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Comparison theorem and estimates for transition probability densities of diffusion processes

In memory of Professor Paul-Andre Meyer

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Abstract. We establish several comparison theorems for the transition probability density $p_b(x, t, y)$ of Brownian motion with drift b, and deduce explicit, sharp lower and upper bounds for $p_b(x, t, y)$ in terms of the norms of the vector field b. The main results are obtained through carefully estimating the mixed moments of Bessel processes. All constants are explicit in our lower and upper bounds, which is different from most of the previous estimates, and is important in many applications for example in statistical inferences for diffusion processes.

1. Introduction

Let $(X_t, \mathbf{P}^x)$ be the weak solution to the stochastic differential equation

$$dX_t = b(X_t)dt + dB_t \ , \quad X_0 = x \tag{1}$$

where b is a bounded vector field on $\mathbf{R}^n$, and $(B_t)_{t\geq 0}$ is a standard n-dimensional Brownian motion. Under standard coordinates $(x^1, \cdots, x^n)$

$$b(x) = \sum_{i=1}^{n} b^i(x)\frac{\partial}{\partial x^i}$$

and, unless otherwise specified, $|x|$ denotes the Euclidean norm $\sqrt{\sum_i |x^i|^2}$.

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The Markov generator associated with the diffusion process $(X_t, \mathbf{P}^x)$ is

$$L = \frac{1}{2}\Delta + b .$$

Denote by $p_b(x,t,z)$ the transition probability density (with respect to the Lebesgue measure), that is, the minimum fundamental solution to the parabolic equation

$$\left(L - \frac{\partial}{\partial t}\right) u(t,x) = 0 , \quad \text{on } (0,+\infty) \times \mathbf{R} .$$

There have been many results about lower and (sharp) upper bounds for the transition probability density $p_b(x,t,z)$ (see for example Aronson [2], Davies [3], Norris-Stroock [11] and the literature therein). When b is not a gradient of a function, L is not symmetrical, and most of the existing estimates for such a non-symmetrical operator are good essentially only for small time. As far as we know, the sharp lower bound is not known even when L is symmetrical. The main goal of the present work is to establish a general comparison theorem for the transition density functions of Brownian motion with drifts (see Theorem 4 below). As a consequence we deduce sharp upper and lower bounds for $p_b(x,t,y)$ under various conditions on the drift vector field b. For one dimensional Brownian motion with drift, a comparison theorem for $p_b(x,t,y)$ has been established in [14]. Indeed it was proved in [14] that the optimal bounds in this one dimensional case are given by

$$H_{-\beta,y}(x,t,y) \le p_b(x,t,y) \le H_{\beta,y}(x,t,y)$$

where $H_{\beta,y}(x,t,z)$ is the transition probability density of the diffusion with two-valued drift

$$d\xi_t = -\beta \operatorname{sgn}(\xi_t - y)dt + dB_t$$

and

$$H_{\beta,y}(x,t,y) = \frac{1}{\sqrt{2\pi t}} \int_{|x-y|/\sqrt{t}}^{\infty} z \mathrm{e}^{-(z-\beta\sqrt{t})^2/2} dz . \tag{2}$$

The latter formula was found in [14], see also [4], [5].

The same approach applies to the multi-dimensional case, and thus leads to the following

Theorem 1. *Let $b = \sum_{i=1}^n b^i \frac{\partial}{\partial x^i}$ be a bounded vector field on $\mathbf{R}^n$. Suppose*

$$|b^i(x)| \le \beta_i \qquad \text{for all } x \in \mathbf{R}^n \tag{3}$$

for some non-negative constants β_i, $i = 1, \cdots, n$, then

$$\frac{1}{(2\pi t)^{n/2}} \prod_{i=1}^n \left(\int_{|x^i-y^i|/\sqrt{t}}^{\infty} z e^{-(z+\beta_i\sqrt{t})^2/2} dz \right) \le p_b(x,t,y) \tag{4}$$

and

$$p_b(x,t,y) \le \frac{1}{(2\pi t)^{n/2}} \prod_{i=1}^n \left(\int_{|x^i-y^i|/\sqrt{t}}^{\infty} z e^{-(z-\beta_i\sqrt{t})^2/2} dz \right) . \tag{5}$$

Proof. Given a vector $\beta = (\beta_i) \in \mathbf{R}^n$, consider the following diffusion process with two-valued drift

$$dX_t = -S_\beta(X_t, y)dt + dB_t \,, \quad X_0 = x \,, \tag{6}$$

where $S_\beta(\cdot, y)$ denotes the vector field

$$S_\beta(x, y) = \sum_{i=1}^{n} \beta_i \operatorname{sgn}\left(x^i - y^i\right) \frac{\partial}{\partial x^i} \,.$$

The only observation is that the components of the above diffusion (6) are independent, and therefore

$$H_{\beta,y}(x, t, y) = \frac{1}{(2\pi t)^{n/2}} \prod_{i=1}^{n} \left(\int_{|x^i - y^i|/\sqrt{t}}^{\infty} z e^{-(z-\beta_i\sqrt{t})^2/2} dz \right)$$

where $H_{\beta,y}$ is the transition probability density of the above diffusion process. It follows from the Cameron-Martin formula (see [14] for detail) that

$$H_{-|b|,y}(x, t, y) \leq p_b(x, t, y) \leq H_{|b|,y}(x, t, y)$$

which in turn yields (4, 5). □

Let us first point out that the bounds in Theorem 1 are optimal under condition (3), as for any fixed y the lower and upper bounds for $p_b(x, t, y)$ are achieved when

$$b(x) = \pm \sum_{i=1}^{n} \beta_i \operatorname{sgn}(x^i - y^i) \frac{\partial}{\partial x^i}$$

respectively. In particular for any fixed x and y, the leading term in large t in both upper and lower bounds are the best possible, coincide with those of Brownian motion with the above extremal drifts, and thus eliminate the additional exponential terms in most results in this respect in literature. For other results concerning large time asymptotic of $p_b(x, t, y)$ under different assumptions on b, see the interesting paper Norris [10]. Another remark is that both condition (3) and the bounds in Theorem 1 depend on the choice of a coordinate system for the Euclidean space $\mathbf{R}^n$, while the density function $p_b(x, t, y)$ does not, and thus, in order to achieve the best possible estimates within the setting of Theorem 1, one should select a orthonormal coordinate system for $\mathbf{R}^n$ so that the non-negative constants β_i are the least possible.

