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ORIGINAL PAPER

Note on Distance Magic Products $G \circ C_4$

Marcin Anholcer · Sylwia Cichacz

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Abstract A distance magic labeling of a graph $G = (V, E)$ of order n is a bijection $l\colon V \to \{1, 2, \ldots, n\}$ with the property that there is a positive integer k (called *magic constant*) such that $w(x) = k$ for every $x \in V$. If a graph G admits a distance magic labeling, then we say that G is a *distance magic graph*. In the case of non-regular graph G, the problem of determining whether there is a distance magic labeling of the lexicographic product $G \circ C_4$ was posted in Arumugam et al. (J Indonesian Math Soc 11–26, 2011). We give necessary and sufficient conditions for the graphs $K_{m,n} \circ C_4$ to be distance magic. We also show that the product $C_3^{(t)} \circ C_4$ of the Dutch Windmill Graph and the cycle C_4 is not distance magic for any $t > 1$.

Keywords Distance magic labeling · Magic constant · Sigma labeling · Graph labeling · Composition of graphs · Lexicographic product of graphs

Mathematics Subject Classification (2010) 05C78

1 Introduction

All graphs considered in this paper are simple finite graphs. Given a graph G, we denote its order by $|G| = n$, its vertex set by $V(G)$ and the edge set by $E(G)$. The

M. Anholcer (✉)
Faculty of Informatics and Electronic Economy, Poznań University of Economics, Al. Niepodległości 10, 61-875 Poznan, Poland
e-mail: m.anholcer@ue.poznan.pl

S. Cichacz
Faculty of Applied Mathematics, AGH University of Science and Technology, Al. Mickiewicza 30, 30-059 Kraków, Poland
e-mail: cichacz@agh.edu.pl

neighborhood $N(x)$ of a vertex x is the set of vertices adjacent to x, and the degree $d(x)$ of x is $|N(x)|$, the size of the neighborhood of x.

Let $w(x) = \sum_{y \in N_G(x)} l(y)$ for every $x \in V(G)$.

A *distance magic labeling* (also called *sigma labeling*) of a graph $G = (V, E)$ of order n is a bijection $l : V \to \{1, 2, \ldots, n\}$ with the property that there is a positive integer k (called *magic constant*) such that $w(x) = k$ for every $x \in V$. If a graph G admits a distance magic labeling, then we say that G is a *distance magic graph* (see [13]).

The concept of distance magic labeling has been motivated by the construction of magic squares. Finding a distance magic labeling of an r-regular graph turns out to be equivalent to finding equalized incomplete tournament EIT(n, r) [4]. In an *equalized incomplete tournament* EIT(n, r) of n teams with r rounds, each team plays with exactly r other teams and the total strength of the opponents that team i plays is k. Thus, it is easy to observe that finding an EIT(n, r) is the same as finding a distance magic labeling of any r-regular graph on n vertices. For a survey, we refer the reader to [2].

The following observations were independently proved:

Observation 1.1 ([8–10,13]) *Let G be an r-regular distance magic graph on n vertices. Then $k = \frac{r(n+1)}{2}$.*

Observation 1.2 ([8–10,13]) *No r-regular graph with r-odd can be a distance magic graph.*

The problem of distance magic labeling of r-regular graphs was studied recently (see [1–4,9,11]). It is interesting that if you blow up an r-regular G graph into some specific p-regular graph, then the obtained graph H is distance magic. More formally, we have the following definition.

Definition 1.3 ([7], p. 185) The lexicographic product $G \circ H$ of two graphs G and H is defined on $V(G \circ H) = V(G) \times V(H)$, two vertices (u, x), (v, y) of $G \circ H$ being adjacent whenever $uv \in E(G)$, or $u = v$ and $xy \in E(H)$.

$G \circ H$ is also called the composition of graphs G and H and denoted by $G[H]$ (see [6]).

Miller at al. [9] proved the following results.

