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ORIGINAL PAPER

Large Sets of Hamilton Cycle and Path Decompositions of Complete Bipartite Graphs

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Abstract In this paper, we determine the existence spectrums for large sets of Hamilton cycle and path (resp. directed Hamilton cycle and path) decompositions of $\lambda K_{m,n}$ (resp. $\lambda K^*_{m,n}$).

Keywords Large set · Hamilton cycle · Hamilton path · Decomposition · Complete automorphism group

1 Introduction

Throughout this paper, let $\lambda K_{m,n}$ (resp. $\lambda K^*_{m,n}$) be the complete bipartite multigraph (resp. multi-digraph) with two partite sets Z_m and $\overline{Z}_n$. Without loss of generality, we suppose $m \geq n$ in $\lambda K_{m,n}$ and $\lambda K^*_{m,n}$. In this paper, we use the convention that if λ is not specified, then $\lambda = 1$. A *k-cycle* (resp. *k-path*) is a subgraph of $K_{m,n}$ with k vertices $x_1, x_2, \ldots, x_k$ and k edges $\{x_1, x_2\}, \ldots, \{x_{k-1}, x_k\}, \{x_k, x_1\}$ (resp. $k-1$ edges $\{x_1, x_2\}, \ldots, \{x_{k-1}, x_k\}$), which is denoted by $(x_1, x_2, \ldots, x_k)$ (resp. $[x_1, x_2, \ldots, x_k]$). A *directed k-cycle* (resp. *directed k-path*) is a subgraph of $K^*_{m,n}$

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with k vertices $x_1, x_2, \ldots, x_k$ and k arcs $(x_1, x_2), (x_2, x_3), \ldots, (x_{k-1}, x_k), (x_k, x_1)$ [resp. $k-1$ arcs $(x_1, x_2), \ldots, (x_{k-1}, x_k)$], which is denoted by $\langle x_1, x_2, \ldots, x_k \rangle$ (resp. $\prec x_1, x_2, \ldots, x_k \succ$). When $k = m + n$, a (directed) k-cycle is called a (*directed*) *Hamilton cycle*, a (directed) k-path is called a (*directed*) *Hamilton path*. It is easy to see that

if there exists a Hamilton cycle (resp. directed Hamilton cycle) in $K_{m,n}$ (resp. $K^*_{m,n}$), then $m = n$;

if there exists a Hamilton path (resp. directed Hamilton path) in $K_{m,n}$ (resp. $K^*_{m,n}$), then $m = n$ or $n + 1$.

A *Hamilton cycle* (resp. *directed Hamilton cycle*) *decomposition* of $\lambda K_{n,n}$ (resp. $\lambda K^*_{n,n}$), HC(n, n, λ) (resp. DHC(n, n, λ)), is a $(Z_n \cup \overline{Z}_n, \mathcal{A})$, where $\mathcal{A}$ is a collection of Hamilton cycles (resp. directed Hamilton cycles), called *blocks*, in $K_{n,n}$ (resp. $K^*_{n,n}$), which form a partition of edge (resp. arc) set of $\lambda K_{n,n}$ (resp. $\lambda K^*_{n,n}$). When $m = n$ or $n + 1$, we can similarly define a *Hamilton path* (resp. *directed Hamilton path) decomposition* of $\lambda K_{m,n}$ (resp. $\lambda K^*_{m,n}$), which is denoted by HP(m, n, λ) (resp. DHP(m, n, λ)). A decomposition is said to be *simple* if it contains no repeated blocks.

A *large set* of Hamilton cycle (resp. directed Hamilton cycle) decomposition of $\lambda K_{n,n}$ (resp. $\lambda K^*_{n,n}$), LHC(n, n, λ) [resp. LDHC(n, n, λ)], is a partition of all Hamilton cycles (resp. directed Hamilton cycles) of $K_{n,n}$ (resp. $K^*_{n,n}$) into HC(n, n, λ)s [resp. DHC(n, n, λ)s]. When $m = n$ or $n + 1$, we can similarly define a *large set* of Hamilton path (resp. directed Hamilton path) decomposition of $\lambda K_{m,n}$ (resp. $\lambda K^*_{m,n}$), which is denoted by LHP(m, n, λ) [resp. LDHP(m, n, λ)]. It is easy to see that every decomposition in a large set is simple.

Let λK_n (resp. λK^*_n) denote the complete multigraph (resp. multi-digraph) on n vertices. A *Hamilton cycle* (resp. *path*) of K_n is a n-cycle (resp. n-path) of K_n. An *almost Hamilton cycle* (resp. *path*) of K_n is a $(n-1)$-cycle [resp. $(n-1)$-path] of K_n. There are similar definitions of directed Hamilton cycle and path of K^*_n. As well, there are similar definitions of (almost) Hamilton cycle and path decomposition of λK_n, of directed Hamilton cycle and path decomposition of λK^*_n.

Lemma 1 [1,10] *There exists a large set of Hamilton cycle (resp. path) decompositions of* $\lambda K_n (\lambda K_{n-1})$ *if and only if* $2|\lambda(n-1)$ *and* $\lambda|(n-2)!$.

Lemma 2 [11] *There exists a large set of almost Hamilton cycle decomposition of* $2K_n$ *for any* $n \equiv 0, 1 (\mathrm{mod} 4)$ *except* $n = 5$.