In terms of Euclidean norms and for large distance, strong and explicit lower and upper bounds are provided in the following

Theorem 2. *Suppose $n \geq 2$. Let*

$$||b|| = \sup_{x \in \mathbf{R}^n} \sqrt{\sum_{i=1}^{n} |b^i(x)|^2} \,.$$

Let $p_b(x,t,y)$ be the transition probability density of the Brownian motion with bounded drift b. Then we have the following lower bound

$$p_b(x,t,y) \geq \frac{1}{(2\pi t)^{n/2}} e^{-\frac{(|x-y|+||b||t)^2}{2t} - ||b||(\zeta_n \sqrt{t} + |x-y|)} \tag{7}$$

where

$$\zeta_n = 2\left(a_{n-1,1} + a_{1,1}\right)$$

and

$$a_{n,p} = \frac{1}{(2\pi)^{n/2}} \int_{\mathbf{R}^n} |x|^p e^{-\frac{|x|^2}{2}} dx\ . \tag{8}$$

For upper bound, for every $1 < q < \frac{n}{n-1}$

$$\begin{aligned} p_b(x,t,y) \leq{} & \frac{1}{(2\pi t)^{n/2}} e^{-\frac{|x-y|^2}{2t}} \\ & \times \left(1 + c_{n,q} ||b|| \left(\kappa_q \sqrt{t} + 2|x-y|\right) e^{\frac{q-1}{2qt}|x-y|^2 + \frac{||b||^2}{2(q-1)} t}\right) \end{aligned} \tag{9}$$

for all x, y and $t > 0$, where

$$c_{n,q} = \frac{2\sqrt{2}q}{n - q(n-1)}\,, \quad \kappa_q = \sqrt[2q]{a_{n-1,q} + 2^{2q-1} a_{1,q}}\ .$$

Besides the interesting fact that both upper and lower bounds contain no unknown constants, the estimates (7, 9) are particularly good for large distance (and for small time as well) as the leading term involving $|x-y|^2$ is exactly the Gaussian decay term. This is obvious for the lower bound, while for the upper bound, thanks to the explicit form of the constant $c_{n,q}$ and the fact that κ_q is bounded in q (as $a_{n,,q}$ are Gaussian moments), one can minimize the expression

$$c_{n,q} \mathrm{e}^{\frac{q-1}{2qt}|x-y|^2 + \frac{||b||^2}{2(q-1)} t}$$

over the range $q \in (1, n/n-1)$ which leads to a bound (though very complicated) with linear term $|x-y|$ in the exponential. Therefore the leading term for large distance in (9) is thus exactly the Gaussian term.

The paper is organized as the following. In Section 2, we prove a comparison theorem (see Theorem 4 below) for the transition probability density $p_b(x,t,y)$ of Brownian motion with drift b to that of a diffusion process with two-valued drift. Let us point out that the heuristics behind such a comparison theorem are obvious from a probabilistic point of view, and the idea has been used in some stochastic control problems. Indeed the present work was motivated by a question of estimating drift parameters of diffusions, in which precise information for $p_b(x,t,y)$ is needed in order to control the convergence rate in terms of the bound of b, for some background in this respect, see [13]. We should also mention that Theorem 4 may be proved by applying the maximum principle for parabolic equations. Although

Theorem 4 is very simple, we are unable to find such a statement in the existing literature. Theorem 4 reduces the problem of estimating $p_b(x, t, y)$ to a problem of explicitly computing some distributions. In particular we prove another comparison theorem (see Corollary 6) for $p_b(x, t, y)$ to that of the Bessel process with a constant drift. In Section 3, by using the comparison theorem, we present a proof of Theorem 2, which in turn relies on careful estimates for some mixed moments of the Bessel process, which we believe have interests of their own.

2. Comparison theorem

We begin with the following

Lemma 3. *Let $y \in \mathbf{R}^n$ be a fixed point, and let $b(x) = \nabla U(|x - y|)$ for some smooth function U on $\mathbf{R}^+$ such that U' is of at most linear growth. Then for all $x \neq y$, $\nabla_x p_b(x, t, y)$ exists and*

$$\nabla_x p_b(x, t, y).\nabla|x - y| \leq 0 .$$

Proof. The diffusion process on $\mathbf{R}^n$ with generator

$$L = \frac{1}{2}\Delta + \nabla U(| \cdot - y|) ,$$

is symmetric with respect to the measure $e^{2U(|x-y|)}dx$. Its symmetric transition density function is denoted by $q_b(x, t, z)$. Then

$$p_b(x, t, z) = q_b(x, t, z)e^{2U(|z-y|)}$$

and in particular

$$p_b(x, t, y) = q_b(x, t, y)e^{2U(0)} .$$

Since the coefficients of L are symmetrical with respect to the center y , its density function $p_b(x, t, y)$ has the same symmetrical center, and therefore $p_b(x, t, y)$ depends on x through $|x-y|$. Thus we may write $p_b(x, t, y) = f(t, |x-y|)$, where f is a continuous function on $(0, \infty) \times \mathbf{R}^+$, as it is well known that $p_b(x, t, y)$ is Hölder continuous [9]. For simplicity, we only consider the case where $n \geq 2$. The case $n = 1$ may be proved similarly by smoothing technique.

It is well known that $q_b(x, t, y) = q_b(y, t, x)$ satisfies (as a function in x and t)

$$\frac{\partial}{\partial t}q_b(x, t, y) = \frac{1}{2}\Delta q_b(x, t, y) + \nabla U(|x - y|).\nabla q_b(x, t, y)$$

in distribution sense, so that

$$\frac{\partial}{\partial t}f(t, r) = \frac{1}{2}\left(\frac{\partial^2}{\partial r^2}f(t, r) + \frac{n-1}{r}\frac{\partial}{\partial r}f(t, r)\right) + U'(r)\frac{\partial}{\partial r}f(t, r) \qquad (10)$$

in distribution sense on $(0,\infty)\times(0,\infty)$. By the standard elliptic theory (see [7]), one may conclude that f is smooth and is a classical solution to (10) as U is smooth. It thus follows that $\nabla_x p_b(x,t,y)$ exists for $x\neq y$. Set for $r>0$,

$$m(r)=\exp\int_1^r 2\left(\frac{1}{2}\frac{n-1}{s}+U'(s)\right)ds\ .$$

Then (10) can be rewritten as

$$\frac{\partial}{\partial t}f(t,r)=\frac{1}{2}\frac{1}{m(r)}\frac{\partial}{\partial r}\left(m(r)\frac{\partial}{\partial r}f(t,r)\right)\ . \tag{11}$$