Theorem 1.4 ([9]) *The cycle C_n of length n is a distance magic graph if and only if $n = 4$.*

Theorem 1.5 ([9]) *Let $r \geq 1$, $n \geq 3$, G be an r-regular graph and C_n the cycle of length n. Then $G \circ C_n$ admits a distance magic labeling if and only if $n = 4$.*

Theorem 1.6 ([9]) *Let G be an arbitrary regular graph. Then $G \circ \overline{K}_n$ is distance magic for any even n.*

Shafiq et al. [12] considered distance magic labeling for disconnected graphs and obtained the following theorems.

Theorem 1.7 ([12]) *Let $m \geq 1$, $n \geq 2$ and $p \geq 3$. Then $mC_p \circ K_n$ has a distance magic labeling if and only if either n is even or mnp is odd or n is odd and $p \equiv 0 (\mathrm{mod}\ 4)$.*

The following problem was posted in [2].

Proposition 1.8 ([2]) *If G is non-regular graph, determine if there is a distance magic labeling of $G \circ C_4$.*

The *Dutch Windmill Graph* $C_3^{(t)}$, also called a *friendship graph*, is the graph obtained by taking $t > 1$ copies of the cycle graph C_3 with a vertex in common [5]. We show that the product $C_3^{(t)} \circ C_4$ is not distance magic for any $t > 1$.

The paper is organized as follows. In the next section we focus on the products of complete bipartite graphs and cycle C_4. In the third section we prove that the product of the Dutch Windmill Graph and the cycle C_4 cannot be distance magic.

2 The Product $K_{m,n} \circ C_4$

Let $K_{m,n}$ have the vertex partite sets $A = \{x_0, x_1, \ldots, x_{n-1}\}$ and $B = \{y_0, y_1, \ldots, y_{m-1}\}$. Let $C_4 = v^0v^1v^2v^3v^0$ and $H = K_{m,n} \circ C_4$. For $0 \leq i \leq n-1$ and $j = 0, 1, 2, 3$, let x_i^j be the vertices of H that replace x_i in A. For $0 \leq i \leq m-1$ and $j = 0, 1, 2, 3$, let y_i^j be the vertices of H that replace y_i in B. Let $A[C_4] = \{x_i^j : i = 0, 1, \ldots, n-1, j = 0, 1, 2, 3, x_i \in A\}$ and $B[C_4] = \{y_i^j : i = 0, 1, \ldots, m-1, j = 0, 1, 2, 3, y_i \in B\}$.

The following Lemma holds true.

Lemma 2.1 *If $H = K_{m,n} \circ C_4$, where $1 \leq m < n$ is a distance magic graph and k is the magic constant, then the following conditions hold:*

(1) *$l(x_i^0) + l(x_i^2) = l(x_i^1) + l(x_i^3) = a$ for some constant a for all $0 \leq i \leq n-1$ and $l(y_i^0) + l(y_i^2) = l(y_i^1) + l(y_i^3) = b$ for some constant b for all $0 \leq i \leq m-1$,*
(2) *$b + 2an = a + 2mb = k$ and $a < b$,*
(3) *$bm + an = (m+n)(4m+4n+1)$,*

Proof (1)
Notice that: $w(x_i^j) = l(x_i^{j+1}) + l(x_i^{j+3}) + \sum_{i=1}^{m} \sum_{j=1}^{4} l(y_i^j)$ for all $0 \leq i \leq n-1$, $j = 0, 1, 2, 3$, where the addition in the superscripts is performed modulo 4. Since the graph H is distance magic we obtain that $l(x_i^0) + l(x_i^2) = l(x_i^1) + l(x_i^3) = a$ for some constant a for all $0 \leq i \leq n-1$. Similarly $l(y_i^0) + l(y_i^2) = l(y_i^1) + l(y_i^3) = b$ for some constant b for all $0 \leq i \leq m-1$.

