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Avoiding and Extending Partial Edge Colorings of Hypercubes

Carl Johan Casselgren[1] · Per Johansson[1] · Klas Markström[2]

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Abstract

We consider the problem of extending and avoiding partial edge colorings of hypercubes; that is, given a partial edge coloring φ of the d-dimensional hypercube Q_d, we are interested in whether there is a proper d-edge coloring of Q_d that *agrees* with the coloring φ on every edge that is colored under φ; or, similarly, if there is a proper d-edge coloring that *disagrees* with φ on every edge that is colored under φ. In particular, we prove that for any $d \geq 1$, if φ is a partial d-edge coloring of Q_d, then φ is avoidable if every color appears on at most $d/8$ edges and the coloring satisfies a relatively mild structural condition, or φ is proper and every color appears on at most $d-2$ edges. We also show that φ is avoidable if d is divisible by 3 and every color class of φ is an induced matching. Moreover, for all $1 \leq k \leq d$, we characterize for which configurations consisting of a partial coloring φ of $d-k$ edges and a partial coloring ψ of k edges, there is an extension of φ that avoids ψ.

Keywords Edge coloring · Hypercube · Precoloring extension · Avoiding edge coloring

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✉ Carl Johan Casselgren
carl.johan.casselgren@liu.se

Per Johansson
perjo018@student.liu.se

Klas Markström
klas.markstrom@umu.se

1 Department of Mathematics, Linköping University, 581 83 Linköping, Sweden

2 Department of Mathematics, Umeå University, 901 87 Umeå, Sweden

1 Introduction

An *edge precoloring* (or *partial edge coloring*) of a graph G is a proper edge coloring of some subset $E' \subseteq E(G)$; *a t-edge precoloring* is such a coloring with t colors. A t-edge precoloring φ is *extendable* if there is a proper t-edge coloring f such that $f(e) = \varphi(e)$ for any edge e that is colored under φ; f is called an *extension* of φ.

Related to the notion of extending a precoloring is the idea of *avoiding* a precoloring: if φ is a t-edge precoloring of a graph G, then a proper t-edge coloring f of G *avoids* φ if $f(e) \neq \varphi(e)$ for every edge e that is colored under φ. More generally, if L is a list assignment for the edges of a graph G, then a proper edge coloring φ of G *avoids* the list assignment L if $\varphi(e) \notin L(e)$ for every edge e of G.

In general, the problem of extending a given edge precoloring is an $\mathcal{NP}$-complete problem, already for 3-regular bipartite graphs [11, 14]. One of the earlier references explicitly discussing the problem of extending a partial edge coloring is [23]; there a necessary condition for the existence of an extension is given and the authors find a class of graphs where this condition is also sufficient. More recently, questions on extending and avoiding a precolored matching have been studied in [12, 16]. In particular, in [12] it is proved that if G is subcubic or bipartite and φ is an edge precoloring of a matching M in G using $\Delta(G) + 1$ colors, then φ can be extended to a proper $(\Delta(G) + 1)$-edge coloring of G, where $\Delta(G)$ as usual denotes the maximum degree of G; a similar result on avoiding a precolored matching of a general graph is obtained as well. Moreover, in [16] it is proved that if φ is a $(\Delta(G) + 1)$-edge precoloring of a distance-9 matching in any graph G, then φ can be extended to a proper $(\Delta(G)+1)$-edge coloring of G; here, by a *distance-k matching* we mean a matching M where the distance between any two edges in M is at least k, and the *distance* between two edges e and e' is the number of edges contained in a shortest path between an endpoint of e and an endpoint of e'. A distance-2 matching is usually called an *induced matching*.

Questions on extending and avoiding partial edge colorings have specifically been studied to a large extent for balanced complete bipartite graphs. In the literature these type of problems and results are usually formulated in terms of completing partial Latin squares and avoiding arrays, respectively. In this form, these type of questions go back to the famous Evans conjecture [13] which states that for every positive integer n, if $n - 1$ edges in the complete bipartite graph $K_{n,n}$ have been (properly) colored, then this partial coloring can be extended to a proper n-edge coloring of $K_{n,n}$. This conjecture was solved for large n by Häggkvist [18] and later for all n by Smetaniuk [24], and independently by Andersen and Hilton [1].

The problem of avoiding partial edge colorings (and list assignments) of complete bipartite graphs was introduced by Häggkvist [17] and has been further studied in e.g. [2, 4, 5]. In particular, by results of [9, 10, 25], any partial proper n-edge coloring of $K_{n,n}$ is avoidable, given that $n \geq 4$. Moreover, a conjecture first stated by Markström suggests that if φ is a partial n-edge coloring of $K_{n,n}$ where any color appears on at most $n - 2$ edges, then φ is avoidable (see e.g. [5]). In [5], several partial results towards this conjecture are obtained; in particular, it is proved that the conjecture holds if each color appears on at most $n/5$ edges, or if the graph is colored by altogether at most $n/2$ colors.

Combining the notions of extending a precoloring and avoiding a list assignment, Andren et al. [3] proved that a "sparse" partial edge coloring of $K_{n,n}$ can be extended to a proper n-edge coloring avoiding a given list assignment L satisfying certain "sparsity" conditions, provided that no edge e is precolored by a color that appears in $L(e)$; we refer to [3] for the exact definition of "sparse" in this context. An analogous result for complete graphs was recently obtained in [8].

The study of problems on extending and avoiding partial edge colorings of hypercubes was recently initiated in the papers [6, 7]. In [6] Casselgren et al. obtained several analogues for hypercubes of classic results on completing partial Latin squares, such as the famous Evans conjecture. Moreover, questions on extending a "sparse" precoloring of a hypercube subject to the condition that the extension should avoid a given "sparse" list assignment were investigated in [7].

In this paper we continue the work on extending and avoiding partial edge colorings of hypercubes, with a particular focus on the latter variant. We obtain a number of results towards an analogue for hypercubes of Markström's aforementioned conjecture for complete bipartite graphs (see Conjecture 3.1), and also prove several related results; in particular, we prove the following.

- For any $d \geq 1$, if φ is a partial d-edge coloring of Q_d where every color appears on at most $d/8$ edges, and φ satisfies a structural condition (described in Theorem 3.6 below), then φ is avoidable.
- For any $d \geq 1$, if φ is a partial proper d-edge coloring of Q_d where every color appears on at most $d-2$ edges, then φ is avoidable.
- If $d = 3k$ and every color class of a partial d-edge coloring φ of Q_d is an induced matching, then φ is avoidable; we conjecture that this holds for any $d \geq 1$.
- For any $d \geq 1$ and any $1 \leq k \leq d$, we characterize for which configurations consisting of a partial coloring φ of $d-k$ edges and a partial coloring ψ of k edges, there is an extension of φ that avoids ψ.

2 Preliminaries

In this paper, all (partial) d-edge colorings use colors $1, \ldots, d$ unless otherwise stated. If φ is an edge precoloring of G and an edge e is colored under φ, then we say that e is *φ-colored*.