Let $u(t,r)=\frac{\partial}{\partial r}f(t,r)$ and take derivatives on the both sides of (11). We obtain

$$\frac{\partial}{\partial t}u(t,\ r)=\frac{\partial}{\partial r}\left[\frac{1}{2}\frac{1}{m(r)}\frac{\partial}{\partial r}\left(m(r)u(t,r)\right)\right]$$

or

$$\frac{\partial}{\partial t}\left(m(r)u(t,\ r)\right)=\frac{1}{2}m(r)\frac{\partial}{\partial r}\left[\frac{1}{m(r)}\frac{\partial}{\partial r}\left(m(r)u(t,r)\right)\right]$$

on $(0,\infty)\times(0,\infty)$. It will follow from the maximum principle that $u(t,r)$ is negative and therefore

$$\nabla_x p_b(x,t,y).\nabla|x-y|\leq 0 \qquad \text{for all } x\neq y\ ,$$

if we can show that for $t>0$ small enough $u(t,\cdot)$ is negative on $(0,\infty)$. The latter claim however comes from the standard off-diagonal asymptotic for heat kernels for small time t (see for example [1]), if our vector field b is smooth. That is, for $t>0$ small enough, the leading term of $\log p_b(x,t,y)$ is the same as that of Gaussian kernel, and the same claim is true for the derivatives of $p_b(x,t,y)$ as well, which in turn implies that $u(t,r)<0$ for $t>0$ small enough. Unfortunately in general the vector field $\nabla U(|x-y|)$ is not smooth. However if we write $U(r)$ as $V(r^2)$, and if V is smooth, then the vector field b will be smooth. Therefore by smoothly approximating the function $U(\sqrt{\cdot})$ we thus can conclude that $\lim_{t\downarrow 0}u(t,r)=-\infty$ for all $r>0$. We therefore complete the proof. □

Consider the following two stochastic differential equations

$$dX_t=b(X_t,t)dt+dB_t\ ,\quad X_0=x$$

and

$$d\tilde{X}_t=\tilde{b}(\tilde{X}_t,t)dt+dB_t\ ,\quad X_0=x$$

where b and $\tilde{b}$ are two measurable vector fields on $\mathbf{R}^n\times\mathbf{R}_+$, $(B_t)_{t\geq 0}$ and $(\tilde{B}_t)_{t\geq 0}$ are standard n-dimensional Brownian motion.

Denote by $p_b(x,t,y)$ and $p_{\tilde{b}}(x,t,y)$ their transition probability density functions with respect to Lebesgue measure.

Theorem 4. *Given* $y \in \mathbf{R}^n$. *If there is a smooth function* U *on* $\mathbf{R}^+$ *such that* $b(x) = \nabla U(|x-y|)$ *(when* $|x-y|>0$*), then we have the following two statements:*

1) if $(\tilde{b}(x,t) - b(x,t), \nabla_x |x-y|) \geq 0$ *for all* (x,t) *such that* $|x-y|>0$, *then*

$$p_{\tilde{b}}(x,t,y) \leq p_b(x,t,y) \quad \text{for all } (x,t)\,;$$

2) if $(\tilde{b}(x,t) - b(x,t), \nabla_x |x-y|) \leq 0$ *for all* (x,t) *such that* $|x-y|>0$, *then*

$$p_{\tilde{b}}(x,t,y) \geq p_b(x,t,y) \quad \text{for all } (x,t).$$

Proof. Let us only prove 1). The proof of 2) is similar. It is sufficient to show the case where b and $\tilde{b}$ are bounded and smooth. The general case can be proved by approximation. From Lemma 3, it is easy to see that $\nabla \log p_b(x, t-s, y)$ is parallel to $-\nabla_x|x-y|$. Thus, by Cameron-Martin's formula (see [8], [14])

$$\begin{aligned}
&\frac{p_{\tilde{b}}(x,t,y)}{p_b(x,t,y)} \\
&\quad = \mathbf{E}\exp\left(\int_0^t (\tilde{b}(X_s,s) - b(X_s,s))\nabla \log p_b(X_s, t-s, y)ds \right.\\
&\qquad \left. + \int_0^t (\tilde{b}(X_s,s) - b(X_s,s))dW_s - \frac{1}{2}\int_0^t |(\tilde{b}(X_s,s) - b(X_s,s))|^2 ds\right) \\
&\quad \leq \mathbf{E}\exp\left(\int_0^t (\tilde{b}(X_s,s) - b(X_s,s))dW_s - \frac{1}{2}\int_0^t |(\tilde{b}(X_s,s) - b(X_s,s))|^2 ds\right) \\
&\quad = 1
\end{aligned}$$

where $\mathbf{E}$ is the law of the diffusion $(X_s)_{0\leq s\leq t}$ with drift b, conditioned on $X_0 = x$ and $X_t = y$. Here we have used under our assumption that

$$(\tilde{b}(X_s,s) - b(X_s,s))\nabla \log p_{\tilde{b}}(X_s, t-s, y) \leq 0\,.$$

Thus the conclusion follows immediately. □

From the above proof, we also have

Corollary 5. *Given* $y \in \mathbf{R}^n$.

1) If $(\tilde{b}(x,t) - b(x,t), \nabla \log p_b(x,t,y)) \leq 0$ *for all* (x,t) *such that* $|x-y|>0$, *then*

$$p_{\tilde{b}}(x,t,y) \leq p_b(x,t,y) \quad \text{for all } (x,t)\,.$$

2) If $(\tilde{b}(x,t) - b(x,t), \nabla \log p_b(x,t,y)) \geq 0$ *for all* (x,t) *such that* $|x-y|>0$, *then*

$$p_{\tilde{b}}(x,t,y) \geq p_b(x,t,y) \quad \text{for all } (x,t).$$

Let $p_{\beta,y}(x,t,z)$ be the transition probability density (with respect to the Lebesgue measure) of the diffusion process

$$dX_t = dB_t + \beta\,(\nabla|X_t - y|)\,dt\ ; \quad X_0 = x \tag{12}$$

where $\beta \in \mathbf{R}$ and $y \in \mathbf{R}^n$ are given. $p_{\beta,y}(x,t,z)$ coincides with $H_{-\beta,y}(x,t,y)$ in the one dimensional case.

Corollary 6. *Let b be a bounded vector field on $\mathbf{R}^n$. Then*

$$p_{||b||,y}(x,t,y) \le p_b(x,t,y) \le p_{-||b||,y}(x,t,y)$$

for all x, y and $t > 0$, where $||b||$ denotes the supremum norm of Euclidean length of the vector field b, that is $||b|| = \sup_{x\in\mathbf{R}^n}\sqrt{\sum_{i=1}^n |b^i(x)|^2}$.

Corollary 6 is different from Theorem 1 only in the case where $n \ge 2$.

Let $(\gamma_t)_{t\ge 0}$ be the one-dimensional diffusion process

$$d\gamma_t = \frac{n-1}{2\gamma_t}dt + \beta dt + dw_t, \quad \gamma_0 = r \ge 0 \tag{13}$$

where $(w_t)_{t\ge0}$ is a standard Brownian motion. $(\gamma_t)_{t\ge0}$ thus is the Bessel process (with parameter n) with a constant drift β. This process was first introduced by D. Kendall in [6], called in Pitman-Yor [12] a Bessel process with a naive constant drift, which is different from the so-called Bessel process with constant drift in [16]. For more information, see also M. Yor [17].