(2) Fact (1) implies that $w(x_i^j) = a + 2bm$ for all $0 \leq i \leq n-1$, $j = 0, 1, 2, 3$ and $w(y_i^j) = b + 2an$ for all $0 \leq i \leq m-1$, $j = 0, 1, 2, 3$. As $m < n$, this implies that $a < b$.

(3) The labeling l is a bijection, so the sum of all labels has to be equal to $\sum_{i=1}^{4m+4n} i$:

$$2an + 2bm = \frac{(4m+4n)(4m+4n+1)}{2}.$$

□

The following theorem completely characterizes the pairs (m, n), for which $K_{m,n} \circ C_4$ is distance magic.

Theorem 2.2 *Let m and n be integers such that $1 \le m < n$. Then $K_{m,n} \circ C_4$ is distance magic if and only if the following conditions hold.*

(1) *The numbers*

$$a = \frac{(m+n)(4m+4n+1)(2m-1)}{4mn-m-n}$$

and

$$b = \frac{(m+n)(4m+4n+1)(2n-1)}{4mn-m-n}$$

are integers.

(2) *There exist integers $p, q, t \ge 1$, such that*

$$\begin{aligned} p+q &= b-a, \\ 4n &= pt, \\ 4m &= qt. \end{aligned}$$

Proof First, let us assume that for given m and n, $1 \le m < n$ there exist a, b, p, q and t with desired properties. Then the following labeling is distance-magic: If $t = 4s$ for some integer s, then let

$$l(x^j_{kp+i}) = \begin{cases} k(2p+2q)+i+1 & \text{for } j=0 \\ k(2p+2q)+p+q+i+1 & \text{for } j=1, \\ a - f(x^{j-2}_{kp+i}) & \text{for } j=2,3, \end{cases}$$

for $0 \le k \le t/4-1$, $i = 0, 1, \dots, p-1$,

$$l(y^j_{kq+i}) = \begin{cases} k(2p+2q)+p+i+1 & \text{for } j=0, \\ k(2p+2q)+2p+q+i+1 & \text{for } j=1, \\ b - f(y^{j-2}_{kq+i}) & \text{for } j=2,3. \end{cases}$$

for $0 \le k \le t/4-1$, $i = 0, 1, \dots, q-1$. Observe that the sets of labels of vertices x^j_i and y^j_i for $j = 0, 1$ do not intersect and their elements are consecutive numbers from the set $\{1, \dots, 2(m+n)\}$. And as $b-a = p+q$, also the sets of labels for $j = 2, 3$ do not intersect and they are consecutive numbers from the set $\{a-(2m+2n)+q, \dots, a+q-1 = b-p-1\}$. In order to prove that l is a bijection it is enough to show that $a+q-1 = 4m+4n$. It is true, as we have:

$$a(n+m) + (b-a)m = an + bm = (m+n)(4m+4n+1).$$

But on the other hand,

$$(b-a)m = (p+q)m = \frac{4(m+n)m}{t} = (m+n)q,$$

so finally

$$a(n+m) + (m+n)q = (m+n)(4m+4n+1)$$

and $a + q = 4m + 4n + 1$.

If $t \not\equiv 0 \pmod 4$, then p and q are even. In such a situation, let

$$l(x^j_{kp/2+i}) = \begin{cases} k(p+q) + 2i + j + 1 & \text{for } j = 0, 1, \\ a - f(x^{j-2}_{kp/2+i}) & \text{for } j = 2, 3, \end{cases}$$

for $0 \le k \le \lfloor t/2 \rfloor - 1, i = 0, 1, \ldots, p/2 - 1$, and for $k = \lfloor t/2 \rfloor = \lceil t/2 \rceil - 1$, $i = 0, 1, \ldots, p/4 - 1$,