If φ is a (partial) proper t-edge coloring of G and $1 \leq a, b \leq t$, then a path or cycle in G is called *(a, b)-colored under φ* if its edges are colored by colors a and b alternately. We also say that such a path or cycle is *bicolored under φ*. By switching colors a and b on a maximal (a, b)-colored path or an (a, b)-colored cycle, we obtain another proper t-edge coloring of G; this operation is called an *interchange*. We denote by $\varphi^{-1}(i)$ the set of edges colored i under φ.

In the above definitions, we often leave out the reference to an explicit coloring φ, if the coloring is clear from the context.

Havel and Morávek [20] (see also [19]) proved a criterion for a graph G to be a subgraph of a hypercube:

Proposition 2.1 *A graph G is a subgraph of Q_d if and only if there is a proper d-edge coloring of G with integers $\{1,...,d\}$ such that*

(i) in every path of G there is some color that appears an odd number of times;
(ii) in every cycle of G no color appears an odd number of times.

A *dimensional matching* M of Q_d is a perfect matching of Q_d such that $Q_d - M$ is isomorphic to two copies of Q_{d-1}; evidently there are precisely d dimensional matchings in Q_d. We state this as a lemma.

Lemma 2.2 *Let $d \geq 2$ be an integer. Then there are d different dimensional matchings in Q_d; indeed Q_d decomposes into d such perfect matchings.*

The proper d-edge coloring of Q_d obtained by coloring the ith dimensional matching of Q_d by color i, $i = 1,...,d$, we shall refer to as the *standard edge coloring* of Q_d.

As pointed out in [6], the colors in the proper edge coloring in Proposition 2.1 correspond to dimensional matchings in Q_d (see also [19]). In particular, Proposition 2.1 holds if we take the dimensional matchings as the color classes. Furthermore we have the following.

Lemma 2.3 *The subgraph induced by r dimensional matchings in Q_d is isomorphic to a disjoint union of r-dimensional hypercubes.*

This simple observation shall be used quite frequently below. In particular, for future reference, we state the following consequence of Lemma 2.3.

Lemma 2.4 *In the standard d-edge coloring, every edge of Q_d is in exactly $d - 1$ 2-colored 4-cycles.*

We shall also need some standard definitions on list edge coloring. Given a graph G, assign to each edge e of G a set $\mathcal{L}(e)$ of colors.

If all lists have equal size k, then $\mathcal{L}$ is called a *k-list assignment*. Usually, we seek a proper edge coloring φ of G, such that $\varphi(e) \in \mathcal{L}(e)$ for all $e \in E(G)$. If such a coloring φ exists, then G is *$\mathcal{L}$-colorable* and φ is called an *$\mathcal{L}$-coloring*. Denote by $\chi'_L(G)$ the minimum integer t such that G is $\mathcal{L}$-colorable whenever $\mathcal{L}$ is a t-list assignment.

A fundamental result in list edge coloring theory is the following theorem by Galvin [15]. As usual, $\chi'(G)$ denotes the chromatic index of a multigraph G.

Theorem 2.5 *For any bipartite multigraph G, $\chi'_L(G) = \chi'(G)$.*

3 Avoiding General Partial Edge Colorings

Most of the results in this paper are partial results towards the following general conjecture for hypercubes. This is a variant of a conjecture for $K_{n,n}$ first suggested by Markström based on unavoidable n-edge colorings of $K_{n,n}$ (see e.g. [5, 22]).

Conjecture 3.1 *For any $d \geq 1$, if φ is a partial d-edge coloring of Q_d where every color appears on at most $d-2$ edges, then φ is avoidable.*

Conjecture 3.1 is best possible: consider the partial coloring of Q_d obtained by coloring $d-1$ edges incident with a vertex u by the color 1, and coloring $d-1$ edges incident with another vertex v by the color 2. This partial coloring is unavoidable if $uv \in E(Q_d)$ and it is uncolored.

Note further that such a statement as in Conjecture 3.1 does not hold for general d-regular (bipartite) graphs. Indeed, we have the following:

Proposition 3.2 *For any $d \geq 1$, there is a d-regular bipartite graph G and a partial proper d-edge coloring with exactly d colored edges that is not avoidable.*

Proof The case when $d = 1$ is trivial, so assume that $d \geq 2$. Let $G_1, \ldots, G_d$ be d copies of the graph $K_{d,d} - e$, that is, the complete bipartite graph $K_{d,d}$ with an arbitrary edge e removed. Denote by $a_i b_i$ the edge that was removed from $K_{d,d}$ to form the graph G_i. From $G_1, \ldots, G_d$, we construct the d-regular bipartite graph G by adding the edges $a_1b_2, a_2b_3, \ldots, a_{d-1}b_d, a_db_1$.

We define a partial d-edge coloring φ of G by coloring a_ib_{i+1} by the color i, $i = 1, \ldots, d$ (where indices are taken modulo d). Now, it is straightforward that any proper d-edge coloring of G uses the same color on all the edges in the set $\{a_1b_2, a_2b_3, \ldots, a_{d-1}b_d, a_db_1\}$; therefore, φ is not avoidable. $\square$

On the other hand, a partial coloring of at most $d-1$ edges of a d-edge-colorable graph is always avoidable:

Proposition 3.3 *Let $k \in \{1, \ldots, d\}$ and let G be a d-edge-colorable graph. If G is precolored with at most k colors and every color appears on at most $d-k$ edges, then there is a proper d-edge coloring of G that avoids the preassigned colors.*

This is a reformulation for general graphs of a theorem in [5] for complete bipartite graphs; the proof is identical to the argument given there; thus, we omit it.

Note further that Proposition 3.3 does not set any restrictions on where colors may appear, so several colors may be assigned to the same edge. Thus, it has a natural interpretation as a statement on list edge coloring.

By the example preceding Proposition 3.3, it is in general sharp; however, by requiring that the colored edges satisfy some structural condition, we can prove that other configurations are avoidable as well.

Proposition 3.4 *Let G be a d-edge colorable graph. If φ is a partial d-edge coloring of G, and there is a set K of k vertices such that every precolored edge is incident to some vertex from K, and every color appears on at most $d-k$ edges, then φ is avoidable.*

The proof of this proposition is similar to the proof of the previous one. The only essential difference is that instead of using the fact that the precoloring uses at most k colors, one employs the property that every matching in a decomposition obtained from a proper k-edge coloring of G contains edges with at most k distinct colors from φ; we omit the details.

Next, we prove the following weaker version of Conjecture 3.1. Following [7], we say that two edges in a hypercube are *parallel* if they are non-adjacent and contained in a common 4-cycle.

We shall use the following simple lemma.

Lemma 3.5 *If φ is a partial d-edge coloring of Q_d, $d \geq 3$, where every color appears on at most one edge, then φ is avoidable.*

Proof Let f be the proper d-edge coloring of Q_d obtained by assigning color i to the ith dimensional matching of Q_d, that is, f is the standard edge coloring of Q_d.

Consider the bipartite graph $B(f)$, with vertices for the colors $\{1, \ldots, d\}$ and for the color classes $f^{-1}(i)$ of f, and where there is an edge between $f^{-1}(i)$ and j if there is no edge colored i under f that is colored j under φ. If there is no set violating Hall's condition for a matching in a bipartite graph, then $B(f)$ has a perfect matching, and by assigning colors to the color classes of f according to this perfect matching, we obtain a proper d-edge coloring of Q_d that avoids φ.