Let $h_\beta(u,t,r)$ denote the transition probability density of the diffusion process (γ_t).

Lemma 7. *For every x, the process $(|X_t - y|, \mathbf{P}^x)$ possesses the same distribution as $(\gamma_t, \mathbf{P}^{|x-y|})$, where $(\gamma_t, \mathbf{P}^r)$ is the diffusion defined by eqn (13).*

Proof. Since $n \ge 2$, by Ito's formula we have

$$|X_t - y| = |x - y| + \int_0^t \frac{n-1}{2|X_s - y|}ds + \beta t + W_t$$

where

$$W_t = \int_0^t \langle \nabla|X_s - y|, dB_s\rangle$$

is a standard Brownian motion. Thus $(|X_t - y|, \mathbf{P}^x)$ is a weak solution to eqn (13) with initial $|x - y|$, and the conclusion follows. □

Theorem 8. *Under the above notations, we have*

$$\frac{1}{c_n}\lim_{\lambda\downarrow 0}\frac{1}{\lambda^{n-1}}h_{||b||}(|x-y|,t,\lambda) \le p_b(x,t,y) \le \frac{1}{c_n}\lim_{\lambda\downarrow 0}\frac{1}{\lambda^{n-1}}h_{-||b||}(|x-y|,t,\lambda)$$

where c_n is the volume of the unit $(n-1)$-sphere.

Proof. Let $(X_t, \mathbf{P}^x)$ be the diffusion defined by (12). Then

$$\mathbf{P}^x(|X_t - y| \leq \lambda) = \int_{|z-y|\leq\lambda} p_{\beta,y}(x,t,z)dz$$
$$= \int_0^\lambda \int_{S^{n-1}} p_{\beta,y}(x,t,y+\rho e^\theta)\rho^{n-1}d\theta d\rho$$

where (ρ, θ) are the polar coordinates around the point y. Therefore

$$\frac{d}{d\lambda}\mathbf{P}^x(|X_t - y| \leq \lambda) = \lambda^{n-1}\int_{S^{n-1}} p_{\beta,y}(x,t,y+\lambda e^\theta)d\theta$$

so that

$$\int_{S^{n-1}} p_{\beta,y}(x,t,y+\lambda e^\theta)d\theta = \frac{1}{\lambda^{n-1}}\frac{d}{d\lambda}\mathbf{P}^x(|X_t - y| \leq \lambda) .$$

On the other hand, by Lemma 7

$$\mathbf{P}^x(|X_t - y| \leq \lambda) = \mathbf{P}^{|x-y|}(\gamma_t \leq \lambda) = \int_0^\lambda h_\beta(|x-y|,t,z)dz ,$$

hence

$$\frac{d}{d\lambda}\mathbf{P}^x(|X_t - y| \leq \lambda) = h_\beta(|x-y|,t,\lambda) .$$

Thus we obtain

$$\int_{S^{n-1}} p_{\beta,y}(x,t,y+\lambda e^\theta)d\theta = \frac{1}{\lambda^{n-1}}h_\beta(|x-y|,t,\lambda)$$

and letting $\lambda \downarrow 0$ we establish

$$p_{\beta,y}(x,t,y) = \frac{1}{c_n}\lim_{\lambda\downarrow 0}\frac{1}{\lambda^{n-1}}h_\beta(|x-y|,t,\lambda) .$$

The existence of the limit follows from the fact that $p_{\beta,y}(x,t,z)$ is continuous and the standard heat kernel estimates [11]. □

3. Estimates on Bessel processes

In this section we prove Theorem 2. By Theorem 8, one has to estimate the transition density function $h_\beta(r,t,\rho)$ of the Bessel process with constant drift β. Unfortunately a closed formula for $h_\beta(r,t,\rho)$ is not known while $\beta \neq 0$ at least to the authors, see however [12]. We will give estimates to

$$\lim_{\rho\downarrow 0}\frac{1}{c_n\rho^{n-1}}h_\beta(r,t,\rho)$$

for $\beta = c_1$ or $-c_1$ (c_1 is a non-negative constant) by using the Cameron-Martin formula, which in turn yields explicit, sharp bounds of $p_b(x,t,y)$ in terms of the bound $||b||$.

Let $h(r, t, \rho)$ denote the transition density function of the Bessel process with parameter n. Then by an elementary computation we have

$$h(r, t, \rho) = \frac{\rho^{n-1}}{(2\pi t)^{n/2}} \int_{S^{n-1}} \exp\left(-\frac{\rho^2 - 2rx_n + r^2}{2t}\right) d\theta \qquad (14)$$

where (ρ, θ) is the polar coordinate of $(x_1, \cdots, x_n)$. In particular

$$\frac{1}{c_n} \lim_{\rho\downarrow 0} \frac{1}{\rho^{n-1}} h(r, t, \rho) = \frac{1}{(2\pi t)^{n/2}} \exp\left(-\frac{r^2}{2t}\right)$$

and it follows that

$$\nabla_r \log h(r, t, \rho) = \frac{\rho^{n-1}}{h(r, t, \rho)} \frac{1}{(2\pi t)^{n/2}} \int_{S^{n-1}} e^{-\frac{\rho^2 - 2rx_n + r^2}{2t}} \left(-\frac{r - x_n}{t}\right) d\theta .$$

Therefore

$$\lim_{\rho\downarrow 0} \nabla_r \log h(r, t, \rho) = -\frac{r}{t} . \qquad (15)$$

Next we consider $h_\beta(r, t, \rho)$ ($\rho > 0$ and $r \geq 0$). Let $(\xi_t, \mathcal{F}_t, \mathbf{P}^r)$ be the n-dimensional Bessel process

$$d\xi_t = \frac{n-1}{2\xi_t} dt + dW_t , \quad \xi_0 = r$$

where $(W_t)_{t\geq 0}$ is a standard Brownian motion. Given $r \geq 0$ and $\rho > 0$ define probability measure $\mathbf{P}^r_\beta$ by the Cameron-Martin density

$$\left.\frac{d\mathbf{P}^r_\beta}{d\mathbf{P}^r}\right|_{\mathcal{F}_t} = \exp\left(\beta W_t - \frac{\beta^2}{2} t\right) .$$

Then $(\xi_t, \mathcal{F}_t, \mathbf{P}^r_\beta)$ is a Bessel process with constant drift β, and therefore for every pair $r \geq 0$ and $\rho > 0$

$$\frac{h_\beta(r, t, \rho)}{h(r, t, \rho)} = \mathbf{P}^{r,\rho} \exp\left(\beta W_t - \frac{\beta^2}{2} t\right)$$

where $\mathbf{P}^{r,\rho} = \mathbf{P}^r(\cdot|\xi_t = \rho)$, and we use the standard notations in the theory of Markov processes. In particular, if X is a random variable and $\mathbf{P}$ a probability measure, then we use both notations $\mathbf{E}X$ and $\mathbf{P}X$ to denote the expectation of X with respect to $\mathbf{P}$.

