* Separation of variables gives an exact form for the Poisson kernel on a rectangle, and the logarithm maps  $\mathbb{D}_{+,r}$  onto a rectangle. Doing this we see, in fact, that

$$H_{\mathbb{D}_{+,r}}(e^{-s+i\theta}, re^{i\varphi}) = \frac{4}{\pi r} \sum_{n=1}^{\infty} \sin(n\theta) \sin(n\varphi) \sinh(ns) \frac{r^n}{1+r^{2n}},$$

from which the estimate comes easily. □

5.2. Bubble measure and loop measure

Suppose  $V_n$  is a sequence of sets in  $\mathbb{H}$  with  $u_n = \text{rad}(V_n) \rightarrow 0$  and such that  $\mathbb{H} \setminus V_n$  is simply connected. Let  $m_n$  be  $\mu_{\mathbb{H}}^{\text{loop}}$  restricted to loops that intersect both  $V_n$  and the unit circle. We set  $h_n = \text{hcap}(V_n)$ .

**Proposition 11.** *When  $n \rightarrow \infty$ ,*

$$m_n = h_n \mu_{\mathbb{H}}^{\text{bub}}(0, 1)(1 + o(1)),$$

where  $o(1)$  is uniformly bounded by a function of  $u_n$  that goes to zero with  $u_n$ .

Note that scaling implies the corresponding results for the measures restricted to paths that intersect any given circle  $r\partial\mathbb{D}$ .

*Proof.* We will use the decomposition of the measures according to the point at which the loop (or the bubble) has maximal absolute value. Recall that

$$m_n = \frac{1}{2\pi} \int_1^\infty \int_0^\pi \mu_{r\mathbb{D}_+}^{\text{bub}}(re^{i\theta} | V_n) d\theta r dr,$$

where  $\mu_{r\mathbb{D}_+}^{\text{bub}}(re^{i\theta} | V_n)$  denotes  $\mu_{r\mathbb{D}_+}^{\text{bub}}(re^{i\theta})$  restricted to loops that intersect  $V_n$ . Recall also that

$$\mu_{\mathbb{H}}^{\text{bub}}(0, 1) = \int_1^\infty \frac{4}{\pi r^3} \int_0^\pi [\mu_{r\mathbb{D}_+}^\#(0, re^{i\theta}) \oplus \mu_{r\mathbb{D}_+}^\#(re^{i\theta}, 0)] \sin^2 \theta d\theta dr. \tag{21}$$

It is not difficult to show that

$$[\mu_{r\mathbb{D}_+}^{\text{bub}}(re^{i\theta} | V_n)]^\# \rightarrow \mu_{r\mathbb{D}_+}^\#(re^{i\theta}, 0) \oplus \mu_{r\mathbb{D}_+}^\#(0, re^{i\theta}),$$

uniformly on  $\{1 \leq r \leq R\}$  and  $\theta \in (0, \pi)$ . (By decomposing the path at the first visit to  $V_n$ , we can write

$$[\mu_{r\mathbb{D}_+}^{\text{bub}}(re^{i\theta} | V_n)]^\# = \int_{\partial V_n} [\mu_{r\mathbb{D}_+ \setminus V_n}^\#(re^{i\theta}, w) \oplus \mu_{r\mathbb{D}_+}^\#(w, re^{i\theta})] dv_n(w),$$

for some (hard to determine) probability measure  $v_n$  supported on  $\partial V_n \cap \mathbb{H}$ . Since there is a conformal transformation  $g : \mathbb{H} \setminus V_n \rightarrow \mathbb{H}$  with  $\max |g(z) - z| \leq c u_n$ , we can show that  $\mu_{r\mathbb{D}_+ \setminus V_n}^\#(re^{i\theta}, w) \rightarrow \mu_{r\mathbb{D}_+}^\#(re^{i\theta}, 0)$  and  $\mu_{r\mathbb{D}_+}^\#(w, re^{i\theta}) \rightarrow \mu_{r\mathbb{D}_+}^\#(0, re^{i\theta})$  uniformly over  $w \in V_n$ .)