Now, if there is such a set violating Hall's condition, then one of the color classes of f contains all φ-colored edges, because every color used by φ appears on just one edge. Without loss of generality, assume that M_1 is such a color class and consider the subgraph $H = Q_d[M_1 \cup M_2]$, where M_2 is another arbitrarily chosen color class of f. By Lemma 2.3, H consists of a collection of bicolored 4-cycles. By interchanging colors on such a bicolored cycle that contains at least one φ-colored edge, we obtain a proper edge coloring f' of Q_d such that the bipartite graph $B(f')$, defined as above, contains a perfect matching. Thus there is a proper d-edge coloring that avoids φ. □

Theorem 3.6 *Let $d \geq 1$, and let φ be a partial d-edge coloring of Q_d. Assume $a(d)$ and $b(d)$ are functions satisfying that $\frac{11}{208}d^2 - 2b(d)\left(a(d) - \frac{7d}{8}\right) \geq 0$ and $a(d) \geq b(d)$.*

(i) *If every color appears on at most $d/8$ edges, for every edge in Q_d there are at most $b(d)$ other parallel φ-colored edges, and every dimensional matching in Q_d contains at most $a(d)$ φ-precolored edges, then φ is avoidable.*

(ii) *For every constant $C_1 \geq 1$, there is a positive constant C_2, such that if every dimensional matching contains at most $C_1 d$ φ-colored edges and every color appears on at most $\frac{d}{C_2}$ edges under φ, then φ is avoidable.*

Before proving Theorem 3.6, allow us to comment on the possible values of $a(d)$ and $b(d)$ for which the inequality in the theorem holds. If we choose $b(d)$ to be as large as possible, that is, $a(d) = b(d)$, then it suffices to require that $a(d) =$

$b(d) \leq \left(\left(\frac{11}{416} + \frac{49}{96} \right)^{1/2} + \frac{7}{16} \right) d \approx 1.17d$ for part (i) of the theorem to hold. On the other hand, if $b(d)$ is a "sufficiently small" linear function of d, then we can pick $a(d)$ to be an arbitrarily large linear function of d.

Proof of Theorem 3.6 We first prove part (i) of the theorem. Let f be the standard edge coloring of Q_d. Similarly to the proof of the preceding lemma, our goal is to transform f into a coloring f' where every color class contains edges of at most $\frac{7}{8}d$ distinct colors under φ. Consider a bipartite graph B with vertices for the colors $\{1, \ldots, d\}$ and for the color classes $f'^{-1}(i)$ of f', and where there is an edge between $f'^{-1}(i)$ and j if there is no edge colored i under f' that is colored j under φ. The condition that every color class in f' contains edges of at most $\frac{7}{8}d$ distinct colors under φ implies that every vertex $f^{-1}(i)$ has degree at least $d/8$ in B. Thus, any subset $S \subseteq \{f^{-1}(1), \ldots, f^{-1}(d)\}$ that violates Hall's condition in B has size at least $d/8 + 1$. However, since every color appears on at most $d/8$ edges under φ, every color in $\{1, \ldots, d\}$ has degree at least $7d/8$ in B. Consequently, $N_B(S) = \{1, \ldots, d\}$, and so, S cannot violate Hall's condition. Hence, B has a perfect matching, and by coloring the color classes of f' according to the perfect matching, we obtain a proper d-edge coloring of Q_d that avoids φ.

We shall use interchanges on 2-colored 4-cycles for transforming the coloring f into a required coloring f'. More precisely, we shall use the following method. Suppose that there is some color class of f that contains at least $\frac{7}{8}d + 1$ edges that are colored under φ; let $M_1 = f^{-1}(1)$ be such a color class. We call such a color class *heavy*; a color class that contains at most $\frac{7}{8}d - 2$ edges that are colored under φ is called a *light* color class.

Since there are at most $\frac{1}{8}d^2$ edges in Q_d that are colored under φ, there must be some light color class of f; without loss of generality assume that $M_2 = f^{-1}(2)$ is such a color class. By Lemma 2.3, the subgraph $Q_d[M_1 \cup M_2]$ of Q_d induced by M_1 and M_2 is a collection of bicolored 4-cycles. Now, since M_1 is heavy and M_2 is light, there is a 4-cycle C in $Q_d[M_1 \cup M_2]$ such that by interchanging colors on C, we obtain a coloring f_1 where the color class $f_1^{-1}(1)$ contains at least one less edge that is colored under φ and $f_1^{-1}(2)$ contains at least one more edge that is colored under φ (but no more than two such additional edges).

We shall apply this procedure iteratively and repeatedly select previously unused edges of a light color class that are not colored under φ (where *unused* means that the edges have not been involved in any interchanges performed by the algorithm before), together with previously unused edges from a heavy color class, at least one of which is colored under φ, which together form a bicolored 4-cycle, and then interchange colors on this 4-cycle. Thus we shall construct a sequence of colorings $f_1, \ldots, f_q$, where f_{i+1} is obtained from f_i by interchanging colors on a bicolored 4-cycle, and f_q is the required coloring f' where every color class contains at most $\frac{7d}{8}$ φ-colored edges. Note that since Q_d contains at most $d^2/8$ φ-colored edges, $q \leq d^2/8$.

We now give a brief counting argument which shows that as long as there is a heavy color class, there is a 4-cycle in the current coloring f_i so that after interchanging colors on this 4-cycle, the obtained coloring f_{i+1} contains fewer or equally many heavy color classes, but in the latter case one heavy color class contains fewer φ-colored edges.

Suppose that Q_d initially contains k heavy color classes under the coloring f, where $k < d$, and that exactly α φ-colored edges are not contained in the heavy k color classes in Q_d, where $\alpha < d^2/8$. Consider a color class M that is heavy under f_i. Suppose that M (initially) contains β φ-colored edges, where $\beta \leq d^2/8$. By Lemma 2.4, every edge in Q_d is contained in $d-1$ 2-colored 4-cycles under f, so initially there are at least

$$\frac{(d-k)}{2}\beta - \alpha$$

4-cycles containing edges from M that may be used by the algorithm, because every φ-colored edge of a heavy color class is contained in $(d-k)$ 4-cycles, where two edges are in a light color class, and up to α such cycles are unavailable since they contain a φ-colored edge of a light color class.

Now, after performing some steps of this algorithm we might have used edges from some of these cycles. Let us estimate how many of these cycles that are unavailable due to this. Suppose that the algorithm has used

- s 4-cycles C with two edges from M, such that both edges from M in C are φ-colored, and
- r 4-cycles C with two edges from M, such that one edge from M in C is φ-colored.

Firstly, Q_d contains at most $d^2/8 - \alpha$ φ-precolored edges that initially do not lie in light color classes, every interchange on a 4-cycle involves two edges from a light color class and α φ-precolored edges are intially in light color classes. Hence we might be unable to use 4-cycles with edges from at most

$$\frac{2\left(\frac{d^2}{8} - \alpha\right) + \alpha}{\frac{7d}{8} - 1} \leq \frac{4d}{13}$$

of the $d-k$ initially light color classes, because such a color class contains at least $\frac{7d}{8} - 1$ φ-colored edges after performing some steps of the algorithm, where the inequality follows from the fact that by Lemma 3.5, we may assume that $d \geq 16$.