On the other hand, it is well known that for $s < t$

$$\left.\frac{d\mathbf{P}^{r,\rho}}{d\mathbf{P}^r}\right|_{\mathcal{F}_s} = \frac{h(\xi_s, t - s, \rho)}{h(r, t, \rho)} . \qquad (16)$$

Let $\mathbf{P}^{r,0}$ be the limit of $\mathbf{P}^{r,\rho}$ as $\rho \downarrow 0$, which may be formally regarded as the conditional distribution $\mathbf{P}^{r,\rho}(\cdot|\xi_t = 0)$. That is, we may define $\mathbf{P}^{r,0}$ (see (14, 16) for justification) by

$$\left.\frac{d\mathbf{P}^{r,0}}{d\mathbf{P}^r}\right|_{\mathcal{F}_s} = \left(\frac{t}{t-s}\right)^{n/2} \exp\left(-\frac{\xi_s^2}{2(t-s)} + \frac{r^2}{2t}\right) \quad \text{for all } s < t\,. \tag{17}$$

In our arguments however we only use the fact that for all $s < t$

$$\lim_{\rho\downarrow 0} \left.\frac{d\mathbf{P}^{r,\rho}}{d\mathbf{P}^r}\right|_{\mathcal{F}_s} = \left.\frac{d\mathbf{P}^{r,0}}{d\mathbf{P}^r}\right|_{\mathcal{F}_s}.$$

Since

$$\exp\left(\beta W_s - \frac{\beta^2}{2}s\right), \quad \text{for } s < t$$

is uniformly integrable with respect to the probability $\mathbf{P}^{r,0}$ (see Appendix for a proof), so that

$$\lim_{\rho\downarrow 0} \frac{h_\beta(r,t,\rho)}{h(r,t,\rho)} = \mathbf{P}^{r,0} \exp\left(\beta W_t - \frac{\beta^2}{2}t\right) \tag{18}$$

and

$$\tilde{W}_s = W_s + \int_0^s \frac{\xi_u}{t-u} du \tag{19}$$

is a standard Brownian motion under $\mathbf{P}^{r,0}$.

Proof of the lower bound in Theorem 2. To show a lower bound, we observe that

$$\begin{aligned}\lim_{\rho\downarrow 0} \frac{h_\beta(r,t,\rho)}{h(r,t,\rho)} &= \mathbf{P}^{r,0} \exp\left(\beta \tilde{W}_t - \beta \int_0^t \frac{\xi_s}{t-s} ds - \frac{\beta^2}{2}t\right) \\ &\geq \exp\left(-\beta \int_0^t \frac{1}{t-s}\left(\mathbf{P}^{r,0}\xi_s\right) ds - \frac{\beta^2}{2}t\right)\end{aligned}$$

where we have used Jensen's inequality.

From (17), it is easily seen that

$$\mathbf{P}^{r,0}\xi_s = \left(\frac{t}{t-s}\right)^{n/2} e^{\frac{r^2}{2t}} \mathbf{P}^r \left(\xi_s e^{-\frac{\xi_s^2}{2(t-s)}}\right)$$

The last integral may be computed explicitly. Indeed

$$\begin{aligned}\mathbf{P}^r\left(\xi_s e^{-\frac{\xi_s^2}{2(t-s)}}\right) &= \frac{1}{(2\pi s)^{n/2}} \int_{\mathbf{R}^n} |z| e^{-\frac{|z|^2}{2(t-s)} - \frac{|z|^2 - 2rz_n + r^2}{2s}} dz \\ &= \frac{1}{(2\pi s)^{n/2}} e^{-\frac{r^2}{2s}} \int_{\mathbf{R}^n} |z| e^{-\frac{|z|^2}{2(t-s)} - \frac{|z|^2}{2s} + \frac{2rz_n}{2s}} dz\,,\end{aligned}$$

so that, after making change of variable

$$\sqrt{\frac{t}{s(t-s)}}z = u \ ,$$

$$\mathbf{P}^r\left(\xi_s \mathrm{e}^{-\frac{\xi_s^2}{2(t-s)}}\right) = \frac{1}{(2\pi s)^{n/2}}\mathrm{e}^{-\frac{r^2}{2s}}\left(\frac{s(t-s)}{t}\right)^{(n+1)/2}\int_{\mathbf{R}^n}|z|\mathrm{e}^{-\frac{|z|^2}{2}+\sqrt{\frac{t-s}{st}}rz_n}dz \ .$$

Write $z = (w, z_n) \in \mathbf{R}^{n-1} \times \mathbf{R}$. Then

$$\begin{aligned}
&\int_{\mathbf{R}^n}|z|e^{-\frac{|z|^2}{2}+\sqrt{\frac{t-s}{st}}rz_n}dz \\
&\quad \leq \int_{\mathbf{R}^n}(|w|+|z_n|)e^{-\frac{|x|^2}{2}+\sqrt{\frac{t-s}{st}}rz_n}e^{-\frac{|w|^2}{2}}dz \\
&\quad = \int_{\mathbf{R}^n}|w|e^{-\frac{|w|^2}{2}}e^{-\frac{|z_n|^2}{2}+\sqrt{\frac{t-s}{st}}rz_n} + \int_{\mathbb{R}^n}|z_n|e^{-\frac{|z_n|^2}{2}+\sqrt{\frac{t-s}{st}}rz_n}e^{-\frac{|w|^2}{2}} \\
&\quad = (2\pi)^{n/2}a_{n-1,1}e^{\frac{t-s}{st}\frac{r^2}{2}} + (2\pi)^{(n-1)/2}e^{\frac{t-s}{st}\frac{r^2}{2}}\int_{-\infty}^{\infty}|z_n + \sqrt{\frac{t-s}{st}}r|e^{-\frac{|z_n|^2}{2}} \ ,
\end{aligned}$$

and therefore

$$\begin{aligned}
\mathbf{P}^{r,0}\xi_s &= \sqrt{\frac{s(t-s)}{t}}\left(a_{n-1,1} + \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\left|x + \sqrt{\frac{t-s}{st}}r\right|\mathrm{e}^{-\frac{|x|^2}{2}}\right) \\
&\leq \sqrt{\frac{s(t-s)}{t}}\left(a_{n-1,1} + a_{1,1} + \sqrt{\frac{t-s}{st}}r\right) \\
&\leq \sqrt{t-s}\left(a_{n-1,1} + a_{1,1} + \frac{r}{\sqrt{t}}\right)
\end{aligned}$$

so that

$$\begin{aligned}
\int_0^t \frac{1}{t-s}\mathbf{P}^{r,0}\xi_s ds &\leq \int_0^t \frac{\sqrt{t-s}}{t-s}\left(a_{n-1,1} + a_{1,1} + \frac{r}{\sqrt{t}}\right)ds \\
&= 2\left(\left(a_{n-1,1} + a_{1,1}\right)\sqrt{t} + r\right).
\end{aligned}$$