We now focus on the total masses. We claim that

$$|\mu_{\mathbb{D}_+}^{\text{bub}}(e^{i\theta} | V_n)| = 4 h_n \sin^2 \theta [1 + O(u_n)]. \tag{22}$$

By the scaling rules for  $\mu^{\text{bub}}$  and  $\text{hcap}$  this implies that for all  $r \geq 1$ ,

$$|\mu_{r\mathbb{D}_+}^{\text{bub}}(re^{i\theta} | V_n)| = 4 r^{-4} h_n \sin^2 \theta [1 + O(u_n)],$$

and the proposition follows, using (21).

To prove (22), we first note that

$$|\mu_{\mathbb{D}_+}^{\text{bub}}(e^{i\theta} | V_n)| = \lim_{\epsilon \rightarrow 0^+} \frac{\pi}{\epsilon} \mu_{\mathbb{D}_+}(\exp(-\epsilon + i\theta), \exp(i\theta))[B \text{ hits } V_n].$$

Define for all  $u$  the hitting time  $\sigma_u$  of the circle of radius  $u$  by  $B$ , and the exit time  $T$  of  $\mathbb{H}$  by  $B$ .

The estimates (19) and (20) show that the following two measures are very close (for all large  $R$ , and  $r$  is small):

- The measure of  $1_{\sigma_u < T} \arg(B_{\sigma_u})$  when  $B$  is defined under the measure  $R\mathbf{P}^{iR}$ .
- The measure of  $1_{\sigma_u < T, \sigma_u < \sigma_1} \arg(B_{\sigma_u})$  when  $B$  is defined under the measure

$$(2 \sin \theta \sinh(\epsilon))^{-1} \mathbf{P}^{(1-\epsilon) \exp(i\theta)}.$$

After these hitting times, it is possible to “couple” the two paths up to their first hitting time  $\rho_{V_n}$  of  $V_n$ . After the hitting of  $V_n$ , we want to estimate the probability that the path go back to the unit circle without hitting  $\mathbb{R}$  and that they hit it in the neighborhood of  $\exp(i\theta)$ . By (18), this will occur with a probability

$$\frac{2}{\pi} \text{Im}(B_{\rho_{V_n}}) \sin \theta d\theta.$$

Hence, we get finally that (recall that the estimates are uniform in  $\theta$ ,  $\epsilon$  and  $R$ )

$$\begin{aligned} |\mu_{\mathbb{D}_+}^{\text{bub}}(e^{i\theta} | V_n)| &\sim \lim_{\epsilon \rightarrow 0, R \rightarrow \infty} \frac{2\pi}{\epsilon} R \sin \theta \sinh \epsilon \mathbf{E}^{iR} \left[ \frac{2}{\pi} \text{Im}(B_{\rho_{V_n}}) \sin \theta \right] \\ &\sim 4 \sin^2 \theta \lim_{R \rightarrow \infty} R \mathbf{E}^{iR} [\text{Im}(B_{\rho_{V_n}})] \\ &\sim 4h_n \sin^2 \theta \end{aligned}$$

when  $u_n \rightarrow 0$ . □

### 5.3. Bubble soup and loop soup

We define a *bubble soup* with intensity  $\lambda \geq 0$  to be a Poisson point process with intensity  $\lambda \mu_{\mathbb{H}}^{\text{bub}}$ . One can also view it as a Poissonian sample from the measure  $\lambda \mu_{\mathbb{H}}^{\text{bub}}(0) \times (\text{length})$  on  $\mathcal{K}_0^0(\mathbb{H}) \times [0, \infty)$ . We can write a realization of the bubble soup as a countable collection  $\mathcal{U} = \{(\gamma_j, s_j)\}$ . Recall that the law of  $\mathcal{U}$  is characterized by the fact that:

- For any two disjoint measurable subsets  $U_1$  and  $U_2$  of  $\mathcal{K}_0^0(\mathbb{H}) \times [0, \infty)$ ,  $\mathcal{U} \cap U_1$  and  $\mathcal{U} \cap U_2$  are independent.
- The law of the number of elements in  $\mathcal{U} \cap U$  is the Poisson law with mean  $\lambda \mu_{\mathbb{H}}^{\text{bub}}(0) \times (\text{length})[U]$  (when this quantity is finite).

We will think of the bubble  $\gamma_j$  as being created at time  $s_j$ . Clearly, with probability one  $s_j \neq s_k$  for  $j \neq k$ .