Lemma 3 [10] *There exists a large set of directed Hamilton cycle (resp. path) decomposition of* λK^*_n *(resp.* λK^*_{n-1}*) for any* $n \geq 3$ *and* $n \neq 4, 6$ *with possible exceptions* $n \in \{p + 1 : \textit{prime } p \geq 23\}$.

Lemma 4 [3] *There exists an LHC(2m,2m,1) for any positive integer m.*

There are many other classical problems about large sets. Please refer [6–8] for *large sets of Steiner triple systems*, [5] for *large sets of Mendelsohn triple systems*, [4] for *large sets of transitive triple systems*, etc. In this paper, we will determine the existence spectrums for large sets of Hamilton cycle and path (resp. directed Hamilton cycle and path) decompositions of $\lambda K_{m,n}$ (resp. $\lambda K^*_{m,n}$).

2 Small Designs

Obviously, an HC(n, n, λ) consists of $\frac{\lambda n^2}{2n} = \frac{\lambda n}{2}$ blocks. Hence,

$$\text{if there exists an HC}(n, n, \lambda)\text{, then } \begin{cases} \text{even } n \geq 2 \ \text{ for any } \lambda; \\ \text{odd } n \geq 3 \ \text{ for even } \lambda. \end{cases}$$

So, the necessary conditions for the existence of a DHC(n, n, λ) are $n > 1$ for any λ.

Lemma 5 *There exists an HC$(2m, 2m, \lambda)$ for positive integers m and λ.*

Proof Define the collection $\mathcal{A}$ of the following m Hamilton cycles

$$C_i = (0, \overline{2i}, 1, \overline{2i+1}, \ldots, 2m-1, \overline{2i+2m-1}), \quad 0 \leq i \leq m-1,$$

where $\overline{2i+j} \in \overline{Z}_{2m}$ for $0 \leq i \leq m-1, 0 \leq j \leq 2m-1$. It is easy to verify that $(Z_{2m} \bigcup \overline{Z}_{2m}, \mathcal{A})$ is an HC$(2m, 2m, 1)$. Repeating every $C_i \lambda$ times, we obtain an HC$(2m, 2m, \lambda)$. □

Lemma 6 *There exists an $HC(2m+1, 2m+1, 2\lambda)$ for positive integers m and λ.*

Proof Define the collection $\mathcal{A}$ of the following $2m+1$ Hamilton cycles

$$D_i = (0, \overline{i}, 1, \overline{i+1}, \ldots, 2m, \overline{i+2m}), \quad 0 \leq i \leq 2m,$$

where $\overline{i+j} \in \overline{Z}_{2m+1}$ for $0 \leq i, j \leq 2m$. It is easy to verify that $(Z_{2m+1} \bigcup \overline{Z}_{2m+1}, \mathcal{A})$ is an HC$(2m+1, 2m+1, 2)$. Repeating every $D_i \lambda$ times, we obtain an HC$(2m+1, 2m+1, 2\lambda)$. □

Lemma 7 *There exists a $DHC(n, n, \lambda)$ for positive integers n and λ, $n > 1$.*

Proof We use the structure in Lemma 1 of [9], define the collection $\mathcal{A}$ of the following n directed Hamilton cycles

$$C_i = \langle 0, \overline{i}, 1, \overline{i+1}, \ldots, n-1, \overline{i+n-1} \rangle, \quad 0 \leq i \leq n-1,$$

where $\overline{i+j} \in \overline{Z}_n$ for $0 \leq i, j \leq n-1$. It is easy to verify that $(Z_n \bigcup \overline{Z}_n, \mathcal{A})$ is a DHC$(n, n, 1)$. Repeating every $C_i \lambda$ times, we obtain a DHC(n, n, λ). □

It is clear that $|\mathcal{A}| = \frac{\lambda n(n-1)}{2n-2} = \frac{\lambda n}{2}$ in an HP$(n, n-1, \lambda)$. Hence,

$$\text{if there exists an HP}(n, n-1, \lambda)\text{, then } \begin{cases} \text{even } n \geq 2 \ \text{ for any } \lambda; \\ \text{odd } n \geq 3 \ \text{ for even } \lambda. \end{cases}$$

Clearly, the necessary conditions for the existence of a DHP$(n, n-1, \lambda)$ are $n > 1$ for any λ. It is easy to see that the existence of an HC(n, n, λ) [resp. DHC(n, n, λ)] is equivalent to the existence of an HP$(n, n-1, \lambda)$ [resp. DHP$(n, n-1, \lambda)$]. In Sects. 3

and 4, we will show that the existence of an LHC(n, n, λ) [resp. LDHC(n, n, λ)] is equivalent to the existence of an LHP$(n, n-1, \lambda)$ [resp. LDHP$(n, n-1, \lambda)$]. So, the following lemma is an immediate consequence of Lemmas 5–7.

Lemma 8 *There exist an* $HP(2m, 2m-1, \lambda)$, *an* $HP(2m+1, 2m, 2\lambda)$ *and a* $DHP(n, n-1, \lambda)$ *for positive integers* m, n *and* $\lambda, n > 1$.