$$l(y^j_{kq/2+i}) = \begin{cases} k(p+q) + p + 2i + j + 1 & \text{for } j = 0, 1, \\ b - f(y^{j-2}_{kq/2+i}) & \text{for } j = 2, 3, \end{cases}$$

for $0 \le k \le \lfloor t/2 \rfloor - 1, i = 0, 1, \ldots, q/2 - 1$, and

$$l(y^j_{kq/2+i}) = \begin{cases} k(p+q) + p/2 + 2i + j + 1 & \text{for } j = 0, 1, \\ b - f(y^{j-2}_{kq/2+i}) & \text{for } j = 2, 3, \end{cases}$$

for $k = \lfloor t/2 \rfloor = \lceil t/2 \rceil - 1, i = 0, 1, \ldots, q/4 - 1$ (observe that in both cases the last range is in use if and only if t is odd and thus p and q are divisible by 4). Also in this case it is straightforward to see that the above labeling is bijective and that in both cases the magic constant equals to $k = a + 2mb = b + 2na$.

Now, let us assume that $K_{m,n} \circ C_4$ is distance magic for some integers m and n, $1 \le m < n$. Let the magic constant be k. From the Lemma 2.1 it follows that the following system of equations must be satisfied:

$$\begin{cases} b + 2an = a + 2mb, \\ bm + an = (m+n)(4m+4n+1). \end{cases}$$

The above system has only one solution with respect to a and b:

$$\begin{cases} a = \frac{(m+n)(4m+4n+1)(2m-1)}{4mn-m-n}, \\ b = \frac{(m+n)(4m+4n+1)(2n-1)}{4mn-m-n}. \end{cases}$$

Obviously, a and b must be integers.

Now we choose any distance magic labeling l of $K_{m,n} \circ C_4$. Let $L_A = \{l(x)|x \in A[C_4]\}$ and $L_B = \{l(y)|y \in B[C_4]\}$. Let us divide the set of all labels $\{1, 2, \ldots, 4m + 4n\}$ into intervals in the following way:

(i) for each interval I, either $I \subseteq L_A$ or $I \subseteq L_B$,
(ii) each interval is maximal, i.e., for any two neighboring intervals I_1, I_2 we have either $I_1 \subseteq L_A$ and $I_2 \subseteq L_B$ or $I_1 \subseteq L_B$ and $I_2 \subseteq L_A$.

In the remainder we will use the notation $I_1 < I_2 \Leftrightarrow \max\{l(x)|l(x) \in I_1\} < \min\{l(y)|l(y) \in I_2\}$. For any interval I and integer c, let $c - I = \{c - l(x)|l(x) \in I\}$ (it is possible that $c - I = I$). From the Lemma 2.1 it follows that for each $x \in \{1, 2, \ldots, 4m + 4n\}$, $l(x) \in L_A \Leftrightarrow a - l(x) \in L_A$ (in fact, $l(x_i^j) = a - l(x_i^{j+2})$, where the addition in the superscripts is performed modulo 4). Similarly, $l(x) \in L_B \Leftrightarrow b - l(x) \in L_B$. This implies that $a - I \subseteq L_A \Leftrightarrow I \subseteq L_A$ and $b - I \subseteq L_B \Leftrightarrow I \subseteq L_B$. Also, if $I_1, I_2 \subseteq L_A$ and $I_1 < I_2$ then $a - I_2 < a - I_1$. Similarly, if $I_1, I_2 \subseteq L_B$ and $I_1 < I_2$ then $b - I_2 < b - I_1$.

Let the first two intervals be $I_1 = \{1, \ldots, r\}$ and $I_2 = \{r + 1, \ldots, r + s\}$ for some $r, s \geq 1$.

Observe first that $I_1 \subseteq L_A$ and $I_2 \subseteq L_B$. Otherwise we would have $b - I_1 = \{b - r, \ldots, b - 1\}$ and $a - I_2 = \{a - r - s, \ldots, a - r - 1\} < b - I_1$. Moreover, as $a - r - 1 < b - r - 1$, this would imply that there is an interval $I \in L_A$, $a - I_2 < I$ and thus $a - I < I_2$, a contradiction (I_2 is the first interval being subset of L_A).