Furthermore, since every 4-cycle that has been used by the algorithm contains two edges from a light color class, at most

$$2\left(\frac{d^2}{8} - \alpha - \frac{7}{8}dk\right)$$

4-cycles C are unavailable because C contains an edge from a light color class that was used previously in another 4-cycle. Similarly, for every edge from M, there are at most $b(d)$ parallel edges that are φ-colored, so at most

$$2b(d)(s+r)$$

4-cycles C are unavailable because it contains an edge from M that was used previously in another 4-cycle by the algorithm.

Consequently, there are at least

$$\frac{\left(d-k-\frac{4d}{13}\right)}{2}(\beta-2s-r)-\alpha-2\left(\frac{d^2}{8}-\alpha-\frac{7}{8}dk\right)-2b(d)(s+r)$$

4-cycles containing two previously unused edges from M, at least one of which is φ-colored, and two previously unused edges from a light color class. So if this quantity is greater than 1, then we can perform all the necessary steps in the algorithm and thus the required coloring f' exists.

Now, since M is a heavy color class, $\beta \geq \frac{7d}{8}$, and by definition $\beta \leq a(d)$. Moreover, since each color class may contain up to $\frac{7d}{8}$ φ-colored edges when the algorithm terminates, we have that $2s+r \leq \beta - \frac{7d}{8}$. Thus

$$\frac{\left(d-k-\frac{4d}{13}\right)}{2}(\beta-2s-r)-\alpha-2\left(\frac{d^2}{8}-\alpha-\frac{7}{8}dk\right)-2b(d)(s+r) \geq 1 \qquad (1)$$

holds if

$$\frac{11d^2}{208}-k\frac{7d}{16}+\alpha+\frac{7}{4}dk-2b(d)\left(a(d)-\frac{7d}{8}\right) \geq 1 \qquad (2)$$

Now, by assumption, $\frac{11}{208}d^2 - 2b(d)\left(a(d)-\frac{7d}{8}\right) \geq 0$, so (1) does indeed hold.

Let us now prove part (ii). The proof of this part is similar to the proof of part (i). We shall prove that we can perform all the necessary steps in the algorithm described above, and choose each 4-cycle C that is used by the algorithm in such a way that for each of the edges of C that belongs to a heavy color class, there are at most $\frac{d}{38C_1}$ parallel unused edges that are φ-colored. Since we will have that $1/C_2 \leq 1/8$, part (ii) of the theorem then holds if (1) is valid under the assumptions that $a(d) = C_1 d$ and $b(d) = \frac{d}{38C_1}$. Since $\frac{11}{208} > \frac{1}{19}$, this, in turn, follows from the fact that (2) holds, given that $a(d) = C_1 d$ and $b(d) = \frac{d}{38C_1}$.

Our task is thus to prove that in each step of the algorithm, we can select a 4-cycle so that each of the edges from the heavy color class is parallel with at most $\frac{d}{38C_1}$ unused φ-colored edges. As we shall see, it shall be sufficient to require that $C_2 - 6 - 100C_1 > 0$ and that d is large enough, $d \geq d_0$ say, for this to hold. Since we can pick C_2 to be smaller than $1/d_0$, and the proof will contain a finite number inequalities involving d and C_2, this suffices for proving part (ii) of the theorem.

So suppose that some steps of the algorithm have been performed and we have selected so far some 4-cycles satisfying this condition. Then, since (1) holds, there is some 4-cycle $C = uvxyu$ that is edge-disjoint from all previously considered 4-cycles and such that uv and xy are edges from some heavy color class, at least one of which is φ-colored, and the edges vx and yu are not φ-colored and lie in a color class that is light under the current coloring f_i. Suppose that one of the edges uv and

xy, uv say, is parallel with at least $\frac{d}{38C_1}$ unused φ-colored edges. Denote by M_1 the dimensional matching containing uv and consider the set $E' \subseteq M_1$ of all these φ-colored edges that are parallel to uv. Now, at most $76C_1^2$ of the edges in E' are parallel with at least $\frac{d}{38C_1}$ φ-colored edges, because any edge (except uv) that is parallel with an edge from E' is parallel with at most one other edge from E', $\frac{1}{2}\frac{d}{38C_1}76C_1^2 = C_1 d$, and M_1 contains altogether at most $C_1 d$ φ-colored edges.

Let $E'' \subseteq E'$ be the set of edges that are parallel with uv and which are parallel with at most $\frac{d}{38C_1}$ φ-colored edges. Then

$$|E''| \geq \frac{d}{38C_1} - 76C_1^2 \geq \frac{d}{50C_1},$$

if d is large enough.

Let $\mathcal{C}$ be a largest set of 4-cycles C with usable edges that contains exactly one edge from E'' and two edges from a light color class, and which satisfies that any two cycles in $\mathcal{C}$ containing different edges from E'' are disjoint.

Let us estimate the size of $\mathcal{C}$. Now, since Q_d contains altogether d^2/C_2 φ-colored edges, certainly at most $2d/C_2$ color classes of f_i are initially heavy before applying the algorithm. Moreover, at most $4d/C_2$ color classes of f_i that are initially light might lose the property of being light after performing some steps of the algorithm. Moreover, since every cycle used by the algorithm at a previous step intersect at most one edge from E', at most d^2/C_2 cycles containing exactly one edge from E'' are unavailable because they contain an edge that was used previously by the algorithm. In conlusion, we have that

$$|\mathcal{C}| \geq \frac{1}{2}\frac{d}{50C_1} d \frac{C_2 - 6}{C_2} - \frac{d^2}{C_2} = d^2 \frac{C_2 - 6 - 100C_1}{100C_1C_2},$$

where the first factor 1/2 in the denominator accounts for the fact that the cycles in $\mathcal{C}$ should be disjoint if they contain different edges from E''.

Now, by definition, all the edges of E'' that are in cycles in $\mathcal{C}$ are parallel with at most $\frac{d}{38C_1}$ unused φ-colored edges. We shall prove that this holds for both edges of M_1 in at least one of the cycles of $\mathcal{C}$.