An HP(n, n, λ) consists of $\frac{\lambda n^2}{2n-1}$ blocks. But, $gcd(n^2, 2n-1) = 1$. Hence,

$$\text{if there exists an HP}(n, n, \lambda), \text{ then}(2n-1)|\lambda.$$

Similarly, the necessary condition for the existence of a DHP(n, n, λ) is also $(2n-1)|\lambda$.

Lemma 9 *There exists an HP*$(n, n, \lambda(2n-1))$ *for positive integers n and* λ.

Proof Define the collection $\mathcal{A}$ of the following n^2 Hamilton paths

$$C_{i,j} = [i, \overline{j}, i+1, \overline{j+1}, \ldots, i+n-1, \overline{j+n-1}], \quad 0 \le i, j \le n-1,$$

where $i + k \in Z_n, \overline{j+k} \in \overline{Z}_n$ for $0 \le i, j, k \le n-1$. It is easy to verify that $(Z_n \bigcup \overline{Z}_n, \mathcal{A})$ is an HP$(n, n, 2n-1)$. Repeating every $C_{i,j}\lambda$ times, we obtain an HP$(n, n, \lambda(2n-1))$. □

Lemma 10 *There exists a* $DHP(n, n, \lambda(2n-1))$ *for positive integers n and* λ.

Proof It is easy to see that the existence of an HP$(n, n, \lambda(2n-1))$ implies the existence of an DHP$(n, n, \lambda(2n-1))$. □

In Lemmas 5−10, when $\lambda > 1$, all decompositions are not simple (i.e., containing repeated blocks). In the following sections, we will mention the simple cases.

3 LHC(n, n, λ) and LHP$(n, n-1, \lambda)$

Let $Sym(S)$ be the symmetric group on a given set S. For a subgroup T of $Sym(S)$, the set of representatives of the right cosets for T in $Sym(S)$ is denoted by $Sym_T(S)$. For any $s \in S$ and two permutations $\xi_1, \xi_2 \in Sym(S)$, define $\xi_1\xi_2(s) = \xi_2(\xi_1(s))$.

Let $C = (x_0, \overline{x}_0, x_1, \overline{x}_1, \ldots, x_{n-1}, \overline{x}_{n-1})$ be a Hamilton cycle of $K_{n,n}$, where $x_i \in Z_n, \overline{x}_i \in \overline{Z}_n$ for $0 \le i \le n-1$. For permutations $\xi \in Sym(Z_n)$ and $\eta \in Sym(\overline{Z}_n)$, denote $\xi C = (\xi(x_0), \overline{x}_0, \xi(x_1), \overline{x}_1, \ldots, \xi(x_{n-1}), \overline{x}_{n-1})$ and $\eta C = (x_0, \eta(\overline{x}_0), x_1, \eta(\overline{x}_1), \ldots, x_{n-1}, \eta(\overline{x}_{n-1}))$, respectively. Take

$$\sigma = (1, n-1)(2, n-2)\cdots\left(\left\lfloor\frac{n-1}{2}\right\rfloor, n-\left\lfloor\frac{n-1}{2}\right\rfloor\right) \in Sym(Z_n),$$

which generates a subgroup $G = \langle\sigma\rangle$ of $Sym(Z'_n)$ with order two, where $Z'_n = Z_n \backslash \{0\}$. Then, $|Sym_G(Z'_n)| = \frac{(n-1)!}{2}$. Let $Sym_G(Z'_n) = \{\sigma_1, \sigma_2, \ldots, \sigma_{(n-1)!/2}\}$. Below, by the shift-equivalence of Hamilton cycles, each Hamilton cycle in $K_{n,n}$ will be denoted by a fixed form as follows.

Under the action of $Sym(\overline{Z}_n)$, all Hamilton cycles in $K_{n,n}$ can be separated into the following $\frac{(n-1)!}{2}$ *orbits*, where $\sigma_i \in Sym_G(Z'_n)$.

$$\mathcal{O}_i = \{(0, \eta(\overline{0}), \sigma_i(1), \eta(\overline{1}), \sigma_i(2), \eta(\overline{2}), \ldots, \sigma_i(n-1), \eta(\overline{n-1})) : \eta \in Sym(\overline{Z}_n)\}.$$

Obviously, $|\mathcal{O}_i| = n!$ for $1 \leq i \leq \frac{(n-1)!}{2}$. So, $|Sym_G(Z'_n)| \cdot |\mathcal{O}_i| = \frac{(n-1)!n!}{2}$ is just the total number of distinct Hamilton cycles in $K_{n,n}$.

Let $\mathcal{A}$ be a collection of Hamilton cycles (resp. directed Hamilton cycles) in $K_{n,n}$ (resp. $K^*_{n,n}$). A subgroup H of $Sym(\overline{Z}_n)$ is called a *complete automorphism group* over $\overline{Z}_n$ of $\mathcal{A}$ if the following conditions are satisfied:

1. $\eta C \in \mathcal{A}$ for any $\eta \in H$ and $C \in \mathcal{A}$;
2. $\forall C, C' \in \mathcal{B}$, if there exists $\eta \in Sym(\overline{Z}_n)$ such that $\eta C = C'$, then $\eta \in H$.

When $\mathcal{A}$ is a collection of Hamilton paths (resp. directed Hamilton paths) in $K_{n,n}$ (resp. $K^*_{n,n}$), we can similarly define the complete automorphism group for $\mathcal{A}$.