Thus it follows that, for $\beta = c_1 > 0$,

$$\begin{aligned}
\lim_{\rho\downarrow 0}\frac{h_\beta(r,t,\rho)}{h(r,t,\rho)} &\geq \exp\left(-\beta\int_0^t \frac{1}{t-s}\mathbf{P}^{r,0}\xi_s ds - \frac{\beta^2}{2}t\right) \\
&\geq \mathrm{e}^{-2c_1\left(\left(a_{n-1,1}+a_{1,1}\right)\sqrt{t}+r\right)-\frac{c_1^2}{2}t}
\end{aligned}$$

and therefore

$$\lim_{\rho\downarrow 0}\frac{1}{c_n\rho^{n-1}}h_{c_1}(r,t,\rho) \geq \frac{1}{(2\pi t)^{n/2}}\mathrm{e}^{-\frac{r^2}{2t}-2c_1\left(\left(a_{n-1,1}+a_{1,1}\right)\sqrt{t}+r\right)-\frac{c_1^2}{2}t} \ .$$

The last estimate together with our comparison theorem 8 yield the lower bound in Theorem 2. □

Proof of the upper bound in Theorem 2. The proof of the upper bound is much more complicated. We need some precise moments estimates for Bessel processes. Thus the proof will be presented through several lemmata.

For simplicity set

$$M_s = \exp\left(\beta W_s - \frac{\beta^2}{2}s\right) \quad \text{for all } s,$$

and for all $s < t$

$$\begin{aligned} N_s &= \left(\frac{t}{t-s}\right)^{n/2} \exp\left(-\frac{\xi_s^2}{2(t-s)} + \frac{r^2}{2t}\right) \\ &= \exp\left(-\int_0^s \frac{\xi_u}{t-u} dW_u - \frac{1}{2}\int_0^s \frac{\xi_u^2}{(t-u)^2} du\right) \end{aligned}$$

where the last equality follows from the Itô formula. Let $G_t(r)$ denote the Gaussian function $G_t(r) = (2\pi t)^{-n/2} \exp\left(-\frac{r^2}{2t}\right)$. Since M and N are non-negative martingales under the probability $\mathbf{P}^r$ (up to time t), thus by the Girsanov theorem

$$M_s = \tilde{M}_s + \langle M, \hat{N}\rangle_s$$

where $\tilde{M}$ is a martingale under the probability $\mathbf{P}^{r,0}$, and

$$\hat{N} = -\int_0^s \frac{\xi_u}{t-u} dW_u$$

To compute $\langle M, \hat{N}\rangle_s$, we notice that by the Itô formula

$$\begin{aligned} M_s &= \exp\left(\beta W_s - \frac{\beta^2}{2}s\right) \\ &= 1 + \beta \int_0^s \exp\left(\beta W_u - \frac{\beta^2}{2}u\right) dW_u \end{aligned}$$

so that

$$\langle M, \hat{N}\rangle_t = -\beta \int_0^t \frac{\xi_s}{t-s} \exp\left(\beta W_s - \frac{\beta^2}{2}s\right) ds \ .$$

Taking expectation, we establish

$$\begin{aligned} &\mathbf{P}^{r,0}\langle M, \hat{N}\rangle_t \\ &= -\beta \int_0^t \frac{1}{t-s} \mathbf{P}^{r,0} \left\{\xi_s \exp\left(\beta W_s - \frac{\beta^2}{2}s\right)\right\} ds \\ &= -\frac{1}{G_t(r)} \frac{\beta}{(2\pi)^{n/2}} \int_0^t \frac{1}{(t-s)^{n/2+1}} \\ &\quad \times \mathbf{P}^r \left\{\xi_s \exp\left(-\frac{\xi_s^2}{2(t-s)} + \beta W_s - \frac{\beta^2}{2}s\right)\right\} ds \ . \end{aligned} \tag{20}$$

□

Lemma 9. *Under above notations, we have*

$$\frac{1}{c_n}\lim_{\rho\downarrow 0} h_\beta(r,t,\rho) = G_t(r)\left(1-\beta \mathbf{P}^{r,0}\left\{\int_0^t \frac{\xi_s}{t-s}e^{\beta W_s-\frac{\beta^2}{2}s}ds\right\}\right) . \tag{21}$$

Proof. It follows from eqns (18, 20) immediately. □

Lemma 10. *Let $(\xi_t, \mathbf{P}^r)$ be the Bessel process (with parameter $n \geq 2$). Then for every $s < t$*

$$\mathbf{P}^r\left(\xi_s^2 e^{-\frac{\xi_s^2}{t-s}}\right) = (\mu_n s + r)\left(\frac{t-s}{t+s}\right)^{(n+2)/2} e^{-\frac{r^2}{t+s}} , \tag{22}$$

where $\mu_n = a_{n-1,2} + a_{1,2}$ and

$$a_{n,p} = \frac{1}{(2\pi)^{n/2}}\int_{\mathbf{R}^n} |x|^p e^{-\frac{|x|^2}{2}}dx .$$

For every $q \in (1/2, 1)$,

$$\begin{aligned}\mathbf{P}^r\left(\xi_s^{2q} e^{-\frac{q\xi_s^2}{t-s}}\right) \leq & 2^q\left(\frac{t-s}{t_q}\right)^{q+\frac{n}{2}} e^{-q\frac{r^2}{t_q}} \\ & \times \max\left\{a_{n-1,2q}s^q, 2^{2q-1}\left(a_{1,2q}s^q + \left(\frac{t-s}{t_q}\right)^q |r|^{2q}\right)\right\}\end{aligned} \tag{23}$$

where $t_q = t + (2q-1)s$.