Consider a cycle $C = abcda \in \mathcal{C}$, where $ab \in E''$, $cd \in M_1 \setminus E''$, the edges ua and bv are contained in the dimensional matching M_i, and the edges bc and ad are contained in the dimensional matching M_j. Now, if cd is parallel with at least $\frac{d}{38C_1}$ unused φ-precolored edges, then there are at least $\frac{d}{38C_1} - 2$ such edges $c'd' \in M_1$, where $cc' \in E(Q_d)$ and $dd' \in E(Q_d)$, such that $cc', dd' \notin M_i$. Suppose, for instance, $cc', dd' \in M_k$, where $k \neq i$. Then, since $i \neq k$, and there are six permutations of the matchings M_i, M_j, M_k, it follows from Proposition 2.1 (ii) (where we take the dimensional matchings as colors), that there are at most 5 other cycles from $\mathcal{C}$ that contain an edge which is parallel with $c'd'$; this is so, because if there is such a cycle $C' \in \mathcal{C}$, then Q_d has a 6-cycle containing the vertices u, a, d, d' and two vertices from C'. Summing up, we conclude that if all cycles in $\mathcal{C}$ contain an edge from M_1 that is parallel with at least $\frac{d}{38C_1}$ unused φ-colored edges, then Q_d contains at least

$$d^2 \frac{C_2 - 6 - 100C_1}{100C_1 C_2} \left(\frac{d}{38C_1} - 2 \right) \frac{1}{6}$$

φ-colored edges. However, if $C_2 - 6 - 100C_1 > 0$ and d is large enough, then this is not possible because Q_d contains at most $\frac{d^2}{C_2}$ precolored edges. We conclude that at least one cycle in $\mathcal{C}$ satisfies that every edge from M_1 is parallel with at most $\frac{d}{38C_1}$ other φ-precolored edges. Consequently, we can perform all the necessary steps in the algorithm to obtain the required coloring f'. □

It is trivial that Conjecture 3.1 is true in the case when only one color appears in the coloring that is to be avoided; the case of two involved colors is also straightforward. We give a short argument showing that Conjecture 3.1 holds in the case when the partial coloring uses at most three colors.

Proposition 3.7 *If φ is a partial edge coloring of Q_d with at most three colors and every color appears on at most $d - 2$ edges, then φ is avoidable.*

Proof Let f be the standard edge coloring of Q_d, and consider the bipartite graph $B(f)$ with parts consisting of the color classes $C(f)$ of f and the colors $\{1, \ldots, d\}$ used in φ, and where an edge appears between a color i of φ and a color class M_j of f if and only if no edge of M_j is colored i under φ.

As in the proof of the preceding theorem, if there is a perfect matching in $B(f)$, then the coloring φ is avoidable; so suppose that this is not the case. Then there is an anti-Hall set $S \subseteq C(f)$, that is, a set $S \subseteq C(f)$, such that $|N(S)| < |S|$. Our goal is to prove that there is a coloring f' that can be obtained from f by interchanging colors on some 4-cycles, so that in the bipartite graph $B(f')$, defined as above, there is a perfect matching.

Now, if S is an anti-Hall set, then since every color in φ appears at most $d - 2$ times, $|S| \leq d - 2$. On the other hand, since at most 3 colors appear in the coloring φ, $|N(S)| \geq d - 3$, so $|S| \geq d - 2$; consequently, $|S| = d - 2$, that is, every dimensional matching in S contains edges of all three colors under φ, and thus there are two dimensional matchings in Q_d where no edges are colored under φ. Without loss of generality, we assume that M_1 is a dimensional matching with color 1 under f that is in S.

We pick a dimensional matching, M_d, with color d under f, say, not contained in the set S. Now, since M_d contains no φ-colored edges and M_1 contains three such edges, there is a 4-cycle in the edge-induced subgraph $Q_d[M_1 \cup M_d]$ containing at least one φ-colored edge. By interchanging colors on this 4-cycle, we obtain the required coloring f'. □

Remark 3.8 We remark that by using the same strategy it is straightforward to prove a version of the preceding result with four instead of three colors, provided that $d \geq 5$; indeed, the only essential difference is that one has to consider two different cases on the size of the anti-Hall set, namely, when it has size $d - 2$ and $d - 3$, respectively. However, for the case when $d = 4$, the only proof we have proceeds by long and detailed case analysis, so we abstain from giving the details in the case when the coloring to be avoided contains four different colors.

As a final observation of this section, let us consider the case when all precolored edges lie in a hypercube of dimension $d-1$ contained in a d-dimensional hypercube.

The following was first conjectured in [21].

Proposition 3.9 *If φ is a partial d-edge coloring of the hypercube Q_d ($d \geq 2$), where all colored edges lie in a subgraph that is isomorphic to Q_{d-1}, then φ is avoidable.*

Proof Let H_1 be a subgraph of Q_d that is isomorphic to Q_{d-1} and contains all precolored edges. Then Q_d consists of the two copies H_1 and H_2 of Q_{d-1} and a dimensional matching M joining vertices of H_1 and H_2.

We define a list assignment L for H_1 by setting $L(e) = \{1,\ldots,d\} \setminus \{\varphi(e)\}$, for every edge $e \in E(H_1)$, where we assume that $\{\varphi(e)\} = \emptyset$ if e is not colored under φ. By Galvin's Theorem 2.5, there is a proper d-edge coloring of H_1 with colors from the lists. Since H_1 and H_2 are isomorphic, this also yields a corresponding d-edge coloring of H_2. By coloring all edges of M by the unique color in $\{1,\ldots,d\}$ missing at its endpoints, we obtain a proper d-edge coloring of Q_d which avoids φ. $\square$

4 Avoiding Partial Proper Edge Colorings

In [21], Johansson presented a complete list of minimal unavoidable partial 3-edge colorings of Q_3, where *minimal* means that removing a color from any colored edge yields an avoidable edge coloring; the list is *complete* in the sense that it contains all such colorings up to permuting colors and/or applying graph automorphisms. There are 29 such configurations, and we refer to [21] for a comprehensive list of all such colorings. Let us here just remark that, based on this list of minimal unavoidable partial edge colorings, it seems to be a difficult task to characterize the family of unavoidable partial edge colorings of Q_d for general d. Note further that a similar investigation for complete bipartite graphs was pursued in [22].

Here, we shall focus on the unavoidable partial proper 3-edge colorings of Q_3. As explained in [21], there are six such minimal configurations.

Proposition 4.1 *The partial edge colorings of Q_3 in Fig. 1 constitute a complete list of minimal unavoidable partial proper 3-edge colorings of Q_3.*

The proof of this proposition is by an exhaustive computer search; we refer to [21] for details.

As in the non-proper case, based on this list of minimal unavoidable partial proper 3-edge colorings of Q_3, it seems difficult to make any specific conjecture as to whether it is possible to characterize the minimal unavoidable partial proper d-edge colorings of Q_d for general d. It is, however, easy to construct infinite families of minimal unavoidable partial (non-proper) 3-edge colorings of hypercubes. For the case when the coloring is required to be proper, this problem appears to be more difficult; in fact, we are interested in whether the following might be true:

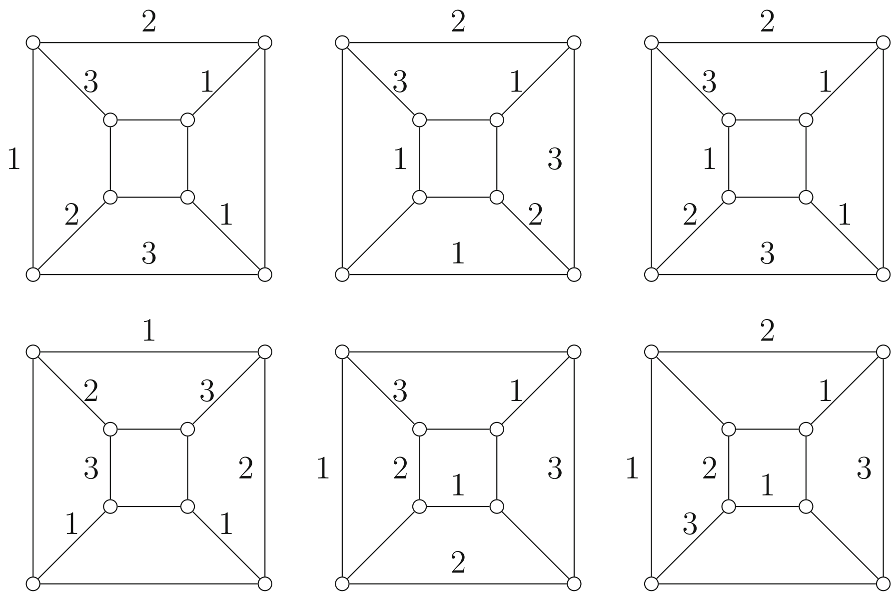

Fig. 1 Minimal unavoidable partial proper 3-edge colorings of Q_3

Problem 4.2 Is there an integer $d_0 \geq 0$ such that every partial proper d-edge coloring of Q_d is avoidable if $d \geq d_0$?