A *Brownian loop soup* with intensity  $\lambda$  is a Poissonian sample from the measure  $\lambda \mu^{\text{loop}}$ . We will use  $\mathcal{L}_{\mathbb{C}}$  to denote a realization of the loop soup. A sample of the

Brownian loop soup is a countable collection of (unrooted) Brownian loops in the plane. We will use  $\mathcal{L}$  to denote the family of loops in  $\mathcal{L}_{\mathbb{C}}$  that are in  $\mathbb{H}$ . This is the Brownian loop soup in the half-plane.

If  $D \subset \mathbb{H}$  is a domain, then we write

- $\mathcal{L}(D)$  for the family of loops in  $\mathcal{L}$  that are in  $D$
- $\mathcal{L}^{\perp}(D)$  for  $\mathcal{L} \setminus \mathcal{L}(D)$ , i.e., the family of loops that intersect  $\mathbb{H} \setminus D$ .

By definition, for any fixed  $D$ , the two random families  $\mathcal{L}(D)$  and  $\mathcal{L}^{\perp}(D)$  are independent.

Note that the (law of the) families  $\mathcal{L}(D)$  inherit the conformal invariance and restriction properties of the Brownian loop measure.

Now suppose that  $\eta : [0, \infty) \rightarrow \mathbb{C}$  is a simple curve with  $\eta(0, \infty) \subset \mathbb{H}$  and that  $\text{hcap}[\eta(0, \infty)] = \infty$  (this conditioned can in fact be weakened). Assume that  $\eta$  is parametrized by capacity, i.e., that  $\text{hcap}[\eta[0, t]] = 2t$ . Let  $H_t = \mathbb{H} \setminus \eta[0, t]$  and let  $g_t$  be the unique conformal transformation of  $H_t$  onto  $\mathbb{H}$  such that  $g_t(\eta(t)) = 0$  and  $g_t(z) \sim z$  as  $z \rightarrow \infty$ . We let  $f_t = g_t^{-1}$  which maps  $\mathbb{H}$  conformally onto  $H_t$  with  $f_t(0) = \eta(t)$ .

Given a realization  $\mathcal{U}$  of the bubble soup, consider the set of loops

$$\mathcal{U}_{\eta,t} = \{f_{s_j} \circ \gamma_j : (\gamma_j, s_j) \in \mathcal{U}, s_j \leq t\}.$$

We consider this as realization of *unrooted* loops by forgetting the root.

**Theorem 12.** *For every  $t < \infty$ , if  $\mathcal{U}$  is a bubble soup with intensity  $\lambda > 0$ , then  $\mathcal{U}_{\eta,t}$ , considered as a collection of unrooted loops, is a realization of  $\mathcal{L}^{\perp}(H_t)$  with intensity  $\lambda$ .*

It is useful to consider this theorem in the other direction, i.e. to see that it is equivalent to Theorem 1. Let  $\mathcal{L}$  be a realization of the loop soup in  $\mathbb{H}$  with intensity  $\lambda$ . We write elements of  $\mathcal{L}$  as  $[\gamma]$  since they are equivalence classes of loops. We write  $V_{\gamma}$  for the hull generated by  $[\gamma]$ , i.e.  $V_{\gamma}$  is the complement of the unbounded component of  $\mathbb{C} \setminus \gamma[0, t_{\gamma}]$  (this does not depend on the choice of representative of  $[\gamma]$ ). Let  $\eta$  be as before and let us write  $\mathcal{L}^{\perp} = \{[\gamma_1], [\gamma_2], \dots\}$  for  $\mathcal{L}^{\perp}(\mathbb{H} \setminus \eta[0, \infty))$ , i.e., for the set of loops in  $\mathcal{L}$  that intersect  $\eta[0, \infty)$ . For every  $[\gamma_j] \in \mathcal{L}$ , let  $r_j$  denote the smallest  $r$  such that  $\eta(r) \in \gamma[0, t_{\gamma}]$ . Note that this does not depend on which representative  $\gamma_j$  of  $[\gamma_j]$  that we choose.

Let us now briefly justify the fact that with probability one, for each  $j$  there is a unique representative of  $[\gamma_j]$ , which we write as just  $\gamma_j$ , such that  $\gamma_j(0) = \eta(r_j)$  and  $\gamma_j(0, t_{\gamma_j}) \subset H_{r_j}$ . It follows for instance readily from the fact that if  $B$  is a Brownian bridge (from  $z$  to  $w$  in time  $t$ ), conditioned to stay in  $\mathbb{H}$ , then for each rational  $0 < q_1 < q_2 < t$ , if one defines the first time  $s(q_1)$  at which  $\eta$  hits  $B[0, q_1]$ , then

$$\mathbf{P}\{s(q_1) < \infty; \eta(s(q_1)) \in B[q_2, t]\} = 0,$$

since complex Brownian motion does not hit points.