In the following discussions, $\mathcal{A}$ consists of all Hamilton cycles in some HC(n, n, λ). We now give a very useful lemma in this paper. The idea of the construction, introduced in [2], is to make use of symmetric groups.

Lemma 11 (1) *If* $(Z_n \bigcup \overline{Z}_n, \mathcal{A})$ *is an* $HC(n, n, \lambda)$ *then so is* $(Z_n \bigcup \overline{Z}_n, \eta\mathcal{A})$ *(resp.* $(Z_n \bigcup \overline{Z}_n, \xi\mathcal{A})$*), where* $\eta \in Sym(\overline{Z}_n), \eta\mathcal{A} = \{\eta C : C \in \mathcal{A}\}$*(resp.* $\xi \in Sym(Z_n), \xi\mathcal{A} = \{\xi C : C \in \mathcal{A}\}$);

(2) *If the system* $\mathcal{A}$ *is simple and has a complete automorphism group* H *over* $\overline{Z}_n$*, then all Hamilton cycles in* $\{\eta\mathcal{A} : \eta \in Sym_H(\overline{Z}_n)\}$ *are pairwise distinct.*

Proof (1) The permutation η on $\overline{Z}_n$ induces a permutation on the set $(\overline{Z}_n \times \overline{Z}_n)\backslash\{(y, y) : y \in \overline{Z}_n\}$. Hence, the system $(Z_n \bigcup \overline{Z}_n, \eta\mathcal{A})$ is also an HC(n, n, λ) by the definition. For $\xi \in Sym(Z_n)$, the proof is similar.

(2) Suppose there exist $C, C' \in \mathcal{A}$ and $\eta_1 \neq \eta_2 \in Sym_H(\overline{Z}_n)$ such that $\eta_1 C = \eta_2 C'$. Then $(\eta_1\eta_2^{-1})C = C'$ and $\eta_1\eta_2^{-1} \in H$ by the definition of complete automorphism group H over $\overline{Z}_n$. This implies $H\eta_1 = H\eta_2$, i.e., η_1 and η_2 belong to the same coset, which is a contradiction. □

An HC(n, n, λ) contains $\frac{\lambda n}{2}$ Hamilton cycles. The total number of distinct Hamilton cycles in $K_{n,n}$ is $\frac{(n-1)!n!}{2}$. Hence, an LHC(n, n, λ) contains $((n-1)!)^2/\lambda$ pairwise disjoint HC(n, n, λ)s. Clearly, there exists an LHC(n, n, λ) only if

$$\lambda|((n-1)!)^2 \text{and} \begin{cases} \text{even}\, n \geq 2 & \text{for any } \lambda; \\ \text{odd}\, n \geq 3 & \text{for even } \lambda. \end{cases}$$

The conditions are also necessary for the existence of LHP$(n, n-1, \lambda)$. Thus, the existence spectrum for LHC(n, n, λ) [resp. LHP$(n, n-1, \lambda)$] only depends on two cases: even $n \geq 2$ for $\lambda = 1$ and odd $n \geq 3$ for $\lambda = 2$.

Lemma 12 *There exists an* $LHC(2m+1, 2m+1, 2)$ *for any positive integer* m.

Proof Take the HC$(2m+1, 2m+1, 2) = (Z_{2m+1} \bigcup \overline{Z}_{2m+1}, \mathcal{A})$ constructed in Lemma 6 as the *base small set*, where $\mathcal{A} = \{D_0, D_1, \ldots, D_{2m}\}$. Let $\tau = (\overline{0}, \overline{1}, \ldots, \overline{2m}) \in Sym(\overline{Z}_{2m+1})$, which generates a subgroup $H = \langle\tau\rangle$ of $Sym(\overline{Z}_{2m+1})$ with order $2m+1$. Clearly, $D_j = \tau^{j-i} D_i$ for $i, j \in Z_{2m+1}$. Now, we have shown that H is a complete automorphism group of $\mathcal{A}$ over $\overline{Z}_{2m+1}$. Let $Sym_H(\overline{Z}_{2m+1}) = \{\tau_1, \tau_2, \ldots, \tau_{(2m)!}\}$, where τ_1 is identical permutation. Let $Sym_G(Z'_{2m+1}) = \{\sigma_1, \sigma_2, \ldots, \sigma_{(2m)!/2}\}$ (refer the beginning of this section).
Define

$$\Omega_{i,j} = \{\sigma_i \tau_j D_0, \sigma_i \tau_j D_1, \ldots, \sigma_i \tau_j D_{2m}\}, \quad 1 \le i \le \frac{(2m)!}{2}, \quad 1 \le j \le (2m)!.$$

Each $\Omega_{i,j}$ is an HC$(2m+1, 2m+1, 2)$ by Lemma 11 (1). Similarly, we can prove that H is a complete automorphism group of $\Omega_{i,1}$, over $\overline{Z}_{2m+1}$, for $1 \le i \le \frac{(2m)!}{2}$. We have the facts:

* all Hamilton cycles in each $\Omega_{i,j}$ fall into orbit $\mathcal{O}_i$, where $1 \le i \le \frac{(2m)!}{2}$, $1 \le j \le (2m)!$;
* for given σ_i, all Hamilton cycles in $\{\Omega_{i,j} : 1 \le j \le (2m)!\}$ are distinct by Lemma 11 (2).