We have $a - I_1 = \{a - r, \ldots, a - 1\}$ and $b - I_2 = \{b - r - s, \ldots, b - r - 1\}$. As $\min\{l(x)|l(x) \in a - I_1\} - 1 = a - r - 1 < b - r - 1 = \max\{l(y)|l(y) \in b - I_2\}$ and the intervals $a - I_1$ and $b - I_2$ are disjoint, it follows that $a - I_1 < b - I_2$. Moreover there is no integer u such that $a - 1 < u < b - r - s$, as it would mean that there is an interval $I \subseteq L_A$, $a - I_1 < I$ and thus $a - I < I_1$, a contradiction. Thus the intervals $a - I_1$ and $b - I_2$ consist of $r + s$ consecutive integers. Moreover, the first entry $b - r - s$ of $b - I_2$ follows immediately after the last entry of $a - I_1$, which is $a - 1$. Hence, we have $b - r - s = a$ and thus $r + s = b - a$. Observe also that the intervals $a - I_1$ and $b - I_2$ are the last ones contained in L_A and L_B respectively.

If $r + s = 4m + 4n$, this proves the hypothesis ($r = 4n$, $s = 4m$, so $t = 1$, $p = r/2$, $q = s/2$). Otherwise let us assume that we are given $d \geq 1$ pairs of intervals (I_1^i, I_2^i), $i = 1, \ldots, d$, such that $I_1^i \subseteq L_A$, $I_2^i \subseteq L_B$, $a - I_1^i \subseteq L_A$, $b - I_2^i \subseteq L_B$, $|I_1^i| = r$, $|I_2^i| = s$, $I_1^i < I_2^i$, $a - I_1^i < b - I_2^i$ for $i = 1, \ldots, d$ and $I_1^i < I_1^{i+1}$, $I_2^i < I_2^{i+1}$, $a - I_1^{i+1} < a - I_j^i$, $b - I_2^{i+1} < b - I_2^i$ for $i = 1, \ldots, d - 1$. Moreover, let us assume that there are no elements $u < \max\{l(y)|l(y) \in I_2^d\}$, $u \notin \bigcup_{i=1}^d (I_1^i \cup I_2^i)$ and no elements $v > \max\{l(x)|l(x) \in a - I_1^d\}$, $v \notin \bigcup_{i=1}^d (a - I_1^i \cup b - I_2^i)$. We are going to prove that we are able to extend this sequence to $d + 1$ pairs of intervals.

Indeed, let us assume, that next two intervals are $I_1 = \{d(r+s)+1, \ldots, d(r+s)+r_1\}$ and $I_2 = \{d(r+s)+r_1+1, \ldots, d(r+s)+r_1+s_1\}$ for some $r_1, s_1 \geq 1$. Obviously $I_1 \subseteq L_A$ and $I_2 \subseteq L_B$. Moreover we have $a - I_1 = \{a - d(r + s) - r_1, \ldots, a - d(r + s) - 1\}$ and $b - I_2 = \{b - d(r + s) - r_1 - s_1, \ldots, b - d(r + s) - r_1 - 1\}$. As $\min\{l(x)|l(x) \in a - I_1\} - 1 < \max\{l(y)|l(y) \in b - I_2\}$, it follows that $a - I_1 < b - I_2$. Moreover there is no integer u between $a - I_1$ and $b - I_2$, as there is no interval $I \subseteq L_A$, $a - I_1 < I$, $I \notin \{I_1^1, \ldots, I_1^d\}$. Thus the intervals $a - I_1$ and $b - I_2$ consist of $r + s$