Proof. By (14) we have

$$\mathbf{P}^r\left(\xi_s^{2q} e^{-\frac{q\xi_s^2}{t-s}}\right) = \frac{1}{(2\pi s)^{n/2}}\int_{\mathbf{R}^n} |z|^{2q}\exp\left(-\frac{q|z|^2}{t-s} - \frac{|z|^2 - 2rz_n + r^2}{2s}\right)dz .$$

In particular if $q = 1$, then with $z = (w, z_n) \in \mathbf{R}^{n-1} \times \mathbf{R}$ we have

$$\begin{aligned}&\mathbf{P}^r\left(\xi_s^{2q} e^{-\frac{q\xi_s^2}{t-s}}\right) \\ &= \frac{1}{(2\pi s)^{n/2}}\int_{\mathbf{R}^n}\left(|w|^2 + |z_n|^2\right)\exp\left(-\frac{|z|^2}{t-s} - \frac{|z|^2 - 2rz_n + r^2}{2s}\right)dz \\ &= \frac{1}{(2\pi s)^{n/2}}\left[\int_{\mathbf{R}^{n-1}} |w|^2 \exp\left(-\frac{|w|^2}{t-s} - \frac{|w|^2}{2s}\right)\right] \\ &\quad \times\left[\int_{\mathbf{R}} \exp\left(-\frac{z_n^2}{t-s} - \frac{z_n^2 - 2rz_n + r^2}{2s}\right)\right] \\ &\quad + \frac{1}{(2\pi s)^{n/2}}\left[\int_{\mathbf{R}^{n-1}} \exp\left(-\frac{|w|^2}{t-s} - \frac{|w|^2}{2s}\right)\right] \\ &\quad \times\left[\int_{\mathbf{R}} z_n^2 \exp\left(-\frac{z_n^2}{t-s} - \frac{z_n^2 - 2rz_n + r^2}{2s}\right)\right].\end{aligned}$$

It is easy to see that

$$\frac{1}{(2\pi s)^{n/2}}\int_{\mathbf{R}^{n-1}}|w|^2\exp\left(-\frac{|w|^2}{t-s}-\frac{|w|^2}{2s}\right)=a_{n-1,2}\frac{\sqrt{s}}{\sqrt{2\pi}}\left(\frac{t-s}{t+s}\right)^{(n+1)/2},$$

$$\int_{\mathbf{R}}\exp\left(-\frac{z_n^2}{t-s}-\frac{z_n^2-2rz_n+r^2}{2s}\right)=\sqrt{2\pi}\sqrt{\frac{s(t-s)}{t+s}}e^{-\frac{r^2}{t+s}},$$

$$\frac{1}{(2\pi s)^{n/2}}\int_{\mathbf{R}^{n-1}}\exp\left(-\frac{|w|^2}{t-s}-\frac{|w|^2}{2s}\right)=\frac{1}{\sqrt{2\pi s}}\left(\frac{t-s}{t+s}\right)^{(n-1)/2}$$

and

$$\int_{\mathbf{R}}z_n^2\exp\left(-\frac{z_n^2}{t-s}-\frac{z_n^2-2rz_n+r^2}{2s}\right)$$
$$=\left[a_{1,2}\sqrt{2\pi}\left(\frac{s(t-s)}{t+s}\right)^{3/2}+\sqrt{s}\left(\frac{t-s}{t+s}\right)^{3/2}r\right]e^{-\frac{r^2}{t+s}}.$$

Thus eqn (22) in this case follows immediately.

Let us consider the case that $1>q>1/2$.

$$\begin{aligned}Q_s&=\frac{1}{(2\pi s)^{n/2}}\int_{\mathbf{R}^n}|z|^{2q}\exp\left(-\frac{q|z|^2}{t-s}-\frac{|z|^2-2rz_n+r^2}{2s}\right)dz\\&\le\left\{\frac{2^q}{(2\pi s)^{n/2}}\int_{\mathbf{R}^n}|w|^{2q}\exp\left(-\frac{q|z|^2}{t-s}-\frac{|z|^2-2rz_n+r^2}{2s}\right)dz\right\}\\&\quad\vee\left\{\frac{2^q}{(2\pi s)^{n/2}}\int_{\mathbf{R}^n}x^{2q}\exp\left(-\frac{q|z|^2}{t-s}-\frac{|z|^2-2rz_n+r^2}{2s}\right)dz\right\}\\&\le\left\{\frac{2^q}{(2\pi s)^{n/2}}\int_{\mathbf{R}^n}|w|^{2q}\exp\left(-\frac{q|w|^2+qz_n^2}{t-s}-\frac{|w|^2+z_n^2-2rz_n+r^2}{2s}\right)dz\right\}\\&\quad\vee\left\{\frac{2^q}{(2\pi s)^{n/2}}\int_{\mathbf{R}^n}z_n^{2q}\exp\left(-\frac{q|w|^2+qz_n^2}{t-s}-\frac{|w|^2+z_n^2-2rz_n+r^2}{2s}\right)dz\right\}.\end{aligned}$$

Set $t_q=t+(2q-1)s$. Then the previous inequality may be restated as

$$\begin{aligned}Q_s\le\max\Bigg\{&\frac{2^q}{(2\pi s)^{n/2}}\left(\int_{\mathbf{R}^{n-1}}|w|^{2q}\exp\left(-\frac{t_q|w|^2}{2s(t-s)}\right)dw\right)\\&\times\left(\int_{\mathbf{R}}\exp\left[-\frac{1}{2}\left(\sqrt{\frac{t_q}{s(t-s)}}z_n-\sqrt{\frac{t-s}{st_q}}r\right)^2-q\frac{r^2}{t_q}\right]dz_n\right),\\&\frac{2^q}{(2\pi s)^{n/2}}\left(\int_{\mathbf{R}^{n-1}}\exp\left(-\frac{t_q|w|^2}{2s(t-s)}\right)dw\right)\\&\times\left(\int_{\mathbf{R}}z_n^{2q}\exp\left[-\frac{1}{2}\left(\sqrt{\frac{t_q}{s(t-s)}}z_n-\sqrt{\frac{t-s}{st_q}}r\right)^2-q\frac{r^2}{t_q}\right]dz_n\right)\Bigg\}.\end{aligned}$$

Evaluating these integral we thus establish

$$Q_s \le 2^q \left(\frac{t-s}{t_q}\right)^{q+\frac{n}{2}} e^{-q\frac{r^2}{t_q}} \times \max\left\{a_{n-1,2q}s^q \ , \ 2^{2q-1}\left(a_{1,2q}s^q + \left(\frac{t-s}{t_q}\right)^q |r|^{2q}\right)\right\} .$$

□

Lemma 11. *Let* $\mathbf{P}^{r,0}$ *be the conditional distribution of the Bessel process* $(\xi_s, \mathbf{P}^r)$ *under* $\xi_t = 0$*, and let* (W_s) *be the martingale part of* ξ_s *under* $\mathbf{P}^r$*, that is*

$$\xi_s = r + \int_0^s \frac{n-1}{2\xi_u} du + W_s .$$

For every $1 < q < \frac{n}{n-1}$ *we have*

$$\mathbf{P}^{r,0}\left\{\int_0^t \frac{\xi_s}{t-s} e^{\beta W_s - \frac{\beta^2}{2}s} ds\right\} \le c_{n,q}\left(\kappa_q\sqrt{t} + 2|r|\right) e^{\frac{2q-1}{4q}r^2 + \frac{\beta^2}{2(q-1)}t} \tag{24}$$

where

$$c_{n,q} = \frac{2\sqrt{2}q}{n-q(n-1)} , \quad \kappa_q = \sqrt[2q]{a_{n-1,q} + 2^{2q-1}a_{1,q}} . \tag{25}$$