As well, $|Sym_G(Z'_{2m+1})| \cdot |Sym_H(\overline{Z}_{2m+1})| = |\bigcup_{i,j} \Omega_{i,j}| = \frac{((2m)!)^2}{2}$, which is just the desired number of disjoint HC$(2m+1, 2m+1, 2)$s in an LHC$(2m+1, 2m+1, 2)$. Therefore, by these facts, an LHC$(2m+1, 2m+1, 2)$ is constructed. □

Theorem 1 *There exists an* $LHC(n, n, \lambda)$ *if and only if* $\lambda | ((n-1)!)^2$ *and*

$$\begin{cases} even\ n \ge 2 & for\ any\ \lambda \\ odd\ n \ge 3 & for\ even\ \lambda \end{cases}.$$

Proof The necessity has been shown before Lemma 12, the sufficiency is proved below.

For even $n \ge 2$, there exists an LHC$(n, n, 1) = \{(Z_n \bigcup \overline{Z}_n, \mathcal{A}_i) : 1 \le i \le ((n-1)!)^2\}$ by Lemma 4. Define

$$\mathcal{B}_k = \bigcup_{i=k\lambda+1}^{(k+1)\lambda} \mathcal{A}_i, 0 \le k \le ((n-1)!)^2/\lambda - 1,$$

then $\{(Z_n \bigcup \overline{Z}_n, \mathcal{B}_k) : 0 \le k \le ((n-1)!)^2/\lambda - 1\}$ is an LHC(n, n, λ), where $\lambda | ((n-1)!)^2$.

For odd $n \ge 3$ and even $\lambda | ((n-1)!)^2$, there exists an LHC$(n, n, 2) = \{(Z_n \bigcup \overline{Z}_n, \mathcal{A}_i) : 1 \le i \le \frac{((n-1)!)^2}{2}\}$ by Lemma 12. Define

$$\mathcal{B}_k = \bigcup_{i=\frac{k\lambda}{2}+1}^{(k+1)\frac{\lambda}{2}} \mathcal{A}_i, \quad 0 \le k \le ((n-1)!)^2/\lambda - 1,$$

then $\{(Z_n \bigcup \overline{Z}_n, \mathcal{B}_k) : 0 \le k \le ((n-1)!)^2/\lambda - 1\}$ is an LHC(n, n, λ). This completes the proof. □

Theorem 2 *There exists an* $LHP(n, n-1, \lambda)$ *if and only if* $\lambda | ((n-1)!)^2$ *and*

$$\begin{cases} even\ n \ge 2 & for\ any\ \lambda \\ odd\ n \ge 3 & for\ even\ \lambda \end{cases}.$$

Proof We start proving the sufficiency first. By Theorem 1, there exists an LHC$(n, n, \lambda) = \{(Z_n \bigcup \overline{Z}_n, \mathcal{A}_i) : 1 \le i \le ((n-1)!)^2/\lambda\}$. Delete the element $\overline{0}$ from the set $\overline{Z}_n$, let $\overline{Z}_n' = \overline{Z}_n \backslash \{\overline{0}\}$. Then, each Hamilton cycle in each $\mathcal{A}_i$ will become a Hamilton path of $K_{n,n-1}$ with two partite sets $Z_n, \overline{Z}_n'$, and each Hamilton cycle decomposition $(Z_n \bigcup \overline{Z}_n, \mathcal{A}_i)$ of $\lambda K_{n,n}$ will become a Hamilton path decomposition $(Z_n \bigcup \overline{Z}_n', \mathcal{A}_i')$ of $\lambda K_{n,n-1}$. It is easy to verify that $\{(Z_n \bigcup \overline{Z}_n', \mathcal{A}_i') : 1 \le i \le ((n-1)!)^2/\lambda\}$ indeed forms an LHP$(n, n-1, \lambda)$. As for the necessity, see before Lemma 12. The conclusion holds. □

Corollary 1 (1) *There exist simple HC*$(2m, 2m, \lambda)$ *and simple HP* $(2m, 2m-1, \lambda)$ *if and only if* $1 \le \lambda \le ((2m-1)!)^2$;

(2) *There exist simple HC*$(2m+1, 2m+1, 2\lambda)$ *and simple HP* $(2m+1, 2m, 2\lambda)$ *if and only if* $1 \le \lambda \le ((2m)!)^2/2$.

4 LDHC(n, n, λ) and LDHP$(n, n-1, \lambda)$

For $\xi \in Sym(Z_n), \eta \in Sym(\overline{Z}_n)$ and a directed Hamilton cycle $C = \langle x_0, \overline{x}_0, \ldots, x_{n-1}, \overline{x}_{n-1}\rangle$ of $K_{n,n}^*$, where $x_i \in Z_n, \overline{x}_i \in \overline{Z}_n$ for $0 \le i \le n-1$, the definitions of ξC and ηC are similar to those introduced in Sect. 3. Let $Z_n' = Z_n \backslash \{0\}$. Then, by the shift-equivalence of directed Hamilton cycles, each directed Hamilton cycle in $K_{n,n}^*$ will be denoted by a fixed form as follows.