From now on we consider  $[\gamma_j]$  as a rooted loop by choosing this representative  $\gamma_j$ . Note that this choice depends on  $\eta$ . The set of times  $\mathcal{T} = \{r_j : \gamma_j \in \mathcal{L}^{\perp}\}$  is

countable and dense in  $[0, \infty)$  since with probability one for each rational  $t$  there exists loops in  $\mathcal{L}$  of arbitrarily small diameter surrounding  $\eta(t)$ . Also,  $r_j \neq r_k$  if  $j \neq k$ . We let  $\mathcal{L}_t^\perp$  denote the set of  $\gamma_j \in \mathcal{L}$  with  $r_j \leq t$ , i.e., the set of loops that intersect  $\eta[0, t]$ . Recall that if  $t < t_1$ , then  $\mathcal{L}_t^\perp$  and  $\mathcal{L}_{t_1}^\perp \setminus \mathcal{L}_t^\perp$  are independent.

*Proof.* For  $r > 0$ , let  $\mathcal{L}_r^\perp$  denote the set of  $\gamma_j \in \mathcal{L}_t^\perp$  such that  $\text{rad}[g_{r_j} \circ \gamma_j] := \sup\{|g_{r_j} \circ \gamma_j(s)| : 0 \leq s \leq t_{g_{r_j} \circ \gamma_j}\} \geq r$ . Note that with probability one  $\mathcal{L}_r^\perp$  is finite for each  $t < \infty$  and  $r > 0$ . It suffices to show that for every  $r > 0$  the set of loops

$$\{g_{r_j} \circ \gamma_j : \gamma_j \in \mathcal{L}_r^\perp\}$$

is a Poissonian realization of the measure  $\lambda t \mu_{\mathbb{H}}^{\text{bub}}(0; r)$ . We only need to do this for the case  $r = 1$ , as the other cases are essentially the same. Let  $\mathcal{A}_t = \{g_{r_j} \circ \gamma_j : \gamma_j \in \mathcal{L}_t(1)^\perp\}$ . We have already noted that for  $\epsilon > 0$ ,  $\mathcal{A}_{t+\epsilon} \setminus \mathcal{A}_t$  is independent of  $\mathcal{A}_t$ . If  $t > 0$ , the curve  $\eta^t(s) = g_t[\eta(t + s)]$ ,  $0 \leq s < \infty$ , is also a simple curve parametrized by capacity. Conformal invariance of  $\mu^{\text{loop}}$  tells us that the distribution of  $g_t \circ [\mathcal{L}_{t+\epsilon}^\perp \setminus \mathcal{L}_t^\perp]$ , derived from the curve  $\eta$ , is the same as the distribution of  $\mathcal{L}_\epsilon^\perp$  derived from the curve  $\eta^t$ . Hence it suffices to prove the two conditions above for  $t = 0$ . But this is the estimate that was done in §5.2 so we have the result.  $\square$

This implies (with (13)) in particular immediately the following fact: Suppose that  $D \subset \mathbb{H}$  is simply connected, and that the curve  $\eta(0, T] \subset D$  is parametrized as before. Define  $D_t = g_t(D)$  (where  $g_t$  is the conformal map from  $\mathbb{H} \setminus \eta[0, t]$  onto  $\mathbb{H}$  with  $g_t(z) \sim z$  at infinity, and  $g_t(\eta_t) = O$ ). As in (13), define also a conformal map  $\phi_t$  from  $D_t$  onto  $\mathbb{H}$  that fixes the origin. Then,

$$\mathbf{P}[\forall \gamma \in \mathcal{L} : \gamma \cap \eta[0, T] = \emptyset \text{ or } \gamma \subset D] = \exp\left(\lambda \int_0^T \frac{S_{\phi_t}(O)}{6} dt\right). \tag{23}$$

### 5.4. Parametrization

Suppose  $\eta : [0, \infty) \rightarrow \mathbb{C}$  is a curve as before, and let  $\mathcal{L}_t^\perp$  denote a realization of the loop soup in  $\mathbb{H}$  restricted to curves that intersect  $\eta[0, t]$ . We are going to show that if the path  $\eta[0, \infty)$  has dimension strictly less than two, then the sum of all the time-lengths of the loops in  $\mathcal{L}_t^\perp$  is almost surely finite. This will imply that one can construct a continuous path by attaching these loops ‘‘chronologically’’ to  $\eta$ .