Proof. Let us denote the left-hand side of eqn (24) by I_t for simplicity. Then

$$\begin{aligned} I_t &= \int_0^t \mathbf{P}^{r,0}\left\{\frac{\xi_s}{t-s}\exp\left(\beta W_s - \frac{\beta^2}{2}s\right)\right\} ds \\ &= \frac{1}{(2\pi)^{n/2}G_t(r)} \int_0^t \frac{1}{(t-s)^{n/2+1}} \\ &\quad \times \mathbf{P}^r\left\{\xi_s \exp\left(-\frac{\xi_s^2}{2(t-s)}\right)\exp\left(\beta W_s - \frac{\beta^2}{2}s\right)\right\} ds \end{aligned}$$

and using Hölder's inequality and Lemma 10, for every $\frac{1}{p} + \frac{1}{2q} = 1$, we have

$$\begin{aligned} &\mathbf{P}^r\left\{\xi_s \exp\left(-\frac{\xi_s^2}{2(t-s)}\right)\exp\left(\beta W_s - \frac{\beta^2}{2}s\right)\right\} \\ &= \left(\mathbf{P}^r\left\{\xi_s^{2q}\exp\left(-\frac{q\xi_s^2}{t-s}\right)\right\}\right)^{1/(2q)} \left(\mathbf{P}^r\left\{\exp\left(p\beta W_s - \frac{p\beta^2}{2}s\right)\right\}\right)^{1/p} \\ &\le \sqrt{2}\left(\frac{t-s}{t_q}\right)^{\frac{1}{2}+\frac{n}{4q}} \exp\left(\frac{(p-1)\beta^2}{2}s - \frac{r^2}{2t_q}\right) \\ &\quad \times \left(\max\left\{a_{n-1,2q}s^q \ , \ 2^{2q-1}\left(a_{1,2q}s^q + \left(\frac{t-s}{t_q}\right)^q |r|^{2q}\right)\right\}\right)^{1/(2q)} . \end{aligned} \tag{26}$$

Choose q such that $\frac{1}{2} < q < \frac{1}{2}\frac{n}{n-1}$, then

$$\delta = \frac{n}{2} + 1 - \left(\frac{1}{2} + \frac{n}{4q}\right) < 1$$

so that

$$\begin{aligned} I_t &\le \frac{\sqrt{2}}{(2\pi)^{n/2} G_t(r)} \int_0^t \exp\left(\frac{(p-1)\,\beta^2}{2} s\right) \frac{1}{(t-s)^\delta} \left(\frac{1}{t_q}\right)^{\frac{1}{2}+\frac{n}{4q}} e^{-\frac{r^2}{2t_q}} \\ &\quad \times \left(\max\left\{a_{n-1,2q} s^q\ ,\ 2^{2q-1}\left(a_{1,2q} s^q + \left(\frac{t-s}{t_q}\right)^q |r|^{2q}\right)\right\}\right)^{1/(2q)} ds \\ &\le \frac{\sqrt{2}}{1-\delta} \max\left\{a_{n-1,2q} t^q\ ,\ 2^{2q-1}\left(a_{1,2q} t^q + |r|^{2q}\right)\right\}^{1/(2q)} \mathrm{e}^{\frac{(p-1)\beta^2}{2} t + \frac{2q-1}{4q} r^2} . \end{aligned}$$

The conclusion follows from the previous inequality (changing $2q$ into q). □

By Lemma 9 and Lemma 11 we have, if $\beta = -c_1 \le 0$ then

$$\frac{1}{c_n} \lim_{\rho \downarrow 0} h_\beta(r, t, \rho) \le G_t(r) \left(1 + c_1 c_{n,q} \left(\kappa_q \sqrt{t} + 2|r|\right) \mathrm{e}^{\frac{2q-1}{4q} r^2 + \frac{c_1^2}{2(q-1)} t}\right)$$

which proves the upper bound in Theorem 2.

4. Appendix

In this section we prove the claim that the family of random variables

$$\exp\left(\beta W_u - \frac{\beta^2}{2} u\right) \ ; \qquad u < t\ ,$$

for any fixed t, is uniformly integrable under the probability $\mathbf{P}^{r,0}$. Clearly we only need to show that for any constant α

$$\sup_{u<t} \mathbf{P}^{r,0} \exp\left(\alpha W_u\right) < \infty\ , \tag{27}$$

or in other words we only need to show that

$$\sup_{u<t} \mathbf{P}^{r,0} \exp\left(\beta W_u - \frac{\beta^2}{2} u\right) < \infty$$

for any β. We have seen from the previous section that for any $u < t$,

$$\begin{aligned} \mathbf{P}^{r,0}\left(\mathrm{e}^{\beta W_u - \frac{\beta^2}{2} u}\right) &= \mathbf{P}^{r,0} \langle M, \hat{N} \rangle_u \\ &= -\frac{1}{G_t(r)} \frac{\beta}{(2\pi)^{n/2}} \int_0^u \frac{1}{(t-s)^{n/2+1}} \mathbf{P}^r \left\{\xi_s \mathrm{e}^{-\frac{\xi_s^2}{2(t-s)} + \beta W_s - \frac{\beta^2}{2} s}\right\} ds \\ &\le \frac{1}{G_t(r)} \frac{|\beta|}{(2\pi)^{n/2}} \int_0^t \frac{1}{(t-s)^{n/2+1}} \mathbf{P}^r \left\{\xi_s \mathrm{e}^{-\frac{\xi_s^2}{2(t-s)} + \beta W_s - \frac{\beta^2}{2} s}\right\} ds \\ &< \infty\ . \end{aligned}$$

Indeed, for the expectation

$$\mathbf{P}^r \left\{ \xi_s \mathrm{e}^{-\frac{\xi_s^2}{2(t-s)} + \beta W_s - \frac{\beta^2}{2} s} \right\}$$

we may use the estimate (26), which thus leads to a similar majorization as I_t:

$$\mathbf{P}^{r,0} \left(\mathrm{e}^{\beta W_u - \frac{\beta^2}{2} u} \right) \leq \frac{\sqrt{2}|\beta|}{1-\delta} \max \left\{ a_{n-1,2q} t^q \,,\; 2^{2q-1} \left(a_{1,2q} t^q + |r|^{2q} \right) \right\}^{1/(2q)} \mathrm{e}^{\frac{(p-1)\beta^2}{2} t + \frac{2q-1}{4q} r^2}$$

(where p, q chosen as in the last section). Hence

$$\sup_{u<t} \mathbf{P}^{r,0} \left(\mathrm{e}^{\beta W_u - \frac{\beta^2}{2} u} \right) < \infty$$

for any β.

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