Under the action of $Sym(\overline{Z}_n)$, all directed Hamilton cycles in $K_{n,n}^*$ can be separated into the following orbits:

$$\mathcal{O}_i' = \{\langle 0, \eta(\overline{0}), \sigma_i(1), \eta(\overline{1}), \ldots, \sigma_i(n-1), \eta(\overline{n-1})\rangle : \eta \in Sym(\overline{Z}_n)\}, \sigma_i \in Sym(Z_n').$$

It is easy to see that $|\mathcal{O}_i'| = n!$ for any $\sigma_i \in Sym(Z_n')$. And, $|Sym(Z_n')| \cdot |\mathcal{O}_i'| = (n-1)!n!$ is just the total number of distinct directed Hamilton cycles in $K_{n,n}^*$.

Similarly to Lemma 11, we can prove the following one which is on oriented cycles.

Lemma 13 (1) *If* $(Z_n \bigcup \overline{Z}_n, \mathcal{A})$ *is a* $DHC(n, n, \lambda)$ *then so is* $(Z_n \bigcup \overline{Z}_n, \eta\mathcal{A})$*[resp.* $(Z_n \bigcup \overline{Z}_n, \xi\mathcal{A})$*], where* $\eta \in Sym(\overline{Z}_n)$*[resp.* $\xi \in Sym(Z_n)$*];*

(2) *If the system* $\mathcal{A}$ *is simple and it has a complete automorphism group* H *over* $\overline{Z}_n$*, then all directed Hamilton cycles in* $\{\eta\mathcal{A} : \eta \in Sym_H(\overline{Z}_n)\}$ *are pairwise distinct.*

A DHC(n, n, λ) contains λn directed Hamilton cycles. The total number of distinct directed Hamilton cycles in $K_{n,n}^*$ is $(n-1)!n!$. Hence, an LDHC(n, n, λ)

contains $((n-1)!)^2/\lambda$ pairwise disjoint DHC(n, n, λ)s. Clearly, there exists an LDHC(n, n, λ) only if $\lambda|((n-1)!)^2$. The conditions are also necessary for the existence of LDHP$(n, n-1, \lambda)$. Therefore, the existence spectrum for LDHC(n, n, λ) and LDHP$(n, n-1, \lambda)$ only depends on one case: $\lambda = 1$ and $n \geq 1$.

Lemma 14 *There exists an $LDHC(n, n, 1)$ for any positive integer n.*

Proof Take the DHC$(n, n, 1) = (Z_n \bigcup \overline{Z}_n, \mathcal{A})$ constructed in Lemma 7 as the base small set, where $\mathcal{A} = \{C_0, C_1, \ldots, C_{n-1}\}$. Let $\tau = (\overline{0}, \overline{1}, \ldots, \overline{n-1}) \in Sym(\overline{Z}_n)$, which generates a subgroup $H = \langle\tau\rangle$ of $Sym(\overline{Z}_n)$ with order n. Clearly, $C_j = \tau^{j-i}C_i$ for $i, j \in Z_n$. Now, we have shown that H is a complete automorphism group of $\mathcal{A}$ over $\overline{Z}_n$. Let $Sym_H(\overline{Z}_n) = \{\tau_1, \tau_2, \ldots, \tau_{(n-1)!}\}$, where τ_1 is identical permutation. Let $Sym(Z'_n) = \{\sigma_1, \sigma_2, \ldots, \sigma_{(n-1)!}\}$. Define

$$\Omega_{i,j} = \{\sigma_i\tau_j C_0, \sigma_i\tau_j C_1, \ldots, \sigma_i\tau_j C_{n-1}\}, \quad 1 \leq i, j \leq (n-1)!.$$

Each $\Omega_{i,j}$ is a DHC$(n, n, 1)$ by Lemma 13 (1). Similarly, we can prove that H is a complete automorphism group of $\Omega_{i,1}$ over $\overline{Z}_n$ for $1 \leq i \leq (n-1)!$. We have the following facts:

- $*$ all directed Hamilton cycles in each $\Omega_{i,j}$ fall into orbit $\mathcal{O}'_i$, where $1 \leq i, j \leq (n-1)!$;
- $*$ all directed Hamilton cycles in $\{\Omega_{i,j} : 1 \leq j \leq (n-1)!\}$ are distinct by Lemma 13 (2).

As well, $|Sym(Z'_n)| \cdot |Sym_H(\overline{Z}_n)| = |\bigcup_{i,j} \Omega_{i,j}| = ((n-1)!)^2$, which is just the number of disjoint DHC$(n, n, 1)$s in an LDHC$(n, n, 1)$. Therefore, an LDHC$(n, n, 1)$ is constructed. □

Similar to Theorems 1, 2 and Corollary 1, we can obtain the following conclusion. The proof is similar.