**Lemma 13.** *Suppose that for some  $\epsilon > 0$  and  $T > 0$ ,*

$$\lim_{\delta \rightarrow 0^+} \delta^{-\epsilon} \text{area}(\{z : \text{dist}[z, \eta[0, T]] \leq \delta\}) = 0. \tag{24}$$

*Then with probability one,*

$$\sum_{\gamma \in \mathcal{L}_T^\perp} t_\gamma < \infty.$$

*Proof.* Fix  $T, \epsilon$ , and let  $r = \text{rad}(\eta[0, T]) < \infty$ . Constants in this proof may depend on  $T, r, \epsilon$ . It suffices to prove two facts:

$$\#\{\gamma \in \mathcal{L}_T^\perp : t_\gamma > 1\} < \infty \text{ a.s., and } \mathbf{E}\left[\sum_{\gamma \in \mathcal{L}_T^\perp} t_\gamma 1_{t_\gamma \leq 1}\right] < \infty.$$

Note that the first one is equivalent to

$$\mu_{\mathbb{H}}^{\text{loop}}[\{\gamma \in \mathcal{L}_T^\perp : t_\gamma > 1\}] < \infty.$$

But on the one hand

$$\begin{aligned} \mu^{\text{loop}}[\{\gamma : \gamma \subset 2r\mathbb{D} : t_\gamma > 1\}] &= \int_{2r\mathbb{D}} \int_1^\infty dA(z) \frac{dt}{2\pi t^2} \mu^\#(z, z; t) \\ &\quad [\{\gamma : \gamma \subset 2r\mathbb{D}\}] \\ &\leq \frac{A(2r\mathbb{D})}{2\pi} < \infty. \end{aligned}$$

On the other hand,

$$\begin{aligned} \mu_{\mathbb{H}}^{\text{loop}}[\{\gamma : t_\gamma > 1, \gamma \not\subset 2r\mathbb{D}, \gamma \cap r\mathbb{D} \neq \emptyset\}] \\ \leq \frac{1}{2\pi} \int_0^r \int_0^\pi \mu_{u\mathbb{D}_+}^{\text{bub}}(u \exp(i\theta)) [\{\gamma : \gamma \not\subset 2r\mathbb{D}\}] du d\theta. \end{aligned}$$

It is easy (using conformal invariance) to see that  $\mu_{u\mathbb{D}_+}^{\text{bub}}(u \exp(i\theta)) [\{\gamma \not\subset 2r\mathbb{D}\}]$  is bounded independently from  $u \leq r$  and  $\theta \in [0, \pi]$ . Hence, the last displayed expression is finite, which completes the proof of the fact that the number of loops in  $\mathcal{L}$  of length greater than one and that do intersect  $\eta[0, T]$  is almost surely finite.

Note that

$$\mathbf{E}\left[\sum_{\gamma \in \mathcal{L}_T^\perp} t_\gamma 1_{t_\gamma \leq 1}\right] = \int_{\mathbb{H}} \int_0^1 |\tilde{\mu}_{\mathbb{H}}(z, z; t)| dt dA(z),$$

where  $\tilde{\mu}_{\mathbb{H}}(z, z; t)$  denotes  $\mu_{\mathbb{H}}(z, z; t)$  restricted to loops that intersect  $\eta[0, T]$  (the  $t_\gamma^{-1}$  in the definition of the loop measure cancels with the  $t_\gamma$  in the expression on the left hand side). Let

$$F(z) = \int_0^1 |\tilde{\mu}_{\mathbb{H}}(z, z; t)| dt,$$

and let  $d_z = \text{dist}(z, \eta[0, T])$ . It is standard to see that there exist constants  $c, a$  such that

$$|\tilde{\mu}_{\mathbb{H}}(z, z; t)| \leq c t^{-1} e^{-a d_z^2/t}.$$

Hence, we get  $F(z) \leq c \log(1/d_z)$  for small  $d_z$ , and

$$\text{area}\{z : F(z) \geq s\} \leq \text{area}\{z : \text{dist}(z, \eta[0, T]) \leq e^{-s/c}\} \leq e^{-s\epsilon/c}. \tag{25}$$