consecutive integers. Thus $a - d(r+s) - 1 = b - d(r+s) - r_1 - s_1 - 1$ and in consequence $r_1 + s_1 = b - a$. Similar reasoning leads us to the conclusion that there are no elements between $b - I_2$ and $a - I_1^d$, so $b - d(r+s) - r_1 - 1 = a - (d-1)(r+s) - r - 1$ and thus $b - a = r_1 + s$. This means that $r_1 = r$ and $s_1 = s$. Obviously the intervals $a - I_1$ and $b - I_2$ are the last ones contained in L_A and L_B, that do not belong to $\{I_1^1, \ldots, I_1^d\}$ and $\{I_2^1, \ldots, I_2^d\}$, respectively.

By induction we obtain that we are able to construct such a sequence of pairs for every d. As the number of pairs (I_1^i, I_2^i) has to be finite, after some number of steps (say t) we exhaust all labels. Obviously

$$rt = \sum_{i=1}^{t} |I_1^i| = 4n,$$

and

$$st = \sum_{i=1}^{t} |I_2^i| = 4m.$$

Putting $p = r$ and $q = s$, we arrive at the hypothesis. □

The pairs (m, n) that satisfy the assumptions of the Theorem 2.2 are very rare. We checked all the pairs where $1 \le m < n \le 80000$ and only for the following ones the graphs $K_{m,n} \circ C_4$ are distance magic: (9, 21), (20, 32), (428, 548), (2328, 2748), (6408, 10368), (7592, 8600), (10098, 24378), (18860, 20840), (39540, 42972), (73808, 79268).

3 The Product $C_3^{(t)} \circ C_4$

Let $C_3^{(t)}$ have the central vertex x and vertices x, y_i, z_i for $i = 1, \ldots, t$ belong to ith copy of cycle C_3. Let $C_4 = v^0 v^1 v^2 v^3 v^0$ and $H = C_3^{(t)} \circ C_4$. For $0 \le i \le t-1$ and $j = 0, 1, 2, 3$, let y_i^j, z_i^j be the vertices of H that replace y_i^j, z_i^j $0 \le i \le t-1$ in $C_3^{(t)}$ and x^0, x^1, x^2, x^3 be the vertices of H that replace x.

Theorem 3.1 *The graph $C_3^{(t)} \circ C_4$ is not distance magic for any $t > 1$.*

Proof Suppose that l is a distance magic labeling of the graph $H = C_3^{(t)} \circ C_4$ and $k = w(x)$, for all vertices $x \in V(H)$. It is easy to observe that there exist natural numbers b, a_y^i and a_z^i, $0 \le i \le t-1$, such that:

- $l(x^0) + l(x^2) = l(x^1) + l(x^3) = b$.
- $l(y_i^0) + l(y_i^2) = l(y_i^1) + l(y_i^3) = a_y^i$ for $0 \le i \le t-1$.
- $l(z_i^0) + l(z_i^2) = l(z_i^1) + l(z_i^3) = a_z^i$ for $0 \le i \le t-1$.

Since $a_y^i + 2a_z^i + 2b = w(y_i^j) = w(z_i^j) = a_z^i + 2a_y^i + 2b$, we obtain that $a_y^i = a_z^i = a^i$. This implies that for any $0 \le i, l \le t-1$ and $0 \le j, h \le 3$, $3a^i + 2b = w(z_i^j) = w(z_l^h) = 3a^l + 2b$, hence $a^i = a^l = a$.

Since $3a + 2b = b + 4ta = k$ and $4ta + 2b = 1 + 2 + \cdots + 4(2t + 1)$ we obtain that $b = \frac{(2t+1)(4t-3)(8t+5)}{6t-3}$. Recall that the biggest label we can use is $4(2t+1)$, hence $b \leq 16t + 7$. One can calculate that the only positive integer t that satisfies the inequality

$$(2t + 1)(4t - 3)(8t + 5) \leq (16t + 7)(6t - 3)$$

is $t = 1$, a contradiction. □

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