As mentioned in the introduction above, for the balanced complete bipartite graphs the answer to the corresponding question is positive and it suffices to require that the graph has at least 8 vertices [9, 10, 25].

Next, we shall deduce some general consequences of Proposition 4.1. We begin by considering the special case of Problem 4.2 when all colored edges are contained in a matching. We shall need the following lemmas, which are immediate from Proposition 4.1.

Lemma 4.3 *If φ is a partial 3-edge coloring Q_3 where all colored edges are contained in a matching, then φ is avoidable.*

We note that an analogous statement does not hold for Q_2, since the partial coloring where two non-adjacent edges of Q_2 are colored by 1 and 2, respectively, is unavoidable.

Lemma 4.4 *If φ is a partial proper 3-edge coloring of Q_3 where all colored edges are contained in two dimensional matchings, then φ is avoidable.*

Corollary 4.5 *If $d = 3k$ and φ is a partial d-edge coloring of Q_d where all colored edges are contained in a matching, then φ is avoidable.*

Proof Let $M_1, \ldots, M_d$ be the dimensional matchings in Q_d. For $i = 1, \ldots, k$, let H_i be the subgraph of Q_d induced by $M_{3i-2} \cup M_{3i-1} \cup M_{3i}$. By Lemma 2.3, each H_i is a collection of disjoint 3-dimensional hypercubes.

Now, by Lemma 4.3, there is a proper edge coloring of H_i using colors $3i-2, 3i-1, 3i$ that avoids the restriction of φ to H_i, for $i = 1, \ldots, k$. Combining such colorings yields a proper d-edge coloring of Q_d that avoids φ. □

Corollary 4.6 *If φ is a partial coloring of Q_d, $d \geq 3$, such that all edges colored i are in the same dimensional matching, $i = 1, \ldots, d$, then φ is avoidable.*

Proof Let $M_1, \ldots, M_d$ be the dimensional matchings of Q_d. If every dimensional matching contains edges of exactly one color from φ, then φ is extendable, and thus avoidable (by permuting colors).

If there is a dimensional matching M on which φ uses $2 \leq k \leq d-1$ colors, then assign one of the other $d-k$ colors to M. The graph $Q_d - M$ consists of two copies of Q_{d-1} and the restriction of φ to these two subgraphs satisfy the hypothesis of the corollary. For $d \geq 3$, proceed by induction; if $d-1=2$, then the restriction of φ to $Q_d - M$ only uses one color, and so it is avoidable.

In the remaining case, one dimensional matching, M_1 say, contains all the φ-colored edges. We assign colors $4, \ldots, d$ to $M_4, \ldots, M_d$, respectively, and consider the subgraph $Q_d - M_4 \cup \ldots \cup M_d$. This subgraph consists of a collection of Q_3's and the restriction of φ to each copy of Q_3 satisfies that all φ-colored edges lie in a matching. Hence, by Lemma 4.3, there is a 3-edge coloring of each copy H of Q_3 that avoids the restriction of φ to H. □

If we insist that all precolored edges are contained in a bounded number of dimensional matchings, then we obtain another family of avoidable partial (not necessarily proper) d-edge colorings of Q_d.

Corollary 4.7 *If φ is a partial d-edge coloring of Q_d where all colored edges are contained in $\lfloor d/3 \rfloor$ dimensional matchings, then φ is avoidable.*

Proof Suppose that $M_1, \ldots, M_a$ are the dimensional matchings that contain edges that are colored under φ, where $a = \lfloor d/3 \rfloor$. As in the proof of the preceding corollary, we decompose Q_d into $a = \lfloor d/3 \rfloor$ subgraphs $H_1, \ldots, H_a$ consisting of 3-dimensional hypercubes, and possibly one subgraph H_{a+1} that consists of disjoint copies of 1- or 2-dimensional hypercubes. Moreover, without loss of generality we assume that M_i is contained in H_i, $i = 1, \ldots, d$. The result now follows from Lemma 4.3 as in the proof of Corollary 4.5. □

If we require that the partial coloring is proper, then we can allow up to $2\lfloor d/3 \rfloor$ dimensional matchings in Q_d containing colored edges, while still being able to avoid the partial coloring.

Corollary 4.8 *If φ is a partial proper d-edge coloring of Q_d where all colored edges are contained in $2\lfloor d/3 \rfloor$ dimensional matchings, then φ is avoidable.*

The only difference in the proof of Corollary 4.8 compared to the proof of Corollary 4.7 is that we use Lemma 4.4 in place of Lemma 4.3; we omit the details.

A weaker and perhaps more tractable version of Problem 4.2 is obtained by requiring that every color class in the partial edge coloring to be avoided is an induced matching.

Conjecture 4.9 *If $d \geq 3$ and φ is a partial d-edge coloring of Q_d where every color class is an induced matching, then φ is avoidable.*

Using Proposition 4.1 and proceeding as in the proofs of the preceding Corollaries, we can prove the following stronger version of Conjecture 4.9 in the case when d is divisible by 3.

Corollary 4.10 *If $d = 3k$ and φ is a partial proper d-edge coloring of Q_d, where every precolored edge is at distance* 1 *from at most one other edge with the same color, then φ is avoidable.*

Finally, we shall prove that Conjecture 3.1 is true in the case when the partial edge coloring is proper. We shall need the following easy lemma.

Lemma 4.11 *If φ is a partial proper* 2*-edge coloring of Q_2, then φ is avoidable unless two non-adjacent edges are colored by different colors.*

Theorem 4.12 *If φ is a partial proper d-edge coloring of Q_d where every color appears on at most $d-2$ edges, then φ is avoidable.*

Proof If $d \leq 3$, then the theorem trivially holds by Proposition 4.1. If $d = 4$, then consider a dimensional matching M not containing edges of all colors under φ; such a matching exists, since every color appears on at most $d-2$ edges under φ. Suppose, for example, that color 4 does not appear on an edge of M.

The graph $Q_d - M$ consists of two disjoint copies H_1 and H_2 of Q_3. Moreover, since every color appears on at most two edges, it follows from Proposition 4.1 that there are proper 3-edge colorings (using colors 1, 2, 3) of H_1 and H_2 that avoid the restrictions of φ to H_1 and H_2, respectively. By coloring all edges of M by color 4, we obtain a proper 4-edge coloring of Q_d that avoids φ.