Theorem 3 *There exists an LDHC(n, n, λ) if and only if $\lambda|((n-1)!)^2$.*

Theorem 4 *There exists an LDHP$(n, n-1, \lambda)$ if and only if $\lambda|((n-1)!)^2$.*

Corollary 2 *There exist simple DHC(n, n, λ) and simple DHP$(n, n-1, \lambda)$ if and only if $1 \leq \lambda \leq ((n-1)!)^2$.*

5 LHP(n, n, λ) and LDHP(n, n, λ)

For $\xi \in Sym(Z_n)$, $\eta \in Sym(\overline{Z}_n)$ and a Hamilton path $C = [x_0, \overline{x}_0, \ldots, x_{n-1}, \overline{x}_{n-1}]$ of $K_{n,n}$, where $x_i \in Z_n, \overline{x}_i \in \overline{Z}_n$ for $0 \leq i \leq n-1$, the definitions of ξC and ηC are similar to those introduced in Sect. 3. Take $\sigma = (0, 1, \ldots, n-1) \in Sym(Z_n)$, which generates a subgroup $G = \langle\sigma\rangle$ of $Sym(Z_n)$ with order n. Then, $|Sym_G(Z_n)| = (n-1)!$ and $Sym(Z_n)$ can be partitioned into $(n-1)!$ right cosets: $Sym(Z_n) = \bigcup_{i=1}^{(n-1)!} G_i$, where $G_i = \{\sigma_{i,0}, \sigma_{i,1}, \ldots, \sigma_{i,n-1}\}$, $1 \leq i \leq (n-1)!$. We can modify the sequence $\sigma_{i,0}, \sigma_{i,1}, \ldots, \sigma_{i,n-1}$ such that $\sigma_{i,j+1} = \sigma\sigma_{i,j}$ for $j \in Z_n$. Furthermore,

$\sigma_{i,j} = \sigma^{j-k}\sigma_{i,k}$ for $j, k \in Z_n$. Let $Sym_G(Z_n) = \{\sigma_{1,0}, \sigma_{2,0}, \ldots, \sigma_{(n-1)!,0}\}$, where $\sigma_{1,0}$ is identical permutation. Below, by the shift-equivalence of Hamilton paths, each Hamilton path in $K_{n,n}$ will be denoted by a fixed form as follows.

Under the action of $Sym(\overline{Z}_n)$, all Hamilton paths in $K_{n,n}$ can be separated into the following *orbit families*:

$$\overline{\mathcal{O}}_i = \{\mathcal{O}_{i,j} : 0 \le j \le n-1\}, \quad 1 \le i \le (n-1)!, \text{ where}$$

$\mathcal{O}_{i,j} = \{[\sigma_{i,j}(0), \eta(\overline{0}), \sigma_{i,j}(1), \eta(\overline{1}), \ldots, \sigma_{i,j}(n-1), \eta(\overline{n-1})] : \eta \in Sym(\overline{Z}_n)\}$. It is easy to see that $|\overline{\mathcal{O}}_i| = n$ and $|\mathcal{O}_{i,j}| = n!$ for $1 \le i \le (n-1)!, 0 \le j \le n-1$. The number of right cosets is $(n-1)!$. Then, $(n-1)! \cdot |\overline{\mathcal{O}}_i| \cdot |\mathcal{O}_{i,j}| = (n!)^2$ is just the total number of distinct Hamilton paths in $K_{n,n}$.

The next lemma is an analog of Lemma 11 too. Its proof is similar.

Lemma 15 (1) *If* $(Z_n \bigcup \overline{Z}_n, \mathcal{A})$ *is an* $HP(n, n, \lambda)$ *then so is* $(Z_n \bigcup \overline{Z}_n, \eta\mathcal{A})$ [*resp.* $(Z_n \bigcup \overline{Z}_n, \xi\mathcal{A})$], *where* $\eta \in Sym(\overline{Z}_n)$ [*resp.* $\xi \in Sym(Z_n)$];

(2) *If the system* $\mathcal{A}$ *is simple and it has a complete automorphism group* H *over* $\overline{Z}_n$, *then all Hamilton paths in* $\{\eta\mathcal{A} : \eta \in Sym_H(\overline{Z}_n)\}$ *are pairwise distinct.*

An HP(n, n, λ) contains $\frac{\lambda n^2}{2n-1}$ Hamilton paths. The total number of distinct Hamilton paths in $K_{n,n}$ is $(n!)^2$. Hence, an LHP(n, n, λ) contains $(2n-1)((n-1)!)^2/\lambda$ pairwise disjoint HP(n, n, λ)s. Clearly, there exists an LHP(n, n, λ) only if $\lambda|(2n-1)((n-1)!)^2$ and $(2n-1)|\lambda$. The conditions are also necessary for the existence of LDHP(n, n, λ). Therefore, the existence spectrum for LHP(n, n, λ) and LDHP(n, n, λ) only depends on one case: $\lambda = 2n-1$ and $n \ge 1$.

Lemma 16 *There exists an* $LHP(n, n, 2n-1)$ *for any positive integer* n.