Also we get  $F(z) \leq ce^{-a|z|^2}$  for  $|z| \geq 3r$ , so that finally,  $\int F(z) dA(z) < \infty$ .  $\square$

*Remark.* From (25) we can see that (24) can be weakened to

$$\text{area}\{z : \text{dist}(z, \eta[0, T]) \leq e^{-s}\} \leq g(s),$$

where  $\int_1^\infty g(s) ds < \infty$ . However, if  $\eta$  is space-filling, then the result does not hold as the following shows:

**Proposition 14.** *If  $D$  is any nonempty open domain, then  $\sum_{\gamma \in \mathcal{L}(D)} t_\gamma = \infty$  almost surely.*

*Proof.* Note first that

$$\mathbf{E}\left[\sum_{\gamma \in \mathcal{L}(D)} t_\gamma\right] = \infty.$$

This can be seen easily from the scaling rule

$$\mathbf{E}\left[\sum_{\gamma \in \mathcal{L}(rD)} t_\gamma\right] = r^2 \mathbf{E}\left[\sum_{\gamma \in \mathcal{L}(D)} t_\gamma\right].$$

For example, if  $D$  is a square, we can divide  $D$  into 4 squares of half the side length,  $D_1, \dots, D_4$ . The scaling rule tells us that

$$\mathbf{E}\left[\sum_{\gamma \in \mathcal{L}(D)} t_\gamma\right] = \sum_{j=1}^4 \mathbf{E}\left[\sum_{\gamma \in \mathcal{L}(D_j)} t_\gamma\right]. \tag{26}$$

But

$$\sum_{\gamma \in \mathcal{L}(D)} t_\gamma = \sum_{j=1}^4 \sum_{\gamma \in \mathcal{L}(D_j)} t_\gamma + \sum_{\mathcal{L}(D) \setminus [\mathcal{L}(D_1) \cup \dots \cup \mathcal{L}(D_4)]} t_\gamma.$$

Since the last term has strictly positive expectation, the expectations in (26) must be infinite.

Furthermore (by dividing the square into  $2^m$  smaller squares),  $\sum_{\gamma \in \mathcal{L}(D)} t_\gamma$  is larger than the mean of the values of  $2^m$  independent copies of itself (i.e. the same random variable with infinite expectation) for any  $m$ . The result follows.  $\square$

With Lemma 13 we can give a Brownian parametrization to the curve “ $\eta$  with the loops added.” Let  $\mathcal{L}$  be a realization of the Brownian loop soup, and let  $\{[\gamma_1], [\gamma_2], \dots\}$  be the (unrooted) loops that intersect  $\eta[0, \infty)$ . As before, choose  $r_j$  and representative  $\gamma_j$  so that  $\gamma_j(0) = \eta(r_j)$  and  $\gamma_j[0, t_{\gamma_j}] \cap \eta[0, r_j) = \emptyset$ . Define

$$S(r-) = \sum_{r_j < r} t_{\gamma_j}, \quad S(r+) = \sum_{r_j \leq r} t_{\gamma_j}.$$

Then  $S(r)$  is an increasing function with jumps at  $r_j$  of size  $t_{\gamma_j}$ . Define the process  $Y_s$  by

$$Y_{S(r-)} = \eta(r),$$

and if  $S(r-) < S(r+)$ ,

$$Y_{S(r-)+s} = \gamma_j(s), \quad 0 \leq s \leq t_\gamma.$$

The density of the loop-soup implies readily that  $t \mapsto Y_t$  is continuous (provided  $\eta$  is a simple curve for instance).

The results of [7] strongly suggest that the following conjecture holds.

**Conjecture 1.** *If the curve  $\eta$  is chordal  $SLE_2$ , and  $\lambda = 1$ , then the law of  $Y$  is  $\mu_{\mathbb{H}}^\#(0, \infty)$ .*

There seem to be different possible ways to prove this. One can use the convergence of loop-erased random walk to the  $SLE_2$  curve [7]. The main missing step is the convergence of discrete bubbles towards the Brownian bubbles.

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