Next, we consider the case when $d \geq 5$. We shall consider two main cases, namely when $d = 2k$, and when $d = 2k + 1$.

Let us first consider the case when $d = 2k \geq 6$. Denote by $M_1, \ldots, M_{2k}$ the dimensional matchings of Q_d and consider the subgraphs $H_1, \ldots, H_k$ of Q_d, where H_i is the subgraph induced by $M_{2i-1} \cup M_{2i}$. We shall partition the colors in $\{1, \ldots, 2k\}$ into 2-subsets $A_1, \ldots, A_k$ and use the colors in A_i for a proper edge coloring of H_i that avoids the restriction of φ to H_i. Combining all these colorings yields a d-edge coloring of Q_d which avoids φ.

In total, there are $\frac{(2k)!}{2^k}$ ordered partitions of $\{1, \ldots, 2k\}$ into k 2-subsets. Now, some of these partitions $A_1 \cup \ldots \cup A_k$ are *forbidden* in the sense that for some i there is no proper edge coloring of H_i using colors from A_i that avoids the restriction of φ to H_i. If a copy of Q_2, which is contained in some subgraph H_i, contains at most three φ-colored edges, then by Lemma 4.11 at most $\frac{(2k-2)!}{2^{k-1}}$ partitions of $\{1, \ldots, 2k\}$ are forbidden due to this coloring of Q_2; similarly, if four edges of Q_2 are colored under φ, then at most $2\frac{(2k-2)!}{2^{k-1}}$ partitions are forbidden.

Now, since Q_d contains at most $d(d-2)$ φ-colored edges, at most $\frac{d(d-2)}{2}\frac{(2k-2)!}{2^{k-1}}$ partitions of $\{1, \ldots, 2k\}$ are forbidden due to the condition that the resulting d-edge coloring of Q_d should avoid φ. Thus, if

$$\frac{(2k)!}{2^k} - \frac{d(d-2)}{2}\frac{(2k-2)!}{2^{k-1}} > 0,$$

then there is a non-forbidden partition of $\{1, \ldots, 2k\}$. Since this inequality holds for any $k \geq 1$, the desired result follows.

Let us now consider the case when $d = 2k + 1$. The argument here is similar to the one given above. We partition Q_d into the subgraphs $H_1, \ldots, H_k$, where H_i is induced by the dimensional matchings $M_{2i-1} \cup M_{2i}$, $i = 1, \ldots, k-1$, and H_k is induced by $M_{2k-1}, M_{2k}, M_{2k+1}$. We now seek a partition of $\{1, \ldots, 2k+1\}$ into sets $A_1 \cup \ldots \cup A_k$, where $|A_i| = 2$, $i = 1, \ldots k-1$, and $|A_k| = 3$, and corresponding proper edge colorings of $H_1, \ldots, H_k$, where a coloring of H_i uses colors from A_i.

In total, there are $\frac{(2k+1)!}{2^{k-1}3!}$ such ordered partitions of $\{1, \ldots, 2k+1\}$. As before, some of these partitions are forbidden due to the fact the resulting edge coloring should avoid φ. We shall need the following claim.

Claim 4.13 *Let φ be a partial edge coloring of a copy H of the 3-dimensional hypercube Q_3 contained in H_k. Let $s(a)$ be the largest number of partitions of $\{1, \ldots, 2k+1\}$ that are forbidden due to the restriction of φ to H being unavoidable when a edges of H are colored. Then*

$$s(a) \leq \begin{cases} 0, & \text{if } a \leq 6, \\ \frac{(2k-2)!}{2^{k-1}}, & \text{if } 7 \leq a \leq 8, \\ 3\frac{(2k-2)!}{2^{k-1}}, & \text{if } a = 9, \\ 4\frac{(2k-2)!}{2^{k-1}}, & \text{if } a = 10, \\ 6\frac{(2k-2)!}{2^{k-1}}, & \text{if } a = 11, \\ 9\frac{(2k-2)!}{2^{k-1}}, & \text{if } a = 12. \end{cases}$$

Proof By Proposition 4.1, Figure 1 constitutes a complete list of minimal unavoidable partial proper 3-edge colorings of Q_3. Note that every such partial coloring contains three edges colored 1, two edges colored 2, and two edges colored 3. Thus, if H contains at most six φ-colored edges, then no partitions of $\{1, \ldots, 2k+1\}$ are forbidden due to the restriction of φ to H being unavoidable; that is, $s(a) = 0$ if $a \leq 6$. Similarly, if at most 8 different φ-colored edges appear in H, then at most $\frac{(2k-2)!}{2^{k-1}}$ partitions are forbidden, because there is at most one set of colors $\{a, b, c\}$ that cannot be used in a proper edge coloring of H that avoids φ.

If H contains 9 φ-colored edges, then at most $3\frac{(2k-2)!}{2^{k-1}}$ partitions are forbidden, since there could be four colors present on edges in H, one of which appears on three edges. Similarly, it is straightforward that $s(10) \leq 4\frac{(2k-2)!}{2^{k-1}}$, $s(11) \leq 6\frac{(2k-2)!}{2^{k-1}}$, and $s(12) \leq 9\frac{(2k-2)!}{2^{k-1}}$. □

Let b be the number of φ-colored edges that appear on edges in H_k. Then by using the same counting arguments as above and invoking Claim 4.13, we deduce that at most

$$(d(d-2)-b)\frac{1}{2}\frac{(2k-1)!}{2^{k-2}3!}+b\frac{9}{12}\frac{(2k-2)!}{2^{k-1}}$$

partitions of $\{1,\ldots,2k\}$ are forbidden due to the condition that the resulting d-edge coloring of Q_d should avoid φ. Thus, if

$$\frac{(2k+1)!}{2^{k-1}3!}-\left(\left(\frac{d(d-2)}{2}-\frac{b}{2}\right)\frac{(2k-1)!}{2^{k-2}3!}+b\frac{9}{12}\frac{(2k-2)!}{2^{k-1}}\right)>0,$$

then there is a non-forbidden partition of $\{1,\ldots,2k+1\}$. This holds if $k\geq 3$, and if $k=2$, then we can select the two dimensional matchings contained in H_1 to be maximal with respect to the property of containing φ-precolored edges. This implies that H_k contains at most nine φ-colored edges; that is, $b\leq 9$, and the required inequality holds. □

5 Extending and Avoiding Edge Colorings Simultaneously

In [6], it was proved that any partial proper coloring of at most $d-1$ edges of Q_d is extendable to a proper d-edge coloring of Q_d. Moreover, it was proved that any partial proper coloring of at most d edges in Q_d is extendable unless it satisfies one of the following conditions:

(C1) there is an uncolored edge uv in Q_d such that u is incident with edges of $r\leq d$ distinct colors and v is incident to $d-r$ edges colored with $d-r$ other distinct colors (so uv is adjacent to edges of d distinct colors);

(C2) there is a vertex u and a color c such that u is incident with at least one colored edge, u is not incident with any edge of color c, and every uncolored edge incident with u is adjacent to another edge colored c;

(C3) there is a vertex u and a color c such that every edge incident with u is uncolored and every edge incident with u is adjacent to another edge colored c;

(C4) $d=3$ and the three precolored edges use three different colors and form a subset of a dimensional matching.