Proof Take the HP$(n, n, 2n-1) = (Z_n \bigcup \overline{Z}_n, \mathcal{A})$ constructed in Lemma 9 as the base small set, where $\mathcal{A} = \{C_{i,j} : 0 \le i, j \le n-1\}$. Let $\tau = (\overline{0}, \overline{1}, \ldots, \overline{n-1}) \in Sym(\overline{Z}_n)$, which generates a subgroup $H = \langle\tau\rangle$ of $Sym(\overline{Z}_n)$ with order n. Clearly, $C_{i,j} = \tau^{j-k}C_{i,k}$ for $i, j, k \in Z_n$. Now, we have shown that H is a complete automorphism group over $\overline{Z}_n$ of $\mathcal{A}$. Let $Sym_H(\overline{Z}_n) = \{\tau_1, \tau_2, \ldots, \tau_{(n-1)!}\}$, where τ_1 is identical permutation. As well, let $Sym_G(Z_n) = \{\sigma_{1,0}, \sigma_{2,0}, \ldots, \sigma_{(n-1)!,0}\}$ (refer the beginning of this section), where $\sigma_{1,0}$ is identical permutation too. Define

$$\Omega_{i,j} = \{\sigma_{i,0}\tau_j C_{k,l} : 0 \le k, l \le n-1\}, \quad 1 \le i, j \le (n-1)!.$$

Each $\Omega_{i,j}$ is an HP$(n, n, 2n-1)$ by Lemma 15 (1). Similarly, we can prove that H is a complete automorphism group of $\Omega_{i,1}$ over $\overline{Z}_n$ for $1 \le i \le (n-1)!$. We have the following facts.

$*$ For given $\sigma_{i,0}$, $1 \le i \le (n-1)!$, all Hamilton paths in $\Omega_{i,j}$ fall into orbit family $\overline{\mathcal{O}}_i$, where $1 \le j \le (n-1)!$. In fact, in $\Omega_{i,j}$, for given $l \in Z_n$,

$$\begin{aligned}
\sigma_{i,0}\tau_j C_{k+1,l} &= \sigma_{i,0}\tau_j(\sigma C_{k,l}) = \sigma\sigma_{i,0}\tau_j C_{k,l} = \sigma_{i,1}\tau_j C_{k,l} \text{for} k \in Z_n, \\
\sigma_{i,0}\tau_j C_{k_2,l} &= \sigma_{i,0}\tau_j(\sigma^{k_2-k_1}C_{k_1,l}) = \sigma^{k_2-k_1}\sigma_{i,0}\tau_j C_{k_1,l} \\
&= \sigma_{i,k_2-k_1}\tau_j C_{k_1,l} \text{for } k_1, k_2 \in Z_n.
\end{aligned}$$

That is to say, the n Hamilton paths $\sigma_{i,0}\tau_j C_{k,0}, \sigma_{i,0}\tau_j C_{k,1}, \ldots, \sigma_{i,0}\tau_j C_{k,n-1}$ belong to orbit $\mathcal{O}_{i,k}$, which is a member of orbit family $\overline{\mathcal{O}}_i$.

$*$ For given $\sigma_{i,0}$, $1 \le i \le (n-1)!$, all Hamilton paths in $\{\Omega_{i,j} : 1 \le j \le (n-1)!\}$ are pairwise distinct by Lemma 15 (2).

As well, $|Sym_G(Z_n)| \cdot |Sym_H(\overline{Z}_n)| = |\bigcup_{i,j} \Omega_{i,j}| = ((n-1)!)^2$, which is just the desired number of disjoint HP$(n, n, 2n-1)$s in an LHP$(n, n, 2n-1)$. Therefore, by the facts, an LHP$(n, n, 2n-1)$ is constructed. □

Theorem 5 *There exists an LHP(n, n, λ) if and only if $\lambda|(2n-1)((n-1)!)^2$ and $(2n-1)|\lambda$.*

Proof Combining Lemma 16 and the necessity for the existence of LHP(n, n, λ), we obtain the conclusion. The proof is similar to that of Theorem 1. □

Theorem 6 *There exists an LDHP(n, n, λ) if and only if $\lambda|(2n-1)((n-1)!)^2$ and $(2n-1)|\lambda$.*

Proof If n, λ satisfy the necessary conditions, then there exists an LHP$(n, n, \lambda) = \{(Z_n \bigcup \overline{Z}_n, \mathcal{A}_i) : 1 \le i \le (2n-1)((n-1)!)^2/\lambda\}$ by Theorem 5. For each Hamilton path

$$C_j = [x_0, \overline{x}_0, x_1, \overline{x}_1, \ldots, x_{n-1}, \overline{x}_{n-1}] \in \mathcal{A}_i,$$

define two directed Hamilton paths in $K^*_{n,n}$:

$$C_{j,1} = \prec x_0, \overline{x}_0, x_1, \overline{x}_1, \ldots, x_{n-1}, \overline{x}_{n-1} \succ,$$
$$C_{j,2} = \prec \overline{x}_{n-1}, x_{n-1}, \overline{x}_{n-2}, x_{n-2}, \ldots, \overline{x}_0, x_0 \succ .$$

Let $\mathcal{A}'_i = \{C_{j,1}, C_{j,2} : C_j \in \mathcal{A}\}$, then each $\{(Z_n \bigcup \overline{Z}_n, \mathcal{A}'_i)$ forms a DHP(n, n, λ). Furthermore, it is easy to verify that $\{(Z_n \bigcup \overline{Z}_n, \mathcal{A}'_i) : 1 \le i \le (2n-1)((n-1)!)^2/\lambda\}$ is an LDHP(n, n, λ). □

Corollary 3 *There exist simple HP$(n, n, \lambda(2n-1))$ and simple DHP$(n, n, \lambda(2n-1))$ if and only if $\le \lambda \le ((n-1)!)^2$.*

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