For $i=1,2,3,4$, we denote by $\mathcal{C}_i$ the set of all colorings of Q_d, $d\geq 1$, satisfying the corresponding condition above, and we set $\mathcal{C}=\cup\mathcal{C}_i$.

Theorem 5.1 [6] *If φ is a partial proper d-edge coloring of at most d edges in Q_d, then φ is extendable to a proper d-edge coloring of Q_d unless $\varphi\in\mathcal{C}$.*

For $1\leq k\leq d$, let φ be a proper precoloring of $d-k$ edges of Q_d and ψ be a partial coloring of k edges in Q_d. Using the preceding theorem, we shall prove that there is a proper d-edge coloring of Q_d that agrees with φ and which avoids ψ unless one of the following conditions are satisfied:

(D1) there is a vertex v such that every edge incident with v is either ψ-colored c, φ-colored by a color distinct from c, or not colored under φ or ψ, but adjacent to an edge with color c under φ; or
(D2) exactly one edge uv is colored under ψ and for every $i \in \{1, \ldots, d\} \setminus \{\psi(uv)\}$ there is an edge incident with u or v that is colored i under φ; or
(D3) $d = 2$ and two non-adjacent edges are colored by different colors under ψ, or there is one edge e colored under φ and another edge e' colored under ψ, such that e and e' have different colors if they are adjacent, and the same color if they are non-adjacent.

Theorem 5.2 *Let φ be a proper d-edge precoloring of $d-k$ edges of Q_d and ψ be a partial coloring of k edges in Q_d, where $1 \le k \le d$. There is an extension of φ that avoids ψ unless some edge of Q_d has the same color under φ or ψ, or the colorings satisfy one of the conditions* (D1)-(D3).

Proof If Q_d contains altogether $d-1$ edges that are colored under φ and ψ (i.e. some edge is colored under both φ and ψ), then since at most $d-1$ edges are colored, we can form a new partial proper edge coloring from φ by greedily assigning some color from $\{1, \ldots, d\} \setminus \psi(e)$ to any edge e that is colored under ψ, but not colored under φ, so that the resulting coloring φ' is proper. By Theorem 5.1, φ' is extendable, so there is an extension of φ that avoids ψ.

Now assume that altogether exactly d edges are colored under φ and ψ, so no edge is colored under both φ and ψ. Let $E_{\varphi,\psi}$ be the set of edges in $E(Q_d)$ that are colored under φ or ψ. The case when $d \le 2$ is trivial, so assume that $d \ge 3$. We shall consider some different cases.

Suppose first that there are two non-adjacent edges e_1 and e_2 that are colored under ψ. Then we consider the coloring φ' obtained from φ by in addition coloring every ψ-colored edge in such a way that the resulting precoloring is proper and avoids ψ; since e_1 and e_2 are non-adjacent, this is possible. At most d edges are colored under the resulting coloring φ', so if it is not extendable, then $\varphi' \in \mathcal{C}$.

If $\varphi' \in \mathcal{C}_1$, then there is an uncolored edge uv in Q_d such that u is incident with edges of $r \le d$ distinct colors under φ' and v is incident to $d-r$ edges φ'-colored with $d-r$ other distinct colors. Suppose without loss of generality that e_1 is incident with u, e_2 is incident with v and that at least two φ'-colored edges are incident with u. Then we can define a new edge coloring of $E_{\varphi,\psi}$ from φ' that avoids ψ by recoloring e_2 by some color that appears at u. The obtained partial edge coloring is not in $\mathcal{C}$, and thus there is an extension of φ that avoids ψ.

If $\varphi' \in \mathcal{C}_3 \cup \mathcal{C}_4$, then since all edges in $E_{\varphi,\psi}$ are non-adjacent, we can recolor the edges that are colored under both φ' and ψ to obtain a proper coloring of $E_{\varphi,\psi}$ that avoids ψ and is extendable to a proper d-edge coloring. Hence, there is an extension of φ that avoids ψ.

Suppose now that $\varphi' \in \mathcal{C}_2$. Since e_1 and e_2 are non-adjacent, at least one of them is not adjacent to any other edge from $E_{\varphi,\psi}$. Thus, we may recolor this edge and a similar argument as in the preceding paragraph shows that there is an extension of φ that avoids ψ.

Suppose now that there are at least two edges colored under ψ and that all such edges are pairwise adjacent. Thus there is some vertex v that is incident with every edge that is colored under ψ. If we cannot define a new proper coloring φ' of $E_{\varphi,\psi}$ from φ by coloring the ψ-colored edges in such a way that φ' avoids ψ, then all ψ-colored edges are colored by the same color. Moreover, if there is no such coloring φ', then all φ-colored edges are incident with v and have colors that are distinct from the ψ-colored edges; that is, (D1) holds.

Let us now consider the case when we can define a coloring φ' as described in the preceding paragraph. Then φ' is extendable, unless $\varphi' \in \mathcal{C}_1 \cup \mathcal{C}_2$.

Suppose first that $\varphi' \in \mathcal{C}_1$. Then, since all colors in $\{1, \ldots, d\}$ appear on edges under φ', there must be some φ-colored edge incident with u; suppose that such an edge has color c under φ. Then no edge incident with v is φ'-colored c, because $\varphi' \in \mathcal{C}_1$. Now, if there is such a color c, such that, in addition, some ψ-colored edge e incident with v is not colored c under ψ, then we can, from φ', define a new coloring φ'' of $E_{\varphi,\psi}$ by recoloring e by the color c. Since $\varphi'' \notin \mathcal{C}$, it is extendable. In conclusion, there is an extension of φ that avoids ψ unless (D1) holds. A similar argument applies if $\varphi' \in \mathcal{C}_2$.

It remains to consider the case when exactly one edge $e = uv$ is colored under ψ. If we cannot pick some color for e that is distinct from $\psi(e)$ and satisfies that this coloring of e taken together with φ is proper, then φ and ψ satisfy (D2). On the other hand, if we can define such a coloring φ' of $E_{\varphi,\psi}$ from φ which avoids ψ, then there is an extension of φ that avoids ψ unless $\varphi' \in \mathcal{C}$.

If $\varphi' \in \mathcal{C}_3$ or $\varphi' \in \mathcal{C}_4$, then since all φ'-colored edges are pairwise non-adjacent, we can define a new proper coloring of $E_{\varphi,\psi}$ from φ' that is extendable, and which avoids ψ.

If $\varphi' \in \mathcal{C}_1$, then we can similarly define a new proper coloring φ'' of $E_{\varphi,\psi}$ that is extendable, unless exactly one φ-colored edge e' is not adjacent to e and $\varphi(e') = \psi(e)$; that is, (D1) holds.

Finally, if φ' satisfies (C2), then a similar argument shows that we can define a new extendable partial edge coloring of $E_{\varphi,\psi}$ that avoids ψ, unless φ and ψ satisfy (D1). $\square$

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Declarations

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